ECE 141: Principles of Feedback Control

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Overview

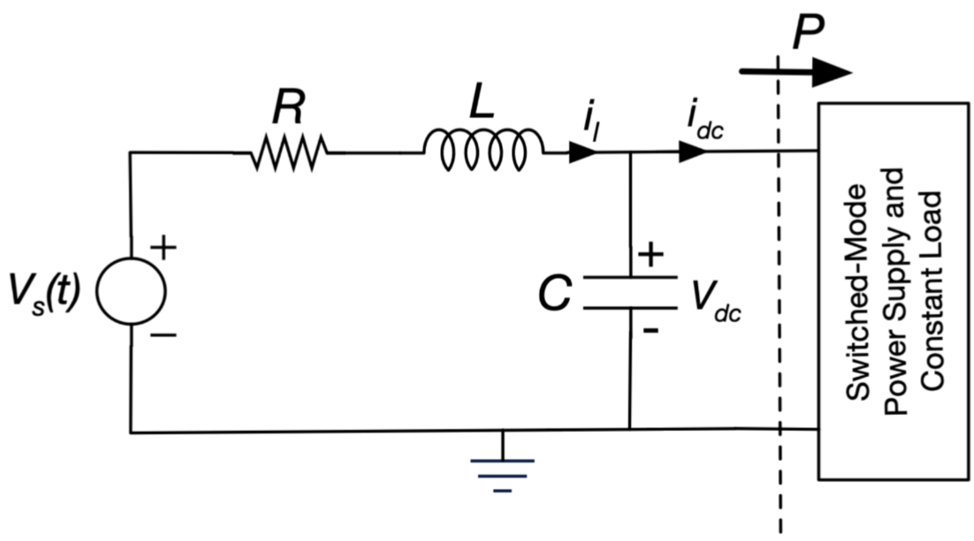

Figure 1: Problem Setup

1) State-Space Representation

Considering the inductor current and the capacitor voltage as your two states (i.e., $x_1(t) = i_l(t)$ and $x_2(t) = v_{dc}(t)$), obtain the state-space representation for this system in the form of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$ where $u(t) = v_s(t)$. Recall that power supplied to the load is the product of the current through and the voltage across it.

From KVL, we know that

$$-v_s + Ri_l + L\frac{di_l}{dt} + v_{dc} = 0$$

Solving for $\frac{di_l}{dt}$, we get

$$\frac{di_l}{dt} = \frac{1}{L}v_s - \frac{1}{L}v_{dc} - \frac{R}{L}i_l$$

Substituting $x_1(t) = i_l(t)$, $x_2(t) = v_{dc}(t)$, and $u(t) = v_s(t)$, the equation becomes

$$\dot{x_1} = \frac{1}{L}u - \frac{1}{L}x_2 - \frac{R}{L}x_1 \tag{1}$$

From KCL, we know that

$$i_2 = i_l - i_{dc}$$

The definition of power tells us that

$$i_{dc} = \frac{P}{v_{dc}}$$

The current through a capacitor is defined as

$$i_2 = C\frac{dv_{dc}}{dt}$$

Rearranging equation and substituting $i_2 = i_l - i_{dc}$, $i_2 = C\frac{dv_{dc}}{dt}$

$$\frac{dv_{dc}}{dt} = \frac{1}{C}i_1 - \frac{P}{C}\frac{1}{v_{dc}}$$

Substituting $x_1(t) = i_l(t)$ and $x_2(t) = v_{dc}(t)$ the equation becomes

$$\dot{x_2} = -\frac{P}{Cx_2} + \frac{1}{C}x_1 \tag{2}$$

Combining equations (1) and (2), we get the state-space representation of the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{L}(u - Rx_1 - x_2) \\ \frac{1}{C}(x_1 - \frac{P}{x_2}) \end{bmatrix} \tag{3}$$

2) Linearization of System

Assume that the system has an equilibrium point at $(i_{l0}, v_{dc0}, v_{s0})$. Obtain the linearized dynamics of this system in the form of

$$\delta\dot{\boldsymbol{x}} = \mathbf{A}\delta\boldsymbol{x} + \mathbf{B}\delta u$$

where

$$\delta x \triangleq \begin{bmatrix} i_l - i_{l0} \\ v_{dc} - v_{dc0} \end{bmatrix} \quad \text{and} \quad \delta u \triangleq v_s - v_{s0}$$

Find matrices **A** and **B**.

Computing the Jacobian Matrices

Let

$$f_1(x_1, x_2, u) = \frac{1}{L}\Big(u - R\,x_1 - x_2\Big)$$

and

$$f_2(x_1, x_2) = \frac{1}{C}\Big(x_1 - \frac{P}{x_2}\Big)$$

The partial derivatives are computed as follows

For f_1:

$$\frac{\partial f_1}{\partial x_1} = -\frac{R}{L}, \quad \frac{\partial f_1}{\partial x_2} = -\frac{1}{L}, \quad \frac{\partial f_1}{\partial u} = \frac{1}{L}$$

For f_2:

$$\frac{\partial f_2}{\partial x_1} = \frac{1}{C}, \quad \frac{\partial f_2}{\partial x_2} = \frac{\partial}{\partial x_2}\left[-\frac{P}{C\,x_2}\right] = \frac{P}{C\,x_2^2}, \quad \frac{\partial f_2}{\partial u} = 0$$

Evaluating at the equilibrium point $(i_{l0}, v_{dc0}, v_{s0})$, the Jacobian matrices become

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(i_{l0}, v_{dc0}, v_{s0})} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & \dfrac{P}{C\,v_{dc_0}^2} \end{bmatrix}$$

and

$$\mathbf{B} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}_{(i_{l0}, v_{dc0}, v_{s0})} = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}$$

Final Linearized Model

Thus, the linearized dynamics of the system is given by

$$\delta\dot{\mathbf{x}} = \mathbf{A}\,\delta\mathbf{x} + \mathbf{B}\,\delta u,$$

or explicitly,

$$\delta\dot{\mathbf{x}} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & \frac{P}{C\,v_{dc_0}^2} \end{bmatrix} \begin{bmatrix} i_l - i_{l_0} \\ v_{dc} - v_{dc_0} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} (v_s - v_{s_0}) \tag{4}$$

This completes the linearization of the nonlinear state-space model around the equilibrium point $\big(i_{l_0}, v_{dc_0}, v_{s_0}\big)$.

3) Open-Loop Transfer Function

Find the open-loop transfer function $H(s)$ from input δv_s to output δv_{dc} in s-domain, assuming zero initial conditions. Find the relationship between R, L, C, and P that would ensure the linearized system would remain stable.

We begin with the linearized state-space model obtained in Part 2:

$$\delta\dot{\mathbf{x}} = \mathbf{A}\,\delta\mathbf{x} + \mathbf{B}\,\delta u, \quad y = \mathbf{C}\delta\mathbf{x},$$

where

$$\delta\mathbf{x} = \begin{bmatrix} i_l - i_{l_0} \\ v_{dc} - v_{dc_0} \end{bmatrix}, \quad \delta u = v_s - v_{s_0}, \quad \text{and} \quad y = v_{dc} - v_{dc0}$$

The matrices are given by

$$\mathbf{A} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & \dfrac{P}{C\,v_{dc_0}^2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The open-loop transfer function from δu (i.e., δv_s) to y (i.e., δv_{dc}) is

$$H(s) = \frac{Y(s)}{\delta U(s)} = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}.$$

Next, we compute the inverse of $(s\mathbf{I} - \mathbf{A})$. First, note that

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s + \dfrac{R}{L} & \dfrac{1}{L} \\ -\dfrac{1}{C} & s - \dfrac{P}{C\,v_{dc_0}^2} \end{bmatrix}.$$

Its determinant is

$$\Delta(s) = \left(s + \frac{R}{L}\right)\left(s - \frac{P}{C\,v_{dc_0}^2}\right) + \frac{1}{L\,C}.$$

Thus, the inverse is given by

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\Delta(s)}\begin{bmatrix} s - \dfrac{P}{C\,v_{dc_0}^2} & -\dfrac{1}{L} \\ \dfrac{1}{C} & s + \dfrac{R}{L} \end{bmatrix}.$$

Multiplying by $\mathbf{B}$ yields

$$(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{\Delta(s)}\begin{bmatrix} \frac{1}{L}\left(s - \dfrac{P}{C\,v_{dc_0}^2}\right) \\ \frac{1}{C\,L} \end{bmatrix}.$$

Taking the product with $\mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$, we find

$$H(s) = \begin{bmatrix} 0 & 1 \end{bmatrix}\frac{1}{\Delta(s)}\begin{bmatrix} \frac{1}{L}\left(s - \frac{P}{C\,v_{dc_0}^2}\right) \\ \frac{1}{C\,L} \end{bmatrix} = \frac{1}{C\,L\,\Delta(s)}.$$

Expanding $\Delta(s)$, we obtain:

$$\Delta(s) = s^2 + s\left(\frac{R}{L} - \frac{P}{C\,v_{dc_0}^2}\right) + \frac{1}{L\,C}\left(1 - \frac{R\,P}{v_{dc_0}^2}\right).$$

Thus, the open-loop transfer function is

$$H(s) = \frac{1}{C\,L\left[s^2 + s\left(\frac{R}{L} - \frac{P}{C\,v_{dc_0}^2}\right) + \frac{1}{LC}\left(1 - \frac{R\,P}{v_{dc_0}^2}\right)\right]} \tag{5}$$

For the linearized system to remain stable, all poles (i.e., the roots of the denominator) must have negative real parts. This requires the coefficients of the characteristic polynomial to be positive. In particular, we require

$$\frac{R}{L} - \frac{P}{C\,v_{dc_0}^2} > 0 \tag{6}$$

and

$$\frac{1}{LC}\left(1 - \frac{R\,P}{v_{dc_0}^2}\right) > 0.$$

Since $L > 0$ and $C > 0$, the second inequality simplifies to

$$1 - \frac{RP}{v_{dc_0}^2} > 0 \quad \Longrightarrow \quad v_{dc_0}^2 > RP \tag{7}$$

These relationships between R, L, C, and P ensure that the characteristic polynomial has all positive coefficients and that the poles lie in the left half of the complex plane, thereby guaranteeing the stability of the linearized system.

4) Equilibrium Values

Assume the wire resistance of $R = 60\Omega$, the CPL power constant of $P = 20$W and, input equilibrium DC voltage of $v_{s0} = 80$V. Find all possible corresponding equilibrium values for the current through the wire and the voltage across the outdoor converter (i.e., i_{l0} and v_{dc0} respectively).

At equilibrium, the time derivatives vanish. For the system we have

$$x_1(t) = i_l(t), \quad x_2(t) = v_{dc}(t), \quad u(t) = v_s(t)$$

with the nonlinear equations:

$$\dot{x}_1 = \frac{1}{L}\Big(v_s - R\,x_1 - x_2\Big) = 0 \tag{8}$$

$$\dot{x}_2 = \frac{1}{C}\Big(x_1 - \frac{P}{x_2}\Big) = 0 \tag{9}$$

At equilibrium the variables take the values:

$$x_1 = i_{l_0}, \quad x_2 = v_{dc_0}, \quad u = v_{s_0}$$

Thus, from (8) we have:

$$\frac{1}{L}\Big(v_{s_0} - R\,i_{l_0} - v_{dc_0}\Big) = 0 \quad \Longrightarrow \quad v_{s_0} - R\,i_{l_0} - v_{dc_0} = 0 \tag{10}$$

From (9):

$$\frac{1}{C}\Big(i_{l_0} - \frac{P}{v_{dc_0}}\Big) = 0 \quad \Longrightarrow \quad i_{l_0} = \frac{P}{v_{dc_0}} \tag{11}$$

Substitute (11) into (10):

$$v_{dc_0} = v_{s_0} - R\,\frac{P}{v_{dc_0}}$$

Rearranging leads to the quadratic equation:

$$v_{dc_0}^2 - v_{s_0}\,v_{dc_0} + R\,P = 0$$

Given $R = 60\,\Omega, P = 20\,\mathrm{W}, v_{s_0} = 80\,\mathrm{V}$, the quadratic becomes:

$$v_{dc_0}^2 - 80\,v_{dc_0} + 60 \times 20 = 0 \quad \Longrightarrow \quad v_{dc_0}^2 - 80\,v_{dc_0} + 1200 = 0$$

The roots are:

$$v_{dc_0} = \frac{120}{2} = 60\,\mathrm{V} \quad \text{or} \quad v_{dc_0} = \frac{40}{2} = 20\,\mathrm{V}$$

Now, using $i_{l_0} = \dfrac{P}{v_{dc_0}}$:

$$\text{For } v_{dc_0} = 60\,\mathrm{V}: \quad i_{l_0} = \frac{1}{3}\,\mathrm{A}$$

$$\text{For } v_{dc_0} = 20\,\mathrm{V}: \quad i_{l_0} = 1\,\mathrm{A}$$

5) BIBO Stability

Consider the system linearized around the equilibrium point with the lowest possible value you obtained for v_{dc0} in Part 4. Assume the same values for R and P as in Part 4 with the addition of $C = 100\mu$F and $L = 100\mu$H. Is this linearized system BIBO stable? Recall Part 3.

Recall from part 3, (6) and (7) must be satisfied for the system to be stable:

$$\frac{R}{L} - \frac{P}{C\, v_{dc_0}^2} > 0 \tag{12}$$

$$v_{dc_0}^2 > R\, P \tag{13}$$

Given $R = 60\Omega$, $P = 20W$, $C = 100\mu F$, $v_{dc_0} = 20V$, and $L = 100\mu H$, (12) and (13) become:

$$\frac{60}{100 \cdot 10^{-6}} - \frac{20}{100 \cdot 10^{-6} \cdot 20^2} > 0 \quad \Longrightarrow \quad 599500 > 0$$

$$20^2 > 60 \cdot 20 \quad \Longrightarrow \quad 400 \not> 1200$$

Therefore, the system is not BIBO stable.

6) Controller Design

Consider the same linearized system as in Part 5, with all the same values. Let $y = \delta x_2$. Recall the open-loop transfer function H. Build the linearized model in Simulink. Use a solver for stiff differential equations (e.g., `ode23s`) when using Simulink. Design a controller such that the closed-loop system's output decays to 0 in no more than $0.1s$ with no overshoot, and the initial conditions are $\delta x(0) = \begin{bmatrix} 8 \\ -8 \end{bmatrix}$.

Recall from part 2, linearized state space representation is:

$$\mathbf{A} = \begin{bmatrix} -\dfrac{R}{L} & -\dfrac{1}{L} \\ \dfrac{1}{C} & \dfrac{P}{C\, v_{dc_0}^2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = 0$$

Given $R = 60\Omega$, $P = 20W$, $C = 100\mu F$, $v_{dc_0} = 20V$, and $L = 100\mu H$, the matrices become

$$\mathbf{A} = \begin{bmatrix} -6 \cdot 10^5 & -10^4 \\ 10^4 & 500 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 10^4 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{D} = 0 \tag{14}$$

Recall from part 3, the open-loop transfer function is:

$$H_{ol}(s) = \frac{1}{C\, L \left[s^2 + s\left(\frac{R}{L} - \frac{P}{C\, v_{dc_0}^2}\right) + \frac{1}{LC}\left(1 - \frac{R\, P}{v_{dc_0}^2}\right)\right]}$$

Plugging the same values into the open loop function, we get:

$$H_{ol}(s) = \frac{10^8}{s^2 + 59000s - 2 \cdot 10^8}$$

Let's try a proportional controller with a gain of K, thus, the close loop transfer function is:

$$H_{cl}(s) = \frac{kH_{ol}(s)}{1 + kH_{ol}(s)} = \frac{10^8 k}{s^2 + 590000s + (2k \cdot 10^8 - 2 \cdot 10^8)} \tag{15}$$

Applying the Routh criteria, we know that:

$$2k \cdot 10^8 - 2 \cdot 10^8 > 0 \quad \Longrightarrow \quad k > 1$$

For starters, let's try k=2 with the block diagram implemented in Figure 2, with the linearized state space matrices derived in (14) and the given initial values for the states. As one can see from the output graph in Figure 3, while the system is BIBO stable with a settling time less than 0.1 seconds and no overshoot, it does not decay to 0.

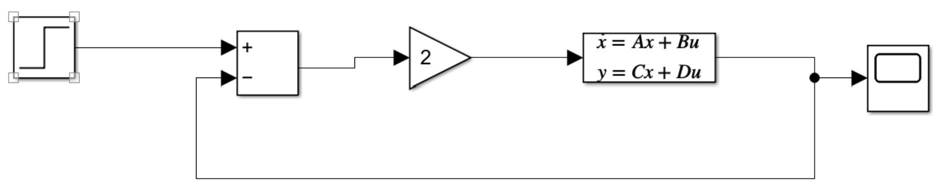

Figure 2: Linearized Block Diagram with k=2

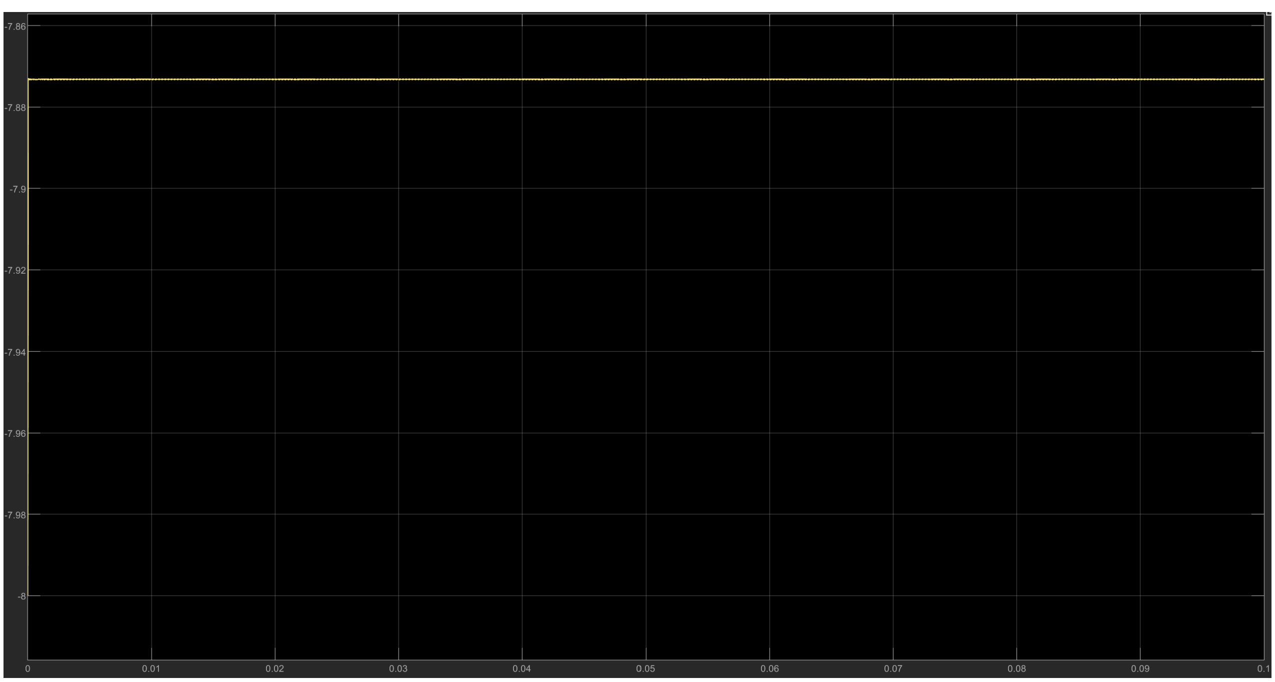

Figure 3: Output Graph for k=2

Therefore, let's change k from 2 to 3, resulting in the graph in Figure 4. In this case, the output of the system decays to zero in less than 0.1 seconds with no overshoot, meeting all the requirements.

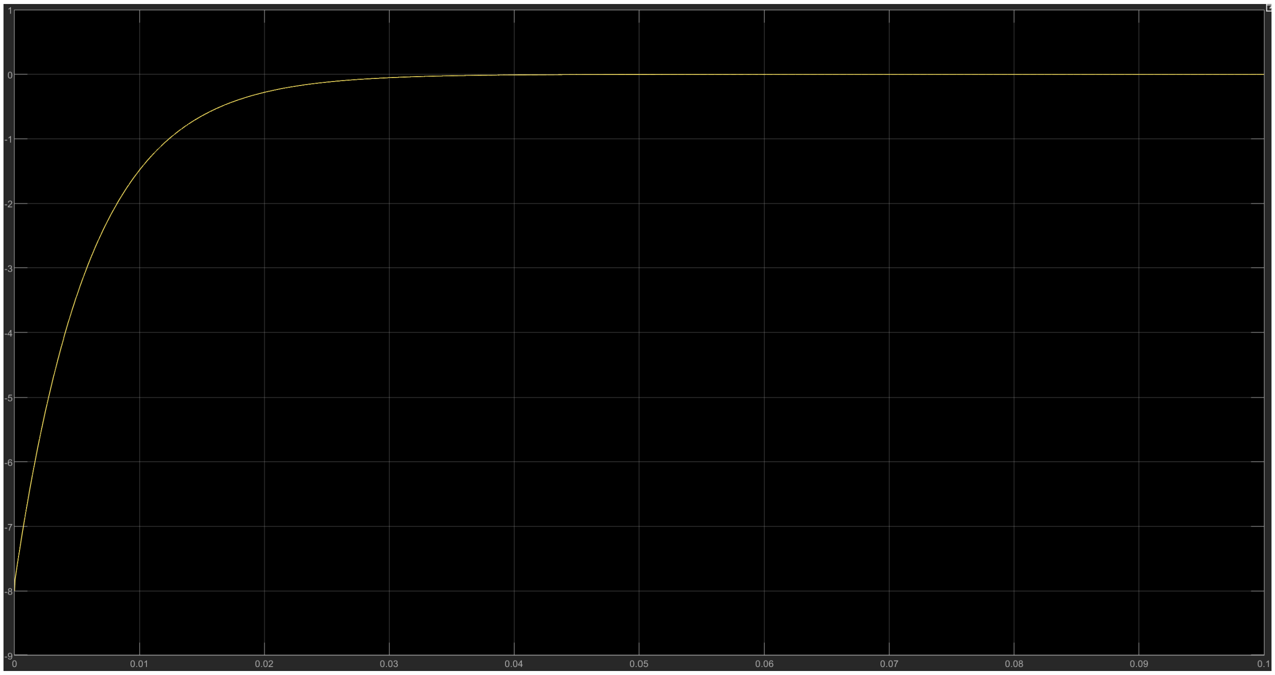

Figure 4: Output Graph for k=3

7) Simulink Implementation

Implement the original nonlinear model in Simulink, assuming the initial states of $\boldsymbol{x(0) = x_0 + \delta x(0)}$ and with the same parameters as Part 6. Keep in mind that the input vs and the states i_l and v_{dc} need to be non-negative at all times, with the additional safety requirement that they shall not exceed 120V, 10A and 40V re-spectively.
Hint: you can use a saturation block in Simulink and set its lower and upper limits to the corresponding safety bounds. Apply the controller found in Part 6 by placing it in the closed-loop with the nonlinear model in Simulink. Since we are translating the system back to the original system, adjust the control input appropri-ately, and the output's settling value. If needed, redesign your controller. Comment on the differences you observe as you implement the controller in the linearized vs the nonlinear model.

Let's first compute the initial values for the states. From the state space representation in Part 1, we know that the state variable:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} i_l \\ v_{dc} \end{bmatrix}$$

since $\boldsymbol{x(0) = x_0 + \delta x(0)}$, we can derive the initial conditions for i_l and v_{dc}:

$$\begin{bmatrix} i_{l0} \\ v_{dc0} \end{bmatrix} = \begin{bmatrix} 1\mathbf{A} \\ 20\mathbf{V} \end{bmatrix} + \begin{bmatrix} 8\mathbf{A} \\ -8\mathbf{V} \end{bmatrix} = \begin{bmatrix} 9\mathbf{A} \\ 12\mathbf{V} \end{bmatrix}$$

Furthermore, notice that the equilibrium value for v_s is 80V.

The block diagram in Figure 5 implements the non-linear model in Simulink with the same conditions as in question 6 and the same proportional controller we used in part 6 with a gain of 3. After running this simulation, we notice that a proportional controller with a gain of 3 cannot be used to stabilize the nonlinear model.

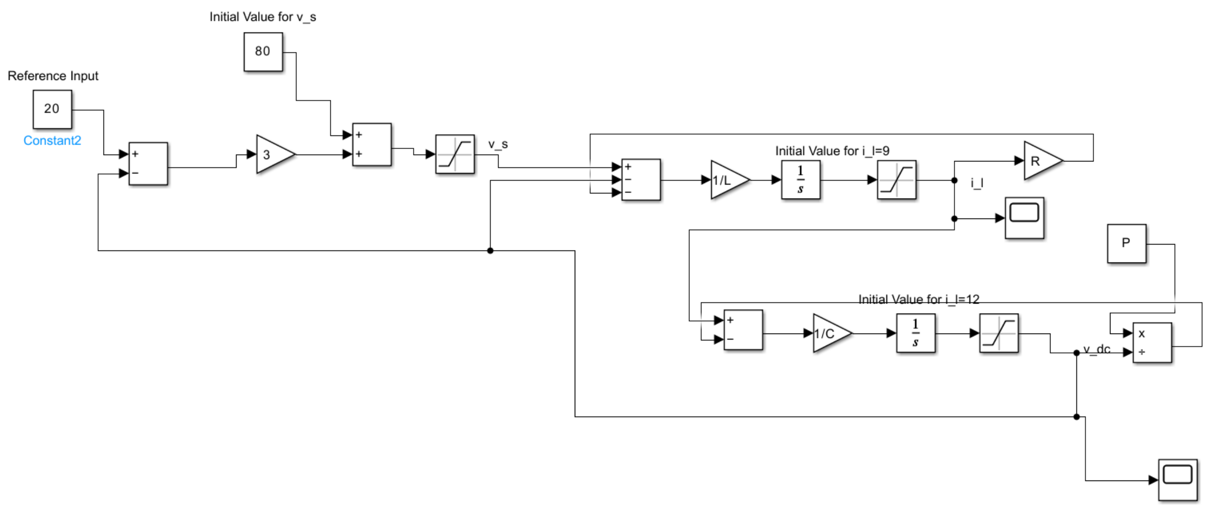

Figure 5: Non-linear Block Diagram with Proportional Controller k=3

Therefore, let's try a proportional controller with a larger gain, k=10, for example. As we can observe in Figure 7 and Figure 8, the output graphs of v_{dc} and i_l are successfully stabilized with no overshoot and a settling time of less than 0.1 seconds. Furthermore, it is shown in the graphs that v_{dc} and i_l converge to 20V and 1A, respectively, which makes sense as those are the equilibrium values we derived in Part 4. Also notice that while the graphs in Figure 7 and Figure 8 look like the response of a second order linear system under over-damp, we can clearly observe some irregularities on the graphs due to the non-linearity of the model.

By comparing the controller for the linearized model in Part 6 and the controller for the original non-linear model in Part 7, we can see that even though the linearized model of a non-linear system can effectively model its behavior close to the chosen equilibrium point, it does not capture all the information about the non-linear system. Therefore, we may need to modify the controller we derive for the linearized model when we are attempting to use it to achieve the same effects on the original non-linear system. In this case, we choose to increase the gain of the controller to better stabilize the original non-linear system.

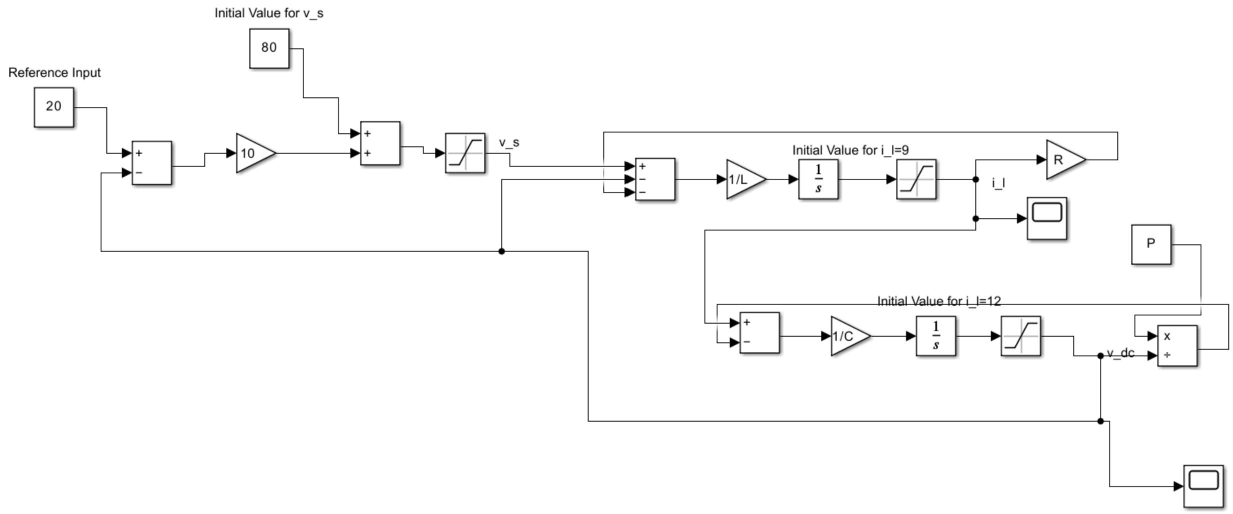

Figure 6: Non-linear Block Diagram with Proportional Controller k=10

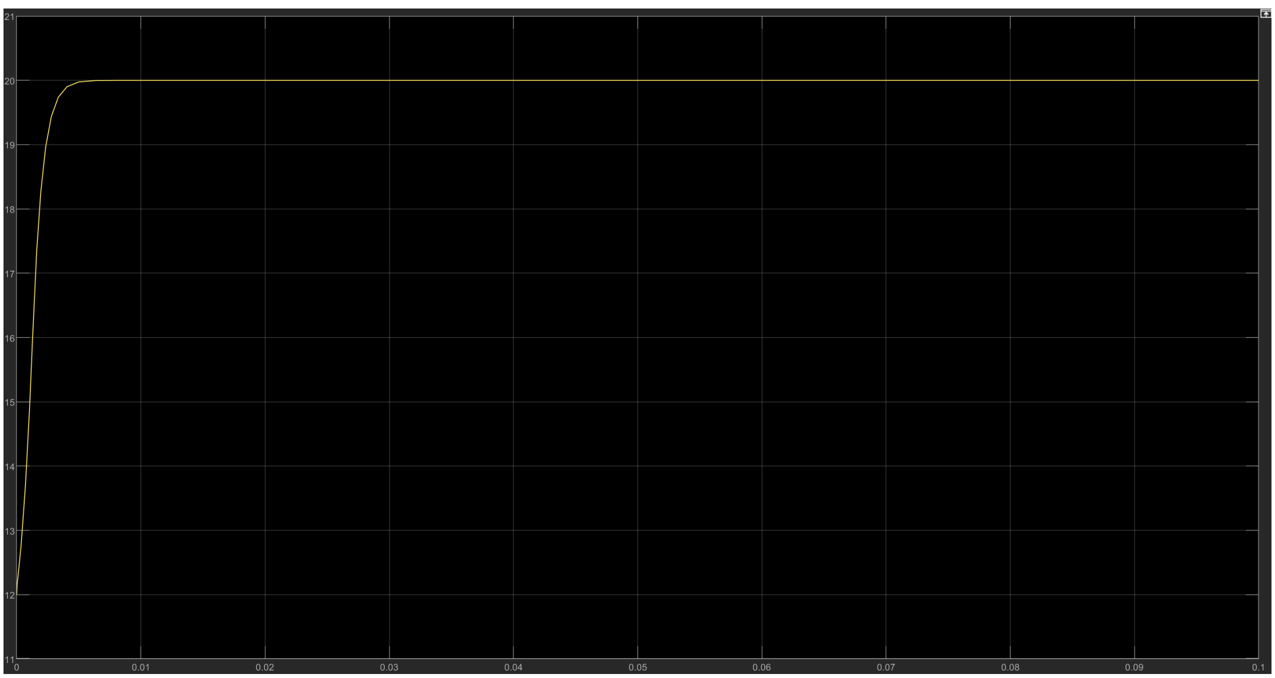

Figure 7: Output graph of v_{dc} with Proportional Controller k=10

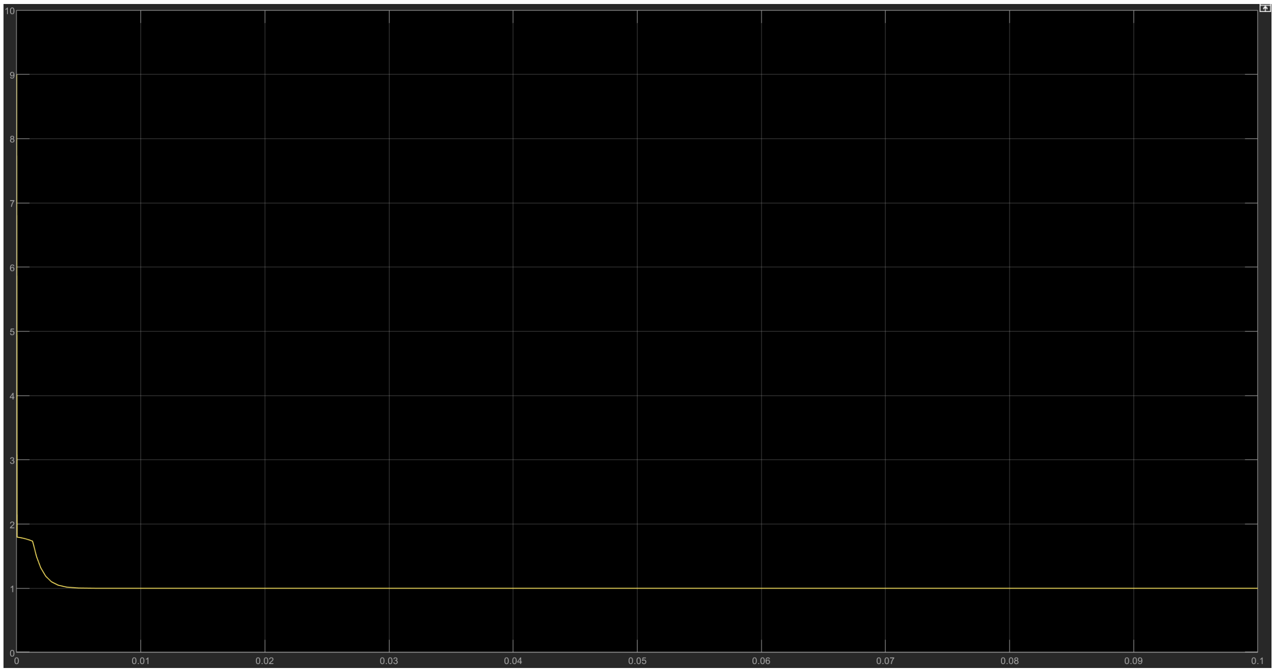

Figure 8: Output graph of i_l with Proportional Controller k=10

Acknowledgments

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