MA1521 CALCULUS FOR COMPUTING

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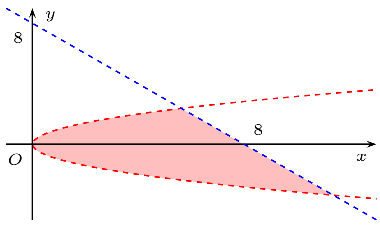
Functions of More than One Variable

- Some objects are determined by more than one **independent variables**:
  - Area of a rectangle: $A = xy$,
    - x is the length, y is the width.
  - Volume of a circular cone: $V = \frac{1}{3}\pi r h^2$,
    - r is the radius and h is the height.
  - Kinetic energy: $E_k = \frac{1}{2}mv^2$,
    - m is the mass, v is the velocity.
  - Potential energy: $E_p = mgh$,
    - m is the mass, h is the height.

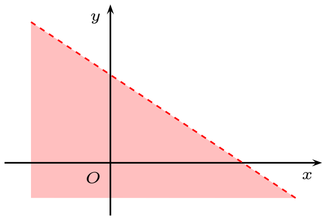
Functions of Two Variables

- Let $z = f(x, y)$ be a function with two variables x, y.
  - Note that x and y are independent.
  - Unless otherwise specified, the **domain** D is the **largest** subset of $\mathbb{R}^2$ on which f is well-defined.
- **Example.** $f(x, y) = \dfrac{x + 2y}{\sqrt{x - y^2}} + \ln(8 - x - y)$.
  - $\dfrac{1}{\sqrt{x - y^2}}$: $\quad x - y^2 > 0 \Leftrightarrow x > y^2$;
  - $\ln(8 - x - y)$: $\quad 8 - x - y > 0 \Leftrightarrow x + y < 8$.

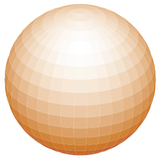


Examples

- Find the domain of $f(x, y) = \ln(10 - 2x - 3y)$.
  - $\ln(10 - 2x - 3y)$: $10 - 2x - 3y > 0 \Leftrightarrow 2x + 3y < 10$.



- Find the domain of $f(x, y, z) = \sqrt{9 - (x^2 + y^2 + z^2)}$.
  - $9 - (x^2 + y^2 + z^2) \geq 0 \Leftrightarrow x^2 + y^2 + z^2 \leq 9 = 3^2$.
  - Solid ball of center O and radius 3.



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Graph of Two-Variable Function

- The **graph** of function $z = f(x, y)$ with domain D:
  - $\{(x, y, z) \in \mathbb{R}^3 \mid (x, y) \in D \text{ and } z = f(x, y)\}$.

  It is a **surface** defined by the function.
- **Question.** How can we plot a surface?
  - For $s = g(t)$, pick points $t_1, \ldots, t_n$, and connect points $(t_1, f(t_1)), \ldots, (t_n, f(t_n))$ with a smooth curve.
  - For $z = f(x, y)$, this method fails. Instead,
    - For each $z = z_0$, draw **level curve** $z_0 = f(x, y)$.

    This is the intersection of the surface $z = f(x, y)$ and the horizontal plane $z = z_0$.
    - $L_{z_0} := \{(x, y) \mid f(x, y) = z_0\}$.

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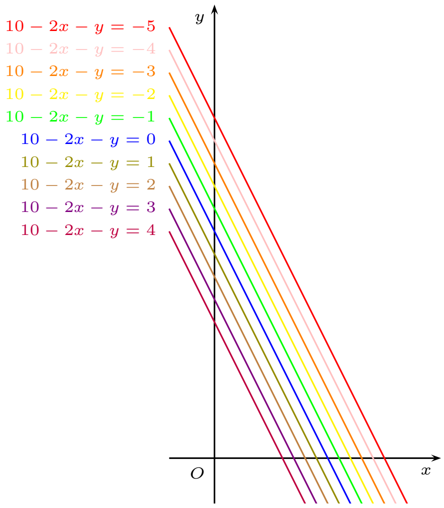
Examples

- $f(x, y) = 10 - 2x - y$.



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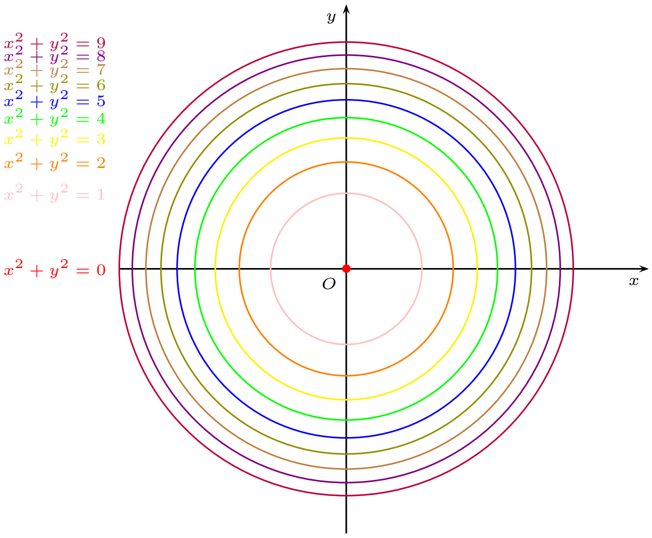
Examples

- $f(x, y) = x^2 + y^2$.



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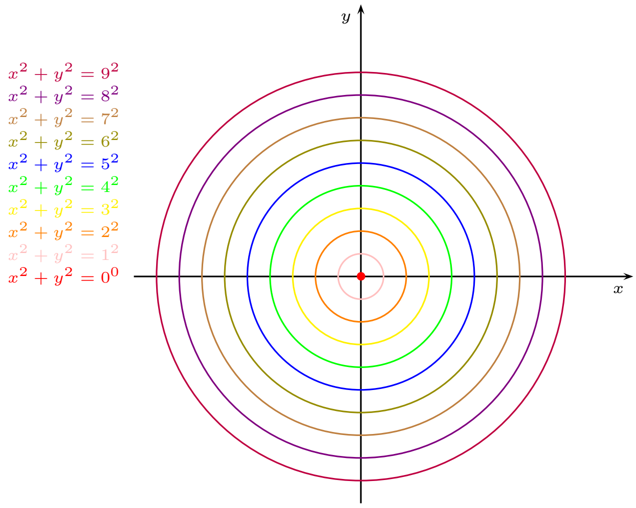
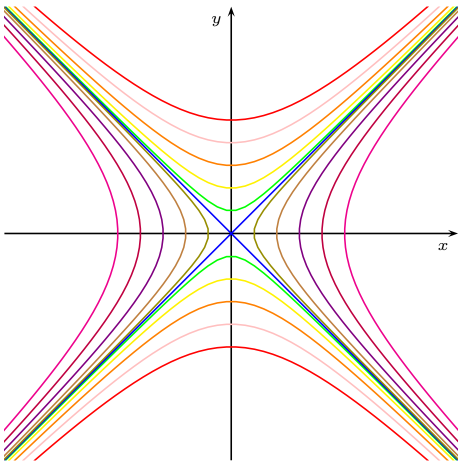
Examples

- $f(x,y) = \sqrt{x^2+y^2}$.



Examples

- $f(x,y) = x^2 - y^2$.



Partial Derivatives

- Let $z = f(x, y)$. How to determine the **rate of change**?
  - If y is fixed, $z = f(x, y)$ is a function in x.
    - The derivative of z with respect to x determines the rate of change of z along the x-direction.
    - It is the **partial derivative** of z with respect to x.
      - $\dfrac{\partial z}{\partial x} = \lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$.
  - If x is fixed, $z = f(x, y)$ is a function in y.
    - The derivative of z with respect to y determines the rate of change of z along the y-direction.
    - It is the **partial derivative** of z with respect to y.
      - $\dfrac{\partial z}{\partial y} = \lim\limits_{\Delta y \to 0} \dfrac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$.

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Partial Derivatives

- Suppose $z = f(x, y)$. We also use the following notations.
  - $\dfrac{\partial z}{\partial x} = \dfrac{\partial f}{\partial x} = f_x(x, y) = D_x f(x, y) = \cdots$,
  - $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial y} = f_y(x, y) = D_y f(x, y) = \cdots$.
- The **gradient** of $z = f(x, y)$ is a vector of functions:
  - $\nabla f(x, y) = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \right) = (f_x(x, y), f_y(x, y))$.
    - $\nabla f(a, b) = \nabla f(x, y) \Big|_{(a,b)}$.
  - Property:
    - $\dfrac{\partial f}{\partial x} = \nabla f(x, y) \bullet (1, 0), \dfrac{\partial f}{\partial y} = \nabla f(x, y) \bullet (0, 1)$.

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Examples

- $f(x,y) = \dfrac{yx}{x^2+1}$. Find f_x and f_y.
  - $f_x(x,y) = \dfrac{y \cdot (x^2+1) - yx \cdot 2x}{(x^2+1)^2} = \dfrac{y(1-x^2)}{(x^2+1)^2}$,
  - $f_y(x,y) = \dfrac{x}{x^2+1}$.
  - $\nabla f(x,y) = \left(\dfrac{y(1-x^2)}{(x^2+1)^2}, \dfrac{x}{x^2+1} \right)$.
- $f(x,y) = x^3 e^y + \ln y + e^x$. Find f_x and f_y.
  - $f_x = 3x^2 e^y + e^x$,
  - $f_y = x^3 e^y + \dfrac{1}{y}$.
  - $\nabla f(x,y) = \left(3x^2 e^y + e^x, x^3 e^y + \dfrac{1}{y} \right)$.

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Directional Derivatives

- Let $z = f(x,y)$. Then
  - f_x is the rate of change along the x-direction;
  - f_y is the rate of change along the y-direction.

  Question. What is the rate of the change of $z = f(x,y)$ along $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, where $\mathbf{u}$ is a unit vector?
  - f_x is the rate of change along $\mathbf{i} = (1,0)$;
  - f_y is the rate of change along $\mathbf{j} = (0,1)$.

  Answer. It is $u_1 f_x + u_2 f_y = (f_x, f_y) \bullet (u_1, u_2) = \nabla f \bullet \mathbf{u}$.
- **Definition.** Suppose f_x and f_y exist. The **directional derivative** of $z = f(x,y)$ along a unit vector $\mathbf{u}$ is given by
  - $D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \bullet \mathbf{u}$.

  In particular, $D_{\mathbf{i}} f = f_x$ and $D_{\mathbf{j}} f = f_y$.

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Directional Derivatives

- **Remark.** In general, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector, then
  - $D_{\mathbf{u}}f(x,y) = \lim_{h\to 0} \dfrac{f(x+u_1h, y+u_2h) - f(x,y)}{h}$.

  This is the original definition. In application, f_x and f_y usually exist.
- **Geometric Meaning.**
  - $D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$,
    - θ is the angle between ∇f and $\mathbf{u}$.

    Therefore,
    - f increases most rapidly in the direction of ∇f;
    - f decrease most rapidly in the direction of $-\nabla f$;
    - Any direction $\mathbf{u}$ orthogonal to ∇f $(\neq 0)$ is a direction of zero change.

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Examples

- Consider the function $f(x,y) = xe^y + \cos xy$.
  - Find the derivative at $(2,0)$ along $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.
    - Gradient: $\nabla f(2,0) = (1,2)$
      - $f_x = e^y - y\sin xy, \quad f_x(2,0) = 1;$
      - $f_y = xe^y - y\sin xy, \quad f_y(2,0) = 2.$
    - Direction: $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$.
    - $D_{\mathbf{u}}f(2,0) = (1,2) \bullet (\frac{3}{5}, -\frac{4}{5}) = -1.$
  - The direction that f increases most rapidly at $(2,0)$:
    - $\dfrac{\nabla f(2,0)}{|\nabla f(2,0)|} = \dfrac{1}{\sqrt{5}}\mathbf{i} + \dfrac{2}{\sqrt{5}}\mathbf{j}$.
  - The direction that f has zero change at $(2,0)$.
    - $\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$ and $-\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}$.

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Examples

- Let $f(x,y) = \tan^{-1}(y/x) + \sqrt{3}\,\sin^{-1}(xy/2)$. Find the derivative at $(1,1)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$.
  - Gradient: $\nabla f(1,1) = (f_x(1,1), f_y(1,1)) = (\frac{1}{2}, \frac{3}{2})$
    - $f_x = -\dfrac{y}{x^2+y^2} + \dfrac{\sqrt{3}\,y}{\sqrt{4-x^2y^2}}$;
    - $f_y = \dfrac{x}{x^2+y^2} + \dfrac{\sqrt{3}\,x}{\sqrt{4-x^2y^2}}$.
  - Direction vector:
    - $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|} = \dfrac{3}{\sqrt{13}}\mathbf{i} - \dfrac{2}{\sqrt{13}}\mathbf{j}$.
  - $D_{\mathbf{u}}f(1,1) = (\frac{1}{2}, \frac{3}{2}) \bullet (\frac{3}{\sqrt{13}}, -\frac{2}{\sqrt{13}}) = -\dfrac{3}{2\sqrt{13}}$.

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Tangent Plane

- Let $z = f(x,y)$ be a surface and (x_0, y_0, z_0) a point on the surface.
  - The tangent line to $z = f(x, y_0)$ at (x_0, y_0, z_0) is
    - $z - z_0 = f_x(x_0, y_0)(x - x_0), \quad y = y_0$.
    - $\mathbf{r}_1 = (x_0, y_0, z_0) + \lambda_1(1, 0, f_x(x_0, y_0))$.
  - The tangent line to $z = f(x_0, y)$ at (x_0, y_0, z_0) is
    - $z - z_0 = f_y(x_0, y_0)(y - y_0), \quad x = x_0$.
    - $\mathbf{r}_2 = (x_0, y_0, z_0) + \lambda_2(0, 1, f_y(x_0, y_0))$.
- There is a unique plane containing both $\mathbf{r}_1$ and $\mathbf{r}_2$.
  - Its normal vector is given by
    - $\begin{pmatrix} 1 \\ 0 \\ f_x(x_0,y_0) \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ f_y(x_0,y_0) \end{pmatrix} = \begin{pmatrix} -f_x(x_0,y_0) \\ -f_y(x_0,y_0) \\ 1 \end{pmatrix}$.

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Tangent Plane

- **Definition.** Let $z = f(x, y)$ be a surface and (x_0, y_0, z_0) a point on the surface. The **tangent plane** of the surface at (x_0, y_0, z_0) is
  - $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
- **Example**. Tangent plane to $z = x \cos y - ye^x$ at $(0, 0, 0)$.
  - Let $f(x, y) = x \cos y - ye^x$.
    - $f_x = \cos y - ye^x, \quad f_x(0, 0) = 1;$
    - $f_y = -x \sin y - e^x, \quad f_y(0, 0) = -1.$

    Then the tangent plane at $(0, 0, 0)$ is given by
    - $z - 0 = 1 \cdot (x - 0) + (-1) \cdot (y - 0);$

    That is, $x - y - z = 0$.

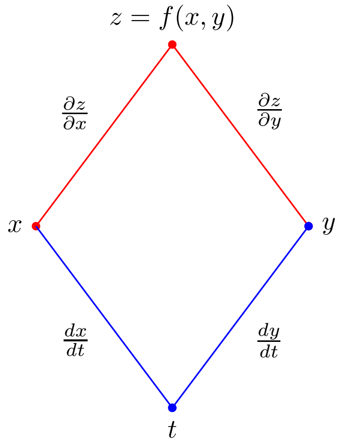
Linear Approximation

- Let $z = f(x)$ be a function. It can be approximated by its **tangent line**:
  - $x \approx x_0 \Rightarrow f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.
- Suppose $z = f(x, y)$. It can be approximated by its **tangent plane**:
  - $(x, y) \approx (x_0, y_0) \Rightarrow$
    $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

  This is called the **linearization** of f at (x_0, y_0).
- **Example.** Estimate $f(x, y) = x \cos y - ye^x$ near $(0, 0)$.
  - Its tangent plane at $(0, 0)$ is $z = x - y$.
    Then $f(x, y) \approx x - y$ near $(0, 0)$.
  - For instance, $f(0.02, 0.01) \approx 0.01$ and $f(0.02, 0.01) = 0.00979698661 \cdots$.

Chain Rule

- Suppose $z = f(x, y)$, where $x = x(t)$ and $y = y(t)$.
    - Then $z = f(x(t), y(t))$ is a function in t. What is $\dfrac{dz}{dt}$?



Chain Rule

- **Chain Rule.** Let $z = f(x, y)$ be a function. Suppose
    - f_x and f_y are continuous, and
    - $x = x(t)$ and $y = y(t)$ are differentiable.

  Then z is differentiable with respect to t.
    - $\dfrac{df}{dt} = f_x \cdot x'(t) + f_y \cdot y'(t)$, or
    - $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$.
- **Example.** Let $z = xy$, where $x = \cos t$ and $y = \sin t$.

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= y \cdot (-\sin t) + x \cdot \cos t \\ &= -\sin^2 t + \cos^2 t = \cos 2t.\end{aligned}$$

Implicit Differentiation Revisited

- Recall the **implicit differentiation**:
  - Suppose $f(x,y)=0$. Find $\dfrac{dy}{dx}$.
    1. Differentiate $f(x,y)=0$ with respect to x to obtain a relation in x, y and $\dfrac{dy}{dx}$;
    2. Solve $\dfrac{dy}{dx}$ in terms of x and y.
- Suppose $y=y(x)$ and let $z=f(x,y(x))$.
  - $0=\dfrac{dz}{dx}=f_x(x,y)\dfrac{dx}{dx}+f_y(x,y)\dfrac{dy}{dx}$.

  Therefore, $\dfrac{dy}{dx}=-\dfrac{f_x(x,y)}{f_y(x,y)}$.

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Examples

- Find $\dfrac{dy}{dx}$ if $x^3+y^3=3xy$.
  - Let $f(x,y)=x^3+y^3-3xy$. Then
    - $f_x(x,y)=3x^2-3y;\quad f_y(x,y)=3y^2-3x$.
  - Then $\dfrac{dy}{dx}=-\dfrac{f_x(x,y)}{f_y(x,y)}=-\dfrac{x^2-y}{y^2-x}$.
- Find $\dfrac{dy}{dx}$ if $y^2=x^2+\sin xy$.
  - Let $f(x,y)=y^2-x^2-\sin xy$. Then
    - $f_x(x,y)=-2x-y\cos xy$;
    - $f_y(x,y)=2y-x\cos xy$.
  - Then $\dfrac{dy}{dx}=-\dfrac{f_x(x,y)}{f_y(x,y)}=-\dfrac{2x+y\cos xy}{2y-x\cos xy}$.

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Geometric Meaning of Gradient

- Let $z = f(x, y)$ be a function.
  - Consider its **level curve**:
    - $f(x, y) = c$, where c is a constant.
  - Suppose the level curve can be parameterized by
    - $\mathbf{r}(t) = (x(t), y(t))$.

    Then $\quad f(x(t), y(t)) = c.$
  - Differentiate with respect to t:
    - $0 = \dfrac{dc}{dt} = \dfrac{d}{dt} f(x(t), y(t)) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt}.$
    - $0 = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}\right) \bullet \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right) = \nabla f(x, y) \bullet \mathbf{r}'(t).$
  - At every point, the gradient is **perpendicular (normal)** to the level curve passing through the point.

Two Independent Variables

- Suppose $z = f(x, y)$, $x = x(s, t)$ and $y = y(s, t)$.
  - $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s};$
  - $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}.$
- **Example.** $\quad z = x^2 + y^2$, $x = s - t$ and $y = s + t$.

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= 2x \cdot 1 + 2y \cdot 1 = 2(s - t) + 2(s + t) = 4s. \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= 2x \cdot (-1) + 2y \cdot 1 = -2(s - t) + 2(s + t) = 4t.
\end{aligned}
$$

Indeed, $z = (s - t)^2 + (s + t)^2 = 2(s^2 + t^2)$.

Higher Order Partial Derivatives

- Let $z = f(x, y)$. Then f_x and f_y are functions in two variables. We can define the **second partial derivatives**:
  - $f_{xx} = (f_x)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial x^2}$;
  - $f_{xy} = (f_x)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial^2 f}{\partial y \partial x}$;
  - $f_{yx} = (f_y)_x = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial x \partial y}$;
  - $f_{yy} = (f_y)_y = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial^2 f}{\partial y^2}$.
- Similarly, we can define **higher order partial derivatives**:
  - $f_{xxx}, f_{xxy}, f_{xyx}, f_{xyy}, f_{yxx}, f_{yxy}, f_{yyx}, f_{yyy}$.
  - $f_{xxxx}, f_{xxxy}, f_{xxyx}, \ldots$.

Second Partial Derivatives

- **Example.** Let $f(x, y) = x \cos y + ye^x$.
  - $f_x = \cos y + ye^x, \quad f_y = -x \sin y + e^x$.
  - $f_{xx} = (f_x)_x = (\cos y + ye^x)_x = ye^x$;
    $f_{xy} = (f_x)_y = (\cos y + ye^x)_y = -\sin y + e^x$;
    $f_{yx} = (f_y)_x = (-x \sin y + e^x)_x = -\sin y + e^x$;
    $f_{yy} = (f_y)_y = (-x \sin y + e^x)_y = -x \cos y$.
  - In this example, $f_{xy} = f_{yx}$.
    Does this relation hold in general?
- **Theorem.** Let $z = f(x, y)$ be a function.
  - Suppose f_x, f_y, f_{xy}, f_{yx} are defined in an open region containing (x_0, y_0) and they are continuous at (x_0, y_0).
  - Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Examples

- Find f_{xy} if $f(x,y) = xy + \dfrac{e^x}{x^2+1}$.
  - Method 1:
    - $f_x = y + \dfrac{e^x \cdot 2x - e^x(x^2+1)}{(x^2+1)^2} = y - \dfrac{e^x(x-1)^2}{(x^2+1)^2}$.
    - $f_{xy} = (f_x)_y = \left(y - \dfrac{e^x(x-1)^2}{(x^2+1)^2}\right)_y = 1$.
  - Method 2:
    - $f_y = x$;
    - $f_{xy} = f_{yx} = (f_y)_x = x_x = 1$.
- **Exercise.** Find all the first and second derivatives of
  - $W(P, V, \delta, v, g) = PV + \dfrac{V\delta v^2}{2g}$.

More Variables

- Let $w = f(x,y,z)$ be a function with three variables.
  - Level surface:
    - $f(x,y,z) = c$, where c is a constant.
  - Partial derivatives:
    - $f_x = \dfrac{\partial w}{\partial x}, \quad f_y = \dfrac{\partial w}{\partial y}, \quad f_z = \dfrac{\partial w}{\partial z}$.
    - Gradient: $\nabla f = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$.
  - Directional derivatives:
    - $D_{\mathbf{u}}f = \nabla f \bullet \mathbf{u}$, where $\mathbf{u}$ is a unit vector.
  - Chain rule:
    - If $x = x(t)$, $y = y(t)$, $z = z(t)$, then
      $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$.

Tangent Plane

- Recall that if $z = f(x, y)$, then the gradient $\nabla f = (f_x, f_y)$ is normal to the level curve $f(x, y) = c$.
- If $w = f(x, y, z)$, we expect that $\nabla f = (f_x, f_y, f_z)$ is normal to the level surface $f(x, y, z) = c$.
  - Then the **tangent plane** of $f(x, y, z) = c$ at $P(x_0, y_0, z_0)$ shall be given by
    - $\nabla f(x_0, y_0, z_0) \bullet (x, y, z) = \nabla f(x_0, y_0, z_0) \bullet (x_0, y_0, z_0)$.

    $f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0$.
- **Definition.** Let $f(x, y, z) = c$ be a surface.
  - It is **nonsingular** (**smooth**) at $P(x_0, y_0, z_0)$ if f_x, f_y, f_z exist at P, and at least one of them is nonzero.
  - If is **singular** at P if it is not nonsingular at P.

Tangent Plane

- Consider the surface $f(x, y, z) = c$, where c is a constant.
  - Suppose it is nonsingular at $P(x_0, y_0, z_0)$.
  - Write $z = z(x, y)$ as a function. Then
    - $f(x, y, z(x, y)) = c$ for all x and y.
  - The partial derivatives are zero:

$$
\begin{aligned}
0 = \frac{\partial c}{\partial x} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \\
&= f_x(P) + f_z(P) \cdot \frac{\partial z}{\partial x}(P), \\
0 = \frac{\partial c}{\partial y} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} \\
&= f_y(P) + f_z(P) \cdot \frac{\partial z}{\partial y}(P).
\end{aligned}
$$

Tangent Plane

- Recall: the tangent plane of $z = z(x, y)$ is given by
  - $\dfrac{\partial z}{\partial x}(P)(x - x_0) + \dfrac{\partial z}{\partial y}(P)(y - y_0) = z - z_0$.
    - $\dfrac{\partial z}{\partial x}(P) = -\dfrac{f_x(P)}{f_z(P)}, \quad \dfrac{\partial z}{\partial y}(P) = -\dfrac{f_y(P)}{f_z(P)}$.
- **Theorem.** Suppose the surface $f(x, y, z) = c$ is nonsingular at point $P(x_0, y_0, z_0)$.
  - Then its tangent plane at P is given by
    - $f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0$.
- **Corollary.** Suppose the curve $f(x, y) = c$ is nonsingular at point $P(x_0, y_0)$.
  - Then its tangent line at P is given by
    - $f_x(P)(x - x_0) + f_y(P)(y - y_0) = 0$.

Example

- Find all the nonsingular points on $x^2 + y^2 = z^2$, and determine the tangent planes.
  - Let $f(x, y, z) = x^2 + y^2 - z^2$. Then
    - $f_x = 2x, \quad f_y = 2y, \quad f_z = -2z$.
  - Let $f = 0$ and $f_x = f_y = f_z = 0$.
    - The only singular point is $(0, 0, 0)$.
  - Suppose $(x_0, y_0, z_0) \neq (0, 0, 0)$.
    The tangent plane at (x_0, y_0, z_0) is given by
    - $2x_0(x - x_0) + 2y_0(y - y_0) - 2z_0(z - z_0) = 0$.

    That is,
    - $x_0 x + y_0 y - z_0 z = x_0^2 + y_0^2 - z_0^2 = 0$.