MA1521 CALCULUS FOR COMPUTING

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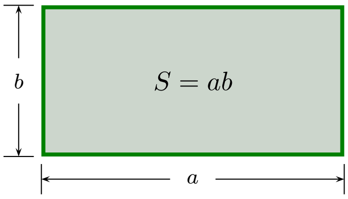
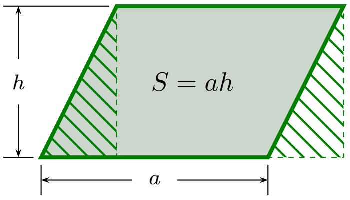
The Area Problem

- How to find the area of a region on a plane?
  - Given a rectangle of length a and width b:

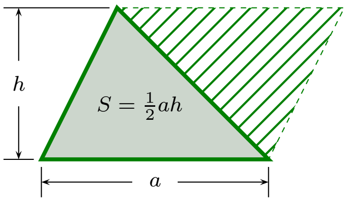


  - Given a parallelogram of base a and height h:

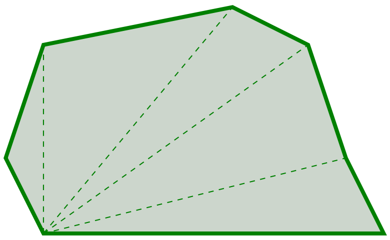


The Area Problem

  - Given a triangle of base a and height h:

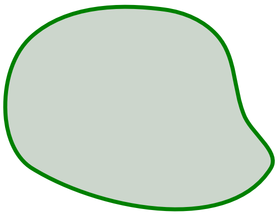


  - In general, given any polygon, we are able to find its area because it can be cut into triangles.



The Area Problem

- However, how about if the area is not bounded by segments but curves, what to do?



There is no formula at this moment. Since we've known polygons, we may use polygons to approximate the area.

- In particular, we can use regular polygons to approximate the **area of the circle**.
  - Archimedes of Syracuse $A\rho\chi\iota\mu\acute{\eta}\delta\eta\zeta$ (287BC–212BC)
    Greek Mathematician, Physicist and Engineer.
    The first person using this method to compute π.

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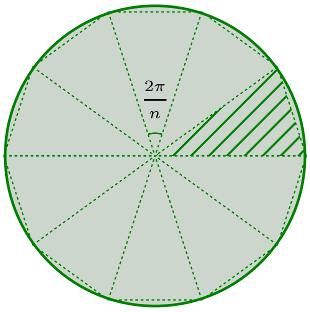
Computation of π: Archimedes' Meththod

- What is π?
  - π is the **ratio** of circle's circumference to its diameter.
    So the circumference of the unit circle is 2π.
  - What is the area of the unit circle?



$$\begin{aligned} A &= \lim_{n\to\infty} \frac{n}{2} \cdot \sin \frac{2\pi}{n} \\ &= \lim_{n\to\infty} \frac{\sin(2\pi/n)}{2\pi/n} \cdot \pi \\ &= \pi. \end{aligned}$$

The area of the n-sided polygon inscribed is $\frac{n}{2} \cdot \sin \frac{2\pi}{n}$.

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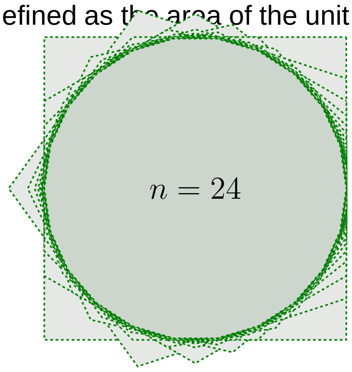
Computation of π: Archimedes' Meththod

- Alternatively, π can be defined as the area of the unit circle.



  - Archimedes: (96-sided polygon) 3.14163.
  - Liu Hui: Chinese Mathematician, (220?–280?), (3079-sided polygon) 3.14159.
  - Zu Chongzhi: Chinese Mathematician, (429–500), (12288-sided polygon) 3.1415926.

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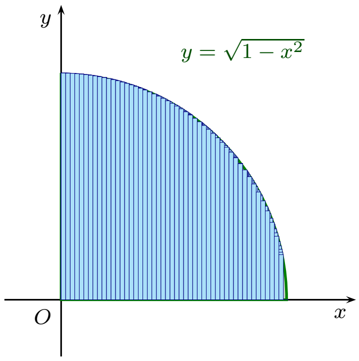
Computation of π: Approximation by Rectangles

- Consider the area of the unit circle in the first quadrant.



$$R_5 = 0.659 < \frac{\pi}{4} \; R_{10} = 0.726 < \frac{\pi}{4} \; R_{15} = 0.747 < \frac{\pi}{4} \; R_{20} = 0.757 < \frac{\pi}{4} \; R_{25} = 0.763 < \frac{\pi}{4}$$
$$R_{30} = 0.767 < \frac{\pi}{4} \; R_{35} = 0.770 < \frac{\pi}{4} \; R_{40} = 0.772 < \frac{\pi}{4} \; R_{45} = 0.773 < \frac{\pi}{4} \; R_{50} = 0.775 < \frac{\pi}{4}$$

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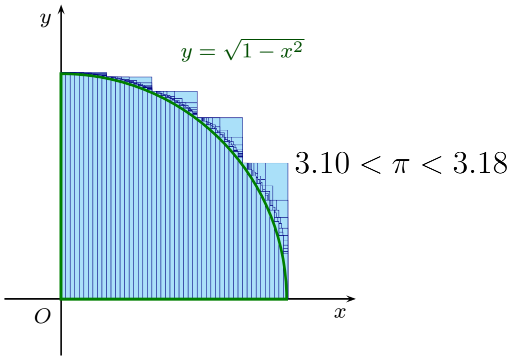
Computation of π: Approximation by Rectangles

- Consider the area of the unit circle in the first quadrant.



$$L_5 = 0.859 > \frac{\pi}{4}\ L_{10} = 0.826 > \frac{\pi}{4}\ L_{15} = 0.814 > \frac{\pi}{4}\ L_{20} = 0.807 > \frac{\pi}{4}\ L_{25} = 0.803 > \frac{\pi}{4}$$
$$L_{30} = 0.800 > \frac{\pi}{4}\ L_{35} = 0.798 > \frac{\pi}{4}\ L_{40} = 0.797 > \frac{\pi}{4}\ L_{45} = 0.796 > \frac{\pi}{4}\ L_{50} = 0.795 > \frac{\pi}{4}$$

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Definite Integral

- Let f be a **continuous** function on $[a, b]$.
  - Divide $[a, b]$ into n equal subintervals, say
    - $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n], \quad \Delta x = \frac{b-a}{n}$
  - Take **sample points** $x_1^*, x_2^*, \ldots, x_n^*$ from subintervals.
    - $x_1^* \in [x_0, x_1], x_2^* \in [x_1, x_2], \ldots, x_n^* \in [x_{n-1}, x_n]$.
  - Compute the **Riemann sum**:
    $[f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)]\, \Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$.
  - The **definite integral** of f from a to b:
    - $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$.
- **Leibnize notation.** $\int$: **integral sign**; $f(x)$: **integrand**;
  a: **lower limit**; b: **upper limit**.

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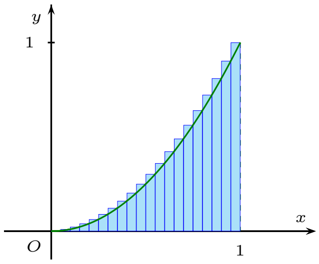
Examples

- Evaluate $A = \int_0^1 x^2\,dx$.

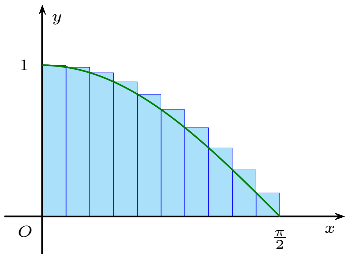


  - Divide $[0,1]$ into n equal subintervals:
    - $[0, \frac{1}{n}], [\frac{1}{n}, \frac{2}{n}], \ldots, [\frac{n-1}{n}, \frac{n}{n}]$, Use $x_i^* = x_i = \frac{i}{n}$.
  - $S_n = (\frac{1}{n})^2 \cdot \frac{1}{n} + (\frac{2}{n})^2 \cdot \frac{1}{n} + \cdots + (\frac{n}{n})^2 \cdot \frac{1}{n}$
    - $S_n = \frac{1}{n^3}(1^2 + 2^2 + \cdots + n^2) = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$.

$A = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} = \frac{1}{3}$.

Examples

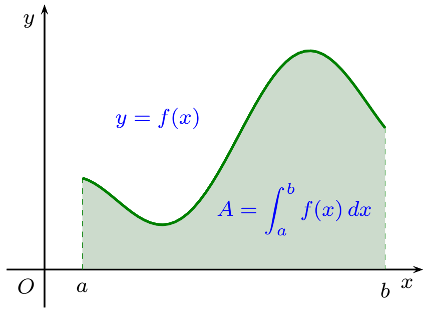
- Evaluate $A = \int_0^{\pi/2} \cos x\,dx$.



  - Divide $[0, \frac{\pi}{2}]$ into n equal subintervals:
    - $[0, \frac{\pi}{2n}], [\frac{\pi}{2n}, \frac{2\pi}{2n}], \ldots, [\frac{(n-1)\pi}{2n}, \frac{n\pi}{2n}]$. $x_i^* = \frac{(i-1)\pi}{2n}$.
  - $S_n = \left[\cos 0 + \cos(\frac{\pi}{2n}) + \cdots + \cos(\frac{(n-1)\pi}{2n})\right] \cdot \frac{\pi}{2n}$.
    - $S_n = \sum_{k=0}^{n-1} \cos(\frac{k\pi}{2n}) \cdot \frac{\pi}{2n}$. $A = \lim_{n\to\infty} S_n = ?$

Geometric Meaning of Integration

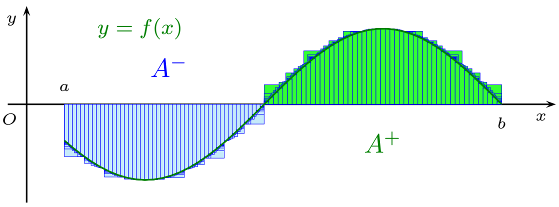
- Let f be a continuous function on $[a, b]$.
  - If $f(x) \geq 0$ for all x, then $\int_a^b f(x)\,dx$ presents the area between the graph of $y = f(x)$ and the x-axis.



  - For example, $\int_0^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{4}$.

Geometric Meaning of Integration

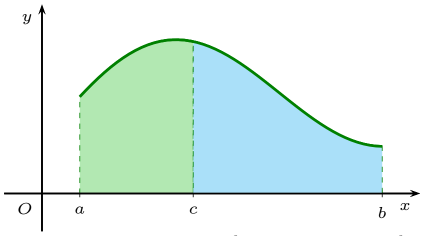
- Let f be a continuous function on $[a, b]$.
  How about if $f(x) < 0$ for some x?



  - A^+: the area below $y = f(x)$ and above the x-axis.
  - A^-: the area above $y = f(x)$ and below the x-axis.
- $\int_a^b f(x)\,dx = A^+ - A^-$, the **net area**.
- In order to find the **area** bounded between $y = f(x)$ and the x-axis, we shall evaluate $\int_a^b |f(x)|\,dx$ instead.

Properties of Definite Integral

- Let f be a continuous function.
  - $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

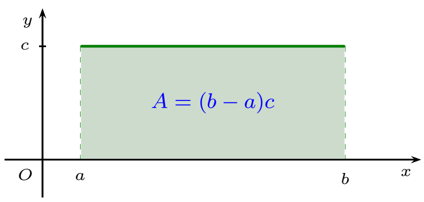


$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

- $\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx = 0.$
  - $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$

Properties of Definite Integral

- $\int_a^b c\,dx = (b-a)c.$

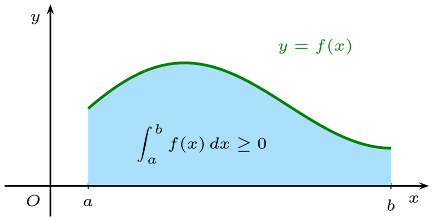


- Let f and g be continuous functions, and c a constant.
  - $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
  - $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  - $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

Properties of Definite Integral

- Let f be a continuous function on $[a, b]$, $(a < b)$.

  Suppose $f(x) \geq 0$ on $[a, b]$. Then $\displaystyle\int_a^b f(x)\,dx \geq 0$.



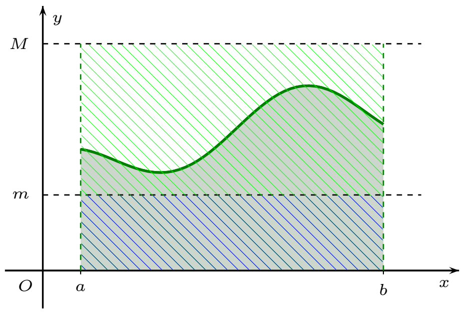
- Let f and g be continuous and $f(x) \geq g(x)$ on $[a, b]$.

  Then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

  Can you prove it?

Properties of Definite Integral

- Let f be continuous and $m \leq f(x) \leq M$ on $[a, b]$.
  - $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b m\,dx = m(b-a)$.
  - $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a)$.



Examples

- Evaluate $\int_0^1 (4+3x^2)\,dx$. Recall that $\int_0^1 x^2\,dx = \frac{1}{3}$.
  - $\int_0^1 (4+3x^2)\,dx = \int_0^1 4\,dx + 3\int_0^1 x^2\,dx$
    $= 4(1-0) + 3\cdot\frac{1}{3} = 5.$
- Show that $\frac{5}{6} \le \int_0^1 \cos x\,dx \le 1$.
  - $\cos x = 1 - 2\sin^2\frac{x}{2} \Rightarrow 1 - \frac{x^2}{2} \le \cos x \le 1.$
  - $\int_0^1 (1-\frac{x^2}{2})\,dx \le \int_0^1 \cos x\,dx \le \int_0^1 dx = 1.$
    - $\int_0^1 (1-\frac{x^2}{2})\,dx = \int_0^1 dx - \frac{1}{2}\int_0^1 x^2\,dx = \frac{5}{6}.$

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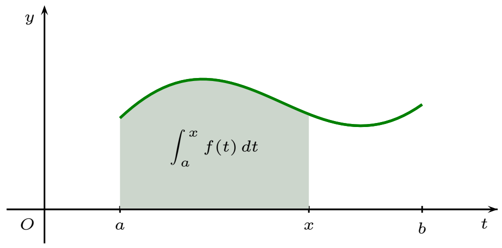
The Fundamental Theorem of Calculus, Part I

- Let f be a **continuous** function defined on $[a,b]$ $(a<b)$.
  - For any $x \in [a,b]$, f is continuous on $[a,x]$.
    - We can evaluate $\int_a^x f(t)\,dt$.



  - $g(x) = \int_a^x f(t)\,dt$ is a function defined on $[a,b]$.
    - Is g **continuous**? Is g **differentiable**?
    - If g is differentiable, what is g'?

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The Fundamental Theorem of Calculus, Part I

- **Theorem**. (Fundamental Theorem of Calculus, Part I)
  Let f be a **continuous** function on $[a, b]$.
  - Define $g(x) = \int_a^x f(t)\,dt$, $(a \le x \le b)$. Then
    - g is **continuous** on $[a, b]$,
    - g is **differentiable** on (a, b), and $g'(x) = f(x)$.
- By Leibniz notation, the theorem can be written as
  - $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x).$
- **Examples**.
  - $\dfrac{d}{dx}\displaystyle\int_0^x \sqrt{1+t^2}\,dt = \sqrt{1+x^2},$
  - $\dfrac{d}{dx}\displaystyle\int_0^x \sin(\cos(\sin t))\,dt = \sin(\cos(\sin x)).$

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Examples

- Find $\dfrac{d}{dx}\displaystyle\int_1^{x^4} \sec t\,dt$. Let $u = x^4$.

$$\begin{aligned}\frac{d}{dx}\int_1^{x^4} \sec t\,dt &= \frac{d}{dx}\int_1^{u} \sec t\,dt = \frac{du}{dx}\cdot\frac{d}{du}\int_1^{u} \sec t\,dt \\ &= 4x^3 \cdot \sec u = 4x^3 \sec(x^4).\end{aligned}$$

- Find the value of a such that $\lim\limits_{x\to 0} \dfrac{\int_0^x \frac{t^2}{\sqrt{a+3t}}\,dt}{x - \sin x} = 1.$

$$\begin{aligned}1 &= \lim_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+3t}}\,dt}{x-\sin x} = \lim_{x\to 0}\frac{\frac{d}{dx}(\int_0^x \frac{t^2}{\sqrt{a+3t}}\,dt)}{\frac{d}{dx}(x-\sin x)} \\ &= \lim_{x\to 0}\frac{\frac{x^2}{\sqrt{a+3x}}}{1-\cos x} = \lim_{x\to 0}\frac{1}{\sqrt{a+3x}}\cdot\lim_{x\to 0}\frac{x^2}{1-\cos x} \\ &= \frac{1}{\sqrt{a}}\cdot\lim_{x\to 0}\frac{2x}{\sin x} = \frac{2}{\sqrt{a}} \Rightarrow a = 4.\end{aligned}$$

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The Fundamental Theorem of Calculus, Part II

- Let f be continuous on $[a, b]$ and $g(x) = \int_a^x f(t)\,dt$.
  - By F.T.C.(I), g is **continuous** on $[a, b]$ and $g' = f$.
  - If we can find a **continuous** function F with $F' = f$,
    - then $F + c = g$ for a constant c.
  - In particular,

$$\begin{aligned} F(a) + c &= g(a) = \int_a^a f(t)\,dt = 0, \\ F(b) + c &= g(b) = \int_a^b f(t)\,dt. \end{aligned}$$

$$\therefore \quad F(b) - F(a) = \left(\int_a^b f(t)\,dt - c\right) - (-c) = \int_a^b f(t)\,dt.$$

  - This formula gives a shortcut to evaluate the **definite integral** without using **Riemann sum**.

The Fundamental Theorem of Calculus

- **Theorem**. (Fundamental Theorem of Calculus, Part II).
  Let f be a **continuous** function on $[a, b]$.
  - If F is **continuous** on $[a, b]$, and $F' = f$ on (a, b),
  - then $\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_{x=a}^{x=b}$.

$$\begin{tikzcd} \boxed{f(x)} \arrow[r, shift left, "f \text{ is the derivative of } F"] & \boxed{F(x)} \arrow[l, shift left, "F \text{ is an antiderivative of } f"] \end{tikzcd}$$

- **Example**. We are now ready to evaluate $\int_0^{\pi/2} \cos x\,dx$.
  - Do we have a function F such that $F'(x) = \cos x$?
    - Use $F(x) = \sin x$.
    - $\int_0^{\pi/2} \cos x\,dx = \sin x\Big|_{x=0}^{x=\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1.$

Indefinite Integral

- **Fundamental Theorem of Calculus (II)** shows that
  - Evaluating **definite integral** may be reduced to finding **antiderivative**.
- An **antiderivative** of a continuous function f is
  - a continuous function F such that $F' = f$.

  We introduce the notation $\int f(x)\,dx$ for such F,
  - it is also called an **indefinite integral** of f.
- By definition, $\dfrac{d}{dx} \displaystyle\int f(x)\,dx = f(x)$.
  - Fundamental Theorem of Calculus (II) now becomes
    $\displaystyle\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_{x=a}^{x=b}$.

Indefinite Integral

- What are the **antiderivatives** (**indefinite integrals**) of f?

  Let F be an antiderivative of f. Then $F' = f$.
  - If G is also an antiderivative of f, i.e., $G' = f$,
    - then $G = F + c$ for a constant c.
  - If c is a constant, then $(F + c)' = F' + c' = f$,
    - then $F + c$ is an antiderivative of f.

$\therefore$ the antiderivatives of f are precisely all the functions of the form $F + c$, where c is a constant.

So we also regard the indefinite integral

$$\int f(x)\,dx = F(x) + c$$

as the **entire family of the antiderivatives**.

Properties of Indefinite Integral

- Recall that $F'(x) = f(x)$ means $\int f(x)\,dx = F(x) + c$.
  - $(kx)' = k \Rightarrow \int k\,dx = kx + c$.
  - $\left(\frac{x^{r+1}}{r+1}\right)' = x^r \Rightarrow \int x^r\,dx = \frac{x^{r+1}}{r+1} + c, (r \neq -1)$.
  - $(-\cos x)' = \sin x \Rightarrow \int \sin x\,dx = -\cos x + c$.
  - $(\sin x)' = \cos x \Rightarrow \int \cos x\,dx = \sin x + c$.
  - $(\tan x)' = \sec^2 x \Rightarrow \int \sec^2 x\,dx = \tan x + c$.
  - $(-\cot x)' = \csc^2 x \Rightarrow \int \csc^2 x\,dx = -\cot x + c$.
  - ....

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Properties of Indefinite Integral

- If $F' = f$ and $G' = g$, then $(aF + bG)' = af + bg$.

$$\begin{aligned}\int (af(x) + bg(x))\,dx &= aF(x) + bG(x) + c \\ &= a\int f(x)\,dx + b\int g(x)\,dx.\end{aligned}$$

- **Example**. Find $\int (10x^4 - 2\sec^2 x)\,dx$.

$$\begin{aligned}\int (10x^4 - 2\sec^2 x)\,dx &= 10\int x^4\,dx - 2\int \sec^2 x\,dx \\ &= 10 \cdot \frac{x^5}{5} - 2\tan x + C \\ &= 2x^5 - 2\tan x + C.\end{aligned}$$

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Example

- Evaluate $\int 2x\sqrt{1-x^2}\,dx$ and $\int x^3\cos(x^4+2)\,dx$.
  - Let $u = 1-x^2$. Then $\frac{du}{dx} = -2x$.

$$\int 2x\sqrt{1-x^2}\,dx = \int (-\sqrt{u})\,du$$
$$= -\frac{u^{3/2}}{3/2} + C = -\frac{2}{3}(1-x^2)^{3/2} + C.$$

  - Let $u = x^4+2$. Then $\frac{du}{dx} = 4x^3$.

$$\int x^3\cos(x^4+2)\,dx = \frac{1}{4}\int \cos u\,du$$
$$= \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C.$$

- **Question**. Does the above method work in general?

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The Substitution Rule

- **The Substitution Rule** (for Indefinite Integral).
  Let $u = g(x)$ be a differentiable function whose range is an interval I.
  - If f is continuous on I, and g' is continuous,
  - then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.
- **Remark**. The conditions are just to make sure that the function to be integrated is continuous on an interval.
- **Proof**. Differentiate the right hand side with respect to x.

$$\frac{d}{dx}\int f(u)\,du = \frac{du}{dx}\cdot\frac{d}{du}\int f(u)\,du$$
$$= g'(x)\cdot f(u) = g'(x)f(g(x)).$$
$$\therefore \quad \int f(u)\,du = \int f(g(x))g'(x)\,dx.$$

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Examples

- Evaluate $\int \frac{dx}{(3-5x)^2}$.
  - Let $u = g(x) = 3 - 5x$. Then $\frac{du}{dx} = -5$. Then

$$\int \frac{dx}{(3-5x)^2} = -\frac{1}{5}\int \frac{du}{u^2} = -\frac{1}{5}\frac{-1}{u} + C = \frac{1}{5(3-5x)} + C.$$

- Evaluate $\int \sqrt{2x+1}\,dx$.
  - Let $u = g(x) = 2x + 1$. Then $\frac{du}{dx} = 2$. Then

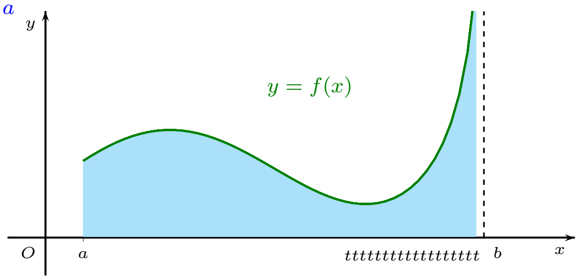
$$\int \sqrt{2x+1}\,dx = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C.$$

Examples

- Evaluate $\int \tan x\,dx$. Let $u = \cos x$. Then $\frac{du}{dx} = -\sin x$.
  - $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-1}{u}\,du = -\ln|u| + C = -\ln|\cos x| + C.$$
  - $$\int_0^{\pi/6} \tan 2x\,dx = \frac{1}{2}\int_0^{\pi/6} \tan 2x \cdot 2\,dx = -\frac{1}{2}\ln|\cos 2x|\Big|_{x=0}^{x=\pi/6} = \frac{1}{2}\ln 2.$$
- $$\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x\ \tan x}{\sec x + \tan x}\,dx = \int \frac{d(\tan x + \sec x)}{\tan x + \sec x} = \ln|\tan x + \sec x| + C.$$

Improper Integral

- Recall in $\displaystyle\int_a^b f(x)\,dx$, f should be continuous on $[a,b]$.
  - However, f may have a discontinuity. What can we do?
- Suppose f is continuous on $[a,b)$ and discontinuous at b.
  - $\displaystyle\int_a^b f(x)\,dx$ is not defined directly by Riemann sum.
  - But we can compute $\displaystyle\int_a^t f(x)\,dx$ for every $t\in[a,b)$.



Improper Integral

- **Improper Integral for Discontinuous Integrands**.
  - Let f be **continuous** on $[a,b)$, and discontinuous at b.
$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx.$$
  - Let f be **continuous** on $(a,b]$, discontinuous at a.
$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx.$$
  - Note that $\displaystyle\int_a^b f(x)\,dx$ is the limit of integrals. We say
    - $\displaystyle\int_a^b f(x)\,dx$ **converges** if the limit exists, and
    - $\displaystyle\int_a^b f(x)\,dx$ **diverges** if the limit does not exist.

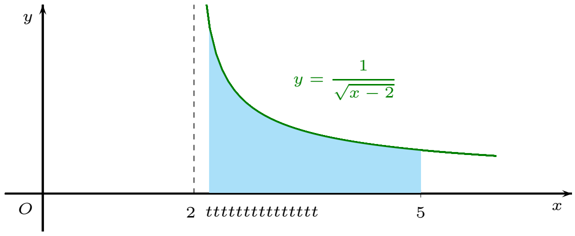
Examples

- Find $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx$. An infinite discontinuity at 2.

1) Find $\displaystyle\int_t^5 \frac{1}{\sqrt{x-2}}\,dx$ for $t \in (2,5]$.

   - $\displaystyle\int_t^5 \frac{1}{\sqrt{x-2}}\,dx = 2\sqrt{x-2}\Big|_{x=t}^{x=5}$.

2) $\displaystyle\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}(2\sqrt{3} - 2\sqrt{t-2}) = 2\sqrt{3}$.



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Examples

- Evaluate $\displaystyle\int_{-1}^8 \frac{1}{\sqrt[3]{x}}\,dx$. An infinite discontinuity at 0.

$$\begin{aligned}
\int_{-1}^8 \frac{dx}{\sqrt[3]{x}} &= \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^8 \frac{dx}{\sqrt[3]{x}} \\
&= \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{\sqrt[3]{x}} + \lim_{t\to 0^+}\int_t^8 \frac{dx}{\sqrt[3]{x}} \\
&= \lim_{t\to 0^-}\frac{x^{2/3}}{2/3}\Big|_{x=-1}^{x=t} + \lim_{t\to 0^+}\frac{x^{2/3}}{2/3}\Big|_{x=t}^{x=8} \\
&= \lim_{t\to 0^-}\left[\frac{t^{2/3}}{2/3} - \frac{(-1)^{2/3}}{2/3}\right] + \lim_{t\to 0^+}\left[\frac{8^{2/3}}{2/3} - \frac{t^{2/3}}{2/3}\right] \\
&= -\frac{1}{2/3} + \frac{4}{2/3} = \frac{9}{2}.
\end{aligned}$$

- This gives the motivation to define the improper integral if the discontinuity is in the interior of the interval.

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Improper Integral

- Suppose f has a discontinuity at $c \in (a, b)$. We define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx,$$

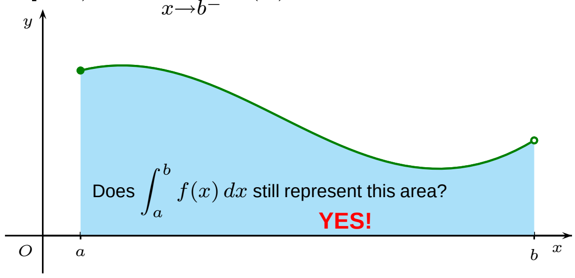
if both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are **convergent**.

  - In other words,
$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

- Graphically, the **improper integral** $\int_a^b f(x)\,dx$ presents the **net area** of the **unbounded region** between the graph of f and the x-axis from a to b.

Examples

- Suppose f is continuous on $[a, b)$, and $\lim_{x\to b^-} f(x)$ exists.

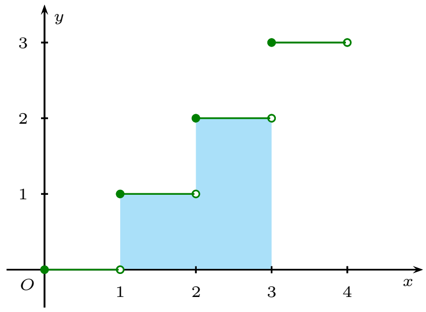


- $g(x) = \begin{cases} f(x), & x \in [a, b) \\ \lim_{x\to b^-} f(x), & x = b \end{cases}$ is continuous on $[a, b]$.

  - $$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$
$$= \lim_{t\to b^-}\int_a^t g(x)\,dx = \int_a^b g(x)\,dx = \text{Area}.$$

Examples

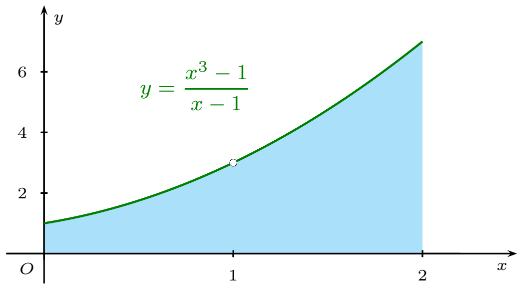
- Find $\displaystyle\int_1^3 \lfloor x \rfloor \, dx$.



  - The integral represents the area of the shaded region.
  - $\displaystyle\int_1^3 \lfloor x \rfloor \, dx = \int_1^2 1 \, dx + \int_2^3 2 \, dx = 1 + 2 = 3.$

Examples

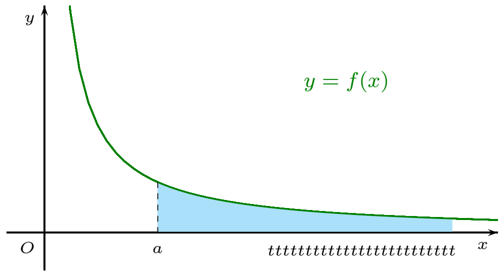
- Find $\displaystyle\int_0^2 \frac{x^3 - 1}{x - 1} \, dx$. $\displaystyle\frac{x^3 - 1}{x - 1} = x^2 + x + 1$ for all $x \neq 1$.



$$\int_0^2 \frac{x^3 - 1}{x - 1} \, dx = \int_0^2 (x^2 + x + 1) \, dx$$
$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + x \right]_{x=0}^{x=2} = \frac{20}{3}.$$

Improper Integral over Infinite Intervals

- Let f be a continuous function on $[a, \infty)$. Then we can compute $A(t) = \int_a^t f(x)\,dx$ for every $t \geq a$.



  - It is natural to define

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

Improper Integral over Infinite Intervals

- If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, the **improper integral** of f from a to ∞ is

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

- If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, the **improper integral** of f from $-\infty$ to b is

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

- They are called **convergent** if the corresponding limits exist, and called **divergent** otherwise.
- $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$, if the two integrals on the right hand side are both convergent.

Examples

- Find $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$. Ans: $\displaystyle\frac{1}{p-1}$ if $p > 1$ and divergent if $p \le 1$.

  1) We first find $\displaystyle\int \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p} + C \quad (p \neq 1)$.

  2) For all $t > 0$, $\displaystyle\int_1^t \frac{1}{x^p}\,dx = \frac{x^{1-p}}{1-p}\Big|_{x=1}^{x=t} = \frac{t^{1-p}-1}{1-p}$.

  3) $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \frac{t^{1-p}-1}{1-p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p < 1. \end{cases}$

  1) Suppose $p = 1$. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.

  2) For all $t > 0$, $\displaystyle\int_1^t \frac{1}{x}\,dx = \ln|x|\Big|_{x=1}^{x=t} = \ln t$.

  3) $\displaystyle\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln t = \infty$.

Substitution Rule

- Evaluate $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx$.
  - Recall the substitution rule: If $u = g(x)$,
    - $\displaystyle\int f(g(x))g'(x)\,dx = \int f(u)\,du$.
  - It is difficult to convert $\displaystyle\frac{1}{1+\sqrt{x}}$ to $f(g(x))g'(x)$.
    - Let $t = \sqrt{x}$. Then $x = t^2$, and $\displaystyle\frac{dx}{dt} = 2t$.

$$\int \frac{1}{1+\sqrt{x}}\,dx = \int \frac{1}{1+t} \cdot 2t\,dt$$
$$= 2\int \left(1 - \frac{1}{1+t}\right) dt = 2(t - \ln|1+t|) + C$$
$$= 2(\sqrt{x} - \ln(1+\sqrt{x})) + C.$$

Substitution Rule

- **Substitution Rule (2nd Version).** Let f be a continuous function, and $x = g(t)$ be a differentiable function.
  - If g' is continuous,
  - then $\displaystyle\int f(x)\,dx = \int f(g(t))g'(t)\,dt$.

  Proof.
$$\frac{d}{dt}\int f(x)\,dx = \frac{dx}{dt}\cdot\frac{d}{dx}\int f(x)\,dx$$
$$= g'(t)f(x) = g'(t)f(g(t)).$$
- **Remark.** Note that the integral in x is converted to an integral in t.
  - After integration we shall convert the function in t back to a function in x.
  - So usually it requires $x = g(t)$ to be one to one.

Example

- Evaluate $\displaystyle\int \frac{1}{x(1+x^4)}\,dx$.
  - Let $t = \dfrac{1}{x}$. Then $x = \dfrac{1}{t}$, and $\dfrac{dx}{dt} = -\dfrac{1}{t^2}$.

$$\int \frac{1}{x(1+x^4)}\,dx = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}(1+\frac{1}{t^4})} = \int -\frac{t^3}{1+t^4}\,dt$$
$$= -\frac{1}{4}\int \frac{d(1+t^4)}{1+t^4} = -\frac{1}{4}\ln(1+t^4) + C$$
$$= -\frac{1}{4}\ln(1+\frac{1}{x^4}) + C.$$

- **Exercise.** Evaluate $\displaystyle\int \sec x\,dx$ using $t = \tan(x/2)$.
  - Answer: $\left(\displaystyle\int \sec x\,dx = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C\right)$.

Example

- Evaluate $\displaystyle\int \frac{1}{(1+x^2)^n}\,dx$, $n \in \mathbb{Z}^+$.
  - We have seen that $(\tan^{-1} x)' = \dfrac{1}{1+x^2}$.
  - Let $x = \tan t, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $\dfrac{dx}{dt} = \sec^2 t$.

$$\begin{aligned}\int \frac{1}{(1+x^2)^n}\,dx &= \int \frac{1}{(1+\tan^2 t)^n} \cdot \sec^2 t\,dt \\ &= \int \frac{\sec^2 t}{\sec^{2n} t}\,dt = \int \cos^{2n-2} t\,dt.\end{aligned}$$

  - When $n = 1$,
    $\displaystyle\int \frac{1}{1+x^2}\,dx = \int 1\,dt = t + C = \tan^{-1} x + C.$

Example

  - When $n = 2$,

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}\,dx &= \int \cos^2 t\,dt = \frac{1}{2}\int (1+\cos 2t)\,dt \\ &= \frac{1}{2}t + \frac{1}{4}\sin 2t + C \\ &= \frac{1}{2}t + \frac{1}{4}\frac{2\tan t}{1+\tan^2 t} + C \\ &= \frac{1}{2}\tan^{-1} x + \frac{x}{2(1+x^2)} + C\end{aligned}$$

  - When $n = 3$,
    $\displaystyle\int \frac{1}{(1+x^2)^3}\,dx = \int \cos^4 t\,dt = \cdots\cdots.$
  - **Problem**. In general, how to evaluate $\displaystyle\int \cos^m t\,dt$?

Integration by Parts

- Recall the product law for differentiation:
  - $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$.

  Equivalently it can be written as
  - $f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx$.

    Or $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$.

  Let $u = f(x)$ and $v = g(x)$.
  - Then $\int u\,dv = uv - \int v\,du$.

  Such method of integration is called **integration by parts**.
  - This is a very useful technique in integration. In particular, it can be used to evaluate $\int \cos^n x\,dx$.

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Examples

- Evaluate $\int x \sin x\,dx$ and $\int \ln x\,dx$.
  - Let $u = x$ and $v = -\cos x$. Then

$$\int x \sin x\,dx = \int x\,d(-\cos x) = x(-\cos x) - \int (-\cos x)\,dx$$
$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

  - Let $u = \ln x$ and $v = x$. Then

$$\int \ln x\,dx = \ln x \cdot x - \int x\,d(\ln x)$$
$$= \ln x \cdot x - \int x(1/x)\,dx = x\ln x - \int dx$$
$$= x\ln x - x + C.$$

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Examples

- Find $\displaystyle\int e^x \sin x\,dx$ and $\displaystyle\int e^x \cos x\,dx$.

$$\begin{aligned}\int e^x \sin x\,dx &= \int \sin x\,d(e^x) = e^x \sin x - \int e^x\,d(\sin x)\\ &= e^x \sin x - \int e^x \cos x\,dx.\end{aligned}$$

$$\begin{aligned}\int e^x \cos x\,dx &= \int \cos x\,d(e^x) = e^x \cos x - \int e^x\,d(\cos x)\\ &= e^x \cos x + \int e^x \sin x\,dx.\end{aligned}$$

$$\therefore \quad \begin{cases} \displaystyle\int e^x \sin x\,dx &= \dfrac{1}{2}\,e^x(\sin x - \cos x) + C,\\ \displaystyle\int e^x \cos x\,dx &= \dfrac{1}{2}\,e^x(\sin x + \cos x) + C. \end{cases}$$

Examples

- Evaluate $\displaystyle\int \cos^n x\,dx$.

$$\begin{aligned}\int \cos^n x\,dx &= \int \cos^{n-1} x \cos x\,dx = \int \cos^{n-1} x\,d(\sin x)\\ &= \cos^{n-1} x \sin x - \int \sin x\,d(\cos^{n-1} x)\\ &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx\\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx\\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx\end{aligned}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

Examples

- $\displaystyle\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$
  - $\displaystyle\int \cos^2 x\,dx = \frac{1}{2}\cos x \sin x + \frac{1}{2}x + C.$
  - $\displaystyle\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + C.$
- Recall $\displaystyle\int \frac{1}{(1+x^2)^3}\,dx = \int \cos^4 t\,dt \quad (x = \tan t).$

  $$= \frac{1}{4}\cos^3 t \sin t + \frac{3}{8}\cos t \sin t + \frac{3}{8}t + C$$
  $$= \frac{1}{4}\frac{x}{(1+x^2)^2} + \frac{3}{8}\frac{x}{1+x^2} + \frac{3}{8}\tan^{-1} x + C.$$
- Similarly, for any $n \in \mathbb{Z}^+$, $\displaystyle\int \frac{1}{(1+x^2)^n}\,dx$ can be evaluated (although it may be very tedious).

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More Examples

- $\displaystyle n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$
  - $\displaystyle -\int \sec x\,dx = \sec^2 x \sin x - 2\int \sec^3 x\,dx.$

    $$\int \sec^3 x\,dx = \frac{1}{2}\sec^2 x \sin x + \frac{1}{2}\int \sec x\,dx$$
    $$= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$
  - $\displaystyle -2\int \sec^2 x\,dx = \sec^3 x \sin x - 3\int \sec^4 x\,dx.$

    $$\int \sec^4 x\,dx = \frac{1}{3}\sec^3 x \sin x + \frac{2}{3}\int \sec^2 x\,dx$$
    $$= \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + C.$$
- **Exercise**. Evaluate $\displaystyle\int \sin^n x\,dx$ and $\displaystyle\int \tan^n x\,dx.$

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Trigonometric Substitution

- **Example**. Evaluate $\int \sqrt{1-x^2}\,dx$.
  - Recall $\sin^2 x + \cos^2 x = 1$. Try $x = \sin t$, $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
    - $\cos t \geq 0 \Rightarrow \sqrt{1-x^2} = \cos t$,
    - $\dfrac{dx}{dt} = \cos t, \quad t = \sin^{-1} x$.
  - $\int \sqrt{1-x^2}\,dx = \int \cos t\ \cos t\,dt = \int \cos^2 t\,dt$

    $= \int \dfrac{1+\cos 2t}{2}\,dt = \dfrac{1}{2}t + \dfrac{1}{4}\sin 2t + C$

    $= \dfrac{1}{2}\sin^{-1} x + \dfrac{1}{2}x\sqrt{1-x^2} + C$.
- **Exercise**. $\int x^2\sqrt{9-x^2}\,dx$. $\left(x = 3\sin t,\ t \in [-\frac{\pi}{2}, \frac{\pi}{2}]\right)$

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Trigonometric Substitution

- **Example**. Evaluate $\int \sqrt{1+x^2}\,dx$.
  - Try $x = \tan t$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then $t = \tan^{-1} x$.
    - $\dfrac{dx}{dt} = \sec^2 t, \quad \sqrt{1+x^2} = \sec t$.
  - $\int \sqrt{1+x^2}\,dx = \int \sec t\ \sec^2 t\,dt = \int \sec^3 t\,dt$

    $= \dfrac{1}{2}\tan t \sec t + \dfrac{1}{2}\ln|\tan t + \sec t| + C$

    $= \dfrac{1}{2}x\sqrt{1+x^2} + \dfrac{1}{2}\ln(x + \sqrt{x^2+1}) + C$.
- **Exercise**. $\int \sqrt{x^2-1}\,dx$.
  - Try $x = \sec t$, $t \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$.

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Trigonometric Substitution

- If we have the **quadratic form** in the integrands, we may try the **trigonometric substitution**:
  - i) $a^2 - x^2$, $\quad x = a\sin t, \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$;
  - ii) $a^2 + x^2$, $\quad x = a\tan t, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2})$;
  - iii) $x^2 - a^2$, $\quad x = a\sec t, \quad t \in [0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$.
- **Example**. $\displaystyle\int \sqrt{1+2x-x^2}\,dx.$
  - ◦ $1 + 2x - x^2 = 2 - (x-1)^2 = (\sqrt{2})^2 - (x-1)^2$.
    Try $\quad x - 1 = \sqrt{2}\,\sin t, \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
  - ◦ $$\int \sqrt{1+2x-x^2}\,dx = \int \sqrt{2}\,\cos t\,\sqrt{2}\,\cos t\,dt$$
    $$= t + \sin t\cos t + C$$
    $$= \sin^{-1}\left(\frac{x-1}{\sqrt{2}}\right) + \frac{1}{2}(x-1)\sqrt{1+2x-x^2} + C.$$

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Integration of Rational Functions

- Is there an algorithm to evaluate any integral?
  No! In fact, only very few integrals can be evaluated.
  - ◦ Polynomial, rational functions, ...
- Recall that a rational function is the ratio of polynomials:
  - ◦ $\dfrac{A(x)}{B(x)}$, where A, B are polynomials.

  We've known how to integrate these rational functions:
  - ◦ $\displaystyle\int \frac{1}{x^n}\,dx. \left(\ln|x| + C,\ n = 1;\ \frac{x^{1-n}}{1-n} + C,\ n \geq 2\right)$
  - ◦ $\displaystyle\int \frac{1}{(1+x^2)^n}\,dx. \left(\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C\right)$
- Indeed, in order to integrate all rational functions, it suffices to know the integration of these two types of functions.

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Integration of Rational Functions

- Let's make some assumptions first.

1. Assume that $A(x)$ and $B(x)$ have no common factor.

   They are called **relatively prime** or **coprime**.

   - Cancellate if they have a common factor.
     $$\frac{x-1}{x^2-1} \rightsquigarrow \frac{1}{x+1}, \quad \frac{x^2(x+1)}{x(x^3+1)} \rightsquigarrow \frac{x}{x^2-x+1}.$$

2. Assume that the leading coefficient of $B(x)$ is 1.

   Such polynomial is called a **monic polynomial**.

   - Divide both $A(x)$ and $B(x)$ by the leading coefficient of $B(x)$ otherwise.
     $$\frac{3x-1}{2x^2+1} \rightsquigarrow \frac{\frac{3}{2}x-\frac{1}{2}}{x^2+\frac{1}{2}}.$$

Integration of Rational Functions

3. Assume that the degree of $A(x)$ is smaller than the degree of $B(x)$, i.e., $\deg A(x) < \deg B(x)$.

   (a) If $\deg A(x) \geq \deg B(x)$, divide $A(x)$ by $B(x)$,

   (b) $A(x) = B(x)Q(x) + R(x), \deg R(x) < \deg B(x)$.

   (c) $\dfrac{A(x)}{B(x)} = Q(x) + \dfrac{R(x)}{B(x)}$.

   (d) To integrate $\dfrac{A(x)}{B(x)}$, it suffices to integrate $\dfrac{R(x)}{B(x)}$.

$$\begin{aligned}\frac{x^3-x^2+2x+2}{x^2+1} &\rightsquigarrow \frac{(x^2+1)(x-1)+(x+3)}{x^2+1}\\ &\rightsquigarrow (x-1)+\frac{x+3}{x^2+1}.\end{aligned}$$

Integration of Rational Functions

- Let's now list two facts. (See Appendix for proof)

Fact 1. Every nonconstant monic polynomial $B(x)$ can be uniquely factorized into the product of **linear factors** and **quadratic factors** (over $\mathbb{R}$).

$$B(x) = (x + a_1)^{k_1}(x + a_2)^{k_2} \cdots (x + a_m)^{k_m}$$
$$\times (x^2 + b_1x + c_1)^{r_1} \cdots (x^2 + b_nx + c_n)^{r_n}.$$

- Here $x^2 + b_ix + c_i$ cannot be factorized further into linear factors; equivalently, $b_i^2 < 4c_i$.

$x^5 - x^3 - x^2 + 1 = (x - 1)^2(x + 1)(x^2 + x + 1).$

$x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2}).$

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Integration of Rational Functions

Fact 2. If $\deg A(x) < \deg B(x)$, $\dfrac{A(x)}{B(x)}$ can be converted into **partial fraction**, which is the sum of rational functions:

(a) For each linear factor $(x + a)^k$ of $B(x)$,

$$\frac{A_1}{x + a} + \frac{A_2}{(x + a)^2} + \cdots + \frac{A_k}{(x + a)^k}.$$

(b) For each quadratic factor $(x^2 + bx + c)^r$ of $B(x)$,

$$\frac{B_1x + C_1}{x^2 + bx + c} + \cdots + \frac{B_rx + C_r}{(x^2 + bx + c)^r}.$$

In general, the number of unknowns A_i, B_j, C_k equals to the degree of $B(x)$.

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Partial Fraction

- **Example**. $\dfrac{4x}{x^3 - x^2 - x + 1}$.
  - Factorize $x^3 - x^2 - x + 1 = (x+1)(x-1)^2$.
  - Then the partial fraction has the form
    - $\dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.
    - $\dfrac{(A+B)x^2 + (-2A+C)x + (A-B+C)}{(x+1)(x-1)^2}$.
  - Compare the coefficients,

$$\begin{cases} 0 = A + B \\ 4 = -2A + C \\ 0 = A - B + C \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 1 \\ C = 2 \end{cases} .$$

$\therefore \quad \dfrac{-1}{x+1} + \dfrac{1}{x-1} + \dfrac{2}{(x-1)^2}$.

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Integration of Rational Functions

- The integration is now reduced to integrating:
  - $\dfrac{1}{(x+a)^k}$.

    $$\int \frac{1}{(x+a)^k}\,dx = \begin{cases} \ln|x+a| + K, & \text{if } k = 1, \\ \dfrac{(x+a)^{1-k}}{1-k} + K, & \text{if } k \geq 2. \end{cases}$$

  - $\dfrac{Bx + C}{(x^2 + bx + c)^r}$, $\quad b^2 < 4c$.

    Note that $x^2 + bx + c = (x + \frac{b}{2})^2 + (c - \frac{b^2}{4})$.

    - Let $u = x + \frac{b}{2}$ and $d = \sqrt{c - \frac{b^2}{4}}$.

$$\begin{aligned} \frac{Bx + C}{(x^2 + bx + c)^r} &= \frac{B(u - \frac{b}{2}) + C}{(u^2 + d^2)^r} \\ &= \frac{Bu}{(u^2 + d^2)^r} + \frac{C - \frac{Bb}{2}}{(u^2 + d^2)^r}. \end{aligned}$$

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Integration of Rational Functions

- Let's consider the two subcases:

(a) $\dfrac{u}{(u^2+d^2)^r}$.

$$\int \frac{u}{(u^2+d^2)^r}\,du = \frac{1}{2}\int \frac{d(u^2+d^2)}{(u^2+d^2)^r}$$
$$= \begin{cases} \frac{1}{2}\ln(u^2+d^2), & \text{if } r=1, \\ \dfrac{(u^2+d^2)^{1-r}}{2(1-r)}, & \text{if } r\geq 2. \end{cases}$$

(b) $\dfrac{1}{(u^2+d^2)^r}$. Let $t=\dfrac{u}{d}$. Then $\dfrac{du}{dt}=d$.

$$\int \frac{1}{(u^2+d^2)^r}\,du = \int \frac{\frac{1}{d^{2r}}}{\left[\left(\frac{u}{d}\right)^2+1\right]^r}\,du$$
$$= \frac{1}{d^{2r-1}}\int \frac{1}{(t^2+1)^r}\,dt.$$

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Examples

- $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx.$

1. Factorize $x^3+4x=x(x^2+4)$.
2. $\dfrac{A}{x}+\dfrac{Bx+C}{x^2+4}=\dfrac{(A+B)x^2+Cx+4A}{x(x^2+4)}$.

$$\begin{cases} 2=A+B \\ -1=C \\ 4=4A \end{cases} \Rightarrow \begin{cases} A=1 \\ B=1 \\ C=-1. \end{cases}$$

3. $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx.$

$$= \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

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Examples

- $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx.$

  1. $\displaystyle\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1) + \frac{4x}{x^3-x^2-x+1}.$
  2. Factorize $x^3-x^2-x+1 = (x+1)(x-1)^2.$
  3. $\displaystyle\frac{4x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$

     $$\begin{cases} 0 = A+B \\ 4 = -2A+C \\ 0 = A-B+C \end{cases} \Rightarrow \begin{cases} A = -1 \\ B = 1 \\ C = 2. \end{cases}$$
  4. $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = \frac{x^2}{2} + x - \ln|x+1| + \ln|x-1| - \frac{2}{x-1} + C.$

Examples

- $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx.$

  1. Factorize $2x^3+3x^2-2x = x(x+2)(2x-1).$
  2. $\displaystyle\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}.$

     $$\begin{cases} 1 = 2A+2B+C \\ 2 = 3A-B+2C \\ -1 = -2A \end{cases} \Rightarrow \begin{cases} A = 1/2 \\ B = -1/10 \\ C = 1/5. \end{cases}$$
  3. $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx = \frac{1}{2}\ln|x| - \frac{1}{10}\ln|x+2| + \frac{1}{10}\ln|2x-1| + C.$

Rational Trigonometric Functions

- Evaluate $\displaystyle\int \frac{dx}{\sin x + 2\cos x + 3}$.

  Recall the "**double-angle**" formulas:

  - $\sin x = \dfrac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$;
  - $\cos x = \dfrac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

  These give the motivation to set $t = \tan\frac{x}{2}$.

  - Let $t = \tan\frac{x}{2}$, $x \in (-\pi, \pi)$.
    - $x = 2\tan^{-1} t$, $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.
    - $\sin x = \dfrac{2t}{1+t^2}$, $\quad \cos x = \dfrac{1-t^2}{1+t^2}$.

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Rational Trigonometric Functions

- Evaluate $\displaystyle\int \frac{dx}{\sin x + 2\cos x + 3}$.

$$\begin{aligned}
\int \frac{dx}{\sin x + 2\cos x + 3} &= \int \frac{1}{\frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2} + 3} \cdot \frac{2\,dt}{1+t^2} \\
&= \int \frac{2\,dt}{2t + 2(1-t^2) + 3(1+t^2)} \\
&= \int \frac{2\,dt}{(t+1)^2 + 2^2} = \int \frac{d\left(\frac{t+1}{2}\right)}{\left(\frac{t+1}{2}\right)^2 + 1} \\
&= \tan^{-1}\left(\frac{t+1}{2}\right) + C \\
&= \tan^{-1}\left(\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2}\right) + C.
\end{aligned}$$

- We can use the **universal trigonometric substitution** $t = \tan\dfrac{x}{2}$, $x \in (-\pi, \pi)$ in the integration of any rational expression in $\sin x$ and $\cos x$.

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