MA1521 CALCULUS FOR COMPUTING

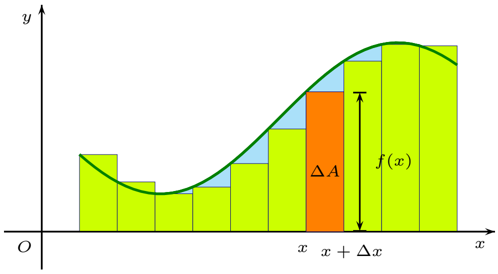
Wang Fei

matwf@nus.edu.sg

Department of Mathematics
Office: S17-06-16
Tel: 6516-2937

The Area Problem

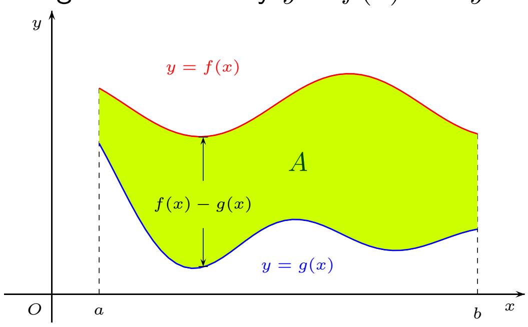
- Recall the geometric meaning of definite integral:
  - Let f be a continuous function on $[a, b]$. Then the (net) area under the curve $y = f(x)$ from a to b is $A = \int_a^b f(x)\,dx$.



  - $\Delta A = f(x)\,\Delta x$. Then $\dfrac{dA}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} = f(x)$.
    - Therefore, $A = \int_a^b f(x)\,dx$.

The Area Problem

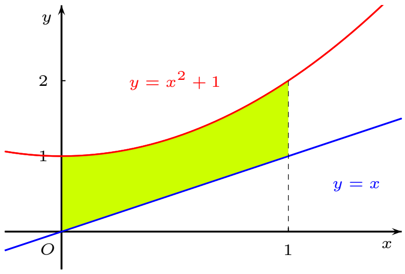
- In general, suppose f and g are continuous functions such that $f(x) \geq g(x)$ for all $x \in [a, b]$.
  - What is the area of the region bounded by $y = f(x)$ and $y = g(x)$ from a to b?



  - Answer: $A = \int_a^b (f(x) - g(x))\,dx$.

Examples

- Find the area of the region bounded above by $y = x^2 + 1$ and bounded below by $y = x$ from $x = 0$ to $x = 1$.

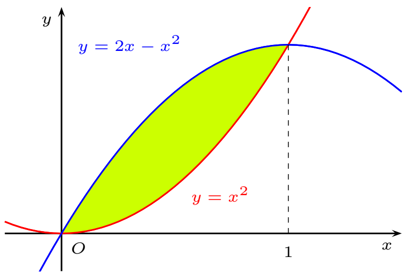


  - $A = \int_0^1 [(x^2 + 1) - x]\, dx = \int_0^1 (x^2 + 1 - x)\, dx.$
    - $A = \left[\frac{x^3}{3} + x - \frac{x^2}{2} \right]_{x=0}^{x=1} = \frac{5}{6} - 0 = \frac{5}{6}.$

5 / 25

Examples

- Find the area of the region enclosed by $y = x^2$ and $y = 2x - x^2$.

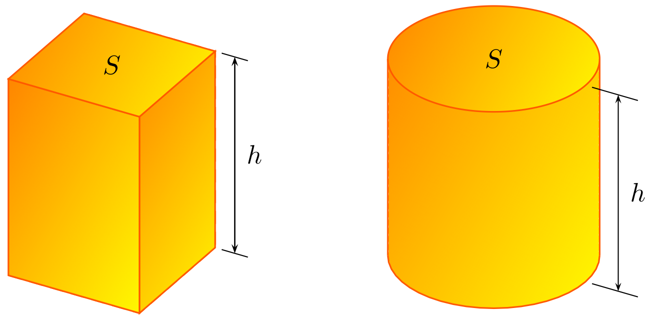


  - Find intersections : Let $x^2 = 2x - x^2$. Then $x = 0$ or $x = 1$.
    - So the integral is from 0 to 1.
    - $A = \int_0^1 [(2x - x^2) - x^2]\, dx = \left[x^2 - \frac{2}{3}x^3 \right]_{x=0}^{x=1} = \frac{1}{3}.$

6 / 25

The Volume Problem

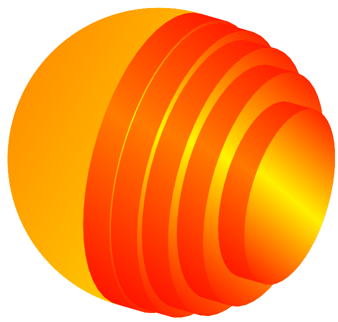
- We now move from $2D$ to $3D$. How to find the volume of a solid?
  - In particular, consider the following:

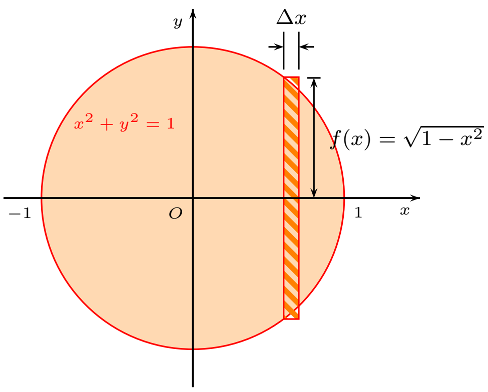


  - If the base area is S and the height is h, then the volume is
$$V = Sh.$$

The Volume of the Unit Sphere

- What is the volume of the unit sphere?

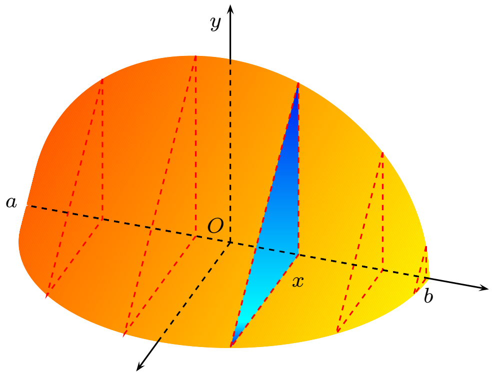




  - $\Delta V = S\,\Delta x = \pi(1 - x^2)\,\Delta x.$
  - $\dfrac{dV}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta V}{\Delta x} = \pi(1 - x^2).$
    - $V = \displaystyle\int_{-1}^{1} \pi(1 - x^2)\,dx = \pi\left[x - \frac{x^3}{3}\right]_{x=-1}^{x=1} = \frac{4\pi}{3}.$

The Volume Problem

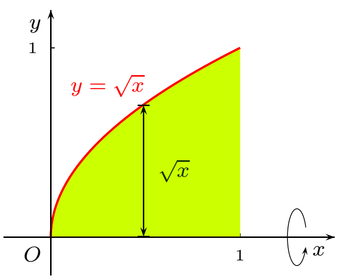
- In general, suppose a solid is put along the x-axis from a to b.

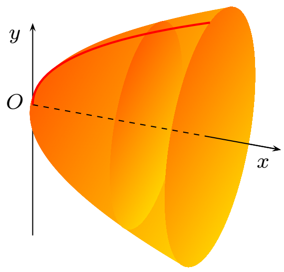


  - Let $A(x)$ be the area of the cross-section perpendicular to the x-axis and passing through the point x.
  - Then the volume of the solid is $V = \int_a^b A(x)\,dx.$

Solids of Revolution

- Suppose the solid is obtained by rotating the region under the curve $y = f(x)$ from a to b about the x-axis.
  - For example, $y = \sqrt{x}$ from 0 to 1:



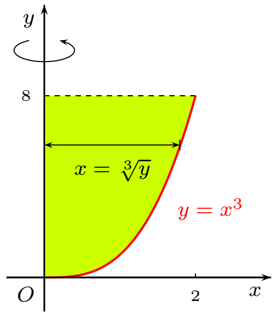


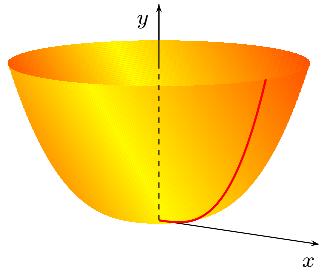
$A(x) = \pi(\sqrt{x})^2$. Then $V(x) = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \frac{\pi}{2}$.

- $A(x) = \pi(f(x))^2$, so $V = \int_a^b \pi(f(x))^2\,dx$. **Washer method**.

Examples

- Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$ and $x = 0$ about the y-axis.

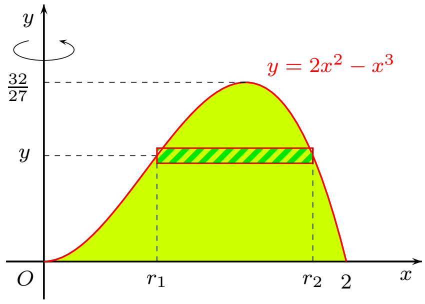


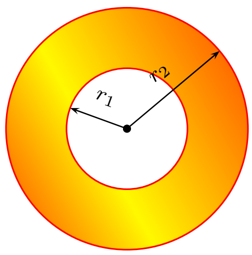


  - Integrate along the y-axis: $y = x^3 \Rightarrow x = \sqrt[3]{y}$. $A(y) = \pi(\sqrt[3]{y})^2$.
    - $V = \displaystyle\int_0^8 \pi(\sqrt[3]{y})^2\, dy = \int_0^8 \pi y^{2/3}\, dy = \frac{96}{5}\pi.$

Volumes by Cylindrical Shells

- Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.

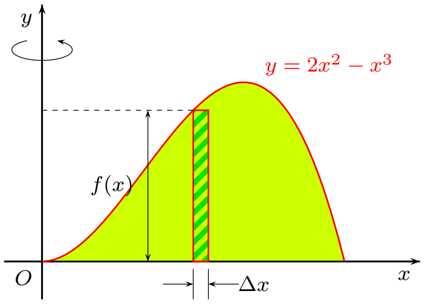


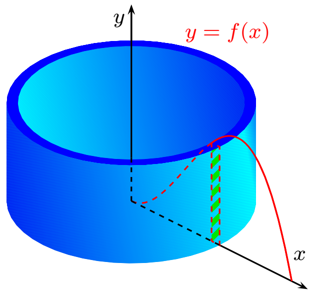


  - $A(y) = \pi(r_2^2 - r_1^2) = \cdots = ?$
  - Let $y' = 4x - 3x^2 = 0$. Then $x = 4/3$. $y = 32/27$.
    - $V = \displaystyle\int_0^{32/27} A(y)\, dy = \cdots\cdots.$
    - It seems that we cannot continue using washer method.

Volumes by Cylindrical Shells

- Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = 2x^2 - x^3$ and $y = 0$.





  - $\Delta V = \underbrace{\pi[(x+\Delta x)^2 - x^2]}_{\text{base area}} \cdot \underbrace{f(x)}_{\text{height}} = \pi(2x+\Delta x)f(x)\cdot\Delta x.$
  - $\dfrac{dV}{dx} = \lim_{\Delta x\to 0}\dfrac{\Delta V}{\Delta x} = \lim_{\Delta x \to 0}\pi(2x+\Delta x)f(x) = 2\pi x\, f(x).$

Volumes of Cylindrical Shells

- In general, let f be a continuous function such that $f(x) \geq 0$ for all $x \in [a,b]$, $(0 \leq a < b)$.
  - Then the volume of the solid obtained by rotating the region under the curve $y = f(x)$ from a to b about the y-axis is

$$V = \int_a^b 2\pi x\, f(x)\, dx.$$

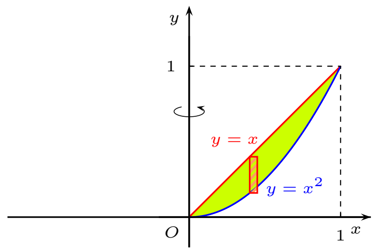
  - This is called the **method of cylindrical shells**.
  - In the previous example, $f(x) = 2x^2 - x^3$ on $[0,2]$.

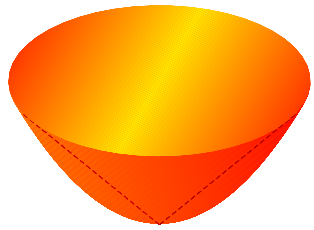
    Then the volume of the solid obtained by rotating the region under the curve $y = f(x)$, $0 \leq x \leq 2$ about the y-axis is

    - $V = \displaystyle\int_0^2 2\pi x(2x^2 - x^3)\, dx = \pi\left[x^4 - \frac{2}{5}x^5\right]_{x=0}^{x=5} = \frac{16\pi}{5}.$

Examples

- Find the volume of the solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$.

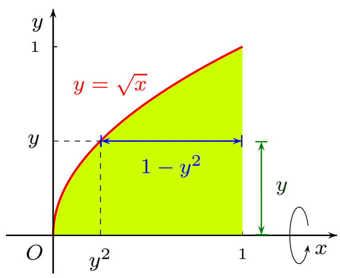


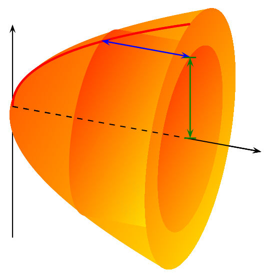


  - Find the lower and upper limit of integral:
    - $x = x^2 \Rightarrow x = 0$ or $x = 1$. The integral is from 0 to 1.
  - $V = \displaystyle\int 2\pi \text{radius} \cdot \text{height}\, dx$.
    - $V = \displaystyle\int_0^1 2\pi x(x - x^2)\, dx = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{\pi}{6}$.

Examples

- Use cylindrical shell method to find the volume of the solid obtained by rotating the region under the curve $y = \sqrt{x}$ from 0 to 1 about the x-axis.

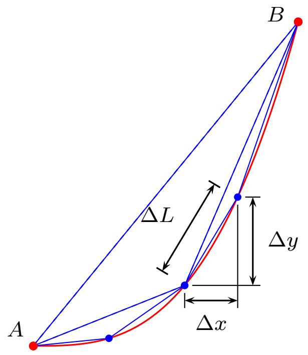




  - $V = \displaystyle\int_0^1 2\pi \underbrace{y}_{\text{radius}} \cdot \underbrace{(1 - y^2)}_{\text{height}}\, dy = \pi \left[y^2 - \frac{1}{2}y^4\right]_{y=0}^{y=1} = \frac{\pi}{2}$.

The Arc Length

- A function f called **smooth** if f' is continuous.
  - How to measure the arc length of a smooth curve?



$$\Delta L = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$\frac{\Delta L}{\Delta x} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

$$\frac{dL}{dx} = \lim_{\Delta x \to 0} \frac{\Delta L}{\Delta x} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

  - The **arc length** of the curve $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

17 / 25

Example

- Find the length of the arc of $y = \sqrt{x}$ from $(0,0)$ to $(1,1)$.
  - $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}$. $L = \displaystyle\int_0^1 \sqrt{1 + \frac{1}{4x}}\, dx = \lim_{a \to 0^+} \int_a^1 \sqrt{1 + \frac{1}{4x}}\, dx.$
    - Use $u = \sqrt{x}$ $(x > 0)$. Then $u^2 = x$.

$$\int \sqrt{1 + \frac{1}{4x}}\, dx = \int \sqrt{1 + \frac{1}{4u^2}}\, 2u\, du = \int \sqrt{1 + 4u^2}\, du.$$

    - Recall $\displaystyle\int \sqrt{1 + x^2}\, dx = \frac{x\sqrt{1+x^2}}{2} + \frac{\ln(x + \sqrt{x^2+1})}{2} + C.$

$$\int \sqrt{1 + 4u^2}\, du = \frac{1}{2} u\sqrt{1 + 4u^2} + \frac{1}{4}\ln(2u + \sqrt{1 + 4u^2}) + C.$$

$$\int \sqrt{1 + \frac{1}{4x}}\, dx = \frac{1}{2}\sqrt{x(1 + 4x)} + \frac{1}{4}\ln(2\sqrt{x} + \sqrt{1 + 4x}) + C.$$

18 / 25

Example

$$
\begin{aligned}
L &= \lim_{a\to 0^+} \int_a^1 \sqrt{1+\frac{1}{4x}}\,dx \\
&= \lim_{a\to 0^+} \left[\frac{1}{2}\sqrt{x(1+4x)} + \frac{1}{4}\ln(2\sqrt{x}+\sqrt{1+4x})\right]_{x=a}^{x=1} \\
&= \frac{1}{2}\sqrt{5} + \frac{1}{4}\ln(2+\sqrt{5}).
\end{aligned}
$$

- Alternatively, $y=\sqrt{x} \Leftrightarrow x = y^2$. Then $\dfrac{dx}{dy} = 2y$.

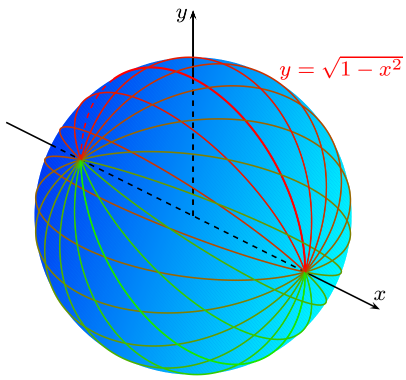
$$
\begin{aligned}
\frac{dL}{dy} &= \lim_{\Delta y\to 0}\frac{\Delta L}{\Delta y} = \lim_{\Delta y\to 0}\frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{\Delta y} \\
&= \lim_{\Delta y\to 0}\sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2+1} = \sqrt{\left(\frac{dx}{dy}\right)^2+1}.
\end{aligned}
$$

- $L = \displaystyle\int_0^1 \sqrt{(2y)^2+1}\,dy = \cdots = \frac{\sqrt{5}}{2} + \frac{1}{4}\ln(2+\sqrt{5}).$

19 / 25

Surface Area of Revolution

- Let's now consider the surface area problem of a $3D$ object.
  - For example, what is the surface area of the **unit sphere**?

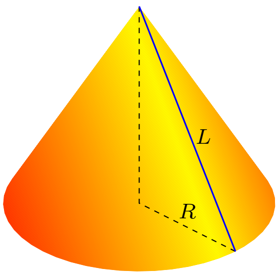


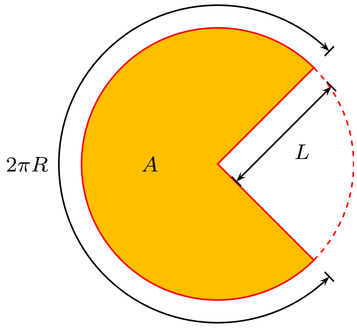
  - The surface of the unit sphere can be viewed as the rotation of the curve $y=\sqrt{1-x^2}$ about the x-axis.

20 / 25

Surface area of a Cone

- **Question**: How to evaluate the surface area of revolution?
  - What is the surface area of a **cone**?



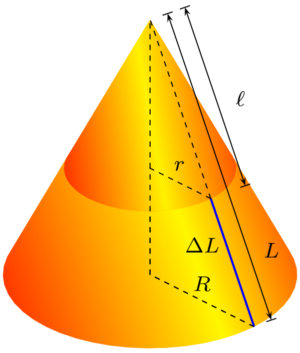


  - $\dfrac{\text{Area of the Wedge}}{\text{Length of the Arc}} = \dfrac{\text{Area of the circle}}{\text{Length of the circle}}$.
  - $\dfrac{A}{2\pi R} = \dfrac{\pi L^2}{2\pi L} = \dfrac{L}{2} \Rightarrow \boxed{A = \pi RL}$.

21 / 25

Frustum of a cone

- What is the surface area of a **frustum of a cone**?

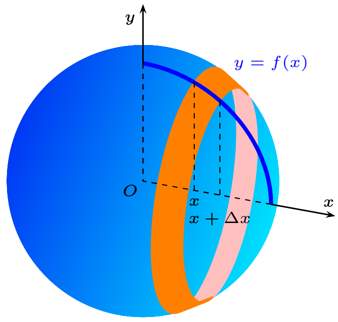


  - $A = \pi(LR - r\ell) = \pi[L(R-r) + Lr - r\ell]$
    $= \pi(R\,\Delta L + r\,\Delta L) = \pi(R + r)\,\Delta L$.
    - $\dfrac{L}{R} = \dfrac{\ell}{r} = \dfrac{L-\ell}{R-r} = \dfrac{\Delta L}{R-r} \Rightarrow L(R-r) = R\,\Delta L$.

22 / 25

Surface Area of Revolution

- We now approximate the surface by the frustums of a cone:



  - $\Delta A = \pi(\underbrace{f(x)}_{\text{base}} + \underbrace{f(x+\Delta x)}_{\text{base}}) \times \underbrace{\Delta L}_{\text{slant height}}$.
  - $\dfrac{\Delta A}{\Delta x} = \pi \cdot (f(x) + f(x+\Delta x))\dfrac{\Delta L}{\Delta x}$.
    - $\dfrac{dA}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta A}{\Delta x} = \pi \cdot 2f(x) \cdot \sqrt{1 + (f'(x))^2}$.

Surface Area of Revolution

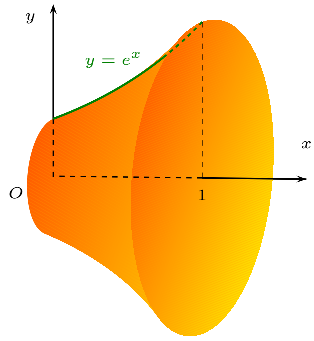
- Let f be a smooth function such that $f(x) \geq 0$ on $[a, b]$. Then the area of the surface obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis is

$$A = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

- In particular, recall that the surface of the unit sphere is obtained by rotating $y = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$, about the x-axis.
  - $f'(x) = -\dfrac{x}{\sqrt{1-x^2}} \Rightarrow \sqrt{1 + (f'(x))^2} = \dfrac{1}{\sqrt{1-x^2}}$.
  - $A = \displaystyle\int_{-1}^{1} 2\pi\sqrt{1-x^2}\,\frac{1}{\sqrt{1-x^2}}\, dx = \int_{-1}^{1} 2\pi\, dx = 4\pi.$
    - Therefore, the surface area of the unit sphere is 4π.

Example

- Find the area of the surface generated by rotating the curve $y = e^x$, $0 \le x \le 1$, about the x-axis.



- $$A = \int_0^1 2\pi\, e^x \sqrt{1 + ((e^x)')^2}\, dx = 2\pi \int_0^1 e^x \sqrt{1 + (e^x)^2}\, dx$$
$$= 2\pi \int_1^e \sqrt{1+u^2}\, du = \pi \left(u\sqrt{1+u^2} + \ln(u + \sqrt{1+u^2}) \right) \Big|_{u=1}^{u=e} = \cdots.$$

25 / 25