MA1521 CALCULUS FOR COMPUTING

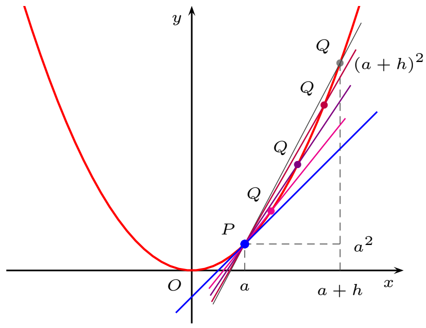
Wang Fei

matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-16
Tel: 6516-2937

The Tangent Line

- Recall that in Chapter 1 we have seen how to find the tangent line of the curve $y = x^2$ at $P(a, a^2)$:



$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{(a+h)^2 - a^2}{h}$$

  - The **slope** of the tangent line can be written as

$$m := \lim_{h \to 0} m_{PQ} = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}.$$

Definition of Derivative

- The **derivative of a function f at a number a** is

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

  - f is **differentiable at a** if $f'(a)$ exists.
  - $f'(a)$ is the **slope** of $y = f(x)$ at $x = a$.
- Let $x = a + h$. Then $h = x - a$, and $h \to 0 \Leftrightarrow x \to a$. We may use an equivalent definition:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

- The **tangent line** to $y = f(x)$ at $(a, f(a))$ is the line passing through $(a, f(a))$ with slope $f'(a)$:

$$y - f(a) = f'(a)(x - a).$$

Examples of Derivatives

- Let $f(x) = x^2 - 8x + 9$. Find $f'(3)$.

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{[(3+h)^2 - 8(3+h) + 9] - (3^2 - 8\cdot 3 + 9)}{h} \\ &= \lim_{h\to 0} \frac{(-6 - 2h + h^2) - (-6)}{h} \\ &= \lim_{h\to 0} \frac{-2h + h^2}{h} = \lim_{h\to 0}(-2+h) = -2. \end{aligned}$$

  - The tangent line of $y = f(x)$ passing through $(3, -6)$:

$$y - (-6) = f'(3)(x-3) = -2(x-3).$$

That is, $\quad 2x + y = 0.$

Examples of Derivatives

- Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$ Find $f'(0)$.

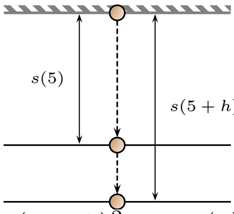
$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h+0) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) \\ &= 0. \end{aligned}$$

$$\begin{array}{lccccccl} \text{Note that} & -|h| & \leq & h\sin\left(\frac{1}{h}\right) & \leq & |h| & \text{for all } h \neq 0. \\ h \to 0 & \downarrow & & \downarrow & & \downarrow & \\ & 0 & \Rightarrow & 0 & \Leftarrow & 0 & \end{array}$$

(Squeeze Theorem)

Velocity

- Let $s = s(t)$ be the position function of a particle.
  - **instantaneous velocity** at time $t = a$: $s'(a)$;
  - **speed** at time $t = a$: $|s'(a)|$.
- **Example.** A ball is dropped from a tower 450m above the ground. Find its velocity after 5 seconds.



  - $s(t) = \frac{1}{2}gt^2 = 4.9t^2$.
  - $v(5) = s'(5) = 49$.
  - $\dfrac{\Delta s}{\Delta t} = \dfrac{4.9(5+h)^2 - 4.9(5)^2}{(5+h)-5} = \dfrac{49h + 4.9h^2}{h}$
  - **Velocity** at $t = 5$: $\lim\limits_{h \to 0} \dfrac{49h + 4.9h^2}{h} = 49$ m/s.

Example

- Let $f(x) = \dfrac{1}{x}$. Find $f'(a)$ at each $a \in \mathbb{R}\backslash\{0\}$.

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(a+h)a}}{h} \\ &= \lim_{h\to 0} \frac{-1}{(a+h)a} \\ &= -\frac{1}{a^2}. \end{aligned}$$

  - f' is therefore a function defined on $\mathbb{R}\backslash\{0\}$.

Derivative as a Function

- The **derivative** of f at point $x = a$:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

- The **derivative** of f as a function:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

  - $f'(x) = y' = \dfrac{dy}{dx} = \dfrac{df}{dx} = \dfrac{d}{dx} f(x) = D_x f(x) = \cdots$
  - $\boxed{\dfrac{dy}{dx} := \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}}$ (Leibniz, 1646–1716, German)
  - $f'(a) = \left.\dfrac{dy}{dx}\right|_{x=a}.$

- **Example.** If $f(x) = \dfrac{1}{x}$, then $f'(x) = -\dfrac{1}{x^2}$.

Examples

- Let $f(x) = \dfrac{1-x}{2+x}$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{1-(x+h)}{2+(x+h)} - \frac{1-x}{2+x}}{h} \\ &= \lim_{h \to 0} \frac{-3h}{h(2+(x+h))(2+x)} \\ &= \lim_{h \to 0} \frac{-3}{(2+(x+h))(2+x)} \\ &= \frac{-3}{(2+x)^2}. \end{aligned}$$

  - Domain of f: $\mathbb{R} \backslash \{-2\}$; Domain of f': $\mathbb{R} \backslash \{-2\}$.

Examples

- Let $f(x) = \sqrt{x}$, $(x \geq 0)$. Find f'.

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad (x > 0).
\end{aligned}
$$

The domain of f' may be $smaller$ than the domain of f.

Differentiable Functions

- **Definition.** (We only consider the differentiability at a point or on **open intervals**)
  - f is **differentiable at** a if
    $$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$
  - f is **differentiable on** (a, b) if it is differentiable at every $c \in (a, b)$.
- **Questions.**
  - What's the relation between differentiability and continuity?
  - What kinds of functions are differentiable?
  - How to construct new differentiable functions?
  - How the derivative affects the original function?

Differentiability Implies Continuity

- **Theorem**. If f is **differentiable** at a, then f is **continuous** at a.
- **Remark**. The converse of the theorem is false.
  - $f(x) = |x|$ is continuous at 0, not differentiable at 0.
- **Proof**. Suppose f is differentiable at a.

  That is, $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} = L$.
  - $\lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a} \cdot (x - a) = L \cdot 0 = 0.$
  - $\lim_{x \to a} f(x) = \lim_{x \to a} [(f(x) - f(a)) + f(a)] = 0 + f(a) = f(a).$

  Therefore, f is continuous at a.

Differentiation Formulas

- Let c be a constant. $(c)' = 0.$
  - $\dfrac{d}{dx}(c) = \lim_{h \to 0} \dfrac{c - c}{h} = \lim_{h \to 0} 0 = 0.$
- Let f be a differentiable function, and c be a constant.

$$\begin{aligned}
(cf)'(x) &= \lim_{h \to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h \to 0} c \left[\frac{f(x+h) - f(x)}{h} \right] \\
&= c \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= cf'(x).
\end{aligned}$$

$$\therefore \quad (cf)' = cf'.$$

Differentiation Formulas

- Let f and g be differentiable functions.

$$
\begin{aligned}
(f+g)'(x) &= \lim_{h\to 0} \frac{(f+g)(x+h)-(f+g)(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\
&= f'(x)+g'(x).
\end{aligned}
$$

$$\therefore \quad (f+g)' = f'+g'.$$

$$
\begin{aligned}
(f-g)' &= [f+(-g)]' = f'+(-g)' = f'+(-g') \\
&= f'-g'.
\end{aligned}
$$

Differentiation Formulas

- Let f and g be differentiable functions. What is $(fg)'$?

$$
\begin{aligned}
\frac{d}{dx}[f(x)]^2 &= \lim_{h\to 0} \frac{[f(x+h)]^2-[f(x)]^2}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]\cdot[f(x+h)+f(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \cdot \lim_{h\to 0}[f(x+h)+f(x)] \\
&= f'(x)\cdot[f(x)+f(x)] \quad (\because f \text{ is continuous}) \\
&= 2f'(x)f(x).
\end{aligned}
$$

$$
\begin{aligned}
(fg)' &= \tfrac{1}{2}[(f+g)^2-f^2-g^2]' \\
&= \tfrac{1}{2}[2(f+g)'(f+g)-2f'f-2g'g] \\
&= (f'+g')(f+g)-f'f-g'g \\
&= f'g+fg'.
\end{aligned}
$$

Differentiation Formulas

- Let f and g be differentiable functions. What is $(f/g)'$?
  Suppose $g(x) \neq 0$.

$$\left(\frac{1}{g}\right)'(x) = \lim_{h\to 0} \frac{\left(\frac{1}{g}\right)(x+h) - \left(\frac{1}{g}\right)(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} = \lim_{h\to 0} \frac{\frac{g(x)-g(x+h)}{g(x)g(x+h)}}{h}$$
$$= \lim_{h\to 0} \left[\frac{g(x+h)-g(x)}{h} \cdot \frac{-1}{g(x)g(x+h)}\right]$$
$$= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \cdot \lim_{h\to 0} \frac{-1}{g(x)g(x+h)}$$
$$= g'(x) \cdot \frac{-1}{[g(x)]^2} = \left(-\frac{g'}{g^2}\right)(x).$$

Differentiation Formulas

- Let f and g be differentiable functions. What is $(f/g)'$?
  Suppose $g(x) \neq 0$.

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \left(\frac{1}{g}\right) + f \cdot \left(\frac{1}{g}\right)'$$
$$= \frac{f'}{g} + \frac{f \cdot (-g')}{g^2}$$
$$= \frac{f'g - fg'}{g^2}.$$

  - $(cf)' = cf'$
  - $(f \pm g)' = f' \pm g'$
  - $(fg)' = f'g + fg'$
  - $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, if $g(x) \neq 0$.

The Chain Rule

- Let $F(x) = \sqrt{x^2+1}$. Find F'.

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{(x+h)^2+1} - \sqrt{x^2+1})(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} \\
&= \lim_{h\to 0} \frac{(x+h)^2+1-(x^2+1)}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} \\
&= \lim_{h\to 0} \frac{2xh+h^2}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} \\
&= \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
&= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}.
\end{aligned}
$$

The Chain Rule

- Let $F(x) = \sqrt{x^2+1}$. Find F'.

  Note that $F = f \circ g$, where
  - $f(x) = \sqrt{x}$, and $g(x) = x^2+1$.

  It is known that
  - $f'(x) = \dfrac{1}{2\sqrt{x}}$, and $g'(x) = 2x$.

  Question. Can we write F' by making use of f' and g'?

- If $y = g(x)$, $z = f(y)$, it seems that

$$
F'(x) = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(g(x))g'(x) = \frac{2x}{2\sqrt{x^2+1}}.
$$

  Question. Can we always differentiate the composite of differentiable functions using this method?

The Chain Rule

- **Theorem**. Let f and g be differentiable functions.
  Then $F = f \circ g$ is differentiable and

$$F' = (f' \circ g)(g').$$

  More precisely,
  if g is differentiable at a and f is differentiable at $b = g(a)$, then $F = f \circ g$ is differentiable at a and

$$F'(a) = f'(b)g'(a) = f'(g(a))g'(a).$$

  In Leibniz notation, if $y = g(x)$ and $z = f(y)$, then

$$\left.\frac{dz}{dx}\right|_{x=a} = \left.\frac{dz}{dy}\right|_{y=b} \left.\frac{dy}{dx}\right|_{x=a}, \quad \text{or in short} \quad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}.$$

Formulas of Derivatives

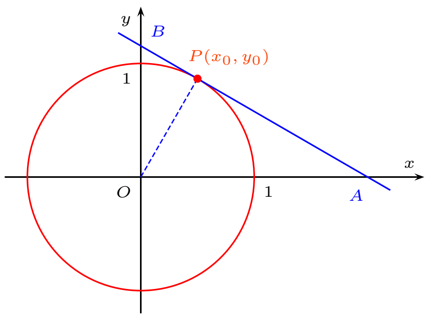
- $\frac{d}{dx}x^a = ax^{a-1}$.
- $\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x;$
  $\frac{d}{dx}\tan x = \sec^2 x, \quad \frac{d}{dx}\cot x = -\csc^2 x;$
  $\frac{d}{dx}\sec x = \sec x \tan x, \quad \frac{d}{dx}\csc x = -\csc x \cot x.$
- $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$
  $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \quad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}.$
  $\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}, \quad \frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}.$
- $\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln|x| = \frac{1}{x}$

Examples

- $\dfrac{d}{dx}\ln\dfrac{\sqrt{x+1}}{e^{2x}\cos^4 5x}$.
  - $\ln\dfrac{\sqrt{x+1}}{e^{2x}\cos^4 5x} = \dfrac{1}{2}\ln(x+1) - 2x - 4\ln|\cos 5x|$.
  - $\dfrac{d}{dx}\ln\dfrac{\sqrt{x+1}}{e^{2x}\cos^4 5x} = \dfrac{1}{2(x+1)} - 2 - \dfrac{4}{\cos 5x}\cdot(-5\sin 5x)$
    $= \dfrac{1}{2(x+1)} - 2 + 20\tan 5x.$
- $\dfrac{d}{dx}\left(x\sin^{-1}\dfrac{1}{x}\right)$.
  - $\dfrac{d}{dx}\left(x\sin^{-1}\dfrac{1}{x}\right) = 1\cdot\sin^{-1}\dfrac{1}{x} + x\cdot\dfrac{1}{\sqrt{1-(\frac{1}{x})^2}}\cdot(-x^{-2})$
    $= \sin^{-1}\dfrac{1}{x} - \dfrac{1}{x\sqrt{1-\frac{1}{x^2}}}.$

Implicit Differentiation

- How to find the **slope** of the tangent line to the unit circle $x^2+y^2=1$ at a point $P(x_0, y_0)$ on the circle?



  - $\overline{AB} \perp \overline{OP} \Rightarrow$ (slope of $\overline{AB}$)(slope of $\overline{OP}$) $= -1$

$$y'\big|_P \cdot \frac{y_0}{x_0} = -1 \Rightarrow \boxed{y'\big|_P = -\frac{x_0}{y_0}}.$$

Implicit Differentiation

- How to find the **slope** of the tangent line to the unit circle $x^2 + y^2 = 1$ at a point $P(x_0, y_0)$ on the circle?
  - Given that $1 = x^2 + y^2$.
    Differentiate both sides with respect to x.

$$\frac{d}{dx}(1) = \frac{d}{dx}(x^2 + y^2).$$

That is,

$$\begin{aligned} 0 &= \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 2x + \frac{dy^2}{dy}\frac{dy}{dx} \\ &= 2x + 2y\frac{dy}{dx}. \end{aligned}$$

$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \Rightarrow \boxed{\left.\frac{dy}{dx}\right|_P = -\frac{x_0}{y_0}}.$$

Implicit Differentiation

- In general, we may not have a **function** $y = f(x)$.
  Instead, it may be an **equation**

$$f(x, y) = 0.$$

  We can still find $y' = \dfrac{dy}{dx}$ as follow:

  1). Differentiate $f(x, y)$ with respect to x;

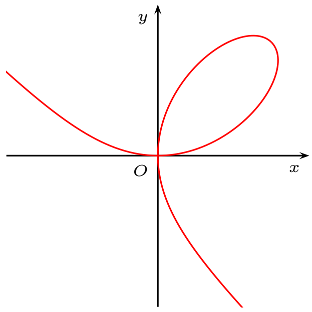
  2). Solve the equation $\dfrac{d}{dx}f(x, y) = 0$ to express $\dfrac{dy}{dx}$ in terms of x and y.

- The method introduced is called **implicit differentiation**.

  Remark. In other to use the method of implicit differentiation, we shall first assume that $\dfrac{dy}{dx}$ exists.

Examples

- Find $\dfrac{dy}{dx}$ if $x^3 + y^3 = 3xy$.



  - Differentiate

$$x^3 + y^3 = 3xy$$

with respect to x:

$$(x^3)' = 3x^2$$
$$(y^3)' = 3y^2y'$$
$$(xy)' = x'y + xy' = y + xy'$$

$$3x^2 + 3y^2y' = 3(y + xy')$$

  - Solve y': $y' = \dfrac{x^2 - y}{x - y^2}$, $\quad (x, y) \neq (0, 0), (\sqrt[3]{4}, \sqrt[3]{2})$.

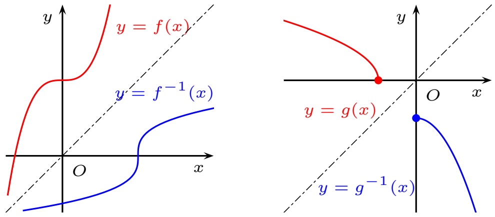
Find the Inverse Function

- Let f be a **one to one** function. Then it admits an inverse function f^{-1}. But how to find f^{-1}?
- Recall that $y = f(x) \Leftrightarrow f^{-1}(y) = x$. We can thus apply the following procedure:
  1. Write $y = f(x)$.
  2. Solve the equation for x in terms of y: $\quad x = f^{-1}(y)$.
  3. Interchanging x and y to express f^{-1} as a function in variable x: $\quad y = f^{-1}(x)$.
- Interchanging x and y has the same effect as the reflection with respect to $y = x$.
  - So the graphs of f and f^{-1} are **symmetric with respect to the straight line** $y = x$.

Examples

- Find inverse of $f(x) = x^3 + 2$ and $g(x) = \sqrt{-1-x}$.
  1. Let $y = f(x) = x^3 + 2$.
  2. Solve x in terms of y: $x = \sqrt[3]{y-2}$.
  3. Interchange x and y: $f^{-1}(x) = y = \sqrt[3]{x-2}$.

  1. Let $y = g(x) = \sqrt{-1-x}$, $(x \leq -1)$.
  2. Solve x in terms of y: $x = -y^2 - 1$, $(y \geq 0)$.
  3. Interchange x and y: $g^{-1}(x) = -x^2 - 1$, $(x \geq 0)$.



Calculus of Inverse Functions

- Let f be a continuous function defined on an interval.
  - If f is increasing, then f is one to one.
    - $a < b \Rightarrow f(a) < f(b) \Rightarrow f(a) \neq f(b)$.

    Similarly, if f is decreasing, then f is one to one.
  - If f is one to one, must f be increasing or decreasing?

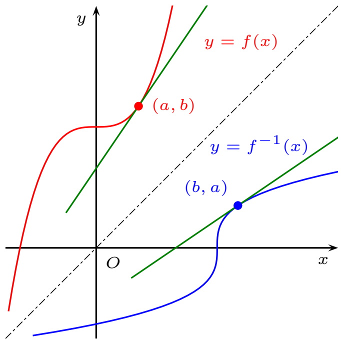
- **Theorem**. Suppose f is a one-to-one and continuous function defined on an interval. Then
  - f is either increasing or decreasing.

- **Theorem**. Suppose f is a one-to-one and continuous function defined on an interval. Then
  - Its inverse function f^{-1} is continuous.

Calculus of Inverse Functions

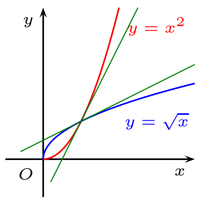
- **Theorem.** Let f be a one to one continuous function defined on an interval.
  - If f is differentiable at a, and $f'(a) \neq 0$,
  - then f^{-1} is differentiable at $b = f(a)$,

    and $(f^{-1})'(b) = \dfrac{1}{f'(a)}$.



Example

- Let $f(x) = x^2$ on $[0, 2]$. Find $(f^{-1})'(1)$.
  - Method 1: $(f^{-1})(x) = \sqrt{x}$. Then $(f^{-1})'(x) = \dfrac{1}{2\sqrt{x}}$.
    - $(f^{-1})'(1) = \dfrac{1}{2\sqrt{x}}\Big|_{x=1} = \dfrac{1}{2}$.
  - Method 2: $f(1) = 1$, and $f'(x) = 2x$. Then
    - $(f^{-1})'(1) = \dfrac{1}{f'(1)} = \dfrac{1}{2}$.



- The advantage of Method 2 is that we can find $(f^{-1})'$ without knowing the explicit expression of f^{-1}.

Derivative of Trigonometric Functions

- Recall the **inverse sine function** $\sin^{-1} x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. What is $(\sin^{-1} x)'$?
  - Let $y = \sin^{-1} x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\sin y = x$.
    $\cos y \geq 0 \Rightarrow \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

$\therefore \ (\sin^{-1} x)' = \dfrac{1}{(\sin y)'} = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{1 - x^2}}, \ -1 < x < 1.$

- The **inverse cosine function** $\cos^{-1} x : [-1, 1] \to [0, \pi]$.
  - Let $y = \cos^{-1} x \in [0, \pi]$. Then $\cos y = x$.
    $\sin y \geq 0 \Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$.

$\therefore \ (\cos^{-1} x)' = \dfrac{1}{(\cos y)'} = \dfrac{-1}{\sin y} = \dfrac{-1}{\sqrt{1 - x^2}}, \ -1 < x < 1.$

- **Exercise.** Prove that $(\tan x)' = \dfrac{1}{1 + x^2}$.

Second Derivative

- Let f be a function. We can **differentiate** it to get f'.
- f' is a function, we can **differentiable** it to get $(f')'$.
  - $f'' := (f')'$, is called the **second derivative** of f.
  - By Leibniz notation:
$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2 y}{dx^2}.$$
  - $f' = D(f) \Rightarrow f'' := D^2(f)$.
- **Geometric meaning**:
  - f' measures the change of f,
  - f'' measures the change of f'.

Physical Meaning of Second Derivative

- Let $s = s(t)$ be the position function of an object along a straight line.
  - $s'(t) = v(t)$: the **instantaneous velocity**,
    it determines the change of the **position**,
  - $s''(t) = v'(t) = a(t)$: the **acceleration**,
    it determines the change of the **velocity**.

- **Example**. Suppose the position of a particle is given by

$$s = s(t) = t^3 - 6t^2 + 9t.$$

  - **Velocity**: $v(t) = s'(t) = 3t^2 - 12t + 9.$
  - **Acceleration**: $a(t) = s''(t) = v'(t) = 6t - 12.$

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Higher Derivatives

- Let f be a function.
  - Differentiate f to get f', the **first derivative**.
  - Differentiate f' to get f'', the **second derivative**.
  - Differentiate f'' to get f''', the **third derivative**.
  - Differentiate f''' to get f'''', the **fourth derivative**.
  - $\cdots\cdots\cdots\cdots$

- In general, we define $f^{(0)} := f$, and for positive integer n,

$$f^{(n)} := \left(f^{(n-1)}\right)',$$

  called the nth **derivative** of f.

  Other notations: if $y = f(x)$, then

$$f^{(n)}(x) = y^{(n)} = \frac{d^n y}{dx^n} = D^n f(x).$$

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Examples

- Let $f(x) = x\cos x$. Find f', f'' and f'''.

$$\begin{aligned} f'(x) &= (x\cos x)' = (x)'\cos x + x(\cos x)' \\ &= \cos x - x\sin x. \end{aligned}$$

$$\begin{aligned} f''(x) &= (\cos x - x\sin x)' = (\cos x)' - (x\sin x)' \\ &= -\sin x - [(x)'\sin x + x(\sin x)'] \\ &= -\sin x - \sin x - x\cos x \\ &= -2\sin x - x\cos x. \end{aligned}$$

$$\begin{aligned} f'''(x) &= (-2\sin x - x\cos x)' = -2(\sin x)' - (x\cos x)' \\ &= -2\cos x - [(x)'\cos x + x(\cos x)'] \\ &= -2\cos x - (\cos x - x\sin x) \\ &= -3\cos x + x\sin x. \end{aligned}$$

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Parametric Equations

- Let x and y be functions of variable t.
  - $x = x(t)$ and $y = y(t)$.

  This is a **parametric equation** of x and y.

- **Examples**.
  - $x^2 + y^2 = 1$ can be parameterized as
    - $x = \cos t$ and $y = \sin t$, $t \in \mathbb{R}$.
  - $y^2 = x^3 + x^2$ can be parameterized as
    - $x = t^2 - 1$ and $y = t^3 - t$, $t \in \mathbb{R}$.

- **Question**. Given a parametric equation, $x = x(t)$ and $y = y(t)$, how to find the derivative of y with respect to x?

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Parametric Equations

- Suppose $x = x(t)$ and $y = y(t)$. Then
  - $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, provided that $\dfrac{dx}{dt} \neq 0$.
- **Example.** Find an expression for $\dfrac{dy}{dx}$ in terms of t if the curve is defined parametrically by
  - $x = \ln t$ and $y = t^2 - e^t$.

  Solution.
  - $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = 2t - e^t$.
  - $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = t(2t - e^t)$

    $\dfrac{dx}{dy} = \dfrac{dx/dt}{dy/dt} = \dfrac{1}{t(2t - e^t)}$.

Parametric Equations

- We can find the second derivative by chain rule:
  - $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dt}{dx} \cdot \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{dx/dt} \cdot \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$.

  Example. $x = \ln t$ and $y = t^2 - e^t$.
  - $\dfrac{dx}{dt} = \dfrac{1}{t}$ and $\dfrac{dy}{dt} = 2t - e^t$.
  - $\dfrac{d^2y}{dx^2} = \dfrac{1}{\frac{dx}{dt}} \cdot \dfrac{d}{dt}\left(\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}\right) = t(4t - e^t - te^t)$.

    $\dfrac{d^2x}{dy^2} = \dfrac{1}{\frac{dy}{dt}} \cdot \dfrac{d}{dt}\left(\dfrac{\frac{dx}{dt}}{\frac{dy}{dt}}\right) = -\dfrac{4t - e^t - te^t}{t^2(2t - e^t)}$.
- **Note.** $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$ but $\dfrac{d^2y}{dx^2} \cdot \dfrac{d^2x}{dy^2} \neq 1$.

Logarithmic Differentiation

- Find $\dfrac{dy}{dx}$ if $y = \dfrac{(x^2+1)\sqrt{x+3}}{x-1}$, $x > 1$.
  - Of course we can use product and quotient law to evaluate. But do we have a shortcut?
    - This is a product of positive functions.
    - Recall that $\ln ab = \ln a + \ln b$ for $a > 0, b > 0$.

  1. Take logarithmic function both sides:
     - $\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$.
  2. Differentiate with respect to x:
     - $\dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{2x}{x^2+1} + \dfrac{1}{2(x+3)} - \dfrac{1}{x-1}$.

  $\dfrac{dy}{dx} = \left[\dfrac{2x}{x^2+1} + \dfrac{1}{2(x+3)} - \dfrac{1}{x-1}\right] \dfrac{(x^2+1)\sqrt{x+3}}{x-1}$.

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Logarithmic Differentiation

- In general, if $y = f_1(x)f_2(x)\cdots f_n(x)$ is a product of nonzero functions, we can find the derivative as following:
  1. Take logarithmic function both sides:
     - $\ln|y| = \ln|f_1(x)| + \ln|f_2(x)| + \cdots + \ln|f_n(x)|$.
  2. Differentiate with respect to x:
     - $\dfrac{1}{y} \cdot \dfrac{dy}{dx} = \dfrac{f_1'(x)}{f_1(x)} + \dfrac{f_2'(x)}{f_2(x)} + \cdots + \dfrac{f_n'(x)}{f_n(x)}$.
  3. $\dfrac{dy}{dx} = \left[\dfrac{f_1'(x)}{f_1(x)} + \dfrac{f_2'(x)}{f_2(x)} + \cdots + \dfrac{f_n'(x)}{f_n(x)}\right] y$

  $= \left[\dfrac{f_1'(x)}{f_1(x)} + \dfrac{f_2'(x)}{f_2(x)} + \cdots + \dfrac{f_n'(x)}{f_n(x)}\right] f_1(x)f_2(x)\cdots f_n(x)$.
- Such method is called **logarithmic differentiation**.

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Logarithmic Differentiation

- **Example**. Find $\dfrac{dy}{dx}$ if $y = \dfrac{x\cos x}{\sqrt{\csc x}}$.
  1. $\ln|y| = \ln|x| + \ln|\cos x| - \frac{1}{2}\ln|\csc x|$.
  2. $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{1}{x} + \dfrac{-\sin x}{\cos x} - \dfrac{1}{2}\cdot\dfrac{-\csc x\ \cot x}{\csc x}$.
  3. $\dfrac{dy}{dx} = \left[\dfrac{1}{x} - \tan x + \dfrac{1}{2}\cot x\right]\dfrac{x\cos x}{\sqrt{\csc x}}$.
- **Example**. Find the derivative of $y = f(x)g(x)$.
  1. $\ln|y| = \ln|f(x)| + \ln|g(x)|$.
  2. $\dfrac{1}{y}\cdot\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}$.
  3. $\dfrac{dy}{dx} = \left[\dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)}\right]y = f'(x)g(x) + f(x)g'(x)$.

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Logarithmic Differentiation

- Evaluate $\dfrac{d}{dx}x^x$. Exercise: $\dfrac{d}{dx}(x^x)^x$ and $\dfrac{d}{dx}x^{(x^x)}$.
  - $\dfrac{d}{dx}x^a = ax^{a-1}$. Is $\dfrac{d}{dx}x^x = x\cdot x^{x-1} = x^x$?
  - $\dfrac{d}{dx}a^x = a^x\ln a$. Is $\dfrac{d}{dx}x^x = x^x\ln x$?
  - Let $y = x^x$. Then $\ln y = x\ln x$.
    - $\dfrac{1}{y}\dfrac{dy}{dx} = 1\cdot\ln x + x\cdot\dfrac{1}{x} = \ln x + 1$.
    - $\dfrac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$.
- In general, if $y = f(x)^{g(x)}$, $(f(x) > 0)$, then
  - $\ln y = g(x)\ln f(x) \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{d}{dx}[g(x)\ln f(x)]$.

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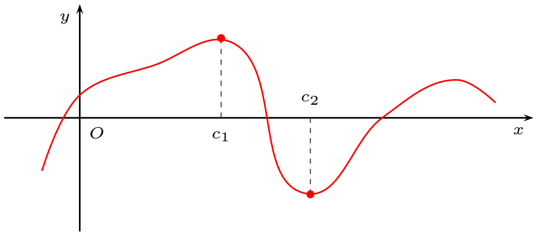
Absolute Maximum and Minimum Values

- **Definition.** Let f be a function, and D be its domain.
  - f has an **global** (or **absolute**) **maximum** at $c \in D$

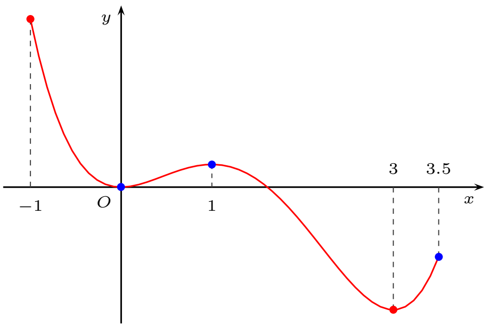
    $\iff f(c) \geq f(x)$ for all $x \in D$.
  - f has an **global** (or **absolute**) **minimum** at $c \in D$

    $\iff f(c) \leq f(x)$ for all $x \in D$.
  - The **absolute maximum** and **absolute minimum** are called the **(absolute) extreme values**.



Example

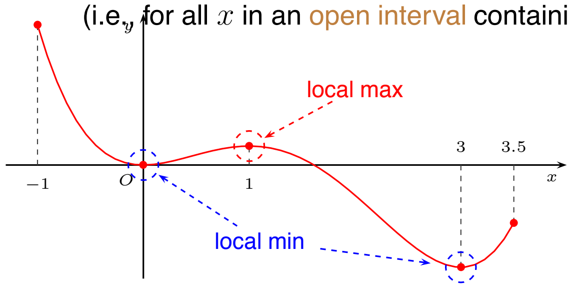
- Let $f(x) = 3x^4 - 16x^3 + 18x^2$ on $[-1, 3.5]$.



  - **global max**: highest point $f(-1) = 37$.
  - **global min**: lowest point $f(3) = -27$.
  - What can we say about other "**turning points**" and "**end points**"?

Local Maximum and Local Minimum

- **Definition.** Let f be a function with domain D.
  - f has a **local** (or **relative**) **maximum** at $c \in D$
    $\iff f(c) \geq f(x)$ for all x near c
    (i.e., for all x in an open interval containing c)
  - f has a **local** (or **relative**) **minimum** at $c \in D$
    $\iff f(c) \leq f(x)$ for all x near c
    (i.e., for all x in an open interval containing c)



Extreme Value Theorem

- **Extreme Value Theorem.**

  If f is **continuous** on a **finite closed interval** $[a, b]$,
  - then f attains **extreme values** on $[a, b]$.

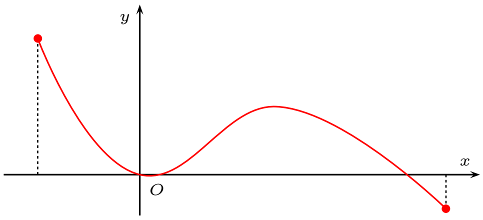
  Precisely, f attains an
  - **absolute maximum** value $f(c)$ at some $c \in [a, b]$,
  - **absolute minimum** value $f(d)$ at some $d \in [a, b]$.

  (The proof requires the "**compactness**" of finite closed interval. It is omitted in our course.)

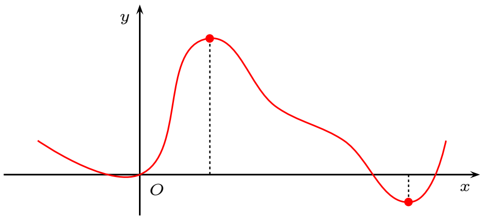
- **Note.** Similarly as the "**Intermediate Value Theorem**", the "**Extreme Value Theorem**" only shows the **existence** of the extreme values.
  We shall introduce a method to find out the exact value of extreme values.

Where are the Extreme Values?

- The extreme value may be obtained at the **endpoints**.



- If the extreme value is not obtained at the end points,



by definition it must occur as a **local max** or a **local min**.

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Finding the Extreme Values

- Let f be a **continuous** function on **closed** interval $[a, b]$.
  1. Compute the values at **endpoints**: $f(a)$, $f(b)$.
  2. Find **local max** and **local min** of f on (a, b).
  3. Compare the values obtained above to seek out the **extreme values**:
     - The **largest** is the **absolute maximum**,
     - The **smallest** is the **absolute minimum**.
- The 1st and the 3rd steps are easy.

  How to find the **local max** and **local min** of f on (a, b)?
  - From the graphs, it seems that the local max and local min always occur at the "**turning points**".

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Fermat's Theorem

- **Fermat's Theorem**.
  - Suppose f has a local maximum or local minimum at c.
    If $f'(c)$ exists, then $f'(c) = 0$.
- Pierre de Fermat (1601–1665), French Lawyer.
  - **Fermat's Last Theorem**: $x^n + y^n = z^n$ *has no nontrivial integer solution for* $n \geq 3$.
  - He wrote: "*I have discovered a truly remarkable proof which this margin is too small to contain.*"
- **Note**. We **CANNOT** find the local maximum and local minimum by simply solving $f'(x) = 0$.
  - Even if $f'(c) = 0$, f may not have a local maximum or local minimum at c.
  - Even if f has a local maximum or a local minimum at c, $f'(c)$ may not exist, and so $f'(c)$ may not be 0.

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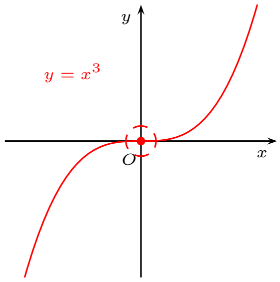
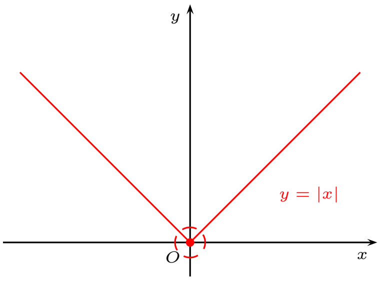
Examples

- "$f'(c) = 0 \not\Rightarrow f$ has local max or local min at c".
  - Let $f(x) = x^3$. Then $f'(x) = 3x^2$ and $f'(0) = 0$.
    But f has no local max or local min at 0.
- "f has local max or local min at $c \not\Rightarrow f'(c) = 0$".
  - Let $g(x) = |x|$. Then f is a local minimum at 0.
    But $f'(0)$ does not exist.





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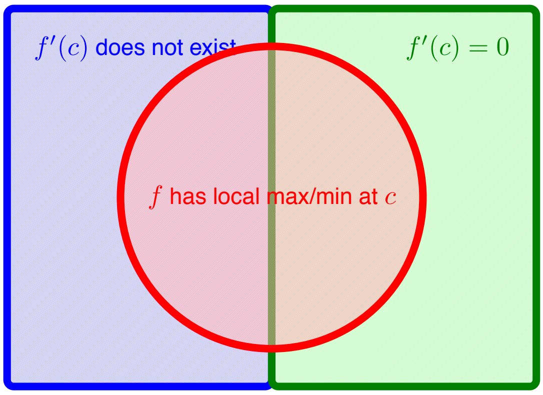
Critical Number

- Consider the following diagram:



- **Definition.** Let f be a function with domain D. Then $c \in D$ is called a **critical number** of f if
  - $f'(c)$ does not exist, or $f'(c)$ exists and equals 0.

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Closed Interval Method

- **Fermat's Theorem (Rephrased).**
  If f has a local maximum or a local minimum at c,
  - then c is a critical number of f.

- **Closed Interval Method**:
  Let f be a **continuous** function on interval $[a, b]$.
  1) Find the values of f at end points: $x = a$, $x = b$,
  2) Find the values of f at critical numbers of f in (a, b):
     - number $c \in (a, b)$ at which $f'(c)$ does not exist, or
     - number $c \in (a, b)$ at which $f'(c) = 0$.
  3) Compare the values of $f(x)$ evaluated in 1) and 2):
     - The largest is the absolute maximum value.
     - The smallest is the absolute minimum value.

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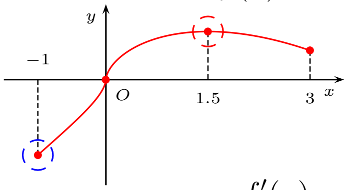
Examples

- Find the extreme values of $f(x) = x^{\frac{3}{5}}(4-x)$ on $[-1, 3]$.



1) End points: -1, 3;

2) Critical numbers: 0,1.5.

$$f'(x) = (x^{\frac{3}{5}})'(4-x) + x^{\frac{3}{5}}(4-x)'$$
$$= \frac{3}{5}x^{-\frac{2}{5}}(4-x) - x^{\frac{3}{5}} = \frac{4(3-2x)}{5x^{\frac{2}{5}}}.$$

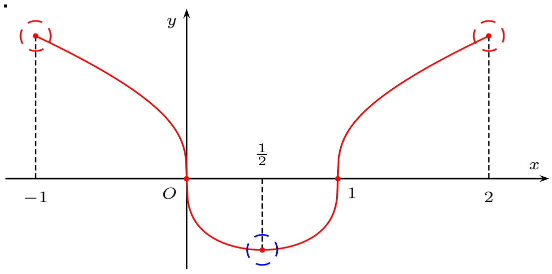
- $f'(x)$ does not exist $\Rightarrow x = 0$,
- $f'(x) = 0 \Rightarrow x = 1.5$.

3) Comparing $f(-1)$, $f(3)$, $f(0)$, $f(1.5)$,

- Absolute maximum: $f(1.5) \approx 3.1886$.
- Absolute minimum: $f(-1) = -5$.

Examples

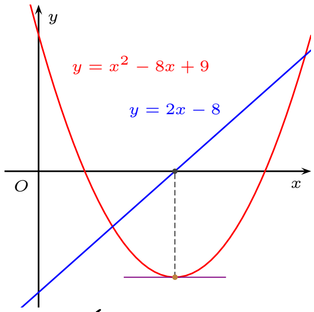
- Let $f(x) = \sqrt[3]{x^2 - x}$ be defined on $[-1, 2]$.
  - $f'(x) = \dfrac{1}{3}(x^2-x)^{-2/3}(2x-1) = \dfrac{2x-1}{3(x^2-x)^{2/3}}$.
    - $f'(x)$ does not exist: $x = 0, x = 1$;
    - $f'(x) = 0$: $x = \frac{1}{2}$.

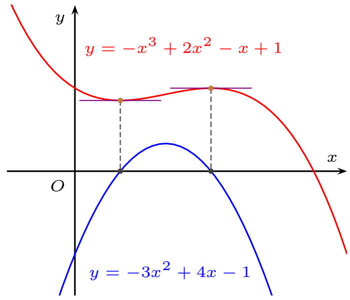


  - Absolute maximum: $f(-1) = f(2) = \sqrt[3]{2}$.
  - Absolute minimum: $f(\frac{1}{2}) = -\dfrac{1}{\sqrt[3]{4}}$.

Increasing/Decreasing Test

- Consider the following functions:





  - It seems that $\begin{cases} f \text{ “turns”} & \Leftrightarrow \quad f' = 0, \\ f \text{ is increasing} & \Leftrightarrow \quad f' > 0, \\ f \text{ is decreasing} & \Leftrightarrow \quad f' < 0. \end{cases}$

Increasing/Decreasing Test

- **Theorem**. Let f be a function such that
  - It is **continuous** on $[a, b]$, **differentiable** on (a, b).

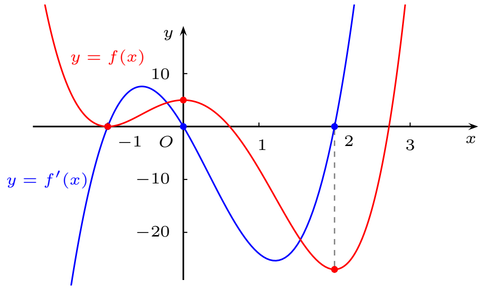
  Then
  - $f'(x) = 0$ on $(a, b) \Leftrightarrow f$ is **constant** on $[a, b]$;
  - $f'(x) > 0$ on $(a, b) \Rightarrow f$ is **increasing** on $[a, b]$;
  - $f'(x) < 0$ on $(a, b) \Rightarrow f$ is **decreasing** on $[a, b]$.
- The **converse** of **Increasing/Decreaing Test** **fails**:
  - f is **increasing** $\not\Rightarrow f'(x) > 0$;
  - f is **decreasing** $\not\Rightarrow f'(x) < 0$.

  Note that f is not necessarily differentiable.
  Even if f is differentiable, f' may be zero at some points.
  - Let $f(x) = x^3$. Then f is increasing on $\mathbb{R}$.
    - $f'(x) = 3x^2 \Rightarrow f'(0) = 0.$

Example

- Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.
  - $f'(x) = 12x^3 - 12x^2 - 24x = 12(x+1)x(x-2)$.

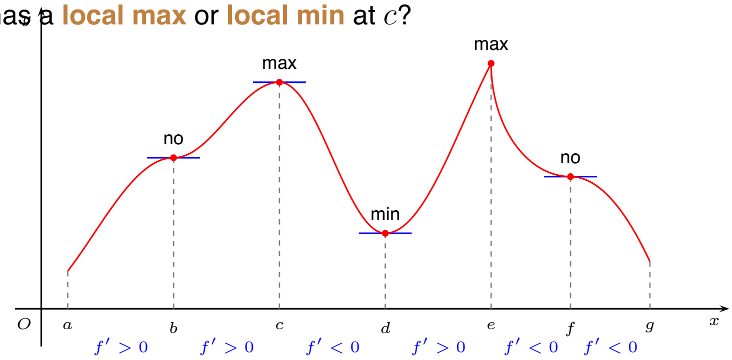
Interval	$x+1$	x	$x-2$	$f'(x)$	$f(x)$
$(-\infty, -1)$	$-$	$-$	$-$	$-$	$\searrow$
$(-1, 0)$	$+$	$-$	$-$	$+$	$\nearrow$
$(0, 2)$	$+$	$+$	$-$	$-$	$\searrow$
$(2, \infty)$	$+$	$+$	$+$	$+$	$\nearrow$



First Derivative Test

- Let f be a continuous function. Recall that
  - if f has a **local max** or **local min** at c,
  - then c is a **critical number** of f.

  Now suppose c is a **critical number** of f.
  How to check if f has a **local max** or **local min** at c?



First Derivative Test

- **First Derivative Test**.

  Let f be **continuous** and c a **critical number** of f.
  Suppose f is **differentiable** near c (except possibly at c).

  - If f' changes from **positive** to **negative** at c,
    then f has a **local maximum** at c.
  - If f' changes from **negative** to **positive** at c,
    then f has a **local minimum** at c.
  - If f' does not change sign at c,
    then f has **no local max/min** at c.

- **Proof**. If f' changes from **positive** to **negative** at c, then

  - f is **increasing** on the left of c, and
  - f is **decreasing** on the right of c.

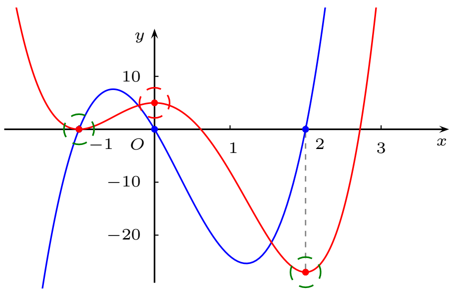
  So f has a **local maximum** at c.
  Other two cases can be shown similarly. (Exercise)

Examples

- $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$
  - $f'(x) = 12x^3 - 12x^2 - 24x = 12(x+1)x(x-2)$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 2)$	$(2, \infty)$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

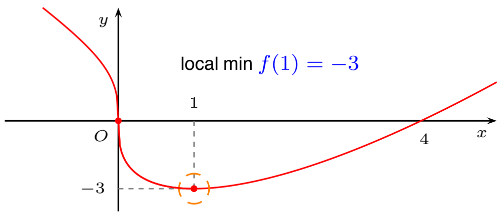
  - local maximum: $x = 0$;
  - local minimum: $x = -1$, $x = 2$.



Examples

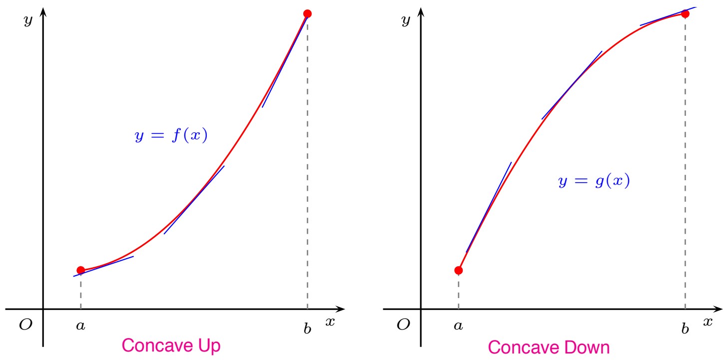
- $f(x) = x^{1/3}(x-4)$. Find its local max and local min.
  - Where are the critical numbers?
    $f'(x) = (x^{4/3} - 4x^{1/3})' = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}\frac{x-1}{x^{2/3}}$.
    - $f'(x)$ does not exists: $x = 0$;
    - $f'(x) = 0$: $x = 1$.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$



Concavity

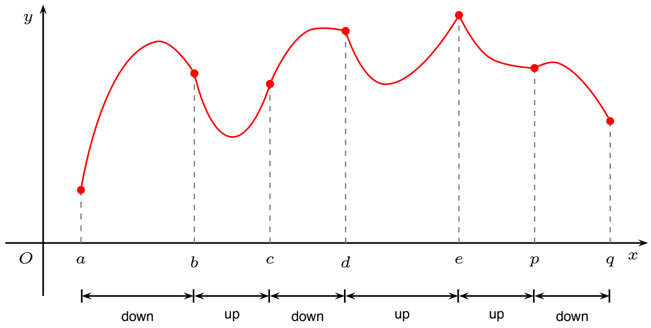
- Consider two graphs with the same end points:



  - They are both increasing functions, but look different.
  - We shall define **concavity** to distinguish the two types of (differentiable) functions.

Concavity

- **Definition.** Let f be differentiable on an open interval I.
  - If the graph lies **above** all its tangent lines on I,
    then it is said to be **concave up**.
  - If the graph lies **below** all its tangent lines on I,
    then it is said to be **concave down**.



Concavity Test

- **Theorem.** Let f be **differentiable** on open interval I.
  - The graph is **concave up** $\Leftrightarrow$ f' is **increasing**.
  - The graph is **concave down** $\Leftrightarrow$ f' is **decreasing**.
- Suppose f is twice differentiable on an open interval I.
  - If $f'' > 0$ on I, by Increasing Test f' is increasing,
    then the graph of f is concave up.
  - If $f'' < 0$ on I, by Decreasing Test f' is decreasing,
    then the graph of f is concave down.
- **The Concavity Test.** Let f be a **twice differentiable** function on an open interval I.
  - If $f'' > 0$ on I,
    then the graph of f is **concave up** on I.
  - If $f'' < 0$ on I,
    then the graph of f is **concave down** on I.

Graph f using f' and f''

- Sketch the graph of $f(x) = x^4 - 4x^3$.
  - $f'(x) = 4x^2(x-3)$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$

So $f(x)$ has local minimum at $x = 3$.

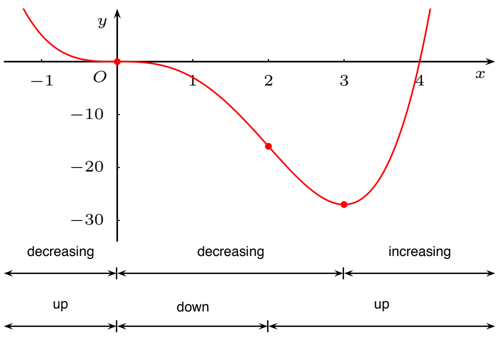
  - $f''(x) = 12x(x-2)$.

Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$f''(x)$	$+$	$-$	$+$
Concavity	Up	Down	Up

Graph f using f' and f''

- Sketch the graph of $f(x) = x^4 - 4x^3$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$
Interval	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
Concavity	Up	Down	Up



Graph f using f' and f''

- Sketch the graph of $f(x) = x^{2/3}(6-x)^{1/3}$.
  - $f'(x) = \dfrac{4-x}{x^{1/3}(6-x)^{2/3}}$.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, \infty)$
$f'(x)$	$-$	$+$	$-$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

  - $f''(x) = \dfrac{-8}{x^{4/3}(6-x)^{5/3}}$.

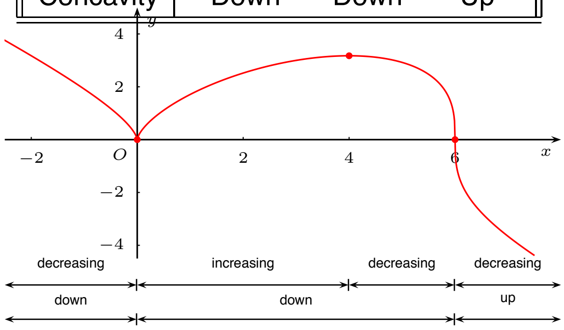
Interval	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
$f''(x)$	$-$	$-$	$+$
Concavity	Down	Down	Up

Graph f using f' and f''

- Sketch the graph of $f(x) = x^{2/3}(6-x)^{1/3}$.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, 6)$	$(6, \infty)$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\searrow$

Interval	$(-\infty, 0)$	$(0, 6)$	$(6, \infty)$
Concavity	Down	Down	Up



Some Inequalities

- Show that for all positive $x \neq 1$, $2\sqrt{x} > 3 - \dfrac{1}{x}$.
  - Let $f(x) = 2\sqrt{x} - \left(3 - \dfrac{1}{x}\right) = 2\sqrt{x} - 3 + \dfrac{1}{x}$.
    - $f'(x) = \dfrac{1}{\sqrt{x}} - \dfrac{1}{x^2} = \dfrac{1}{x^2}\left(\sqrt{x^3} - 1\right)$.
    - $\begin{cases} f'(x) > 0, & \text{if } x > 1, \\ f'(x) < 0, & \text{if } 0 < x < 1. \end{cases}$
    - $\begin{cases} f \text{ is increasing on } [1, \infty), \\ f \text{ is decreasing on } (0, 1]. \end{cases}$
  - Then for any positive $x \neq 1$, $f(x) > f(1) = 0$.
    That is,
$$2\sqrt{x} > 3 - \frac{1}{x}.$$

Some Inequalities

- We have seen that $\sin x < x$ for all $0 < x < \dfrac{\pi}{2}$.
  Show that $\dfrac{2}{\pi}x < \sin x$ when $0 < x < \dfrac{\pi}{2}$.
  - Let $g(x) = \dfrac{\sin x}{x}$ on $(0, \dfrac{\pi}{2}]$.
    - $g'(x) = \left(\dfrac{\sin x}{x}\right)' = \dfrac{\cos x(x - \tan x)}{x^2} < 0$.
    - By Increasing Test, g is decreasing on $(0, \dfrac{\pi}{2}]$.
  - For any $0 < x < \dfrac{\pi}{2}$, $g(x) > g(\dfrac{\pi}{2}) = \dfrac{\sin(\pi/2)}{\pi/2} = \dfrac{2}{\pi}$.
    That is,
$$\sin x > \frac{2x}{\pi}.$$

Some Inequalities

- Recall that $\sin x < x < \tan x$ for all $x \in (0, \frac{\pi}{2})$.

  $\tan x + \sin x$ and $2x$, which one is bigger on $(0, \frac{\pi}{2})$?
- Let $f(x) = \tan x + \sin x - 2x$.
  - $f'(x) = \sec^2 x + \cos x - 2$.

    $f''(x) = 2\sec^2 x \tan x - \sin x > 0$
  - f' is increasing on $[0, \frac{\pi}{2})$.

    Then for any $x \in (0, \frac{\pi}{2})$, $f'(x) > f'(0) = 0$.
    - f is increasing on $[0, \frac{\pi}{2})$.

      Then for any $x \in (0, \frac{\pi}{2})$, $f(x) > f(0)$.

$$\therefore \quad \tan x + \sin x > 2x.$$

Approximation

- Suppose f is continuous. Then $x \to a \Rightarrow f(x) \to f(a)$.
  - In other words, if $x \approx a$, then $f(x) \approx f(a)$.
  - For example, $\sqrt{1.1} \approx \sqrt{1} = 1$.
- **Question.** Do we have a better approximation under some stronger assumptions?
- Suppose f' is continuous. Then $x \to a \Rightarrow f'(x) \approx f'(a)$.
  - f can be approximated by the tangent line at a.
    - $f(x) \approx f(a) + f'(a)(x - a)$.
  - For example, let $f(x) = \sqrt{x}$, then $f'(x) = \dfrac{1}{2\sqrt{x}}$.
    - $f(1.1) \approx f(1) + f'(1)(1.1 - 1) = 1.05$.

Approximation

- Suppose f'' is continuous. Then
  - $x \to a \Rightarrow f'(x) \to f'(a)$ and $f''(x) \to f''(a)$.
  - Approximate f by a quadratic function P:
    - $P(a) = f(a)$, $P'(a) = f'(a)$, $P''(a) = f''(a)$.

    Let $P(x) = p + q(x-a) + r(x-a)^2$. Then

$$P(a) = f(a) \Rightarrow p = f(a);$$
$$P'(a) = f'(a) \Rightarrow q = f'(a);$$
$$P''(a) = f''(a) \Rightarrow 2r = f''(a).$$

$\therefore \quad f(x) \approx f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2}(x-a)^2.$

Approximation

- Let $f(x) = \sqrt{x}$. $f'(x) = \dfrac{1}{2}x^{-1/2}$, $f''(x) = -\dfrac{1}{4}x^{-3/2}$.
  - $f(x) \approx f(1) + f'(1)(x-1) + \dfrac{f''(1)}{2}(x-1)^2$.
  - $\sqrt{1.1} \approx 1 + \dfrac{1}{2} \times 0.1 - \dfrac{1}{8} \times 0.1^2 = 1.04875$.
- In general, assume that $f^{(n)}$ is continuous, then f can be approximated by a polynomial $P(x)$ of degree n:
  - $f(x) \approx f(a) + f'(a)(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \dfrac{f'''(a)}{3!}(x-a)^3 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$.
  - $P(a) = f(a)$, $P'(a) = f'(a)$, $\ldots$, $P^{(n)}(a) = f^{(n)}(a)$.

Optimization Problems

- What is **optimization problem**?
  - Finding **extreme values** in practical application.
    - **Maximize** areas, volumes, profits, . . . ,
    - **Minimize** distances, costs, times, ...
- How to **optimize**?
  - Understand the problem.
  - Draw a diagram.
  - Introduce notations.
  - Find relations among the variables.
  - Express the problem as finding the absolute maximum or minimum of a function $f(x)$ on a specified domain.
  - Find the absolute maximum and minimum.
    - Closed Interval Method (on finite closed interval),
    - Increasing/Decreasing Test (works for all cases).

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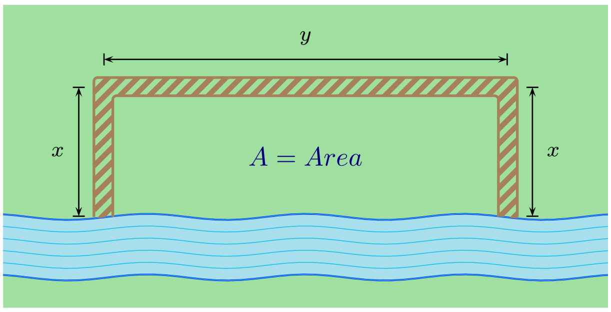
Example 1

- **Example.** A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the **largest** area?



  - Aim: Maximize $A = xy$, where
    - $2x + y = 2400, \quad x, y \geq 0.$

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Example 1

- Maximize $A = xy, \quad$ where $2x + y = 2400, \quad x, y \geq 0.$
  - $2x + y = 2400 \Rightarrow y = 2400 - 2x.$
    - $y \geq 0 \Rightarrow x \leq 1200.$
- It is equivalent to
  Finding maximum of $A(x) = x(2400 - 2x)$ on $[0, 1200]$.
  - Critical number:
    - $A'(x) = 2400 - 4x.$
    - $A'(x) = 0 \Rightarrow x = 600. \quad A(600) = 720\,000.$
  - Endpoints: $\quad x = 0, 1200. \quad A(0) = A(1200) = 0.$
  - $A(x)$ has maximum value $720\,000$ when $x = 600$.
- Conclusion: the field has the largest area $720\,000$ ft^2,

  when it has width 600 ft, and length 1200 ft.

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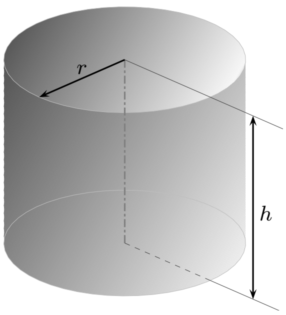
Example 2

- A cylindrical can is to be made to hold 1 liter of oil. Find the dimensions that will **minimize** the cost of the metal to manufacture the can.



  - Minimize $S = 2\pi rh + 2\pi r^2$, where
    - $V = \pi r^2 h = 1, \quad r, h > 0.$

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Example 2

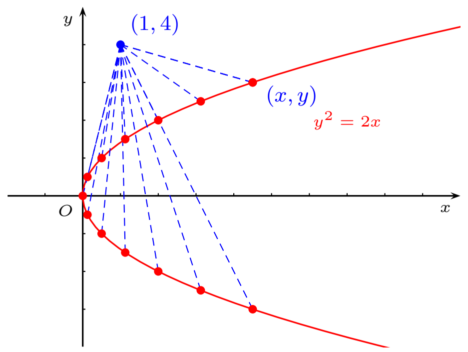
- Minimize $S = 2\pi r^2 + 2\pi rh, \quad V = \pi r^2 h = 1, \quad r, h > 0.$
  - $h = \dfrac{1}{\pi r^2} \Rightarrow S = 2\pi r^2 + \dfrac{2\pi r}{\pi r^2} = 2\pi r^2 + \dfrac{2}{r}.$

  Find the minimum of $S(r) = 2\pi r^2 + \dfrac{2}{r}$ for $r > 0$.
  - $S'(r) = 4\pi r - \dfrac{2}{r^2} = \dfrac{2}{r^2}(2\pi r^3 - 1).$
  - $S'(r) = 0 \Rightarrow r = \dfrac{1}{\sqrt[3]{2\pi}} = r_0.$
    $0 < r < r_0 \Rightarrow S'(r) < 0; \quad r > r_0 \Rightarrow S'(r) > 0.$
    $S(r)$ is decreasing on $(0, r_0]$, is increasing on $[r_0, \infty)$.

  $\therefore$ $S(r)$ has the absolute minimum at $r = r_0$.
- The cost is minimized when we choose
  radius $r = r_0 = \dfrac{1}{\sqrt[3]{2\pi}}$, and height $h = \sqrt[3]{\dfrac{4}{\pi}}$.

Example 3

- Find the point on the parabola $y^2 = 2x$ that is **closest** to the point $(1, 4)$.



  - Minimize $d = \sqrt{(x-1)^2 + (y-4)^2}$,
    - where $y^2 = 2x$.

Example 3

- Minimize $d = \sqrt{(x-1)^2 + (y-4)^2}$, with $y^2 = 2x$.
  - $d(y) = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2}, \quad (x = \frac{y^2}{2})$.

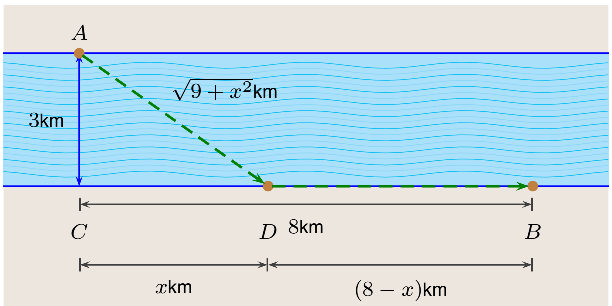
  It is equivalent to minimizing
  - $f(y) = (d(y))^2 = \left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2$ on $\mathbb{R}$.
    - $f'(y) = y^3 - 8$. Then $f'(y) = 0 \Rightarrow y = 2$.
    - If $y < 2$, $f'(y) < 0$; f is decreasing on $(-\infty, 2]$.
    - If $y > 2$, $f'(y) > 0$; f is increasing on $[2, \infty)$.
  - So $f(y)$ attains the absolute minimum at $y = 2$.

  $\therefore$ $d(y)$ attains the absolute minimum at $y = 2$. $(x = 2)$
  - Therefore, the point on $y^2 = 2x$ which is closest to $(4, 1)$ is $(2, 2)$. Moreover, the distance is

    $$d = \sqrt{(2-4)^2 + (2-1)^2} = \sqrt{5}.$$

Example 4

- A man launches his boat from point A on a bank of a straight river, 3km wide, and wants to reach point B, 8km downstream on the opposite bank, **as quick as possible**. If he can row 6km/h and run 8km/h, where should he land?



  - Minimize $T(x) = \dfrac{\sqrt{9+x^2}}{6} + \dfrac{8-x}{8}$, $0 \le x \le 8$.

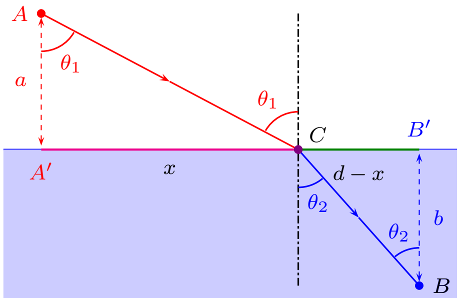
Example 4

- Minimize $T(x) = \dfrac{\sqrt{9+x^2}}{6} + \dfrac{8-x}{8}$ on $[0, 8]$.
  - $T'(x) = \dfrac{x}{6\sqrt{9+x^2}} - \dfrac{1}{8}$.
    $T'(x) = 0 \Rightarrow 8x = 6\sqrt{9+x^2} \Rightarrow 16x^2 = 81 + 9x^2$
    $\Rightarrow 7x^2 = 81 \Rightarrow x = \dfrac{9}{\sqrt{7}}\ (x > 0)$.
  - Compare the values $T(0)$, $T(8)$ and $T(\frac{9}{\sqrt{7}})$:
    - $T(0) = 1.5, \quad T(8) = \dfrac{73}{6} \approx 1.42,$ and
      $T(\frac{9}{\sqrt{7}}) = 1 + \dfrac{\sqrt{7}}{8} \approx 1.33.$
  - Therefore, he should land at $9/\sqrt{7}$ km away downstream from the starting point.

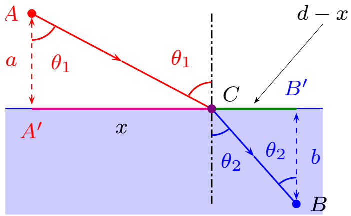
Example 5: Fermat's Principle and Snell's Law

- **Fermat's Principle**. The light travels along a path for which the time is minimized.
- **Snell's Law**. Let v_1 and v_2 be the velocity of light in air and in water respectively. Use Fermat's Principle to show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$



Example 5: Fermat's Principle and Snell's Law



- Minimize $T(x) = \dfrac{\sqrt{a^2+x^2}}{v_1} + \dfrac{\sqrt{b^2+(d-x)^2}}{v_2}$ on $[0,d]$.

$$T'(x) = \frac{x}{v_1\sqrt{a^2+x^2}} - \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}.$$

  - As x moves from 0 to d smoothly,
    - $\theta_1 \nearrow$ and $\theta_2 \searrow \quad \Rightarrow \quad T'(x) \nearrow$.
    - $T'(0) < 0$, $T'(d) > 0$, T' is continuous on $[0,d]$.
      $\Rightarrow$ there is a unique $x_0 \in (0,d)$ with $T'(x_0) = 0$.

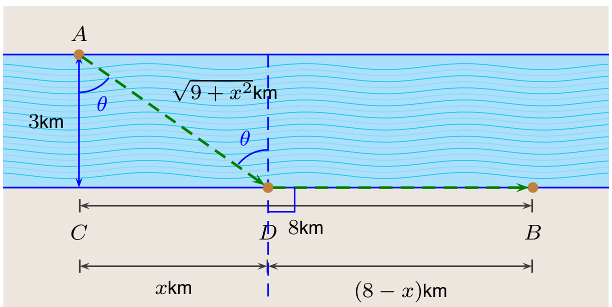
    $T'(x)$ increases smoothly from negative to positive.

    - $T'(x) < 0$ on $(0,x_0) \Rightarrow T(x) \searrow$ on $[0,x_0]$,
    - $T'(x) > 0$ on $(x_0,d) \Rightarrow T(x) \nearrow$ on $[x_0,d]$.

$\therefore$ $T(x)$ attains the min if $x = x_0$, at which $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$.

Example 5: Fermat's Principle and Snell's Law

- Recall Example 4:



- By Snell's Law, the time is minimized when $\dfrac{\sin\theta}{v_1} = \dfrac{\sin\frac{\pi}{2}}{v_2}$.
  - $\dfrac{x/\sqrt{9+x^2}}{6} = \dfrac{1}{8} \Rightarrow x = \dfrac{9}{\sqrt{7}}$.

Limits of Indeterminate Forms

- How do we compute the following limits?
  - $\lim_{x\to 0} \dfrac{1-\cos x}{x+x^2}$, $\lim_{x\to 0} \dfrac{x-\sin x}{x^3}$, $\lim_{x\to 0} \dfrac{\sqrt{1-x}-1+\frac{x}{2}}{x^2}$.

  Both the numerator and denominator tend to 0 as $x \to 0$.
  They have the $0/0$ **Indeterminate Form**.
- How to compute the following?
  - $\lim_{x\to\frac{\pi}{2}} \dfrac{\sec x}{1+\tan x}$, $\lim_{x\to\infty} \dfrac{x^2+3x}{3x^2+1}$.

  Both the numerator and denominator tend to $\pm\infty$ as $x \to \frac{\pi}{2}$ or $x \to \infty$.
  They have the ∞/∞ **Indeterminate Form**.
- **Question**. Can we evaluate the limits without using ϵ, δ-definition?
  - We may use **differentiation**.

Example

- $\lim_{x\to 0} \dfrac{1-\cos x}{x+x^2}$.
  - Let $f(x) = 1-\cos x$ and $g(x) = x+x^2$.
    - $f(0) = g(0) = 0$.
    - $f'(x) = \sin x$, and $g'(x) = 1+2x$.

$$\begin{aligned}\lim_{x\to 0}\frac{f(x)}{g(x)} &= \lim_{x\to 0}\frac{f(x)-f(0)}{g(x)-g(0)}\\ &= \lim_{x\to 0}\frac{[f(x)-f(0)]/(x-0)}{[g(x)-g(0)]/(x-0)}\\ &= \frac{\lim_{x\to 0}[f(x)-f(0)]/(x-0)}{\lim_{x\to 0}[g(x)-g(0)](x-0)}\\ &= \frac{f'(0)}{g'(0)} = \frac{0}{1+2\cdot 0} = 0.\end{aligned}$$

- However, this method does not work if $g'(0) = 0$.

l'Hôpital's Rule

- **l'Hôpital's Rule.** Let f and g be functions such that
  - $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, and
  - f and g are **differentiable** near a (except at a), and
  - $g'(x) \neq 0$ near a (except at a).

  Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided that the limit on the right hand side exists or equals $\pm\infty$.
- Guillaume Françis Antoine, Marquis de l'Hôpital (1661–1704) French Mathematician.
  - l'Hôpital's rule is published in his "*Analysis of the infinitely small to understand curves*", the first book on differential calculus.
  - The rule is discovered by Johann Bernoulli (1667–1748), a Swiss Mathematician.

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Examples

- Find $\lim_{x\to 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2}$.

$$\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1-x} - 1 + \frac{x}{2}}{x^2} &= \lim_{x\to 0} \frac{\left(\sqrt{1-x} - 1 + \frac{x}{2}\right)'}{(x^2)'} \\
= \lim_{x\to 0} \frac{\frac{-1}{2\sqrt{1-x}} + \frac{1}{2}}{2x} &= \lim_{x\to 0} \frac{\left(\frac{-1}{2\sqrt{1-x}} + \frac{1}{2}\right)'}{(2x)'} \\
&= \lim_{x\to 0} \frac{\frac{-1}{2} \cdot \frac{1}{2\sqrt{(1-x)^3}}}{2} \\
&= -\frac{1}{8}.
\end{aligned}$$

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Examples

- Find $\lim_{x \to 0} \frac{x - \sin x}{x^3}$.

$$\begin{aligned}
\lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \frac{(x - \sin x)'}{(x^3)'} \\
&= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \\
&= \lim_{x \to 0} \frac{(1 - \cos x)'}{(3x^2)'} \\
&= \lim_{x \to 0} \frac{\sin x}{6x} \\
&= \lim_{x \to 0} \frac{(\sin x)'}{(6x)'} \\
&= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}.
\end{aligned}$$

Remarks on l'Hôpital's Rule

- **Remark**.
  - The condition $x \longrightarrow a$ may be replaced by $x \longrightarrow a^+$ or $x \longrightarrow a^-$. In other words, l'Hôpital's Rule also holds for one sided limit.
  - l'Hôpital's Rule holds if $x \longrightarrow a$ if replaced by $x \longrightarrow \infty$ or $x \longrightarrow -\infty$. In other words, it holds for limit at infinity.
    - If f and g are differentiable for large x, $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} g(x) = 0$, $g'(x) \neq 0$ for large x.
      Then $\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}$, if the limit on the right hand side exists or equals $\pm\infty$.
  - l'Hôpital's Rule holds if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ is replaced by $\lim_{x \to a} |f(x)| = \lim_{x \to a} |g(x)| = \infty$.

I'Hôpital's Rule (∞/∞)

- **I'Hôpital's Rule.** Suppose that
  - $\lim_{x\to a} |f(x)| = \lim_{x\to a} |g(x)| = \infty$,
  - f and g are differentiable near a (except at a),
  - $g'(x) \neq 0$ for all x near a (except at a).

  Then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$, provided that the limit on the right side exists or equals $\pm\infty$.

- **Example.** Find $\lim_{x\to\pi/2} \dfrac{\sec x}{1+\tan x}$.

$$\begin{aligned}\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x} &= \lim_{x\to\pi/2} \frac{(\sec x)'}{(1+\tan x)'} \\ &= \lim_{x\to\pi/2} \frac{\sec x \tan x}{\sec^2 x} \\ &= \lim_{x\to\pi/2} \sin x = 1.\end{aligned}$$

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Examples

- Find $\lim_{x\to\infty} \dfrac{x^2+3x}{3x^2+1}$.

$$\begin{aligned}\lim_{x\to\infty} \frac{x^2+3x}{3x^2+1} &= \lim_{x\to\infty} \frac{(x^2+3x)'}{(3x^2+1)'} = \lim_{x\to\infty} \frac{2x+3}{6x} \\ &= \lim_{x\to\infty} \frac{(2x+3)'}{(6x)'} = \frac{2}{6} = \frac{1}{3}.\end{aligned}$$

- Find $\lim_{x\to 1} (1-x^2) \tan \dfrac{\pi x}{2}$.

$$\begin{aligned}\lim_{x\to 1} (1-x^2) \tan \frac{\pi x}{2} &= \lim_{x\to 1} \frac{1-x^2}{\cot \frac{\pi x}{2}} = \lim_{x\to 1} \frac{(1-x^2)'}{(\cot \frac{\pi x}{2})'} \\ &= \lim_{x\to 1} \frac{-2x}{-\frac{\pi}{2} \csc^2 \frac{\pi x}{2}} = \frac{-2}{-\frac{\pi}{2}\cdot 1} = \frac{4}{\pi}.\end{aligned}$$

- Convert $0\cdot\infty$ or $\infty-\infty$ indeterminate forms to $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ indeterminate forms, then apply l'Hôpital's rule.

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Correct or Wrong?

- Evaluate $\lim_{x\to 1} \dfrac{x^2+1}{2x+1}$.
  - $\lim_{x\to 1} \dfrac{x^2+1}{2x+1} = \lim_{x\to 1} \dfrac{(x^2+1)'}{(2x+1)'} = \lim_{x\to 1} \dfrac{2x}{2} = \lim_{x\to 1} x = 1$.
  - We **cannot** apply l'Hôpital's rule unless the limits of numerator and denominator are both 0 or both $\pm\infty$.
- Evaluate $\lim_{x\to\infty} \dfrac{x+\sin x}{x}$.
  - $\lim_{x\to\infty} \dfrac{x+\sin x}{x} = \lim_{x\to\infty} \dfrac{(x+\sin x)'}{x'}$
    $= \lim_{x\to\infty} (1+\cos x)$. So the limit does not exist.
  - l'Hôpital's rule is **inconclusive** if $\lim_{x\to a} \dfrac{f'(x)}{g'(x)} \neq L, \pm\infty$. We shall use squeeze theorem for this question.

Correct or Wrong?

- Evaluate $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$.

$$\begin{aligned}
\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} &= \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2+1}} \\
&= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \\
&= \lim_{x\to\infty} \frac{x/\sqrt{x^2+1}}{1} \\
&= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \cdots .
\end{aligned}$$

The l'Hôpital's rule is useful only when the evaluation of $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ is simpler than the evaluation of $\lim_{x\to a} \dfrac{f(x)}{g(x)}$.

Properties of the Exponential Function

- We can compute the number e numerically as a limit.
- **Theorem**. $e = \lim_{x\to 0}(1+x)^{1/x}$.

$$\begin{aligned}\lim_{x\to 0}(1+x)^{1/x} &= \lim_{x\to 0}\exp\left(\frac{1}{x}\ln(1+x)\right) \\ &= \exp\left(\lim_{x\to 0}\frac{\ln(1+x)}{x}\right) \\ &= \exp\left(\lim_{x\to 0}\frac{(\ln(1+x))'}{(x)'}\right) \\ &= \exp\left(\lim_{x\to 0}\frac{1/(1+x)}{1}\right) \\ &= \exp(1) = e.\end{aligned}$$

  - **Remark**. Let $y = 1/x$. Then $x \to 0^+ \Leftrightarrow y \to \infty$.
    - $e = \lim_{y\to\infty}(1+1/y)^y = \lim_{n\to\infty}(1+1/n)^n$.

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Example

- Evaluate $\lim_{x\to 0^+} x^x$.

$$\begin{aligned}\lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{x\ln x} = \exp\left(\lim_{x\to 0^+} x\ln x\right) \\ &= \exp\left(\lim_{x\to 0^+}\frac{\ln x}{1/x}\right) = \exp\left(\lim_{x\to 0^+}\frac{1/x}{-1/x^2}\right) \\ &= \exp\left(\lim_{x\to 0^+}(-x)\right) = \exp(0) = 1.\end{aligned}$$

- In general, in order to evaluate $\lim_{x\to a}\left(f(x)^{g(x)}\right)$, we use

$$\begin{aligned}\lim_{x\to a}\left(f(x)^{g(x)}\right) &= \lim_{x\to a}\exp\left(g(x)\ln(f(x))\right) \\ &= \exp\left(\lim_{x\to a} g(x)\ln f(x)\right) = \cdots .\end{aligned}$$

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Continuously Compounded Interest

- Initial deposit: A dollars; Interest rate (year): r.
  - Suppose the interest is credited n times per year.
    - After one year, we have $A\left(1+\frac{r}{n}\right)^n$ dollars.

    It seems that we will get more if n gets larger.
- **Question.** What will we get after one year if $n \to \infty$, in other words, if the **interest** is **continuously compounded**?
  - $$\lim_{n\to\infty} A\left(1+\frac{r}{n}\right)^n = \lim_{x\to\infty} A\left(1+\frac{r}{x}\right)^x$$
    $$= A\lim_{x\to\infty}\exp\left(x\ln\left(1+\frac{r}{x}\right)\right) = A\exp\left(\lim_{x\to\infty}\frac{\ln\left(1+\frac{r}{x}\right)}{\frac{1}{x}}\right)$$
    $$= A\exp\left(\lim_{x\to\infty}\frac{\left(1+\frac{r}{x}\right)^{-1}\frac{r}{-x^2}}{-\frac{1}{x^2}}\right)$$
    $$= A\exp\left(\lim_{x\to\infty} r\left(1+\frac{r}{x}\right)^{-1}\right) = A\exp(r) = Ae^r.$$

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