MA1521 CALCULUS FOR COMPUTING

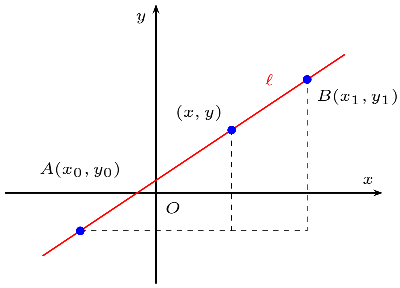
Wang Fei

matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-16
Tel: 6516-2937

The Straight Line

- A **straight line** is uniquely determined by 2 distinct points.
  - Let $A(x_0, y_0)$ and $B(x_1, y_1)$ be distinct points on $\mathbb{R}^2$.
    Consider the line ℓ passing through A and B:

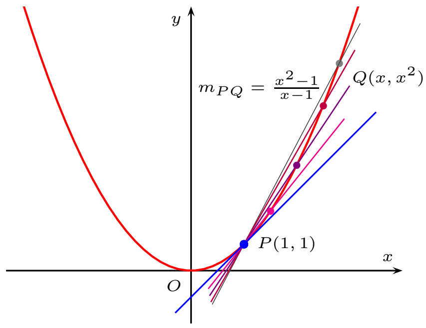


Let (x, y) be a point on ℓ. Then $\dfrac{y - y_0}{x - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0}$.

- $m = \dfrac{y_1 - y_0}{x_1 - x_0}$ is called the **slope** (or **gradient**) of ℓ.

The Tangent Line

- Consider the parabola $y = x^2$:

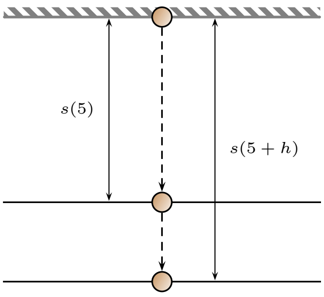


  - Fix $P(1, 1)$. Let Q be another point on the curve.
  - Connect P and Q, and let Q approach P.
    The resulting line is the **tangent line** of $y = x^2$ at P.
    - Slope $m = \lim\limits_{Q \to P} m_{PQ} = \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.

Example

- A ball is dropped from a tower 450 m above the ground. Find its **instantaneous velocity** after 5 seconds.

$s(t) = \frac{1}{2}gt^2 = 4.9t^2$



  - $\overline{V}_{[5,5+h]} = \dfrac{\Delta s}{\Delta t} = \dfrac{4.9(5+h)^2 - 4.9(5)^2}{(5+h)-5}$.
  - **Instantaneous velocity** at $t = 5$: $v = \lim_{h \to 0} \overline{V}_{[5,5+h]}$.

Definition of Limit

- **Definition.** If $f(x)$ is **arbitrarily close** to L by taking x to be **sufficiently close** (but not equal) to a, then we write

$$\lim_{x \to a} f(x) = L.$$

We say the **limit** of $f(x)$, as x approaches a, equals L.

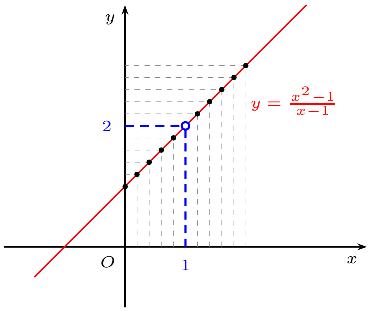
  - Sometimes we may simply write:

$$x \to a \quad \Rightarrow \quad f(x) \to L.$$

- **Note.** The limit $\lim_{x \to a} f(x)$
  - **depends** only on the values of $f(x)$ for "x near a".
  - The limit is **independent** to the value of $f(x)$ "at a".

Example

- Find $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$.
  - Consider the graph of $f(x) = \dfrac{x^2 - 1}{x - 1}$:



  - $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x \to 1} f(x) = 2$.

Limit Laws

- Let c be a real number. Then the constant function $f(x) = c$ is not affected by the behavior of x.
  - Let $a \in \mathbb{R}$. $\quad x \to a \Rightarrow c \to c$.

  $\therefore$ $\lim_{x \to a} c = c$.
- Let $a \in \mathbb{R}$. It is trivial that
  - $x \to a \Rightarrow x \to a$.

  $\therefore$ $\lim_{x \to a} x = a$.
- Suppose $\lim_{x \to a} f(x) = L$. $\quad$ Let c be a constant.
  - $x \to a \Rightarrow f(x) \to L \Rightarrow cf(x) \to cL$.

  $\therefore$ $\lim_{x \to a} (cf(x)) = cL = c \lim_{x \to a} f(x)$.

Limit Laws

- Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.
  - $x \to a \Rightarrow \begin{cases} f(x) \to L \\ g(x) \to M \end{cases} \Rightarrow f(x) + g(x) \to L + M$

  $\therefore \lim_{x\to a}(f(x) + g(x)) = L + M = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

  Similarly, we have
  - $\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$,
  - $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$,
  - $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided that $\lim_{x\to a} g(x) \neq 0$.

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Limit Laws

- The product law $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$ can be generalized to the product of n functions:

$$\begin{aligned}&\lim_{x\to a} [f_1(x)f_2(x)\cdots f_n(x)] \\ &\quad = \lim_{x\to a} f_1(x) \lim_{x\to a} f_2(x) \cdots \lim_{x\to a} f_n(x).\end{aligned}$$

  In particular, if $f := f_1 = f_2 = \cdots = f_n$, it becomes

$$\lim_{x\to a} f(x)f(x)\cdots f(x) = \lim_{x\to a} f(x) \lim_{x\to a} f(x) \cdots \lim_{x\to a} f(x)$$

$$\lim_{x\to a} \underbrace{f(x)f(x)\cdots f(x)}_{n \text{ times}} = \underbrace{\lim_{x\to a} f(x) \lim_{x\to a} f(x) \cdots \lim_{x\to a} f(x)}_{n \text{ times}}$$

  That is, $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$.

- We can show: $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, $f(x) \geq 0$.

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Examples

- Find $\lim_{x\to 1}(2x^2-3x+4)$ and $\lim_{x\to 5}(2x^2-3x+4)$.

$$\begin{aligned}\lim_{x\to 1}(2x^2-3x+4) &= \lim_{x\to 1}2x^2-\lim_{x\to 1}3x+\lim_{x\to 1}4\\ &= 2\lim_{x\to 1}x^2-3\lim_{x\to 1}x+4\\ &= 2\cdot 1^2-3\cdot 1+4\\ &= 3.\\ \lim_{x\to 5}(2x^2-3x+4) &= \lim_{x\to 5}2x^2-\lim_{x\to 5}3x+\lim_{x\to 5}4\\ &= 2\lim_{x\to 5}x^2-3\lim_{x\to 5}x+4\\ &= 2\cdot 5^2-3\cdot 5+4\\ &= 39.\end{aligned}$$

- Let $f(x)=2x^2-3x+4$. It seems that for any $a\in\mathbb{R}$

$$\lim_{x\to a}f(x)=f(a).$$

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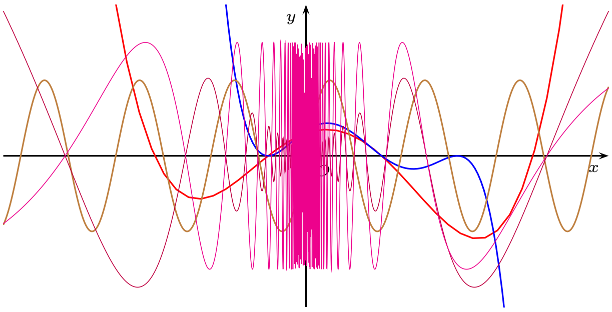
Substitution Property

- **Theorem**. Let f be a polynomial or a rational function. If a is in the domain of f, then

$$\lim_{x\to a}f(x)=f(a).$$

- Do the following functions have the same property?



  - **Algebraically**:
    - the value of $f(x)$, as x tends to a, is close to $f(a)$.
  - **Geometrically**:
    - the graph of $f(x)$ has no interruption at a.
  - A function which satisfies the above condition is said to be **continuous**.

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Definition of Continuity

- A function f is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a).$$

  If f is not continuous at a, we say f is **discontinuous** at a.
- **Remark.** The definition consists of the 3 properties:

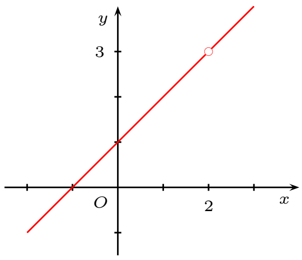
  i) f is defined at a (i.e., a is in the domain of f), and

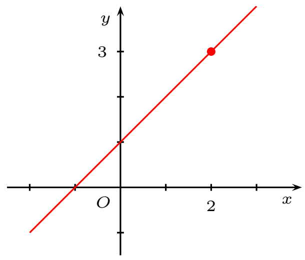
  ii) $\lim_{x \to a} f(x)$ exists, and

  iii) $\lim_{x \to a} f(x) = f(a)$.

Examples of Discontinuity

- $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.
  - For $x \neq 2$, $f(x) = \dfrac{(x-2)(x+1)}{x-2} = x + 1$.





  Since $\lim_{x \to 2} f(x)$ exists, we can redefine $f(2) = \lim_{x \to 2} f(x) = 3$ to remove the discontinuity at 2.
  - Such discontinuity is a **removable discontinuity**.

Properties of Continuous Function

- Suppose f and g are continuous at a.
  - Let c be a constant.

$$\lim_{x\to a}(cf(x)) = c \cdot \lim_{x\to a} f(x) = cf(a).$$

$\therefore \quad cf$ is continuous at a.

$$\begin{aligned}\lim_{x\to a}(f+g)(x) &= \lim_{x\to a}(f(x)+g(x)) \\ &= \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\ &= f(a)+g(a) = (f+g)(a).\end{aligned}$$

$\therefore \quad f+g$ is continuous at a.

  - Similarly, replacing "$+$" by "$-$" or "$\times$", we can show that $f-g$ and fg are continuous at a as well.

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Properties of Continuous Function

- Suppose f and g are continuous at a, $\quad (g(a) \neq 0)$.

$$\begin{aligned}\lim_{x\to a}(f/g)(x) &= \lim_{x\to a}(f(x)/g(x)) \\ &= \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{f(a)}{g(a)} = (f/g)(a).\end{aligned}$$

- **Theorem.** Let f and g be functions continuous at a. Then
  - cf is continuous at a, where c is a constant,
  - $f+g$ is continuous at a,
  - $f-g$ is continuous at a,
  - fg is continuous at a,
  - f/g is continuous at a, provided that $g(a) \neq 0$.

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Continuous Functions

- The following classes of functions are continuous:
  - Polynomials (on $\mathbb{R}$);
  - Rational functions (on its domain);
  - Algebraic functions (on the interior of domain);
  - Trigonometric functions (on its domain);
  - Exponential functions (on $\mathbb{R}$);
  - Logarithmic functions (on $\mathbb{R}^+$).
- Suppose $\lim\limits_{x\to a} f(x) = b$, and g is continuous at b.
  - Then $\lim\limits_{x\to a} g(f(x)) = g(b) = g\left(\lim\limits_{x\to a} f(x)\right)$.

  In other words, the $\lim$ operator commutes with continuous functions.
- **Theorem**. If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

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Examples

- $\lim\limits_{x\to 1} \dfrac{\sec(x^2-1)}{\sqrt{5-x}}$.
  - $\lim\limits_{x\to 1} \dfrac{\sec(x^2-1)}{\sqrt{5-x}} = \dfrac{\sec(1^2-1)}{\sqrt{5-1}} = \dfrac{1}{2}$.
- $\lim\limits_{x\to 0} \dfrac{\ln(2x^2+x+1)}{3e^{2x}+2}$.
  - $\lim\limits_{x\to 0} \dfrac{\ln(2x^2+x+1)}{3e^{2x}+2} = \dfrac{\ln(2\cdot 0^2+0+1)}{3\cdot e^{2\times 0}+2} = 0$.
- $\lim\limits_{x\to 1} \tan^2\left(\dfrac{\pi}{\sqrt{x}+3}\right)$.
  - $\lim\limits_{x\to 1} \tan^2 \dfrac{\pi}{\sqrt{x}+3} = \tan^2 \dfrac{\pi}{\sqrt{1}+3} = \tan^2 \dfrac{\pi}{4} = 1$.

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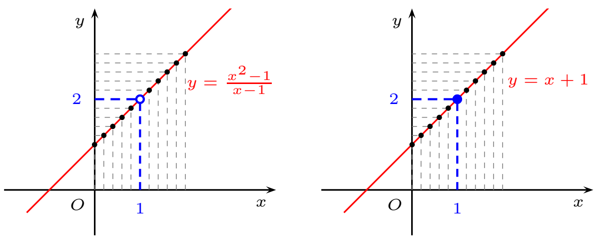
Example

- Let $f(x) = \dfrac{x^2-1}{x-1}$. Find $\lim\limits_{x\to 1} f(x)$.
  - $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \dfrac{\lim_{x\to 1}(x^2-1)}{\lim_{x\to 1}(x-1)} = \dfrac{0}{0}$. ×
  - The **direct substitution** does not work since
    - $x = 1$ is **NOT** in the domain of $f(x)$.
  - Recall that $\lim\limits_{x\to 1} f(x)$ only depends on the value of $f(x)$ when x is "near" 1, not "at" 1.
    - If $x \neq 1$, $f(x) = \dfrac{(x-1)(x+1)}{x-1} = x+1$.
    - $f(x)$ and $x+1$ are the same near 1.
      Can we say that they have the same limits at $x = 1$?

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x+1)\,?$$

Example

- Let $f(x) = \dfrac{x^2-1}{x-1}$. Find $\lim\limits_{x\to 1} f(x)$.
  Let $g(x) = x+1$. Then $f(x) = g(x)$ for all $x \neq 1$.



  - Therefore,
    - $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} g(x) = \lim\limits_{x\to 1}(x+1) = 1+1 = 2$.

An Intuitive Conclusion

- **Theorem**. If $f(x) = g(x)$ for all x near a except possibly at a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

- **Example**. Let $f(x) = \dfrac{\sqrt{x^2+9}-3}{x^2}$. Find $\lim_{x \to 0} f(x)$.
  - Find a function which agrees with f for $x \neq 0$.

$$\frac{\sqrt{x^2+9}-3}{x^2} = \frac{\left(\sqrt{x^2+9}-3\right)\left(\sqrt{x^2+9}+3\right)}{x^2\left(\sqrt{x^2+9}+3\right)}$$
$$= \frac{(x^2+9)-3^2}{x^2\left(\sqrt{x^2+9}+3\right)} = \frac{1}{\sqrt{x^2+9}+3}.$$

  - $\lim_{x \to 0} f(x) = \lim_{x \to 0} \dfrac{1}{\sqrt{x^2+9}+3} = \dfrac{1}{\sqrt{0^2+9}+3} = \dfrac{1}{6}.$

Limits of Rational Functions

- We use the method of **factorization**:
  1. Given a rational function, factorize both numerator and denominator.
  2. Cancellate common factors.
  3. Reduce to the limit of a continuous function.
- **Example.** $\lim_{x \to 3} \dfrac{x^2+x-12}{9-x^2}$.
  - $\lim_{x \to 3} \dfrac{x^2+x-12}{9-x^2} = \lim_{x \to 3} \dfrac{(x+4)(x-3)}{(3-x)(3+x)}$
    $= \lim_{x \to 3} \dfrac{x+4}{-(3+x)} = \dfrac{3+4}{-(3+3)} = -\dfrac{7}{6}.$

Limits of Algebraic Functions

- We use the method of **rationalization**.
  1. Given an algebraic function, multiply the **conjugate** of the numerator or/and the denominator both sides.
  2. **rationalize** the numerator or/and the denominator.
     - $(\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}) = a-b$;
     - $(\sqrt[3]{a}-\sqrt[3]{b})(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}) = a-b; \ldots$
  3. Factorize the numerator and/or the denominator if necessary.
  4. Cancellate common factors.
  5. Reduce to the limit of a continuous function.

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Examples

- $\lim\limits_{x\to -1} \dfrac{\sqrt{x+10}-\sqrt{8-x}}{x^2-1}$.
  - $$\begin{aligned}
&\lim_{x\to -1} \frac{\sqrt{x+10}-\sqrt{8-x}}{x^2-1} \\
&= \lim_{x\to -1} \frac{(\sqrt{x+10}-\sqrt{8-x})(\sqrt{x+10}+\sqrt{8-x})}{(x^2-1)(\sqrt{x+10}+\sqrt{8-x})} \\
&= \lim_{x\to -1} \frac{(x+10)-(8-x)}{(x^2-1)(\sqrt{x+10}+\sqrt{8-x})} \\
&= \lim_{x\to -1} \frac{2(x+1)}{(x+1)(x-1)(\sqrt{x+10}+\sqrt{8-x})} \\
&= \lim_{x\to -1} \frac{2}{(x-1)(\sqrt{x+10}+\sqrt{8-x})} \\
&= \frac{2}{((-1)-1)(\sqrt{(-1)+10}+\sqrt{8-(-1)})} = -\frac{1}{6}.
\end{aligned}$$

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Examples

- $\lim_{x\to 2} \dfrac{\sqrt[3]{x}-\sqrt[3]{2}}{x-2}$.
  - $$\begin{aligned}\lim_{x\to 2} \frac{\sqrt[3]{x}-\sqrt[3]{2}}{x-2} &= \lim_{x\to 2} \frac{(\sqrt[3]{x}-\sqrt[3]{2})(\sqrt[3]{x^2}+\sqrt[3]{2x}+\sqrt[3]{2^2})}{(x-2)(\sqrt[3]{x^2}+\sqrt[3]{2x}+\sqrt[3]{2^2})}\\ &= \lim_{x\to 2} \frac{x-2}{(x-2)(\sqrt[3]{x^2}+\sqrt[3]{2x}+\sqrt[3]{2^2})}\\ &= \lim_{x\to 2} \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{2x}+\sqrt[3]{2^2}}\\ &= \frac{1}{\sqrt[3]{2^2}+\sqrt[3]{2\times 2}+\sqrt[3]{2^2}} = \frac{1}{3\sqrt[3]{4}}.\end{aligned}$$

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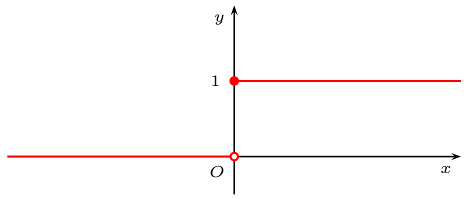
Heaviside Function

- Define the **Heaviside function**:

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$



  - As $x \to 0$ **from the left**, $H(x) \to 0$,
    - $\lim_{x\to 0^-} H(x) = 0$.
  - As $x \to 0$ **from the right**, $H(x) \to 1$.
    - $\lim_{x\to 0^+} H(x) = 1$.

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One-Sided Limits

- **Definition.** If as x is close to a from the **right** $f(x)$ is close to L, or simply

$$x \to a^+ \Rightarrow f(x) \to L,$$

we say that the **right-hand limit** of f as x approaches a equals L, and write

$$\lim_{x \to a^+} f(x) = L.$$

- If as x is close to a from the **left** $f(x)$ is close to L, or

$$x \to a^- \Rightarrow f(x) \to L,$$

then the **left-hand limit** of f as x approaches a equals L,

$$\lim_{x \to a^-} f(x) = L.$$

Limit & One-Sided Limits

- The limit laws also hold for one-sided limits. For example,
  - $\lim\limits_{x \to a^+} (f(x) + g(x)) = \lim\limits_{x \to a^+} f(x) + \lim\limits_{x \to a^+} g(x)$.
- **Theorem**.
  - $\lim\limits_{x \to a} f(x) = L \iff \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = L$.

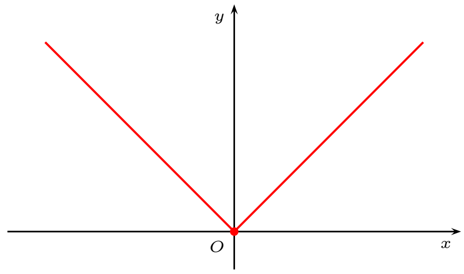
$$f(x) \to L \Leftarrow x \to a \iff \begin{cases} x \to a^+ & \Rightarrow f(x) \to L \\ x \to a^- & \Rightarrow f(x) \to L \end{cases}$$

- **Example**. Recall that $H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$ and that
  - $\lim_{x \to 0^-} H(x) = 0$,
  - $\lim_{x \to 0^+} H(x) = 1$.

$\lim\limits_{x \to 0^+} H(x) \neq \lim\limits_{x \to 0^-} H(x)$, So $\lim\limits_{x \to 0} H(x)$ **does NOT exist**.

Examples

- Find $\lim\limits_{x \to 0} |x|$.



  - If $x > 0$, $|x| = x$, then $\lim\limits_{x \to 0^+} |x| = \lim\limits_{x \to 0^+} x = 0$;
  - If $x < 0$, $|x| = -x$, then $\lim\limits_{x \to 0^-} |x| = \lim\limits_{x \to 0^-} (-x) = 0$.

  $\therefore \lim\limits_{x \to 0} |x| = 0$.

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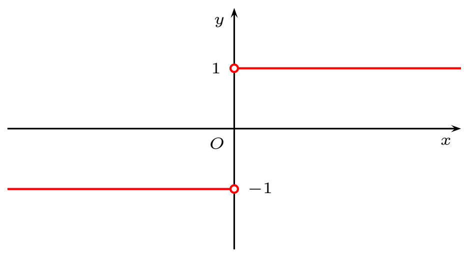
Examples

- Find $\lim\limits_{x \to 0} \dfrac{|x|}{x}$.



If $x > 0$, $\dfrac{|x|}{x} = \dfrac{x}{x} = 1$, then $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = \lim\limits_{x \to 0^+} 1 = 1$.

If $x < 0$, $\dfrac{|x|}{x} = \dfrac{-x}{x} = -1$, $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = \lim\limits_{x \to 0^-} (-1) = -1$.

$\because \lim\limits_{x \to 0^+} \dfrac{|x|}{x} \neq \lim\limits_{x \to 0^-} \dfrac{|x|}{x}$, $\quad \therefore \lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.

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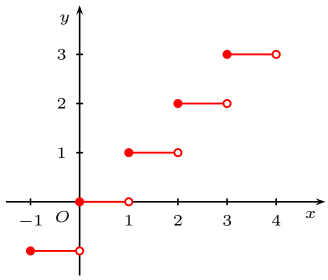
One-Sided Continuity

- A function f is **continuous from the right at a** if
$$\lim_{x \to a^+} f(x) = f(a),$$
and f is **continuous from the left at a** if
$$\lim_{x \to a^-} f(x) = f(a).$$
Recall that
$$\lim_{x \to a} f(x) = L \Leftrightarrow \begin{cases} \lim_{x \to a^+} f(x) = L \\ \lim_{x \to a^-} f(x) = L \end{cases}.$$
By letting $L = f(a)$, we have the following conclusion:
- **Proposition**. f is continuous at a *if and only if* f is continuous from the left at a and continuous from the right at a.

Jump Discontinuity

- Let $f(x) = \lfloor x \rfloor$ be the **floor function**, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.
  - $\lfloor 1.2 \rfloor = 1, \quad \lfloor 3 \rfloor = 3, \quad \lfloor -3.14 \rfloor = -4.$



At each $n \in \mathbb{Z}$, $\lim_{x \to n^-} f(x) = n - 1$, $\lim_{x \to n^+} f(x) = n$.
There is a "**jump**" from the left to the right.
  - Such discontinuity is a **jump discontinuity**.

Continuity of a Function on an Interval

- **Definition.** A function is **continuous on an interval** if it is **continuous at every number** in the interval.
  - f is continuous on open interval (a, b)
    $\Leftrightarrow f$ is continuous at every $x \in (a, b)$.
  - f is continuous on closed interval $[a, b]$
    $$\Leftrightarrow \begin{cases} f \text{ is continous at every } x \in (a, b), \\ f \text{ is continuous from the right at } a, \\ f \text{ is continuous from the left at } b. \end{cases}$$
  - f is continuous on $[a, b) \Leftrightarrow \cdots\cdots\cdots$
  - f is continuous on $(a, b] \Leftrightarrow \cdots\cdots\cdots$

  Example. The **floor function** $f(x) = \lfloor x \rfloor$ is continuous on $[n, n + 1)$ for each $n \in \mathbb{Z}$.

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Example

- Find the value of c such that the function
  - $f(x) = \begin{cases} x^3 + 2 & \text{if } -2 \leq x \leq 2 \\ (x - 2)^2 + c & \text{if } 2 < x \leq 4 \end{cases}$

  is continuous on $[-2, 4]$.
- **Solution.**
  - f is continuous on $[-2, 2)$ and on $(2, 4]$.
  - It suffices to make the function continuous at $x = 2$. That is, $\lim_{x\to 2} f(x) = f(2)$.
    - $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^3 + 2) = 2^3 + 2 = 10.$
    - $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} ((x - 2)^2 + c) = (2 - 2)^2 + c = c.$
    - $f(2) = 2^3 + 2 = 10.$
  - Hence, f is continuous at $x = 2 \Leftrightarrow 10 = c = 10$.

  Therefore, f is continuous on $[-2, 4] \Leftrightarrow c = 10$.

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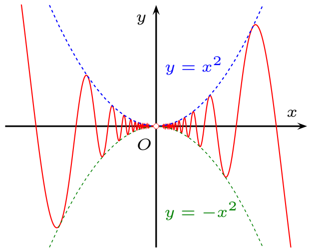
Squeeze Theorem

- **Problem**. Find $\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right)$.



  - It seems that $\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0$.
  - $-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$ for all $x \neq 0$.
- **Question**. Can use the inequalities to compute the limit?

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Squeeze Theorem

- **Theorem**. Let f, g, h be functions such that
  - $f(x) \leq g(x) \leq h(x)$ for all x near a (except at a), and
  - $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.

  Then $\lim_{x \to a} g(x)$ exists and equals L.
- **Example**. Find $\lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right)$. Let $x \to 0$.

$$\begin{array}{ccccc} -x^2 & \leq & x^2 \sin \frac{1}{x} & \leq & x^2 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \Rightarrow & 0 & \Leftarrow & 0 \end{array}$$

$$\therefore \lim_{x \to 0} \left(x^2 \sin \frac{1}{x} \right) = 0.$$

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Examples

- Suppose that $\lim_{x \to a} f(x) = 0$. Prove that
  - $\lim_{x \to a} f(x)\, \sin g(x) = 0$.
- **Proof**. For all x near a (except possible at a),
  - $-1 \le \sin g(x) \le 1$.
  - $-|f(x)| \le f(x)\, \sin g(x) \le |f(x)|$.

  Note that
  - $\lim_{x \to a} |f(x)| = \left|\lim_{x \to a} f(x)\right| = |0| = 0$,
  - $\lim_{x \to a}(-|f(x)|) = -\lim_{x \to a} |f(x)| = -0 = 0$.

  By Squeeze Theorem,
  - $\lim_{x \to a} f(x)\, \sin g(x) = 0$ exists and equals 0.

Examples

- Evaluate $\displaystyle\lim_{x \to 0^+} \frac{2 \sec \sqrt{x} + x^3 \sin \frac{2}{\sqrt{x}}\, \cos(\ln x)}{4x + 3 \cos x^3 + 2}$.
- **Solution.** For all $x > 0$,
  - $-1 \le \sin \frac{2}{\sqrt{x}} \le 1$ and $-1 \le \cos(\ln x) \le 1$,
  - $-x^3 \le x^3 \sin \frac{2}{\sqrt{x}}\, \cos(\ln x) \le x^3$.

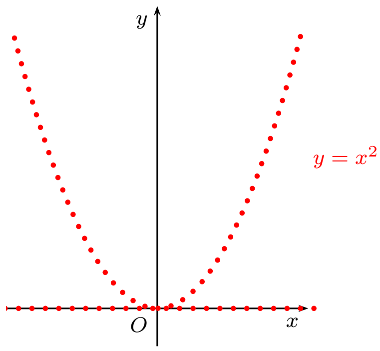
  We can check that
  - $\lim_{x \to 0^+} x^3 = 0$ and $\lim_{x \to 0^+} (-x^3) = 0$,

  By Squeeze Theorem,
  - $\lim_{x \to 0^+} x^3 \sin \frac{2}{\sqrt{x}}\, \cos(\ln x) = 0$.

  Hence,
  - $\displaystyle\lim_{x \to 0^+} \frac{2 \sec \sqrt{x} + x^3 \sin \frac{2}{\sqrt{x}}\, \cos(\ln x)}{4x + 3 \cos x^3 + 2} = \frac{2+0}{0+3+2} = \frac{2}{5}$.

Examples

- Let $f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$ Find $\lim_{x\to 0} f(x)$.
- Note that the set of rational numbers are discrete points, and the set of irrational numbers are also discrete.



  - If x is rational, then $0 \le f(x) = x^2$;
  - If x is irrational,then $0 = f(x) \le x^2$.

  Hence, for every $x \in \mathbb{R}$, $0 \le f(x) \le x^2$.

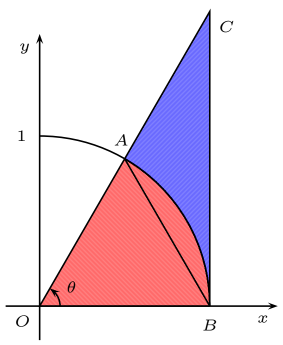
  - $\lim_{x\to 0} 0 = 0$ and $\lim_{x\to 0} x^2 = 0$.

  By Squeeze Theorem, $\lim_{x\to 0} f(x)$ exists and equals 0.

- **Remark.** For $x = 0 \in \mathbb{Q}$, $f(0) = 0^2 = 0$.
  - $\lim_{x\to 0} f(x) = f(0) = 0$.

  This shows that f is continuous at $x = 0$, although there is no piece of “curve” on its graph.

$\lim_{\theta\to 0} \sin\theta/\theta$



- Let $0 < \theta < \frac{\pi}{2}$. Then

$$\triangle AOB < \triangleleft AOB < \triangle BOC$$

- That is,

$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}.$$

- $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$ for all $0 < \theta < \pi/2$.
  - $\lim_{\theta\to 0^+} \cos\theta = 1$and $\lim_{\theta\to 0^+} 1 = 1 \Rightarrow \lim_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1.$

$\lim_{\theta\to 0}\sin\theta/\theta$

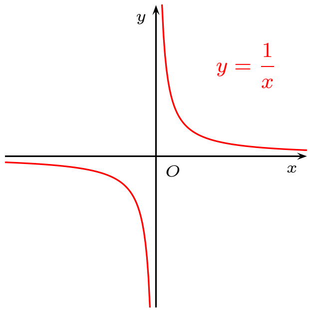
- Similarly we can prove $\lim\limits_{\theta\to 0^-}\dfrac{\sin\theta}{\theta}=1$.
- **Theorem.** $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ and $\lim\limits_{\theta\to 0}\dfrac{\tan\theta}{\theta}=1$.
- **Examples.**
  - $\lim\limits_{x\to 0}\dfrac{\sin 4x^2}{x\sin 2x}=\lim\limits_{x\to 0}\left(2\dfrac{\sin 4x^2}{4x^2}\cdot\dfrac{2x}{\sin 2x}\right)=2\cdot 1\cdot 1=2.$
  - $\lim\limits_{x\to -2^-}\dfrac{\sqrt{x^2+2x}}{\tan\sqrt{x^2-4}}=\lim\limits_{x\to -2^-}\dfrac{\sqrt{x^2-4}}{\tan\sqrt{x^2-4}}\cdot\dfrac{\sqrt{x^2+2x}}{\sqrt{x^2-4}}$
    $=1\cdot\lim\limits_{x\to -2^-}\sqrt{\dfrac{x(x+2)}{(x+2)(x-2)}}$
    $=\lim\limits_{x\to -2^-}\sqrt{\dfrac{x}{x-2}}=\dfrac{\sqrt{2}}{2}.$

Examples

- $\lim\limits_{x\to 1}\left(\dfrac{\sin\frac{x-1}{2}}{x^2+2x-3}\right)^5.$
  - $\lim\limits_{x\to 1}\dfrac{\sin\frac{x-1}{2}}{x^2+2x-3}=\lim\limits_{x\to 1}\dfrac{\sin\frac{x-1}{2}}{\frac{x-1}{2}}\,\dfrac{\frac{x-1}{2}}{(x-1)(x+3)}$
    $=1\cdot\lim\limits_{x\to 1}\dfrac{1}{2(x+3)}=\dfrac{1}{8}.$
  - $\lim\limits_{x\to 1}\left(\dfrac{\sin\frac{x-1}{2}}{x^2+2x-3}\right)^5=\dfrac{1}{8^5}=\dfrac{1}{2^{15}}.$
- $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}.$
  - $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2\frac{x}{2}}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2\frac{x}{2}}{4\left(\frac{x}{2}\right)^2}=\dfrac{1}{2}.$

Limit at Infinity

- Consider the graph of $f(x) = \dfrac{1}{x}$:



  - When x gets larger and larger, $\dfrac{1}{x}$ is close to 0.
  - It is denoted by $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$.

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Limit at Infinity

- **Definition**. If $f(x)$ is **arbitrarily close** to L by taking x **sufficiently large**, then we write

$$\lim_{x\to\infty} f(x) = L.$$

  We say the **limit** of $f(x)$, as x approaches infinity, equals L.

- **Definition**. If $f(x)$ is **arbitrarily close** to L by taking x **sufficiently negatively large**, then we write

$$\lim_{x\to-\infty} f(x) = L.$$

  We say the **limit** of $f(x)$, as x approaches negative infinity, equals L.

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Examples

- $\lim_{x\to\infty} \sqrt{\sin \frac{3}{\sqrt{x}}}$.
  - $\lim_{x\to\infty} \sqrt{\sin \frac{3}{\sqrt{x}}} = \sqrt{\sin\left(3\sqrt{\lim_{x\to\infty} \frac{1}{x}}\right)} = \sqrt{\sin(3\cdot\sqrt{0})} = 0.$
- $\lim_{x\to-\infty} \sec(e^{3+2x})$.
  - $\lim_{x\to-\infty} \sec(e^{3+2x}) = \sec\left(\lim_{x\to-\infty} e^{3+2x}\right) = \sec 0 = 1.$
- **Some Useful Facts**.
  - $\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \frac{1}{x} = 0.$
  - $\lim_{x\to-\infty} e^x = 0;$
  - $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}, \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$

Examples

- $\lim_{x\to\infty} \frac{3x^2-1}{2x^2+5}$.
  - $\lim_{x\to\infty} \frac{3x^2-1}{2x^2+5} = \lim_{x\to\infty} \frac{3-\frac{1}{x^2}}{2+\frac{5}{x^2}} = \frac{3-0}{2+0} = \frac{3}{2}.$
- $\lim_{x\to-\infty} \frac{(4x^3-x^2-1)^2}{(x^4+5)(3-2x^2)}$.
  - $$\lim_{x\to-\infty} \frac{(4x^3-x^2-1)^2}{(x^4+5)(3-2x^2)} = \lim_{x\to-\infty} \frac{(4-\frac{1}{x}-\frac{1}{x^3})^2}{(1+\frac{5}{x^4})(\frac{3}{x^2}-2)}$$
    $$= \frac{(4-0-0)^2}{(1+0)(0-2)} = -8.$$

Examples

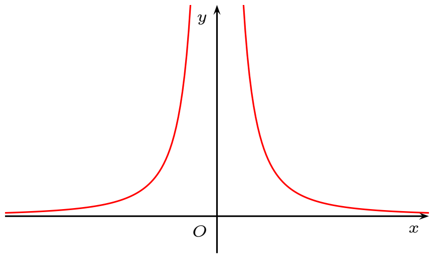
- In general, if we want to evaluate the limit of an **algebraic function** of the form $\frac{f(x)}{g(x)}$ at $\pm\infty$, divide both the numerator and denominator by the **highest degree** of x.
- $\lim_{x\to\infty}\sqrt{\frac{9x^6+3x-5}{(4x^4+1)(x-1)^2}} = \lim_{x\to\infty}\sqrt{\frac{9+\frac{3}{x^5}-\frac{5}{x^6}}{(4+\frac{1}{x^4})(1-\frac{1}{x})^2}}$
$= \sqrt{\frac{9+0-0}{(4+0)(1-0)^2}} = \frac{3}{2}.$
- **Facts.**
  - If $\deg f(x) = \deg g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is a nonzero constant.
  - If $\deg f(x) < \deg g(x)$, then $\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = 0$.

More Examples

- $\lim_{x\to\infty} x^3\left(\tan\frac{1}{x}\right)\left(\sin\frac{3}{x^2}\right)$.
  - $\lim_{x\to\infty} x^3\left(\tan\frac{1}{x}\right)\left(\sin\frac{3}{x^2}\right) = \lim_{x\to\infty} 3\,\frac{\tan\frac{1}{x}}{\frac{1}{x}}\,\frac{\sin\frac{3}{x^2}}{\frac{3}{x^2}}$
  $= 3\cdot 1\cdot 1 = 3.$
- $\lim_{x\to\infty}(1+2^x+3^x)^{4/x}$.
  - For all $x > 0$,
    - $3^x < 1+2^x+3^x < 3\cdot 3^x$,
    - $3^4 < (1+2^x+3^x)^{4/x} < 3^{4/x}\cdot 3^4$.
  - We can check that
    - $\lim_{x\to\infty} 3^4 = 81,\ \lim_{x\to\infty} 3^{4/x}\cdot 3^4 = 1\cdot 3^4 = 81.$
  - By Squeeze Theorem, $\lim_{x\to\infty}(1+2^x+3^x)^{4/x} = 81$.

Example

- Let $f(x) = \dfrac{1}{x^2}$.



  - As x approaches 0, $f(x)$ gets arbitrarily large.

$$\text{We write:} \quad \lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Infinite Limits

- **Definition**. Suppose f is defined on both sides of a (except possibly at a).
  - If $f(x)$ is **arbitrarily large** by taking x **sufficiently close** to a, we write
$$\lim_{x \to a} f(x) = \infty.$$
  - If $f(x)$ is **arbitrarily negatively large** by taking x **sufficiently close** to a, we write
$$\lim_{x \to a} f(x) = -\infty.$$
- **Note**. If $\lim_{x \to a} f(x) = \infty$ or $-\infty$, it **makes sense**. However, neither ∞ nor $-\infty$ is a number.For instance,
  - $1 + \infty = 2 + \infty = \infty \not\Rightarrow 1 = 2.$

  We still say that an infinite limit **does NOT exist**.

Infinite Limits

- Similarly, we can define the **one-sided infinite limits**:
  - $\lim\limits_{x\to a^+} f(x)=\infty$ $\lim\limits_{x\to a^+} f(x)=-\infty$
  - $\lim\limits_{x\to a^-} f(x)=\infty$ $\lim\limits_{x\to a^-} f(x)=-\infty$

  the **infinite limit at infinity**:
  - $\lim\limits_{x\to \infty} f(x)=\infty$ $\lim\limits_{x\to \infty} f(x)=-\infty$
  - $\lim\limits_{x\to -\infty} f(x)=\infty$ $\lim\limits_{x\to -\infty} f(x)=-\infty$

  Examples.
  - $\lim\limits_{x\to 0^+} \dfrac{1}{x}=\infty$, $\lim\limits_{x\to 0^+} \ln x=-\infty$.
  - $\lim\limits_{x\to -\infty} x^3=-\infty$, $\lim\limits_{x\to \infty} \ln x=\infty$.

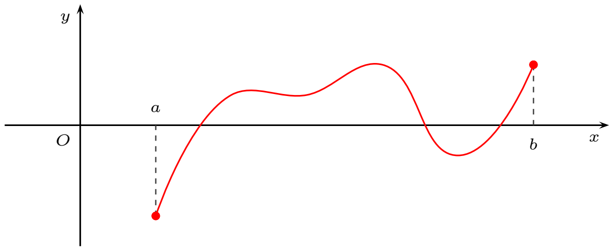
Example

- Determine the infinite limits:
  - $\lim\limits_{x\to 5^-} \dfrac{6}{x-5}=-\infty$
    - $x\to 5^- \Rightarrow \begin{cases} 6\to 6\neq 0 \\ x-5\to 0 \end{cases} \Rightarrow \left|\dfrac{6}{x-5}\right|\to\infty.$
    - $x\to 5^- \Rightarrow \begin{cases} 6>0 \\ x-5<0 \end{cases} \Rightarrow \dfrac{6}{x-5}<0.$
  - $\lim\limits_{x\to 1^+} \dfrac{x+1}{x\sin\pi x}=-\infty$
    - $x\to 1^+ \Rightarrow \begin{cases} x+1\to 2\neq 0 \\ x\sin\pi x\to 0 \end{cases} \Rightarrow \left|\dfrac{x+1}{x\sin\pi x}\right|\to\infty.$
    - $x\to 1^+ \Rightarrow \begin{cases} x+1>0 \\ x>0 \\ \sin\pi x<0 \end{cases} \Rightarrow \dfrac{x+1}{x\sin\pi x}<0.$

Intermediate Value Theorem

- Let f be a function **continuous on** $[a, b]$.

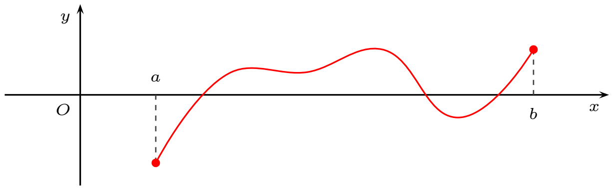
  Suppose $f(a) < 0$ and $f(b) > 0$.



  - As x moves from a to b,
    $f(x)$ moves smoothly from **negative** to **positive**.
  - f is continuous $\Rightarrow$ the graph has **no break**.
- $\boxed{\text{The graph cuts the } x\text{-axis somewhere between } a \text{ and } b}$.

Intermediate Value Theorem

- **Intermediate Value Theorem** (Simple Version)
  Let f be a function **continuous on** $[a, b]$.
  - If $f(a)f(b) < 0$, then
    there exists a number $c \in (a, b)$ such that $f(c) = 0$.
- **Remark.** The proof of IVT requires the "completeness of real numbers". We will not prove IVT in our course.
  - It **DOES NOT** tell us the exact value of the solution.
    It shows only the existence of solution.
  - It **DOES NOT** show the number of the solutions.
    There may be more than one root for $f(x) = 0$.



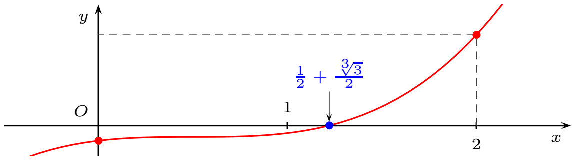
Examples

- Show that there is a real root to $4x^3 - 6x^2 + 3x - 2 = 0$.
  - Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
    - f is a polynomial $\Rightarrow f$ is continuous on $\mathbb{R}$.

    In order to use **Intermediate Value Theorem**,
    we shall find two numbers a and b such that

    - $f(a) < 0$ and $f(b) > 0$.

$$\left.\begin{array}{r} f(0) = -2 < 0 \\ f(2) = 12 > 0 \\ f \text{ is continuous on } [0,2] \end{array}\right\} \Rightarrow \begin{array}{l} \text{there is a } c \in (0,2) \\ \text{such that } f(c) = 0. \end{array}$$



Examples

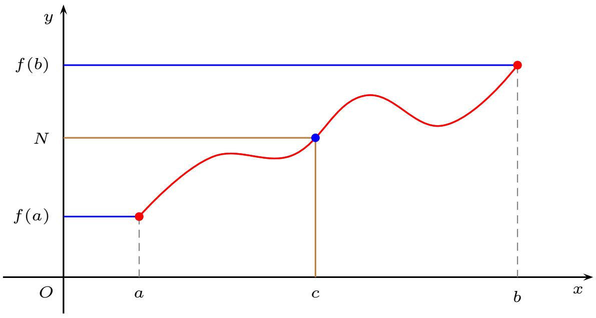
- Show that there is a real root to $3\ln x + x^3 = 7e^{-x}$.
  - Let $f(x) = 3\ln x + x^3 - 7e^{-x}$.
    - f is the sum of continuous functions $\Rightarrow f$ is continuous on $\mathbb{R}^+$.
  - In order to use **Intermediate Value Theorem**,
    we shall find two numbers a and b such that
    - $f(a) < 0$ and $f(b) > 0$.

$$\left.\begin{array}{r} f(1) = 1 - 7e^{-1} < 0 \\ f(2) = 3\ln 2 + 8 - 7e^{-2} > 0 \\ f \text{ is continuous on } [1,2] \end{array}\right\} \Rightarrow \begin{array}{l} \text{there is a } c \in (1,2) \\ \text{such that } f(c) = 0. \end{array}$$

- **Note**. We must make sure that the function is continuous entirely on the interval. The argument will be invalid if there is any discontinuity in the interval.

Intermediate Value Theorem (General Version)

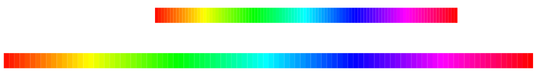
- **Intermediate Value Theorem**
  Let f be a function continuous on $[a, b]$ with $f(a) \neq f(b)$.
  - Let N be a number between $f(a)$ and $f(b)$,
  - Then there exists $c \in (a, b)$ such that $f(c) = N$.



- The proof is to use the Simple Version of Intermediate Value Theorem.

Examples

- Stretch a rubber band by moving one end to the right and the other to the left.



  It seems that some point of the rubber band will end up in its original position.
- **Solution**. Suppose the rubber band has length 1, and it is put on the interval $[0, 1]$.
  - Let $f(x)$ be the position of x after stretching. Then
    - f is continuous on $[0, 1]$, $f(0) < 0$, $f(1) > 1$.
  - Define $g(x) = f(x) - x$. Then

$$\left.\begin{array}{l} g \text{ is continuous on } [0,1] \\ g(0) < 0 \text{ and } g(1) > 0 \end{array}\right\} \Rightarrow g(c) = 0 \text{ for some } c \in (0, 1).$$

i.e., $f(c) = c$. The rubber band has a fixed point.