Halts

correctness

efficiency

Algo for arithmetic

I. Addition

Algo1

Algo2.

II. Multiplication

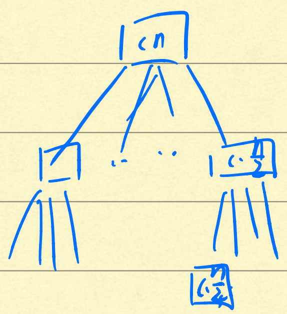
$$x = x_h \cdot 10^{\frac{n}{2}} + x_l, \quad y = y_h \cdot 10^{\frac{n}{2}} + y_l$$

Algo1

Algo2

Algo3. $x \cdot y$ $\quad \underbrace{x_h}_{\frac{n}{2}} \underbrace{x_l}_{\frac{n}{2}} \quad y_h \; y_l$

$$x \cdot y = \overset{A}{x_h \cdot y_h} \cdot 10^n + (\overset{D-A-B}{x_l \cdot y_h + x_h \cdot y_l}) \cdot 10^{\frac{n}{2}} + \overset{B}{x_l y_l}.$$




$cn$

$4 \cdot c \frac{n}{2}$

$4^2 \cdot c \cdot \frac{n}{4}$

$\vdots$

$k = \log_2 n$

$$T(n) = cn(1 + 2 + 2^2 + \cdots 2^k)$$

$$= cn \cdot \frac{1 \cdot (1 - 2^{k+1})}{1 - 2} = c \cdot n \cdot (2^{k+1} - 1)$$

$$= c \cdot n (2n - 1) = \Theta(n^2)$$

# Karatsuba Algorithm

$A = x_h \cdot y_h$

$B = x_l \, y_l$

$D = (x_h + x_l)(y_h + y_l)$

$$T(n) \le \begin{cases} 3 \cdot T(\frac{n}{2}) + c \cdot n & n > 1 \\ 1 & n = 1 \end{cases}$$

$$T = cn + 3 \cdot c \cdot \frac{n}{2} + 3^2 \cdot c \cdot \frac{n}{4} + \cdots \; 3^k \cdot c \cdot \frac{n}{2^k}$$

$$= cn\left(1 + \frac{3}{2} + \frac{3^2}{4} + \cdots \frac{3^k}{2^k}\right) \qquad k = \log_2 n$$

$$= cn \cdot \frac{1 \cdot \left(\left(\frac{3}{2}\right)^{k+1} - 1\right)}{\frac{3}{2} - 1} = cn \cdot \left[2 \cdot \left(\frac{3}{2}\right)^{k+1} - 2\right]$$

$$= cn \cdot \left[\cdot \frac{3^{k+1}}{2^k} - 2\right] = cn\left[\cdot \frac{3^{k+1}}{n} - 2\right]$$

$$= cn\left[\frac{3^{\log_2 n + 1}}{n} - 2\right]$$

$$\le 3c \cdot 3^{\log_2 n}$$

$$= 3 \cdot c \cdot \left[2^{\log_2 3}\right]^{\log_2 n}$$

$$= 3 \cdot c \cdot n^{\log_2 3}$$

$$= O(n^{1.585})$$

'66 Toom, Cook : $O(n^{\log_3 5})$, $O(n^{\log_k 2k-1})$

'71 Schonhage-Strassen : $O(n \log n \cdot \log\log n)$

$\downarrow$ $5431 = P(x)$

$p(x) = 1 \cdot x^0 + 3 \cdot x^1 + 4 \cdot x^2 + 5 \cdot x^3$

'07 Furer : $O(n \cdot \log n \cdot 8^{\log^* n})$

'19 Harvey, Vander Hoeven. $O(n \cdot \log n)$