Robotics

Estimation and Learning
with Dan Lee

Basic Intro to Probability

Penn Engineering

Why Learn About Probability?

- The real world has huge aspects of randomness and uncertainty.
- Still, we hope to make useful predictions and inferences.
- Randomness often follows reliable laws.
- The language of these laws is the language of probability (and statistics).

Contents

1. Definition of Probability

- Consider an (potentially abstract) experiment
  - A **sample space** Ω is the set of *all possible* outcomes of that experiment
  - An **elementary event** ω is a single outcome of the set.
- Example (Rolling two dice):
  - Sample space $\Omega = \{(1,1), (1,2), (1,3), \dots, (6,5), (6,6)\}$
  - Each member $\omega \in \Omega$ is an elementary event
  - Example of non-elementary event : Rolling doubles

    $B = \{(1,1), (2,2), (3,3), \dots, (6,6)\}$

1. Definition of Probability

- Consider a finite* set Ω
- A **probability space** (Ω, P) is a sample space Ω, together with a function P, satisfying the following:
  i. $0 \leq P(\omega) \leq 1$ for all $\omega \in \Omega$
  ii. $\sum_{\omega \in \Omega} P(\omega) = 1$
  iii. For any event $A \subseteq \Omega$, $P(A) = \sum_{\omega \in \mathrm{A}} P(\omega)$
- The function P is called **probability measure**.

*To deal with countably infinite or uncountable space, we need third element called sigma algebra, but here we are simplifying.

1. Definition of Probability

- Basic Consequences

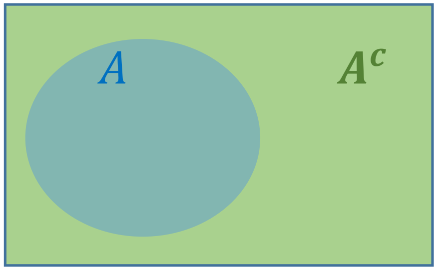
$$P(\emptyset) = 0$$

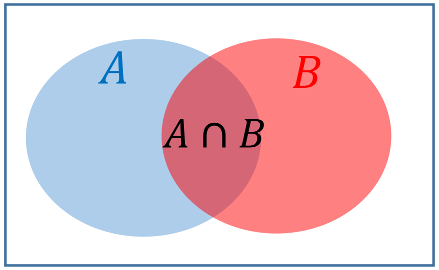
$$P(\Omega) = 1$$

$$P(A^C) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$





2. Independence

- Given a probability space, two events A and B are **independent** if and only if:

$$P(A \cap B) = P(A)P(B)$$

- Two events are dependent if they are not independent.

2. Independence

- Example (two coin flip): $\Omega = \{HH, HT, TH, TT\}$
  - Assume $\forall \omega \in \Omega, P(\omega) = \frac{1}{|\Omega|} = \frac{1}{4}$.
  - Define event A: "First flip is H." $A = \{HH, HT\}$
  - Define event B: "Second flip is H." $B = \{HH, TH\}$
  - Are A and B Independent?

    i) $P(A \cap B) = P(\{HH\}) = 1/4$

    ii) $P(A)P(B) = 1/2 * 1/2 = 1/4$ → Yes

2. Independence

- Example (two coin flip): $\Omega = \{HH, HT, TH, TT\}$
  - Assume $\forall \omega \in \Omega, P(\omega) = \frac{1}{|\Omega|} = \frac{1}{4}$.
  - Define event A: "First flip is H." $A = \{HH, HT\}$
  - Define event B: "Contains a T." $B = \{HT, TH, TT\}$
  - Are A and B Independent?

    i) $P(A \cap B) = P(\{HT\}) = 1/4$

    ii) $P(A)P(B) = 1/2 * 3/4 = 3/8$ → No

3. Conditional Probability

- Given some probability space (Ω, P), for any two events A and B, if $P(B) \neq 0$, then we define the **conditional probability** $P(A|B)$ that A occurs given that B occurs as,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

- From this follows **Chain Rule**:

$$P(A \cap B) = P(A|B)\, P(B)$$
$$\begin{aligned} P(A \cap B \cap C) &= P(A|B \cap C)\, P(B \cap C) \\ &= P(A|B \cap C)\, P(B|C) P(C) \end{aligned}$$

and so on..

3. Conditional Probability

- Example (two coin flip): What is the probability that both are head, GIVEN at least one is head?
- Probability Problem:

$\Omega = \{HH, HT, TH, TT\}$
$B = \{HH, HT, TH\}$ → "At least one is head."
$A = \{HH\}$ → "Both are heads."
$P(A \cap B) = P(\{HH\}) = 1/4$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{3/4} = \frac{1}{3}$$

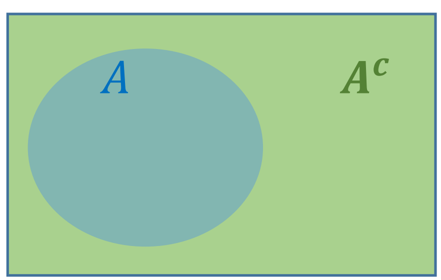
3. Conditional Probability

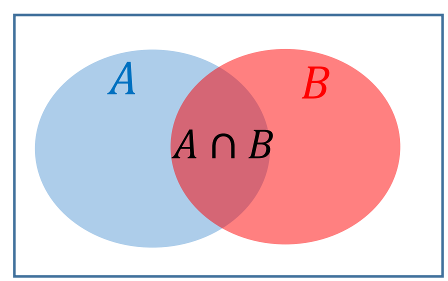
- Consequences

$$P(\emptyset|B) = 0$$

$$P(\mathrm{B}|\mathrm{B}) = 1$$

$$P(A|B) = 1 - P(A^C|B)$$





4. Bayes Rule

- From chain rule,
  $P(A \cap B) = P(A|B)\, P(B)$
  $P(B \cap A) = P(B|A)\, P(A)$
- Intersections are commutative, $A \cap B = B \cap A$.

$$\begin{array}{cccc} P(A \cap B) & = & P(A|B) & P(B) \\ \| & & \| & \\ P(B \cap A) & = & P(B|A) & P(A) \end{array}$$

- As a result, we have **Bayes Rule**:

$$\boxed{P(A|B) = \frac{P(B|A)P(A)}{P(B)}}$$

4. Bayes Rule

- Each term is often called:

$$\overbrace{P(A|B)}^{\text{Posterior}} = \frac{\overbrace{P(B|A)}^{\text{Likelihood}}\overbrace{P(A)}^{\text{Prior}}}{\underbrace{P(B)}_{\text{Evidence}}}$$

5. Random Variables

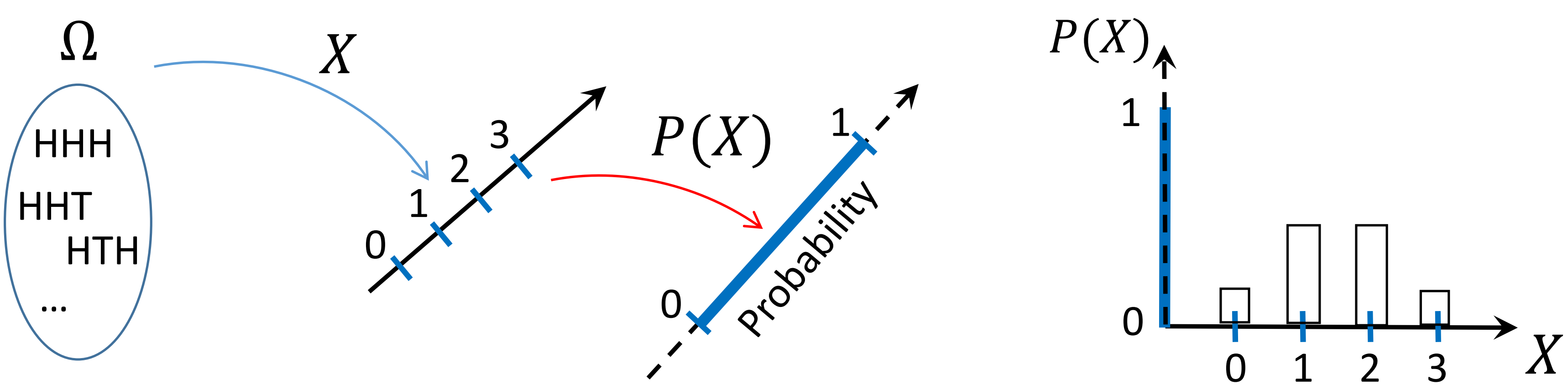
- Given some probability space (Ω, P),
  a **random variable** $X: \Omega \rightarrow R$ is a *function* that maps the sample space to the reals.
- When we say $P(X = a)$, we actually mean the probability of the inverse image $X^{-1}(a)$. That is,
  $$P(X = a) = P\left(X^{-1}(a)\right) = P(\{\omega \in \Omega | X(\omega) = a\})$$
- Example (single coin flip): $\Omega = \{Head, Tail\}$
  $X(Head) = 1, X(Tail) = 0$
  $P(X = 1) = P\left(X^{-1}(1)\right) = P(Head)$

5. Random Variables

- Example: 3 coin flips
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - Let us define $X(\omega)$ to be the number of Heads in a given flip. Then,

    $X(HHH) = 3, X(HHT) = X(HTH) = 2, \ldots, X(TTT) = 0$

    $P(X = 3) = 1/8, P(X = 2) = P(X = 1) = 3/8, P(X = 0) = 1/8$

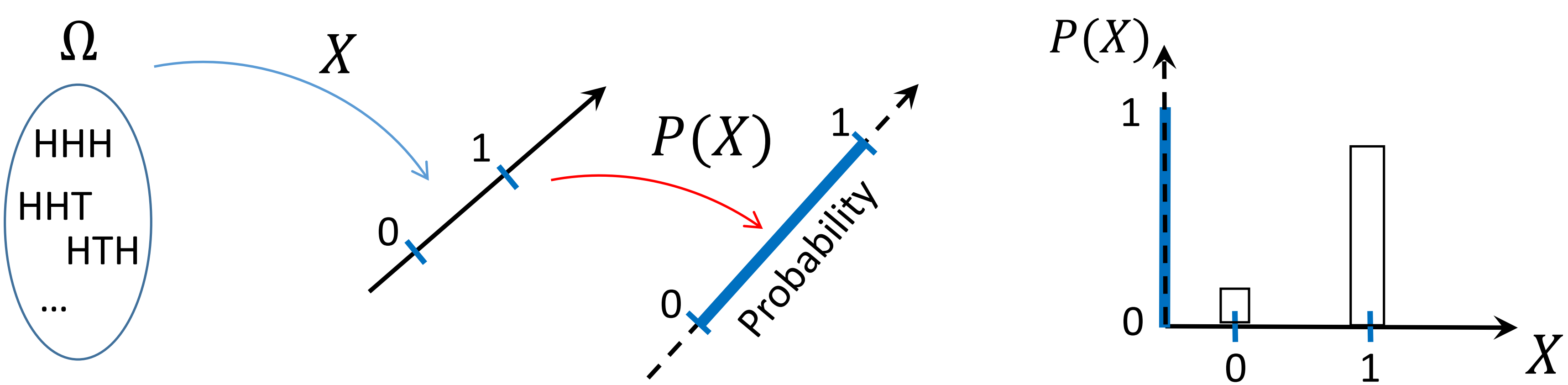


5. Random Variables

- Example: 3 coin flips
  - $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  - This time, let us define $X(\omega)$ to be 1 if H appears in a given flip, otherwise is 0. Then,

    $X(HHH) = X(HHT) = \cdots = 1\,, X(TTT) = 0$

    $P(X = 1) = 7/8, P(X = 0) = 1/8$

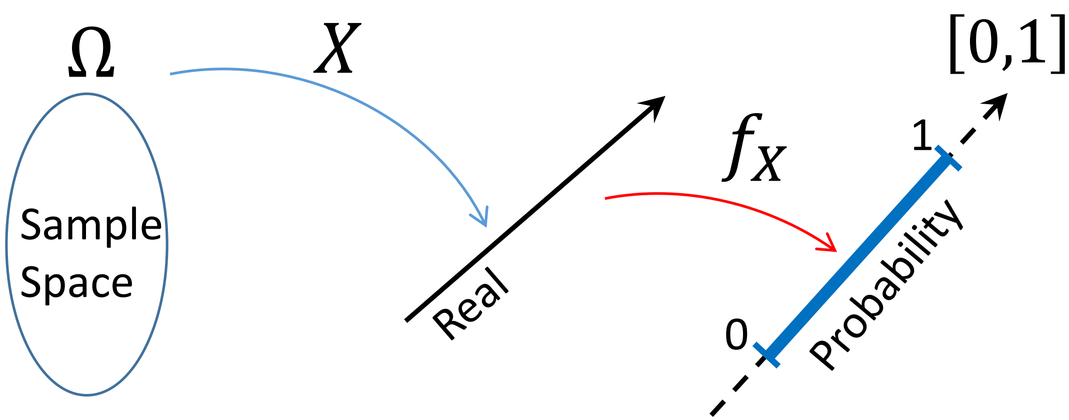


6. Density/Distribution Functions

- **Probability mass function** (pmf) of **discrete** RVs
- **Probability density function** (pdf) of **continuous** RVs

$f: R \rightarrow [0,1]$

$\forall a \in R, f_X(a) = P(X = a)$



6. Density/Distribution Functions

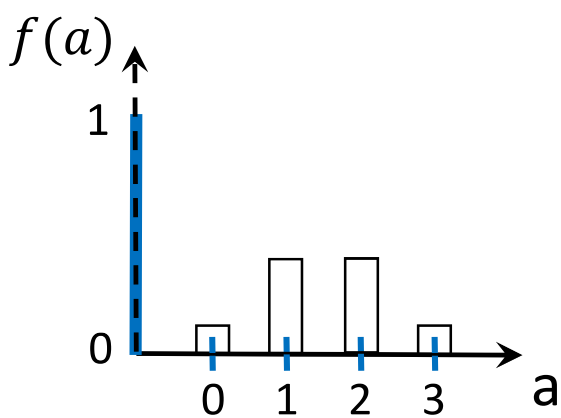
- **Cumulative distribution functions** (cdf)

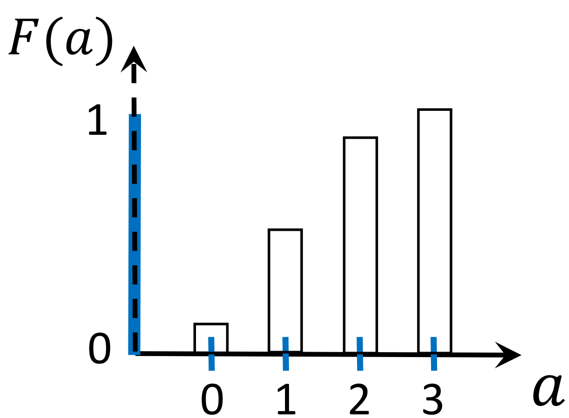
  $F: R \to [0,1]$

  $\forall a \in R, F_X(a) = P(X \leq a)$

- A cdf is a monotonic nondecreasing function, i.e.,

  $\forall x \leq y, F(x) \leq F(y)$





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