MA1521 CALCULUS FOR COMPUTING

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0: Pre-Calculus

- Understand the basic properties of the following functions:
  - Polynomials;
  - Rational functions;
  - Root functions;
  - Trigonometric and inverse trigonometric functions;
  - Logarithm and exponential functions.
- Know how to compute
  - The domain of these functions;
  - The composite of these functions.
- The graphs of simple functions.
  - $y = ax + b, \quad y = ax^2 + bx + c,$
  - $y = \sin x, \quad y = \cos x, \quad y = \tan x,$
  - $y = \sin^{-1} x, \quad y = \tan^{-1} x,$
  - $y = e^x, \quad y = \ln x, \quad \ldots$

0: Pre-Calculus

- Products of Vectors in $\mathbb{R}^3$.
  - Dot product: $\mathbf{u} \bullet \mathbf{v}$.
    - Example: determine the angle between two vectors.
  - Cross product: $\mathbf{u} \times \mathbf{v}$.
    - Example: evaluate the normal vector of a plane.
- Write the equation of a line.
  - It has direction vector $\mathbf{u}$ and passes through point A.
  - It passes through two points A and B.
  - The intersection of two planes.
- Write the equation of a plane.
  - It has normal vector $\mathbf{u}$ and passes through point A.
  - It passes through three points A, B and C.
  - It contains a line ℓ and passes through point A.

0: Pre-Calculus

- The complex numbers $\mathbb{C}$.
  - $z = x + iy, \quad x, y \in \mathbb{R}$ and $i^2 = -1$.
  - Correspondence between $\mathbb{R}^2$ and $\mathbb{C}$: $(x, y) \leftrightarrow x + iy$.
- Given complex numbers, evaluate
  - Sum, substraction, multiplication, division.
  - Absolute value: $|z| = \sqrt{x^2 + y^2}$.
  - Argument: $\theta = \arg z, \quad \sin\theta = \dfrac{y}{|z|}, \cos\theta = \dfrac{x}{|z|}$.
- Polar form of complex numbers, $z = |z|e^{i\theta}$, $\theta = \arg z$.
  - Multiplication: $z_1 z_2 = |z_1|\,|z_2|\,e^{i(\theta_1+\theta_2)}$.
  - Power: $z^n = |z|^n e^{in\theta}$.
  - Trigonometric form: $z = |z|(\cos\theta + i\sin\theta)$.

1: Limit and Continuity

- Limit: $\lim\limits_{x\to a} f(x) = L$.
  - $x \to a \ (x \neq a) \Rightarrow f(x) \to L$.
- f is said to be continuous at a if $\lim\limits_{x\to a} f(x) = f(a)$.
  - Removable and jump discontinuities.
- Find limits:
  - If f is continuous at a, then $\lim\limits_{x\to a} f(x) = f(a)$.
  - Factorization: $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x - 1}$.
  - Rationalization: $\lim\limits_{x\to 0} \dfrac{\sqrt{x+4} - 2}{x}$.
  - Left- and right-hand limits:
    - $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L \Leftrightarrow \lim\limits_{x\to a} f(x) = L$.

1: Limit and Continuity

- Find limits: (Cont'd)
  - Squeeze thm: $f(x) \le g(x) \le h(x)$ for all x near a.
    - $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} g(x) = L$.
  - l'Hôpital's rule: $0/0$ or ∞/∞ form:
    - $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ (with lots of restrictions).
  - $\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$ if f is continuous.
    - $$\begin{aligned}\lim_{x\to a} f(x)^{g(x)} &= \lim_{x\to a} \exp(g(x)\ln f(x)) \\ &= \exp\left(\lim_{x\to a} g(x)\ln f(x)\right) \\ &= \exp\left(\lim_{x\to a} \frac{\ln f(x)}{1/g(x)}\right) = \cdots\end{aligned}$$

2: Derivatives

- Definition of derivative:
  - $\dfrac{df}{dx} = f'(x) = \lim_{h\to x} \dfrac{f(x+h)-f(x)}{h}$.
- Differentiation formulas:
  - $(cf)' = cf'$, $(f\pm g)' = f' \pm g'$,$(fg)' = f'g + fg'$.
  - $(f/g)' = (f'g - fg')/g^2$.
  - Chain rule: $\dfrac{dz}{dx} = \dfrac{dz}{dy}\,\dfrac{dy}{dx}$.
- Differentiable functions:
  - Polynomials, rational functions.
  - Power functions: $(x^r)' = rx^{r-1}$.
  - Trigonometric functions: $(\sin x)' = \cos x, \ldots$
  - Inverse trigonometric functions:
    - $(\sin^{-1} x)' = \cdots$, $(\tan^{-1} x)' = \cdots, \ldots$
  - Logarithm function $(\ln x)' = 1/x$, $\quad (a^x)' = a^x \ln a$.

2: Derivatives

- Implicit differentiation: $\dfrac{d}{dx}F(x,y)=0$ to find $\dfrac{dy}{dx}$.
    - Using multi-variable calculus: $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$.

  Logarithmic differentiation:
    - $y=f(x)g(x)\Rightarrow \ln|y|=\ln|f(x)|+\ln|g(x)|$.
    - $y=f(x)^{g(x)}\Rightarrow \ln y=g(x)\ln f(x)$.
- Derivative of inverse function:
    - $(f^{-1})'(b)=\dfrac{1}{f'(a)}$ if $f(a)=b$ and $f'(a)\neq 0$.
- Parametric equations: $x=x(t)$ and $y=y(t)$.
    - $\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt},\quad \dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy/dt}{dx/dt}\right)=\cdots.$

2: Applications of Derivative

- Extreme values:
    - global max and min, local max and min.
- Fermat's theorem:
    - If f has a local max/min at c and $f'(c)$ exists, then $f'(c)=0$.
- Closed interval method: Find the global max/min of a continuous function f on a finite closed interval $[a,b]$.
    - Evaluate $f(x)$ at end points $x=a$, $x=b$.
    - Evaluate $f(x)$ at critical points in (a,b):
        - number $c\in(a,b)$ such that $f'(c)$ does not exist,
        - number $c\in(a,b)$ such that $f'(c)=0$.
    - Compare $f(x)$ at end points and critical points.
        - Largest $\Rightarrow$ max; smallest $\Rightarrow$ min.

2: Applications of Derivative

- Increasing Test: Suppose f is continuous on $[a, b]$ and differentiable on (a, b).
  - $f'(x) > 0$ on $(a, b) \Rightarrow f$ is increasing on $[a, b]$.
  - $f'(x) < 0$ on $(a, b) \Rightarrow f$ is decreasing on $[a, b]$.
  - $f'(x) = 0$ on $(a, b) \Leftrightarrow f$ is constant on $[a, b]$.
- Optimization problem: (Single-variable)
  - Express the problem as finding global max/min of $y = f(x)$ on domain A.
  - How to maximize or minimize $y = f(x)$ on A?
    - If A is a finite closed interval $[a, b]$, use the closed interval method.
    - If A is not a finite closed interval, use increasing/decreasing test.

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2: Applications of Derivative

- Concavity:
  - f is concave up (resp. down) on interval I
    $\Leftrightarrow f$ is above (resp. below) all tangent lines on I
    $\Leftrightarrow f'$ is increasing (resp. decreasing) on I.
  - Concavity test:
    - $f''(x) > 0$ on $I \Rightarrow f$ is concave up on I.
    - $f''(x) < 0$ on $I \Rightarrow f$ is concave down on I.
- First derivative test: Let f be continuous at a critical pt c.
  - f' changes from $+$ to $-$: local max at c;
  - f' changes from $-$ to $+$: local min at c;
  - f' does not change sign: neither at c.
- Second derivative test: (see Chapter 5).
  - $f''(c) = 0$ and $f'(c) > 0 \Rightarrow$ local min at c;
  - $f''(c) = 0$ and $f'(c) < 0 \Rightarrow$ local max at c.

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3: Sequences and Series

- Power series representation:
  - $f(x) = \sum_{n=0}^{\infty} c_n x^n$. e.g., $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$.
  - Radius of convergence: $R = L^{-1}$.
    - $L = \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ or $L = \lim_{n\to\infty} \sqrt[n]{|c_n|}$.

    $\sum_{n=0}^{\infty} c_n x^n$ converges if $|x| < R$ and diverges if $|x| > R$.
- Taylor's theorem:
  - If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then $c_n = \frac{f^{(n)}(0)}{n!}$.

  $\therefore$ $f^{(n)}(0) = n!\, c_n$.

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3: Sequences and Series

- How to determine whether $\sum_{n=0}^{\infty} a_n$ converges or diverges?
  - Suppose $a_n \geq 0$ for all n.
    - If $\lim_{n\to\infty} a_n \neq 0$, then it diverges.
    - If $\lim_{n\to\infty} a_n = 0$, use (limit) comparison test.
      Compare with a known series, such as geometric series or p-series.
  - Suppose a_n are not always positive.
    - If $\lim_{n\to\infty} a_n \neq 0$, then it diverges.
    - Suppose $\lim_{n\to\infty} a_n = 0$.

      If $\sum_{n=0}^{\infty} a_n$ is an alternating series, use alternating series test.
      Otherwise, use absolute convergence test.

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4: Partial Derivatives

- Multi-variable function $z = f(x, y)$.
  - Partial derivatives: $f_x(x, y) = \dfrac{\partial z}{\partial x}$, $f_y(x, y) = \dfrac{\partial z}{\partial y}$.
    - Gradient vector: $\nabla f(x, y) = (f_x(x, y), f_y(x, y))$.
  - Directional derivative along unit vector $\mathbf{u}$: $(|\mathbf{u}| = 1)$
    - $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \bullet \mathbf{u}$.
  - Tangent plane at (x_0, y_0, z_0):
    - $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
  - Linearization of f at (x_0, y_0):
    - $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

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4: Partial Derivatives

- Chain rule: $z = f(x, y)$ and $x = x(t)$, $y = y(t)$.
  - $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial z}{\partial y}\dfrac{dy}{dt}$.

  Suppose $z = f(x, y)$ and $x = x(s, t)$, $y = y(s, t)$.
  - $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$.
  - $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$.
- Implicit differentiation (Revisited):
  - $f(x, y) = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{f_x(x, y)}{f_y(x, y)}$.

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4: Partial Derivatives

- Second Partial Derivatives. $z = f(x, y)$.
  - $f_{xx} = \dfrac{\partial^2 z}{\partial x^2}$, $f_{yy} = \dfrac{\partial^2 z}{\partial y^2}$, $f_{xy} = f_{yx} = \dfrac{\partial^2 z}{\partial x \partial y}$.
- Functions of three variables: $w = f(x, y, z)$.
  - Partial and second partial derivatives,
    - $f_x, f_y, f_z, f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yz}, f_{zx}$.
  - Gradient $\nabla f(x, y, z) = (f_x, f_y, f_z)$,
  - Directional derivative $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \bullet \mathbf{u}$.
  - Chain rule: Suppose $x = x(t)$, $y = y(t)$, $z = z(t)$.
    - $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial w}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial w}{\partial z}\dfrac{dz}{dt}$.
  - Tangent plane to $f(x, y, z) = c$ at $P(x_0, y_0, z_0)$:
    $f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0$.

5: Optimization

- Definition of extreme values:
  - Global maximum/minimum, local maximum/minimum.
- First derivative test for local extreme values:
  - Suppose $f(x, y)$ has local extreme value at (a, b).
    - If f_x and f_y exist, then $f_x(a, b) = f_y(a, b) = 0$.
  - A point (a, b) in the domain is a critical point if
    - $f_x(a, b) = f_y(a, b) = 0$, or
    - at least one of $f_x(a, b)$ and $f_y(a, b)$ does not exist.
  - If $f(x, y)$ has local extreme value at (a, b),
    - then (a, b) is a critical point of f.
  - (a, b) is said to be a saddle point of f if
    - (a, b) is a critical point of f, but f does not have local extreme value at (a, b).

5: Optimization

- Second derivative test for two-variable functions.
  - Let $H(x, y) = f_{xx}f_{yy} - (f_{xy})^2$.

  Suppose $f_x(a, b) = f_y(a, b) = 0$.
  - $H(a, b) < 0 \Rightarrow$ saddle point at (a, b).
  - $H(a, b) > 0$ & $f_{xx}(a, b) > 0 \Rightarrow$ local min at (a, b).
  - $H(a, b) > 0$ & $f_{xx}(a, b) < 0 \Rightarrow$ local max at (a, b).
- Lagrange multiplier: Find the local maximum and minimum of $z = f(x, y)$ subject to the restriction $g(x, y) = 0$.
  - Solve $f_x = \lambda g_x$, $f_y = \lambda g_y$ and $g(x, y) = 0$.
- Lagrange multiplier: Find the local maximum/minimum of $w = f(x, y, z)$ subject to $g(x, y, z) = h(x, y, z) = 0$.
  - Solve $f_x = \lambda g_x + \mu h_x$, $f_y = \lambda g_y + \mu h_y$, $f_z = \lambda g_z + \mu h_z$, $g(x, y, z) = 0$, $h(x, y, z) = 0$.

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6: Integrals

- Definite integral defined using Riemann sum
  - $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$

  It represents the net area bounded between the curve $y = f(x)$, and the x-axis from a to b.
- Basic properties of definite integrals.
  - $\displaystyle\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$
  - $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$
- Fundamental Theorem of Calculus (Part I).
  - Let f be continuous on $[a, b]$, $g(x) = \displaystyle\int_a^x f(t)\,dt.$
    - g is continuous on $[a, b]$ and $g'(x) = f(x)$ on (a, b).

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6: Integrals

- Fundamental Theorem of Calculus (Part II).
  - Let f be a continuous function. If F is continuous on $[a, b]$ and $F'(x) = f(x)$ on (a, b), then
    - $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$.
- Indefinite integral:
  - $\displaystyle\int f(x)\,dx = F(x) + C \Leftrightarrow f(x) = F'(x)$.
- Substitution rule (I).
  - Let $u = g(x)$. $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$.

    Examples: $\displaystyle\int \tan x\,dx, \int \sec x\,dx, \ldots$

6: Integrals

- Improper integrals.
  - $\displaystyle\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$.
  - $\displaystyle\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$.
- Substitution rule (II).
  - Let $x = g(t)$. $\displaystyle\int f(x)\,dx = \int f(g(t))g'(t)\,dt$.

  Note: $x = g(t)$ must be a 1-1 function.
  - $t = x^{1/n}$, $t = \ln x$, $t = e^x$, $\ldots$
  - $t = \sin^{-1} x$, $t = \tan^{-1} x$, $t = \sec^{-1} x$, $t = \tan\dfrac{x}{2}$, $\ldots$
- Integral by parts:
  - $\displaystyle\int u\,dv = uv - \int v\,du$.

6: Integrals

- Integration of rational functions $\int \frac{A(x)}{B(x)}\,dx$.
  1. If $\deg A \geq \deg B$, write $A(x) = B(x)Q(x) + R(x)$.
     - $\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)}, \quad \deg R < \deg B.$
  2. Factorize $B(x)$ into linear and quadratic factors.
  3. Convert $\frac{A(x)}{B(x)}$ into its partial fraction form.
     - Note that the number of terms of the partial fraction form equals the degree of B.
  4. Integrate term by term.

6: Applications of Integration

- Washer method: Suppose the solid is formed by rotating the region bounded between $y = f(x)$ and the x-axis from a to b about the x-axis. Then
  - $V = \int_a^b \pi(\text{radius})^2\,dx = \int_a^b \pi(f(x))^2\,dx.$

  Suppose the solid is formed by rotating the region bounded between $y = f(x)$ and $y = g(x)$, $f(x) \geq g(x)$, from a to b about the x-axis. Then
  - $V = \int_a^b \pi[(\text{outer radius})^2 - (\text{inner radius})^2]\,dx.$
- Cylindrical shell method: Suppose the solid is formed by rotating the region bounded between $y = f(x)$ an the x-axis from a to b about the y-axis. $(f(x) \geq 0,\ a \geq 0)$
  - $V = \int_a^b 2\pi(\text{radius})(\text{height})\,dx = \int_a^b 2\pi x f(x)\,dx.$

6: Applications of Integration

- Arc length: Suppose f is smooth (i.e., f' is continuous). Then the arc length of $y = f(x)$, $a \leq x \leq b$, is
  - $L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$
- Surface of revolution: Suppose f is smooth. Then the surface formed by rotating the arc $y = f(x)$, $a \leq x \leq b$, about the x-axis is
  - $A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$

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7: Ordinary Differential Equations

- Separable equation: $\dfrac{dy}{dx} = f(x)g(y)$.
  - $\displaystyle\int \frac{1}{g(y)}\, dy = \int f(x)\, dx.$
- Homogeneous of degree zero: $\dfrac{dy}{dx} = F(x, y)$,

  $F(tx, ty) = F(x, y)$ for all $t \neq 0$.

  Let $z = \dfrac{y}{x}$. Then the equation becomes
  - $x\dfrac{dz}{dx} + z = F(1, z) \Rightarrow \dfrac{dz}{dx} = \dfrac{F(1, z) - z}{x}$,

  which is separable in x and z.

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7: Ordinary Differential Equations

- First order linear equation: $\dfrac{dy}{dx} + p(x)y = q(x)$.
  - Find an integrating factor $v(x) = \exp\left(\int p(x)\,dx\right)$.
  - $y = \dfrac{1}{v(x)} \int v(x)q(x)\,dx$.
- Bernoulli's equation: $\dfrac{dy}{dx} + p(x)y = q(x)y^n$, $(n \neq 0, 1)$.
  - Let $z = y^{1-n}$. Then the equation becomes
    - $\dfrac{dz}{dx} + (1-n)p(x)z = (1-n)q(x)$.

    This is a linear equation in y.

7: Ordinary Differential Equations

- Second Order linear equation: $\dfrac{d^2y}{dx^2} + p\dfrac{dy}{dx} + qy = r(x)$.

  1. Characteristic equation $\lambda^2 + p\lambda + q = 0$.
     - $\lambda_1 \neq \lambda_2$ are real: $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_2 x}$.
     - $\lambda_1 = \lambda_2 = \lambda$: $y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$.
     - $\lambda_{1,2} = a \pm bi$: $y_1 = e^{ax}\cos bx$, $y_2 = e^{ax}\sin bx$.
  2. $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$.
  3. $v_1 = \displaystyle\int \frac{-y_2r(x)}{W(y_1,y_2)}\,dx$, $v_2 = \displaystyle\int \frac{y_1r(x)}{W(y_1,y_2)}\,dx$.
  4. $y = C_1y_1 + C_2y_2 + v_1y_1 + v_2y_2$.
  5. Determine C_1 and C_2 using initial conditions.