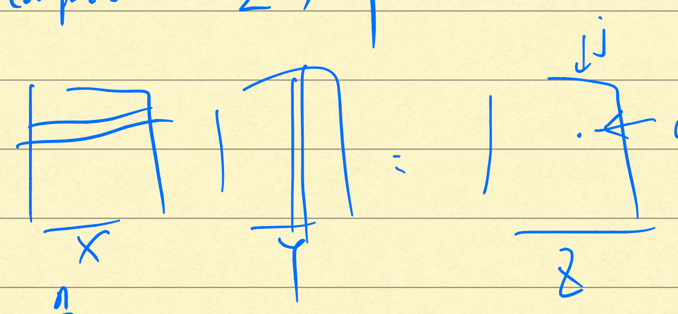
Divide and Conquer II

- Matrix Multiplication
- Sorting
- Median / Selection

MM: Input two $n \times n$ matrixes X, Y

Goal: compute $Z = X \cdot Y$



$\Theta(n^3)$

$$\begin{array}{c} \begin{array}{cc} \frac{n}{2} & \frac{n}{2} \end{array} \\ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{array}{c} \frac{n}{2} \\ \frac{n}{2} \end{array} \\ X \end{array} \begin{array}{c} \\ \begin{bmatrix} E & F \\ G & H \end{bmatrix} \\ Y \end{array} = \begin{array}{c} \\ \begin{bmatrix} A\cdot E + B\cdot G & A\cdot F + B\cdot H \\ C\cdot E + D\cdot G & C\cdot F + D\cdot H \end{bmatrix} \\ Z \end{array}$$

$$T(n) \le 8 \cdot T\left(\frac{n}{2}\right) + C \cdot n^2$$

$$a = 8, \ b = 2, \ d = 2, \Rightarrow \frac{8}{2^2} > 1 \Rightarrow T(n) = O(n^{\log_2 8}) = O(n^3)$$

Strassen '69

$P_1 = A(F - H)$

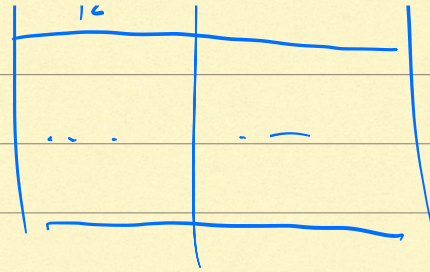
$P_2 = (A+B) \cdot H$

$$\Rightarrow Z = \begin{bmatrix} P_4 + P_5 + P_6 - P_2 & P_1 + P_2 \end{bmatrix}$$

$P_3 = (C+D)E$

$\vdots$

$P_7 = (A-C)\cdot(E+F)$



$T(n) \le 7\cdot T(\frac{n}{2}) + c\cdot n^2$

$\Rightarrow T(n) = O(n^{\log_2 7}) = O(n^{2.807\ldots})$

Coppersmith-Winograd '87 : $O(n^{2.3754770})$

Stothers '10 : $O(n^{2.37368\ldots})$

Vassilevska-Williams : $O(n^{2.372873\ldots})$

Le Gall '14 : $O(n^{2.3728639\ldots})$

Sorting (Mergesort)

$A = [\boxed{A_1, A_2, \ldots A_{\frac{n}{2}}}, \boxed{\ldots A_n}]$

$B = 1 \; 5 \; 7 \; 8 \; 15 \; 17 \ldots$

$C = 2 \; 3 \; 6 \; 9 \; 14 \ldots$

MergeSort(A[1...n])

  B ← MS(A[1...$\frac{n}{2}$])

  C ← MS(A[$\frac{n}{2}$+1,...n])

  return Merge(B, C)

$T(n) \le 2\cdot T(\frac{n}{2}) + Cn$

$a=2,\ b=2,\ d=1 \Rightarrow T(n) = O(n\log n)$

8 7 6 5 ... 1

Non-Recursive Version

Q = | $\boxed{8}$ $\boxed{7}$ ... $\boxed{1}$ $\boxed{7,8}$ |

Han '02: Sorting in $O(n\log\log n)$ (Deterministic)

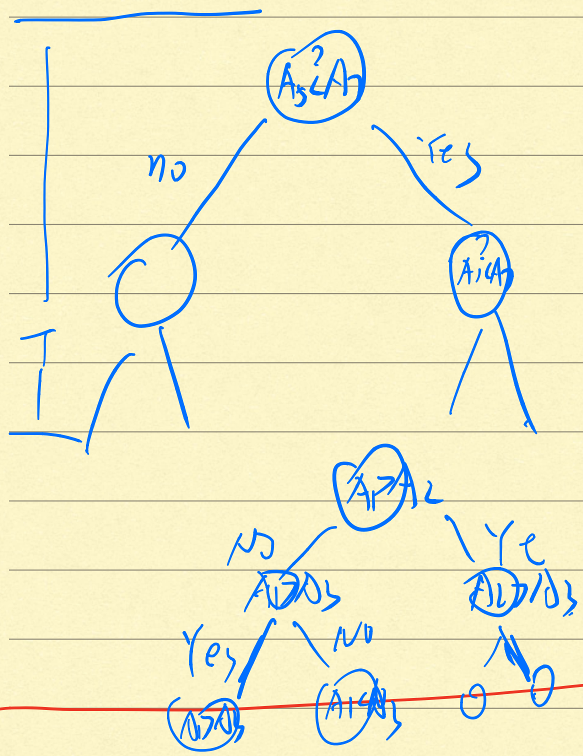
Han-Thor. '02: ... $O(n\sqrt{\log\log n})$ (Rand)

No compare Sort:

- Count Sort
- radix sort

Compare Model: $O(n\log n)$ is the best.

Input: some unknown permutation σ of $1,2,\dots n$



$\#$ leaves $\geq$ $\#$ perms

$2^T \geq n!$

$T \geq \log_2 n!$

$\geq n \ln n - O(n)$

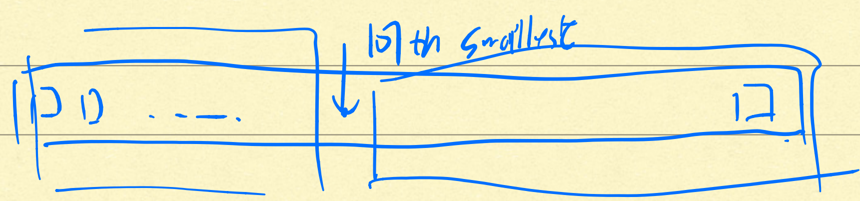
$= \Omega(n \log n)$

Median / Select

Given $A[1, \ldots n]$, $1 \leq k \leq n$

Goal: output kth smallest entry of A (median is $k = \frac{n}{2}$)

Quick select

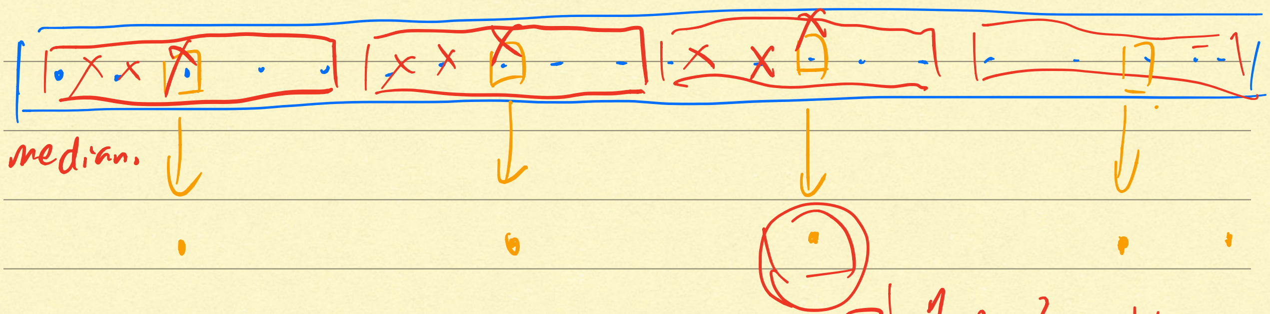


(Desire) $T(n) \leq T(\frac{n}{2}) + cn$

Without loss of generality: A ~~has no~~ duplicate entries

$B = [(A_1, 1), (A_2, 2) \ldots (A_n, n)]$

From now on: A has distinct entries.



$\geq \frac{1}{2}\cdot\frac{n}{5}\cdot 3 = \frac{3}{10}n$ elts

Select$(A[1,\dots n], k)$

- Break A into groups of size of 5 each.
- median: $B \leftarrow$ array of median elt in each group
- $p \leftarrow$ select$(B, \frac{n}{10})$
- $L \leftarrow \{<p\}$, $R \leftarrow \{>p\}$
- if $k = |L|+1$: return p.
- elif $k \leq |L|$: return Select(L, k)

  else: return Select$(R, k-|L|-1)$.

$$T(n) \leq O(n) + \frac{n}{5}\cdot O(1) + T(\frac{n}{5}) + O(n) + T(\frac{7}{10}n).$$

$$\leq T(\frac{n}{5}) + T(\frac{7}{10}n) + C\cdot n$$

Guess: $T(n) \leq B\cdot n$

Now try to prove via induction:

Base case: ✓ (as long as $B \geq 1$)

Inductive step: $B \cdot \frac{n}{5} + B \cdot \frac{7}{10}n + Cn \leq B \cdot n$.

$B(1 - \frac{1}{5} - \frac{7}{10}) \geq C \Rightarrow B \geq 10C$