MA1521 CALCULUS FOR COMPUTING

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Roots of Quadratic Equations

- Solve the **quadratic equation**: $x^2 + bx + c = 0$.
  1. $\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0.$
  2. $\left(x + \frac{b}{2}\right)^2 = \frac{b^2 - 4c}{4}.$
  3. $x + \frac{b}{2} = \pm\frac{\sqrt{b^2 - 4c}}{2}.$
  4. $x = -\frac{b \pm \sqrt{b^2 - 4c}}{2}.$

  - Remove the **linear term** by **substitution**.
    - Let $t = x + \frac{b}{2}$, i.e., $x = t - \frac{b}{2}$.
    - The equation becomes $t^2 + d = 0$.

Roots of Cubic Equations

- Solve the **cubic equation**: $x^3 + bx^2 + cx + d = 0$.
  1. Use **Cardano's method** (1545): $x = t - b/3$.
     - $t^3 + pt + q = 0$.
  2. Set $t = u + v$:
     - $(u^3 + v^3) + (3uv + p)(u + v) + q = 0.$
  3. Suppose $3uv + p = 0$.
     - $u^3 + v^3 = -q, \quad u^3v^3 = -p^3/27.$
  4. u^3 and v^3 are roots of
     - $z^2 + qz - p^3/27 = 0$.
  5. Solve the equation above to get u and v.
     - $x = t - b/3 = u + v - b/3$.

Roots of Cubic Equations

- Solve the **cubic equation**: $x^3 + bx^2 + cx + d = 0$.
  1. Set $x = t - b/3$.
     - $t^3 + pt + q = 0$.
  2. Solve $z^2 + qz - p^3/27 = 0$.
     - $z_1 = u^3$ and $z_2 = v^3$.
  3. $x = u + v - b/3 = \sqrt[3]{z_1} + \sqrt[3]{z_2} - b/3$.
- $t^3 + pt + q = 0$ is called the **depressed form**.
  - The **discriminant** $\Delta = 4p^3 + 27q^2$.
    - $\Delta > 0$: 3 distinct real roots.
    - $\Delta = 0$: multiple real roots.
    - $\Delta < 0$: 1 real and 2 nonreal conjugate roots.

Example

- Solve $x^3 - 6x^2 + 9x - 4 = 0$.
  1. $b = -6$. Set $x = t - b/3 = t + 2$.
     - Depressed form: $t^3 - 3t - 2 = 0$.
  2. $p = -3$ and $q = -2$. Solve $z^2 + qz - p^3/27 = 0$:
     - $z^2 - 2z + 1 = 0 \Rightarrow z_1 = z_2 = 1$.
  3. $u = \sqrt[3]{z_1} = 1$ and $v = \sqrt[3]{z_2} = 1$.
     - $x_1 = u + v + 2 = 1 + 1 + 2 = 4$.
  4. Factorize $x^3 - 6x^2 + 9x - 4 = (x - 4)(x^2 - 2x + 1)$.
  5. Solve $x^2 - 2x + 1 = 0$: $x_2 = x_3 = 1$.
  6. Therefore, the roots are $x_1 = 4, x_2 = 1, x_3 = 1$.
     - $x^3 - 6x^2 + 9x - 4 = (x - 4)(x - 1)^2$.

Roots of Quartic Equations

- **Quartic equation**: $x^4 + bx^3 + cx^2 + dx + e = 0$.
  1. **Ferrari method** (1522–1565): Set $x = t - b/4$.
     - **Depressed form**: $t^4 + pt^2 + qt + r = 0$.
  2. Solve a **cubic equation** in z:
     - $z^3 + \frac{5}{2}pz^2 + (2p^2 - r)z + \left(\frac{p^3}{2} - \frac{pr}{2} - \frac{q^2}{8}\right) = 0$.
  3. $x = \dfrac{\pm\sqrt{p+2z} \pm \sqrt{-\left(3p + 2z \pm \frac{2q}{\sqrt{p+2z}}\right)}}{2} - \dfrac{b}{4}$.
     - The first and the third $\pm$ are both positive or negative.

Example

- Solve $x^4 - 12x^3 + 43x^2 - 24x - 80 = 0$.
  1. Set $x = t + 3$.
     - $t^4 - 11t^2 + 18t - 8 = 0$.
  2. Solve a cubic equation in z:
     - $z^3 - \frac{55}{2}z^2 + 250z - 750 = 0$.
       - Use Cardano's method to get $z = 15/2$.
  3. $z = \dfrac{\pm 2 \pm \sqrt{-(-18 \pm 18)}}{2}$.
     - $z = 1, 1, -4, 2$.

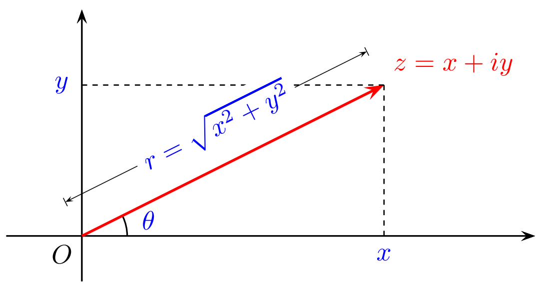
Roots of Quintic Equations

- **Quintic equation**: $x^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$.
  - Évariste Galois (1811 – 1832) French
  - Niels Henrik Abel (1802 – 1829) Norwegian
- **Abel's Impossibility Theorem**.
  - There is **no** general **algebraic** solution
    - that is, expression using $+, -, \times, \div, \sqrt[n]{\ }$,

    to **polynomial equations** of **degree** ≥ 5.
- **Remarks**.
  - Approximated solution may be found using **Newton-Raphson's method**.
  - Johann Carl Friedrich Gauss (1777 – 1855)
    - Fundamental Theorem of Algebra (1799): A polynomial equation of degree n has n roots in complex numbers.

Complex Numbers

- **Complex numbers**: $\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\}$, $i^2 = -1$.
  - $\mathbb{C}$ is identified with the **Cartesian plane** $\mathbb{R}^2$:
    - $x + iy \leftrightarrow (x, y)$.
  - **Polar coordinate**: $z = r(\cos\theta + i\sin\theta)$.
    - $z = re^{i\theta}, \qquad r = |z|, \quad \theta = \arg z$.



Complex Functions

- For any **real** number x,
  - $e^{ix} = \cos x + i\sin x$,
  - $e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i\sin x$.

  We solve that
  - $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, $x \in \mathbb{R}$.
- **Definition**. For any **complex** number z,define
  - $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.

  We can verify that all the **trigonometric identities** still hold:
  - $\cos^2 z + \sin^2 z = 1$;
  - $\sin 2z = 2\sin z \cos z$;
  - $\cos 2z = \cos^2 z - \sin^2 z$;
  - ....

Complex Functions

- Recall that $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.
  - $\cos^2 z + \sin^2 z = \left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2$.
    - $\left(\dfrac{e^{iz} + e^{-iz}}{2}\right)^2 = \dfrac{e^{i2z} + e^{-i2z} + 2}{4}$.
    - $\left(\dfrac{e^{iz} - e^{-iz}}{2i}\right)^2 = \dfrac{e^{i2z} + e^{-i2z} - 2}{-4}$.
  - $\cos^2 z + \sin^2 z = 1$.
- **Definition**. $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{i(e^{iz} - e^{-iz})}{e^{iz} + e^{-iz}}$.
  - $\cot z = \dfrac{\cos z}{\sin z}$, $\sec z = \dfrac{1}{\cos z}$, $\csc z = \dfrac{1}{\sin z}$.

Complex Functions

- Recall that $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.
  - $\cos i = \dfrac{e^{i \cdot i} + e^{-i \cdot i}}{2} = \dfrac{e^{-1} + e}{2} = \dfrac{e + e^{-1}}{2} \approx 1.543.$
  - $\sin i = \dfrac{e^{i \cdot i} - e^{-i \cdot i}}{2i} = \dfrac{e^{-1} - e}{2i} = \dfrac{i(e - e^{-1})}{2} \approx 1.175i.$
- The **exponential function** a^x is extendable for any $a, x \in \mathbb{C}$.
  - Examples.
    - $(-1)^i = e^{-\pi}; \quad i^i = e^{-\pi/2}.$
- The **logarithmic function** $\ln z$ is also extendable to $\mathbb{C} \setminus \{0\}$.
  - Examples.
    - $\log(-1) = \pi i; \quad \log i = \dfrac{\pi i}{2}; \quad \log(-i) = -\dfrac{\pi i}{2}.$

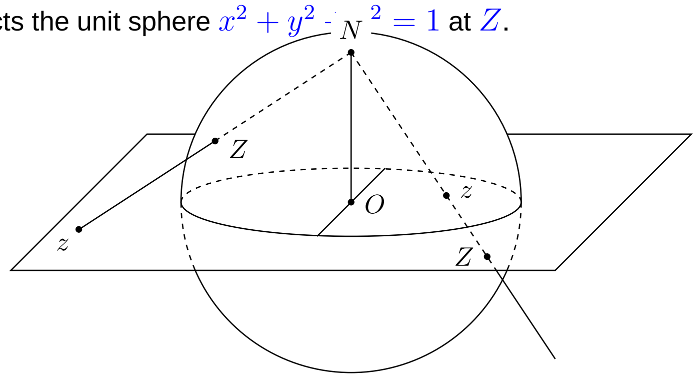
Example

- Let $z = \cos x + i \sin x = e^{ix}$. Let
  - $S = 1 + z + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}.$

$$
\begin{aligned}
S &= \frac{1 - [\cos(n+1)x + i\sin(n+1)x]}{1 - (\cos x + i \sin x)} \\
&= \frac{1}{2} + \frac{1}{2}\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} + \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}} i \\
\mathrm{Re}(S) &= 1 + \cos x + \cos 2x + \cdots + \cos nx \\
&= \frac{1}{2} + \frac{1}{2}\frac{\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \\
\mathrm{Im}(S) &= \sin x + \sin 2x + \cdots + \sin nx \\
&= \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}.
\end{aligned}
$$

Stereographic Projection

- For any $z = x + iy \leftrightarrow (x, y)$, connect z to $N(0, 0, 1)$.
  - ☐ Line Nz intersects the unit sphere $x^2 + y^2 + z^2 = 1$ at Z.



  - ☐ $S^2 := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ the **unit sphere**.
    - $\mathbb{C} \leftrightarrow S^2 \setminus \{N\}$ via $z \leftrightarrow Z$.
    - $\mathbb{C} \cup \{\infty\} \leftrightarrow S^2$ via $z \leftrightarrow Z$ and $\infty \leftrightarrow N(0, 0, 1)$.

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Matrix Representation

- A **complex number** can be identified with a 2×2 real **matrix**:
  - ☐ $z = x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = M_z$.
- All the **arithmetic properties** are preserved:
  - ☐ **Addition**: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
    - $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$.
    - $M_{z_1} + M_{z_2} = \begin{pmatrix} x_1 + x_2 & -(y_1 + y_2) \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix} = M_{z_1 + z_2}$
  - ☐ **Multiplication**: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
    - $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$.

$$
\begin{aligned}
M_{z_1} M_{z_2} &= \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 x_2 - y_1 y_2 & -(x_1 y_2 + y_1 x_2) \\ x_1 y_2 + y_1 x_2 & x_1 x_2 - y_1 y_2 \end{pmatrix} = M_{z_1 z_2}
\end{aligned}
$$

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Matrix Representation

- A **complex number** can be identified with a 2×2 real **matrix**:
  - $z = x + iy \leftrightarrow \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = M_z.$
- All the **arithmetic properties** are preserved:
  - **Modulus**: $|z|^2 = x^2 + y^2 = \det(M_z).$
  - **Conjugate**: $z^* = x - iy \leftrightarrow \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = (M_z)^T.$
  - **Quotient**: Let $z = x + iy$. Then $1/z = z^{-1} = \dfrac{x - iy}{x^2 + y^2}.$
    - $M_{z^{-1}} = \dfrac{1}{x^2 + y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \cdots = (M_z)^{-1}.$
  - **Power**: $M_{z^n} = (M_z)^n.$
  - ....

Evaluation of π

Liu Hui's Algorithm

- Recall that Archimedes, Liu Hui and Zu Chongzhi used **regular polygons** to approximation π, the area of **unit circle**.
- **Liu Hui's** (220 – ?) **algorithm** for π:
  1. Let $M_N =$ length of the side of regular N-sided polygon inscribed in the unit circle. $M_6 = 1.$
  2. Let $A_N =$ area of the regular N-sided polygon inscribed in the unit circle. $A_6 \approx 3M_6 = 3.$
  3. Let $L(n) = 2 - M^2_{3\times 2^n}.$
     - $L(n+1) = \sqrt{2 + L(n)}, \quad L(1) = 2 - 1^2 = 1.$
  4. $\pi \approx A_{3\times 2^n} \approx 3 \times 2^{n-1} \times M_{3\times 2^n}.$
- **Remark**.
  - Liu Hui evaluated up to 96-sided polygon, and used a shortcut to generate the result for 1536-sided polygon.

Liu Hui's Algorithm

- **Liu Hui's Algorithm** for π:

Iteration	Sides	Approximation of π
1	6	3.
2	12	3.1
3	24	3.13
4	48	3.14
5	96	3.141
9	1536	3.14159
12	12288	3.1415926
15	98304	3.141592653
20	1572864	3.141592653589
30	1610612736	3.141592653589793238

- **Remarks**: $\pi \approx 22/7$ and $\pi \approx 355/113$.

Integration

- If $f(x) \leq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
- For all $0 \leq x \leq 1$, $\displaystyle\frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$.
  - $\displaystyle\frac{1}{2}x^4(1-x)^4 \leq \frac{x^4(1-x)^4}{1+x^2} \leq x^4(1-x)^4$.
  - $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{2}\,dx \leq \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx \leq \int_0^1 x^4(1-x)^4\,dx$.
    - $\displaystyle\frac{1}{1260} \leq \frac{22}{7} - \pi \leq \frac{1}{630}$.

$\therefore$ $\displaystyle\frac{22}{7}$ is a bigger approximation of π with error $\displaystyle\leq \frac{1}{630} < 0.0016$.

Integration

- $$\int_0^1 \frac{x^8(1-x)^8}{8}\,dx \le \int_0^1 \frac{x^8(1-x)^8}{4(1+x^2)}\,dx \le \int_0^1 \frac{x^8(1-x)^8}{4}\,dx.$$
  - □ $\dfrac{1}{1750320} \le \pi - \dfrac{47171}{15015} \le \dfrac{1}{875160}.$

$\therefore$ $\dfrac{47171}{15015} \approx 3.14159174\cdots$ is a smaller approximation for π

with error $\le \dfrac{1}{875160} \approx 10^{-6}.$

- $$\int_0^1 \frac{x^{12}(1-x)^{12}}{32}\,dx \le \int_0^1 \frac{x^{12}(1-x)^{12}}{16(1+x^2)}\,dx \le \int_0^1 \frac{x^{12}(1-x)^{12}}{16}\,dx.$$
  - □ $\dfrac{1}{2163324800} \le \dfrac{431302721}{137287920} - \pi \le \dfrac{1}{1081662400}.$

$\therefore$ $\dfrac{431302721}{137287920} \approx 3.141592654\cdots$ is a bigger approximation for π

with error $\le \dfrac{1}{1081662400} \approx 10^{-9}.$

Integration

- Approximate π using
  - □ $$\int_0^1 \frac{x^{4n}(1-x)^{4n}}{4^{n-1}(1+x^2)}\,dx \le \int_0^1 \frac{x^{4n}(1-x)^{4n}}{4^{n-1}}\,dx < \frac{1}{2^{10n-2}}$$

n	Fraction	Decimal
1	$\dfrac{22}{7}$	3.14
3	$\dfrac{431302721}{137287920}$	3.141592654
5	$\dfrac{26856502742629699}{8548690331301120}$	3.141592653589793
10	$\dfrac{8929347825205334111475899568201677 3}{28422996899365886608045972478361600}$	3.14159265358979323846264338 3279

Power Series

- Recall that a **power series** has the form:
  - $\displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots.$

  It plays an important role in **approximation theory**.

- **Examples**.
  - $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots.$
    - $e \approx 1 + 1 + \dfrac{1}{2!} + \dfrac{1}{3!} + \cdots + \dfrac{1}{10!} \approx 2.718281801.$
  - $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots.$
    - $\sin 2 \approx 2 - \dfrac{2^3}{3!} + \dfrac{2^5}{5!} - \dfrac{2^7}{7!} + \dfrac{2^9}{9!} \approx 0.9093474427.$

Integration

- Approximate $\displaystyle\int_0^1 \sin(x^2)\,dx$.

1. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \dfrac{x^{11}}{11!} + \cdots.$
2. $\sin(x^2) \approx x^2 - \dfrac{x^6}{3!} + \dfrac{x^{10}}{5!} - \dfrac{x^{14}}{7!} + \dfrac{x^{18}}{9!} - \dfrac{x^{22}}{11!}.$
3. Approximate $\displaystyle\int_0^1 \sin(x^2)\,dx$ by
   - $\displaystyle\int_0^1 x^2\,dx - \int_0^1 \frac{x^6}{3!}\,dx + \cdots - \int_0^1 \frac{x^{22}}{11!}\,dx$
4. $\displaystyle\int_0^1 \sin(x^2)\,dx \approx 0.3102683017174579.$

- The exact value is $0.310268301723338110\cdots$.

Ordinary Differential Equation

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by term differentiation**: $f'(x)$:
    - $c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots$.
- **Example**. Suppose $\dfrac{dy}{dx} = y$ and $y = 1$ at $x = 0$.
  - Let $f(x) = y = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$.
    - $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots$.
  - Compare coefficients:
    - $c_0 = 1$, $c_1 = c_0$, $2c_2 = c_1$, $3c_3 = c_2$, $4c_4 = c_3$, $\ldots$.
    - $c_0 = 1$, $c_1 = 1$, $c_2 = 1/2$, $c_3 = 1/6$, $c_4 = 1/24$, $\ldots$
  - $y = f(x) = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{6} + \dfrac{x^4}{24} + \cdots$.

Term by Term Integration

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by term integration**: $\displaystyle\int f(x)\,dx$:
    - $c_0 x + \dfrac{c_1}{2} x^2 + \dfrac{c_2}{3} x^3 + \cdots + \dfrac{c_n}{n+1} x^{n+1} + \cdots$
- **Examples**.
  - $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$.
    - $\displaystyle\int \frac{1}{1-x}\,dx = -\ln|1-x| + C$.
    - $x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots + \dfrac{x^n}{n} + \dfrac{x^{n+1}}{n+1} + \cdots$.
  - $-\ln|1-x| = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{4} + \cdots + \dfrac{x^n}{n} + \dfrac{x^{n+1}}{n+1} + \cdots$.

Term by Term Integration

- Suppose $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + \cdots$.
  - **Term by term integration**: $\displaystyle\int f(x)\,dx$:
    - $c_0 x + \dfrac{c_1}{2}x^2 + \dfrac{c_2}{3}x^3 + \cdots + \dfrac{c_n}{n+1}x^{n+1} + \cdots$
- **Examples**.
  - $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$.
    - $\displaystyle\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$.
    - $x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + \cdots$.
  - $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots + (-1)^n \dfrac{x^{2n+1}}{2n+1} + \cdots$

Some Special Series

- $-\ln|1-x| = x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots + \dfrac{x^n}{n} + \cdots$, $R = 1$.
  - Let $x = 1/2$. Then
    - $\ln 2 = \dfrac{1}{2} + \dfrac{1}{2 \cdot 2^2} + \dfrac{1}{3 \cdot 2^3} + \cdots + \dfrac{1}{n \cdot 2^n} + \cdots$
    - $\ln 2 \approx \dfrac{1}{2} + \dfrac{1}{2 \cdot 2^2} + \cdots + \dfrac{1}{20 \cdot 2^{20}} = 0.6931471\cdots$
  - Let $x = -1$. Then
    - $-\ln 2 = -1 + \dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{4} - \cdots + \dfrac{(-1)^n}{n} + \cdots$.
    - $\ln 2 = 1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{(-1)^{n-1}}{n} + \cdots$.

  $\therefore$ The **alternating Harmonic series** converges to $\ln 2$.

  Warning: The convergency of alternating Harmonic series is very slow!

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $R = 1$. Take $x = \frac{1}{\sqrt{3}}$:
    - $\frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} - \frac{1}{7(\sqrt{3})^7} + \cdots$
    - $\frac{\pi}{6} \approx \frac{1}{\sqrt{3}} - \frac{1}{3(\sqrt{3})^3} + \frac{1}{5(\sqrt{3})^5} + \cdots + \frac{1}{41(\sqrt{3})^{41}}$
      - $\frac{\pi}{6} \approx 0.52359877559927\cdots$
      - $\pi \approx 3.14159265359563\cdots$
  - $\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-3)^{-n}}{2n+1}$ is the **Madhava-Leibniz series**.
    - It is **efficient** in evaluating π.

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $R = 1$. Take $x = 1$:
    - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots.$

    This is known as the **Leibniz formula** for π.
  - **Warning**: The convergency is very slow.
- Srinivasa Ramanujan (1887–1920) Indian mathematician.
  - $\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 394^{4n}}.$
    - The first term gives $\frac{1}{\pi} \approx \frac{2\sqrt{2}}{9801} \cdot 1103.$
    - $\pi \approx \frac{9801\sqrt{2}}{4412} = 3.14159273\cdots.$

Some Special Series

- $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots$
  - Note that $R = 1$. Take $x = 1$:
    - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots$.

    This is known as the **Leibniz formula** for π.
  - **Warning**: The convergency is very slow.
- Srinivasa Ramanujan (1887–1920) Indian mathematician.
  - $\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!(13591409 + 545140134n)}{(3n)!(n!)^3 640320^{3n+3/2}}$.
    - First term gives $\frac{1}{\pi} \approx 12 \cdot \frac{13591409}{640320^{3/2}}$.
    - $\pi \approx \frac{640320^{3/2}}{12 \cdot 13591409} = 3.141592653589793 \cdots$.