Chapter 2

Continuum Mechanics of Soft Solids

From a macroscopic point of views, solids and liquids are usually seen as continuous entities that can deform over time. While in classical engineering applications, the deformation of solids remains small enough so that their geometry is not dramatically affected (deformation of aircraft wings, buildings and bridges), soft materials can exhibit very large distortions. It is therefore necessary to distinguish the deformed solid's morphology from its reference (or original) state. In the following, we provide a summary of the kinematics of a soft body under finite (or large) deformation and further introduce the concept of stress, a key indicator of the internal forces acting within its inner structure.

2.1 Preliminaries

To understand the mechanical behavior of a soft solid under the effect of external forces, let us first concentrate on one of its simplest forms: the force-extension behavior of a one-dimensional specimen under small deformation. For this, consider the cylindrical specimen shown on Fig. 2.1 before and after is is subjected to a normal force f. Under this condition, the specimen extends by an amount δ from its original length L. To understand the material's response from this test, we can characterize the internal forces acting in the specimens cross-section via the **normal stress** σ and the linera strain ϵ defined as:

$$\sigma = \frac{f}{A} \quad \text{and} \quad \epsilon = \frac{\delta}{L} \tag{2.1}$$

With these definitions, we indeed ensure that the material response is independant of specimen size and shape. Although the above expression are only valid for small deformations ($\delta << L$), this simple example may be used to introduce a variety of responses displayed by soft solids. We here briefly review some common behavior encountered in soft matter and soft materials.

Elasticity. Elastic solids are those that can be deformed back and forth without dissipating energy (i.e. the stress is only a function of strain). Examples of elastic behaviors are shown in Fig. 2.1 for a solids that exhibit a linear (red) and nonlinear (black) response. In the stress-strain space, an elastic solid is characterized by a unique path, regardless of the loading

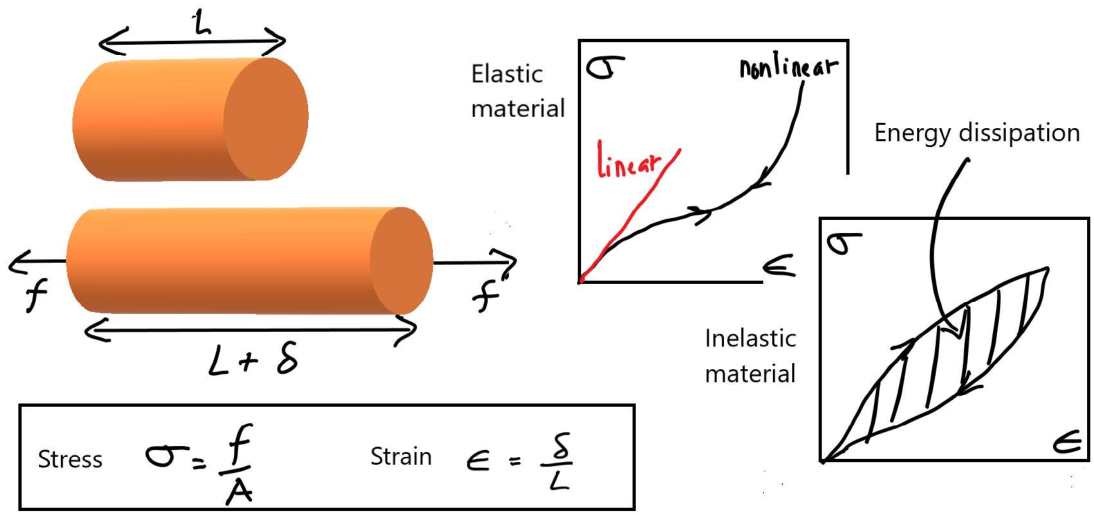


Figure 2.1: Stress, strain and typical mechanical behavior exhibited by soft materials

history (cycles, strain rate, ...). The simplest form of the constitutive relation for an elastic solid is given by **Hooke's law** in the form:

$$\sigma = E\epsilon \tag{2.2}$$

where E is the **Young's modulus**, measuring the material's stiffness.

Fluidity. Other materials such as polymer melts or glues exhibit a response that are more reminiscent of viscous fluids. In this case, instead of being a function of strain, the stress becomes a function of strain rate $\dot{\epsilon}$ where a superimposed dot denotes the time derivative. We may therefore write a relation of the form:

$$\sigma = \nu\dot{\epsilon} \tag{2.3}$$

Here, the symbol ν represents the viscosity of the material.

Visco-elasticity and inelasticity. Some solids may finally exhibit combinations of fluidic and elastic response. Such behavior, usually qualified as viscoelastic may take different formes. One of the simplest model is the Kelvin-Voigt model that combines the previous two models as follows:

$$\sigma = E\epsilon + \nu\dot{\epsilon} \tag{2.4}$$

Clearly this type of material will show different apparent stiffness according to the rate of deformation and will dissipate energy. There are other types of viscoelastic models (such as the Maxwell model and the standard models), but their constitutive relation take a form that is a bit mode complex than (2.4). There are finally other types of inelastic material behavior that include plasticity, in which permanent deformation is observed if the applied stress exceeds a critical, or yield stress. Damage and fatigue also constitute more complex behaviors, in which the materials deteriorate as a results of excessive loads and cycles.

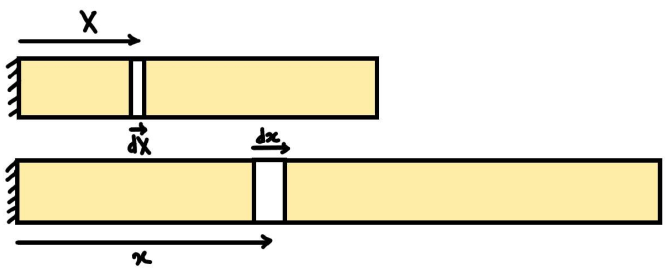
2.2 Kinematics of solids in large deformation

2.2.1 A one-dimensional situation: stretch of a rubber band

To introduce the notion of large deformation, let us start be a simple, one-dimensional problem in the form of the stretch of a rubber band. For this, let us consider the strip shown in Fig. **??** in its reference (top) and deformed (bottom) configurations. We localize a particle P (line element) on this strip by its location X and infinitesimal length dx in the reference state. After deformation, the coordinate of this particle P becomes x, with infinitesimal length dx. The overall deformation of the strip is then entirely characterized if we know the mapping

$$x = x(X) \tag{2.5}$$

between the reference and deformed configurations.



Let us now describe the distortion of a material element. In this one-dimensional situation, distortion can be measured with the stretch ratio $\lambda = \ell/\ell 0$ where ℓ and ℓ_0 are the deformed and reference length of a line element. Applying this to the strip element, we find that:

$$\lambda(X) = \lim_{dX \to 0} \frac{dx}{dX} = \frac{\partial x}{\partial X} \tag{2.6}$$

where we used the common definition of the partial derivative. In other words, we find that the stertch of an element along the bar is found by differentiating the mapping function with respect to the reference coordinates X. This expression may also be written such that the relationship between the length of the element in its deformed and reference state is given by:

$$dx = \lambda(X)dX \tag{2.7}$$

Problem: Consider a strip of reference length L, for which the mapping is given by $x(X) = aX + bX^2$. **(a)** Sketch the coordinate and deformation of a material element located at $X = 0$, $X = L/2$ and $X = L$ along the strip. **(b)** Determine the stretch ratio $\lambda(X)$ of an element along the bar. **(c)**, Explain what happens if $a = 1$ and $b = 0$.

2.2.2 Extension to three-dimensional bodies

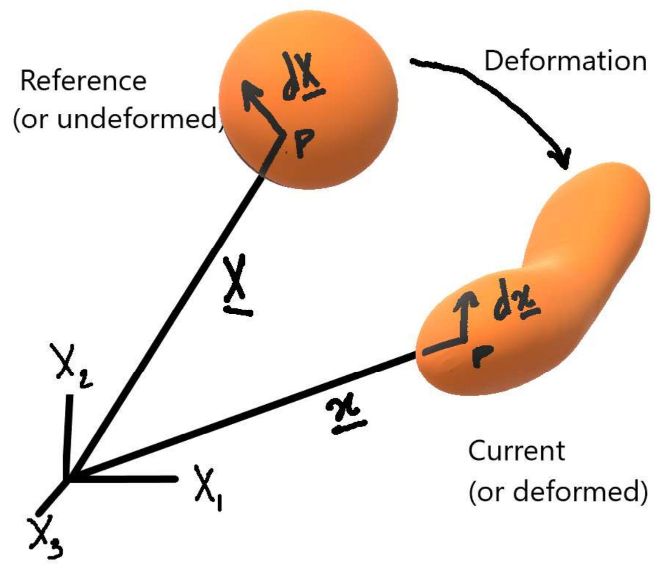
To extend the above concepts to a three-dimensional solid, we first need an orthonormal cartesian coordinate system made of three basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. The solid domain, in its undeformed state may then be defined as the collection of points comprised in a volume Ω_0, surrounded by its boundary Γ_0. The location of a material point in this undeformed (or reference) state is typically denoted as $\boldsymbol{X}$ and it expressed as:

$$\boldsymbol{X} = X_i \boldsymbol{e}_i \tag{2.8}$$

where an implicit summation is assumed on the repeated index i. After deformation, the domain changes shape and can be redefined by its volume Ω and boundary Γ. The new location of the material point P originally located in $\boldsymbol{X}$ is now represented by the smaller case symbol $\boldsymbol{x}$. The mapping between reference and deformed domain is thus represented by the vector function:

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) \tag{2.9}$$

Let us consider a material domain deforming from its reference (or stress-free) state Ω_0 to



a deformed state Ω and aim to represent the deformation undergone by a material volume centered around point P. For this, let us imagine that we can draw a small segment (or vector) $d\boldsymbol{X}$ that originates at point P in the reference configuration. We then look to understand how this segment transforms under deformation. It is clear that this segment will undergo a combination of rotation and stretch so that it becomes a new vector $d\boldsymbol{x}$. The mapping between the reference and the deformed segment is linear and given by the **deformation gradient** tensor $\boldsymbol{F}$ as follows:

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X} \qquad or \qquad \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{pmatrix} dX_1 \\ dX_2 \\ dX_3 \end{pmatrix} \tag{2.10}$$

We see here that in 3D, the deformation gradient is represented by a $3x3$ matrix, with 9 components. This tensor is representative of the material deformation at a point and is therefore a very important quantities in solid mechanics. From (2.28), it is clear that $\boldsymbol{F}$ is computed by taking the gradient of the mapping function shown in (2.11) with respect to the reference coordinates:

$$\boldsymbol{F}(\boldsymbol{X},t) = \frac{\partial \boldsymbol{x}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \qquad or \qquad F_{ij} = \frac{\partial x_i}{\partial X_j} \tag{2.11}$$

Notes

- The deformation gradient contains information on both rotation and stretch of the lines. For instance, the solid undergoes a rigid body rotation (no stretch), it can be shown that $\boldsymbol{F}$ becomes the orthogonal rotation matrix $\boldsymbol{R}$ (i.e. $\boldsymbol{R}^{-1} = \boldsymbol{R}^{T}$ where the superscripts -1 and T denote the inverse and the transpose operations, respectively).

- The deformation gradient only represent the linear deformation of an infinitesimal line around a material point. Quadratic and higher order terms are neglected. This means that a circle will always deform locally into an ellipse.

- When a solid does not undergo any deformation, $d\boldsymbol{x} = d\boldsymbol{X}$ and the deformation gradient becomes the identity tensor $\boldsymbol{F} = \boldsymbol{I}$ (i.e. a matrix with ones on the diagonal and zeros everywhere else).

Computing the deformation gradient for a deforming unit square

To better understand the concept behind the deformation gradient, we consider here the linear deformation of a two-dimensional rectangle and determine the corresponding components of **F**. To simplify the approach, let us consider that the unit square has its principal axes aligned with the basis vectors $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ with reference dimension $\Delta X = 1$ and $\Delta Y = 1$ in these directions, respectively. As the rectangle is subjected to a linear deformation, we follow the deformation of the two line elements (see Fig.):

$$\Delta \boldsymbol{X} = [1, 0] \quad \text{and} \quad \Delta \boldsymbol{Y} = [0, 1] \tag{2.12}$$

After deformation, these vectors become:

$$\Delta \boldsymbol{x} = [\Delta x|_X, \Delta y|_X] \quad \text{and} \quad \Delta \boldsymbol{y} = [\Delta x|_Y, \Delta y|_Y] \tag{2.13}$$

Following (2.28), the deformation gradient for this deformation can be determined as:

$$\boldsymbol{F} = \begin{bmatrix} \Delta x|_X & \Delta x|_Y \\ \Delta y|_X & \Delta y|_Y \end{bmatrix} \tag{2.14}$$

Problem 1: Compute the deformation gradient for the 2D deformation represented in Fig. **??**

Solution: Using the methodology described above, we find that the deformation gradient is represented by the following matrix in the (x-y) coordinate system:

$$\boldsymbol{F} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \tag{2.15}$$

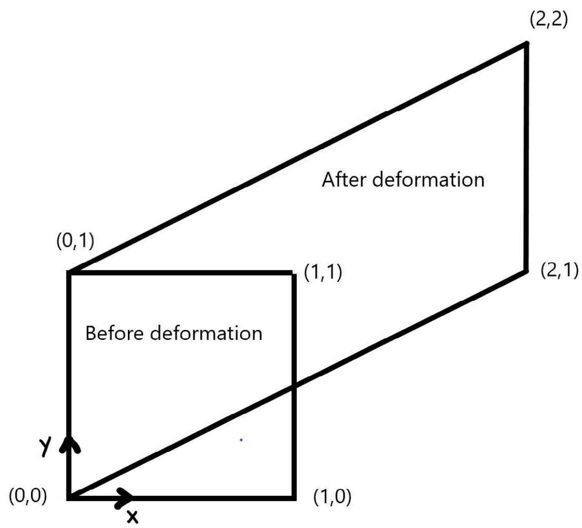


Figure 2.2: Deformation of a unit square into a parallelogram

Problem 2: Show that the rotation of a unit square by angle θ leads to the following expression for $\boldsymbol{F}$:

$$\boldsymbol{F} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \tag{2.16}$$

Change of volume

It is often useful to evaluate the change of volume experienced by a material volume experiencing a deformation gradient $\boldsymbol{F}$ (or alternatively a deformation $\boldsymbol{C}$). If we consider a unit cube and place ourselves in the principal directions, the deformation gradient will take the form:

$$\boldsymbol{F} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{2.17}$$

This means that the unit cube will be deformed into a parallelepiped of dimensions λ_1, λ_2, and λ_3. The volume ratio J of this domain after deformation is therefore:

$$J = \frac{V}{V_0} = \frac{\lambda_1\lambda_2\lambda_3}{1} \tag{2.18}$$

Noting that $\lambda_1\lambda_2\lambda_3 = det(\boldsymbol{F})$ and Since the determinant of a tensor is an invariant (i.e. it is independent of the basis in which it is expressed), the above result can be generalized as:

$$J = \frac{V}{V_0} = det(\boldsymbol{F}) \tag{2.19}$$

2.3 The Cauchy-Green Deformation measure

While the deformation gradient is a very good candidate to represent the deformation of soft solids, it suffers from an important limitation. Indeed, the concept of deformation (or distortion) implies that a solid changes its shape, and as a consequence, mechanical energy should be provided from external forces. Rigid body rotation therefore does not satisfy this requirement, despite the fact that it is an integral component of $\boldsymbol{F}$. A new measure of deformation must therefore be defined such that it is independent of rigid body rotation.

A solution to this issue can be found by extraccting the rotation component from $\boldsymbol{F}$. Indeed, it can be shown that the deformation gradient can be decomposed into a rotation (represented by the orthogonal tensor $\boldsymbol{R}$ and a symetric stretch tensor (represented by the tensor $\boldsymbol{U}$ such that:

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} \tag{2.20}$$

In other words, the stretch tensor $\boldsymbol{U}$ is an excellent candidate to measure deformation. Its computation however requires the independent determinations of $\boldsymbol{F}$ and $\boldsymbol{R}$ so that $\boldsymbol{U} = \boldsymbol{R}^T\boldsymbol{F}$. This calculation is generally tedious, and we will see that a more convenient deformation measure may instead be provided by the **right Cauchy-Green deformation tensor $\boldsymbol{C}$** defined as follows:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \tag{2.21}$$

It can easily be shown that this tensor can be related to $\boldsymbol{U}$ as follows:

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} = (\boldsymbol{R}\boldsymbol{U})^T(\boldsymbol{R}\boldsymbol{U}) = \boldsymbol{U}^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{U} = \boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{U}^2 \tag{2.22}$$

where we used $\boldsymbol{R}^T = \boldsymbol{R}^{-1}$. This confirms that $\boldsymbol{C}$ is an equally good candidate to compute deformation, and is fairly easy to compute. To check this statement, one sees that of a small volume undergoes a rigid body rotation ($i.e.\boldsymbol{F} = \boldsymbol{R}$), the Cauchy-Green tensor becomes $\boldsymbol{C} = \boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$. In other words, it is not affected by rotation, as expected.

We note here that the definition of $\boldsymbol{C}$ implies that $\boldsymbol{C} = \boldsymbol{I}$ in the absence of deformation. This is not consistent with the traditional definition of small strain, for which $\epsilon = 0$ when no deformation occurs. To adress this issue, an alternative measure of finite strain, known as the *Green-Lagrange strain tensor* can be invoked in the form:

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) \tag{2.23}$$

where $\boldsymbol{I}$ is the identity tensor. With this measure, it can be shown (see exercise 1) that the change in the square length $dx^2 - dX^2$ of an infinitesimal vector $d\boldsymbol{X}$ is given by:

$$dx^2 - dX^2 = 2d\boldsymbol{X} \cdot \boldsymbol{E} \cdot d\boldsymbol{X} \tag{2.24}$$

Principal stretches

As discussed previously, the Cauchy-Green deformation tensor is represented by a symmetric matrix. In other words, it is represented by 6 independent components that are representative of combined stretch and shear deformations in any arbitrary coordinate system:

$$\boldsymbol{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ \star & C_{22} & C_{23} \\ \star & \star & C_{33} \end{bmatrix} \tag{2.25}$$

The stars $\star$ are used here to indicate symmetrical components. Note here that these components depend on the chosen coordinate system, which tends to overshadow a clear interpretation. Using ..., we know that for a positive definite, symmetrical matrix, one can find a special coordinate system in which the tensor $\boldsymbol{C}$ becomes diagonal. This system is made of the principal basis and we will see that the diagonal components can be easily interpreted. In these principal basis, one can therefore write:

$$\boldsymbol{C}^* = \begin{bmatrix} (\lambda_1)^2 & 0 & 0 \\ 0 & (\lambda_2)^2 & 0 \\ 0 & 0 & (\lambda_3)^2 \end{bmatrix} \tag{2.26}$$

where the quantities $\lambda_1, \lambda_2, \lambda_3$ are known as the principal stretches.

To better understand the physical meaning of these stretches, Let us consider the deformation of a unit cube in its principal directions, if the sides of the cube are aligned with the direction of the bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ of an orthornmal coordinate system, the stretches will also be applied along the basis vectors. Noting that in the principal axes, there is no shear deformations (there are no off-diagonal terms in (2.26)), the cube becomes a brick with dimensions λ_1, λ_2, λ_3 as shown in Fig. XX. These quantities (or stretch ratio) are defined as the ratio of the deformed length and the reference length of one of the bases in the principal directions.

Notes

- The strech ratio are always positive. A stretch ratio < 0 would imply that, under deformation, a solid would see its volume become negative.
- When there is no deformation, the stretch ratios are equal to one.
- A stretch ratio is less than one ($\lambda < 1$) for compression and more than one ($\lambda < 1$) for tension.

To summarize, any state of deformation can be represented, in the appropriate coordinate system, by three principal stretches (or stretch ratio). In this system, the Cauchy-Green tensor is represented by a diagonal tensor shown in (2.26). The square of these stretches are the eigenvalues of the Cauchy-Green tensor and may therefore be determine by solving the characteristic equation:

$$det(\boldsymbol{C} - \mu \boldsymbol{I}) = 0 \tag{2.27}$$

which, for a 3D problem becomes a third-order polynomial for μ. This typically leads to 3 real and positive solutions for μ_i. The stretch ratios are then found as $\lambda_i = \sqrt{\mu_i}$ $(i = 1, 2, 3)$.

Problem 3: Compute the principal stretches for the 2D deformation represented by the deformation gradient:

$$\boldsymbol{F} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \tag{2.28}$$

Solution: For this deformation the Cauchy-Green deformation tensor can be computed as:

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \tag{2.29}$$

We can check here that $\boldsymbol{C}$ is symmetric as expected. To find the principal stretches, we solve the equation:

$$0 = det(\boldsymbol{C} - \mu \boldsymbol{I}) = det\left(\begin{bmatrix} 5-\mu & 1 \\ 1 & 1-\mu \end{bmatrix} \right) = (5-\mu)(1-\mu) - 2 \tag{2.30}$$

This equation has two roots $\mu_1 = 3 + \sqrt{5}$ and $\mu_2 = 3 - \sqrt{5}$. The principal stretches $\lambda_1 = \sqrt{\mu_1}$ and $\lambda_2 = \sqrt{\mu_2}$ can therefore be approximated as:

$$\lambda_1 \approx 2.29 \quad \text{and} \quad \lambda_2 \approx 0.87 \tag{2.31}$$

We can now find the principal axes by finding the eigenvectors corresponding to the stretches λ_1 and λ_2. For this, we first note that the two eigen vectors are orthogonal for a symmetric matrix; i.e. we may only determine the first $\boldsymbol{u} = [u_x, u_y]$; the second may then be found as $\boldsymbol{u} = [-u_y, u_x]$, which ensures $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

The eigenvector corresponding to λ_1 (or μ_1 is determined by solving the system:

$$\begin{bmatrix} 5-\mu_1 & 1 \\ 1 & 1-\mu_1 \end{bmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.32}$$

By definition, the above matrix is singular, which means that the first and second row are linear combinations of one-another. We therefore here seek a class of solution that verify only one of the row equations (say the first). In other words, we seek a solution to the equation $(5-\mu_1)u_x + u_y = 0$. Substituting the value $\mu_1 = 3 - \sqrt{5}$, this equation becomes:

$$(2 - \sqrt{5})u_x + u_y = 0 \tag{2.33}$$

As expected, there are many solutions to this equation and to simplify our calculation, let us choose the particular value $u_y = -2 + \sqrt{5})$; which leads to a value of u_x equal to 1. The family of eigenvectors is thus given by the vector:

$$\boldsymbol{u} = [1, -2 + \sqrt{5}] \tag{2.34}$$

or any vector generated by multiplying $\boldsymbol{u}$ by an arbitrary non-zero constant. Using the orthogonality condition, we then find the second principal direction as:

$$\boldsymbol{v} = [2 - \sqrt{5}, 1] \tag{2.35}$$

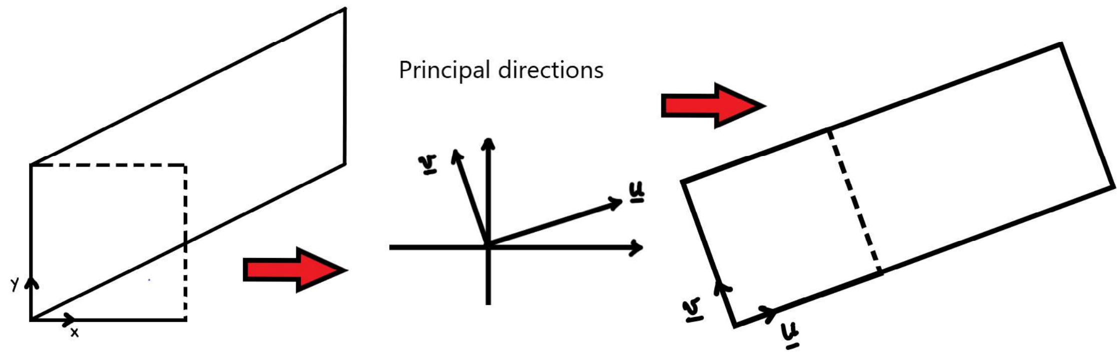


Figure 2.3: Deformation of a material element in arbitrary and principal directions.

2.4 Characterizing the rate of deformation

We have seen earlier that a number of materials exhibit a rate-dependent mechanical response (in the case of viscous fluids and viscoelastic materials). In this case, the relevant measure of deformation is the rate of strain. To evaluate the rate of deformation at a material point, we concentrate on a material point P whose current coordinate is $\boldsymbol{x}(\boldsymbol{X}, t)$ and follow the evolution of an infinitesimal vector $d\boldsymbol{x}$ passing through this point. The time derivative $d\boldsymbol{x}/dt$ is mapped to the vector $d\boldsymbol{x}$ through the tensor $\boldsymbol{L}$ by:

$$\frac{d(d\boldsymbol{x})}{dt} = \boldsymbol{L} d\boldsymbol{x} \tag{2.36}$$

Realizing that $d\boldsymbol{x}/dt$ is the velocity $\boldsymbol{v}$ of the particle P and using the definition of the partial derivative, $\boldsymbol{L}$ can be interpreted as velocity gradient:

$$\boldsymbol{L} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \tag{2.37}$$

where we note that the derivative is taken with respect to the current coordinates. Similarly to the deformation gradient, it is a good candidate to represent strain rate, but unfortunately suffers from the fact it it includes the rate of rotation, which should not be accounted for in our measure of distortion. To remedy this, the velocity gradient can be made symmetric (and thus independent of rotation) through the following operation:

$$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L} + \boldsymbol{L}^T\right) \tag{2.38}$$

	Total	Rate
Distortion + Rotation	$\boldsymbol{F} = \frac{d\boldsymbol{x}}{d\boldsymbol{X}}$ (Deformation gradient)	$\boldsymbol{L} = \frac{d\boldsymbol{v}}{d\boldsymbol{X}}$ (Velocity gradient)
Distortion only	$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ (Cauchy-Green deformation)	$\boldsymbol{D} = \frac{1}{2}\left(\boldsymbol{L}^T + \boldsymbol{L}\right)$ (Rate-of-deformation)

Table 2.1: Summary of finite deformation measures in total and rate forms.

where the superscript T is used for the transpose operation. This new tensor $\boldsymbol{D}$ is known as *the rate-of-deformation tensor* is can be used to characterize strain rate. For instance, it can be shown (see exercise 1) that the change in the square length dx^2 of an infinitesimal vector $d\boldsymbol{x}$ is given by:

$$\frac{\partial}{\partial t}(dx^2) = 2d\boldsymbol{x} \cdot \boldsymbol{D} \cdot d\boldsymbol{x} \tag{2.39}$$

This equation is comparable to equation (2.24) in the context of rate of changes (instead of total change). A summary of the deformation measure in their total and rate forms is provided in table **??**.

Problem 4: Using the definitions of the Green-Lagrange strain tensor and the rate-of-deformation tensors, prove equations (2.24) and (2.39).

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2.5 Stress measures

When a soft solid is subjected to external forces,

Let us consider a three-dimensional body and consider a surface element ΔS centered around a material point P. Imagine that this surface element constitutes a cut on which an internal force ΔF is applied to maintain equilibrium. On this cut, the stress vector $\boldsymbol{t}_n$ isexpressed in the orthonormal bases $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$:

$$\boldsymbol{t}_n = \lim_{\Delta S \to 0} \frac{\Delta \boldsymbol{F}_n}{\Delta S} = [t_{n1}\ t_{n2}\ t_{n3}] \tag{2.40}$$

where ΔS is the current material area. Note here that this vector depends on the direction $\boldsymbol{n}$ of the surface. To overcome this dependence, it is more convenient to work with the stress tensor, which expresses the component of the stress vectors along the basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$.

$$\boldsymbol{\sigma} = \begin{bmatrix} \boldsymbol{t}_1 \\ \boldsymbol{t}_2 \\ \boldsymbol{t}_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{2.41}$$

From this expression ,the component σ_{ij} is interpreted as the force component (per unit surface) in the j-direction on a surface whose unit normal is $\boldsymbol{e}_i$. A graphical representation of these components on a unit cube is given by Fig. XX.

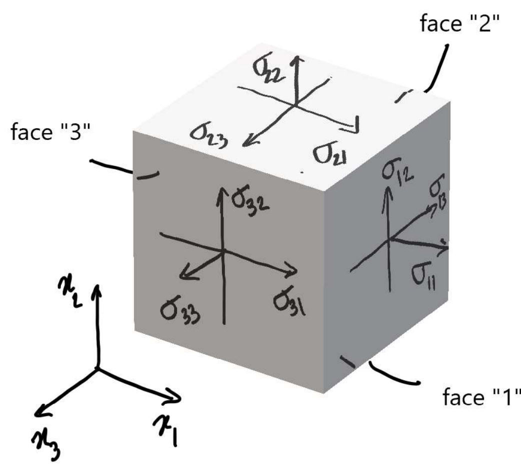

Figure 2.4: Illustration of the stress components on a unit volume

Equilibrium and Symmetry of the Cauchy stress tensor. A volume element in a solid at equilibrium must verify at least two balance laws: first the balance of force and second, the balance of moments. It is straighforard to show that the first leads to a differential equation for the stress of the form:

$$\frac{\partial \sigma_{ij}}{\partial x_i} + f_j = 0 \quad \text{or} \quad \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = 0 \tag{2.42}$$

where $\boldsymbol{f} = f_i \boldsymbol{e}_i$ is an external body force vector applied on the volume (such as gravity for instance). The balance of moment implies that:

$$\sigma_{ij} = \sigma_{ji} \quad \text{or} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{2.43}$$

This implies the symmetry of the Cauchy stress tensor.

The nominal stress. Soft materials such as polymers and gels typically undergo large deformations, making it difficult to follow the surface area on which a stress is computed. For this reason, other stress measures, an notably the nominal stress have been introduced. This stress, usually represented by the symbol $\boldsymbol{P}$ is defined as the ratio of current forces and the **undeformed** surface area. In other words, the corresponding stress vector $\boldsymbol{T}_n$ and stress tensor $\boldsymbol{P}$ are defined in a similar fashion as the true stress as:

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \\ \boldsymbol{T}_3 \end{bmatrix} \quad \text{where} \quad \boldsymbol{T}_n = \lim_{\Delta S_0 \to 0} \frac{\Delta \boldsymbol{F}_n}{\Delta S_0} \tag{2.44}$$

This stress is usually more convenient to calculate from an experimental view point, but we should be careful that it is not a physical stress and may not be directly used to assess the internal forces experienced by a solid. Furthermore, this stress is **not symmetric** and is therefore represented by all 9 components of the tensor. It is however possible to recover the Cauchy stress from $\boldsymbol{P}$ from the transformation formula:

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{P}\boldsymbol{F}^T \quad \text{where} \quad J = det(\boldsymbol{F}) \tag{2.45}$$

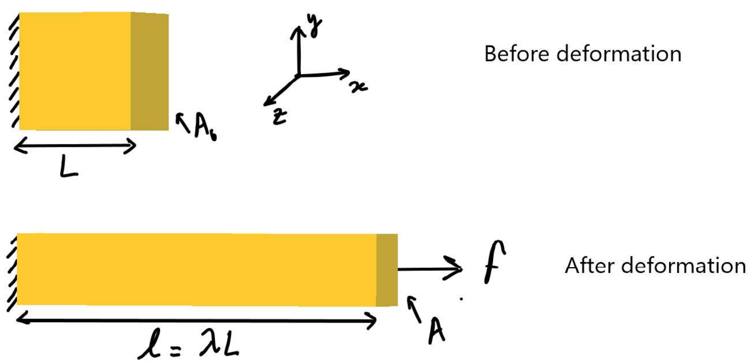

Figure 2.5: Unaxial deformation of a soft elastomer under an applied force f.

Example: Consider the tensile deformation of a soft rubber sample as shown in Fig. 2.5. In it undeformed state, the sample has a length L and square cross-section of area A_0. After being subjected to a force f, normally applied to the surface A_0, the specimen stretches to a new length $\ell = \lambda L$ and cross-sectional area A. Knowing that the rubber is isotropic and incompressible (its volume remains constant),

Determine the deformation gradient in the specimen. It is clear here that the specimen does not undergo any shear deformation (characterized by changes in angles). This means that there are no off-diagonal terms in $\boldsymbol{F}$, which allows us to write:

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda_T & 0 \\ 0 & 0 & \lambda_T \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix} \tag{2.46}$$

where λ_T is the transverse stretch ratio, taking the same value in the 2 and 3 direction due to the rubber's isotropy. To determine these values, we use the incompressibility assumption that reads (from (2.19)) $det(\boldsymbol{F}) = \lambda\lambda_T^2 = 1$. This thus yields $\lambda_T = 1/\sqrt{\lambda}$ and the result in the right end side of (2.46).

Find the nominal stress tensor in the specimen. To find the nominal stress, we need to look at forces exerted on all faces of the specimen. Clearly, the only non-zero force is acting on the face whose normal is in the 1-direction and is applied in the 1-direction. This component is therefore P_{11} and the nominal stress tensor is represented by:

$$\boldsymbol{P} = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f/A_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.47}$$

The value of this component is the magnitude of the force f divided by the undeformed area of the surface it is applied on. This yields $P_{11} = f/A_0$.

Find the Cauchy stress tensor in the specimen. Using a similar approach as for the nominal stress, we can find that the Cauchy stress as form that is identical to (2.48), but by replacing the component f/A_0 by f/A since the Cauchy stress is defined per current area. This result can also by found using (2.45). To show this, we use the fact that $J = 1$ (incompressibility) and the expressions for $\boldsymbol{F}$ and $\boldsymbol{P}$ found previously to find:

$$\boldsymbol{\sigma} = \begin{bmatrix} f/A_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{\lambda} & 0 \\ 0 & 0 & 1/\sqrt{\lambda} \end{bmatrix} = \begin{bmatrix} f/A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.48}$$

where we used the fact that $J = \lambda A/A_0 = 1$.

2.6 Relating stress and deformation: the constitutive relation

The concepts of stress and deformation described above are generic and can be applied to any material (especially those undergoing large deformations). We now discuss the concept of constitutive relation, which defines the mechanical behavior of a specific material. As discussed earlier in this chapter, these relation generally takes the form of a relationship between stress and strain (such as Hooke's law). Deriving an explicit relation between stress $\boldsymbol{\sigma}$ and deformation $\boldsymbol{C}$ is however challenging for two reasons. First, these quantities are tensors and generalization of Hooke's law is possible but only applicable to small deformation. Second, for large deformation, the combined tensorial form and nonlinearity of the response precludes writing explicit relationships.

For elastic solids, a more appropriate approach consist in evaluating the stored elastic energy at a point in terms of the level of deformation $\boldsymbol{C}$. This energy usually expresses the elastic energy per unit volume of material in its undeformed state and is represented by the scalar ψ:

$$\psi = \psi(\boldsymbol{C}) = \psi(\lambda_1, \lambda_2, \lambda_3) \tag{2.49}$$

From the knowledge of this function, the nominal stress $\boldsymbol{P}$ can be found as:

$$\boldsymbol{P} = 2\boldsymbol{F}\frac{\partial \psi}{\partial \boldsymbol{C}} \tag{2.50}$$

The derivation of the scalar function $\psi(\boldsymbol{C})$ is therefore a key ingredient in understanding and predicting the behavior of elastic soft matter.

Example. Consider the uniaxial deformation of a bar whose length changes from ℓ_0 to ℓ. The elastic potential is given by:

$$\psi(\lambda_1, \lambda_2, \lambda_3) = \frac{K}{2}\left[(\lambda_1 - 1)^2 + (\lambda_2 - 1)^2 + (\lambda_3 - 1)^2\right] \tag{2.51}$$

where K is the material's stiffness.

(**a**) Estimate the uniaxial Cauchy(or true) stress in the bar if its diameter does not change during deformation.

Since the bar is stretched in the x-direction, the stretch ratio is $\lambda_1 = \lambda = \ell/\ell_0$. Furthermore, because the diameter remains unchanged during deformation, the stretch ratio λ_2 and λ_3 are equal to one and the deformation gradient and the Cauchy-Green deformation tensor become:

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.52}$$

Using formula (2.50), one can estimate the normal component $P = P_11$ of the nominal stress along the x-direction as:

$$P = 2\lambda\frac{\partial \psi}{\partial(\lambda^2)} = 2\lambda\frac{\partial \psi}{\partial \lambda}\frac{\partial \lambda}{\partial(\lambda^2)} = \frac{\partial \psi}{\partial \lambda} \tag{2.53}$$

where we used the fact that $\partial\lambda/\partial(\lambda^2) = 1/(2\lambda)$. Using the expression of the elastic potential ψ, we therefore obtain:

$$P = K(\lambda - 1) \tag{2.54}$$

From this expression, one can evaluate the Cauchy stress by invoking equation (2.45). In this situation, $J = det(\boldsymbol{F}) = \lambda$ and the component $\sigma_11 = \sigma$ simply becomes:

$$\sigma = P = K(\lambda - 1) \tag{2.55}$$

We see here that the stress increases linearly with stretch ratio λ and vanishes when $\lambda = 1$, as expected.

(**b**) Estimate the uniaxial Cauchy (or true) stress in the bar if the material is incompressible (i.e. the volume does not change during deformation).

In this situation, the diameter of the bar should decrease as it extends in order to keep its volume constant. We should therefore have $\lambda_2 = \lambda_3 = lambda_T < 1$, where the subscript T is used for "transversal". To compute its value, we use the fact $J = det(\boldsymbol{F}) = 1$ from the incompressibility assumption. This means that:

$$det(\boldsymbol{F}) = \lambda\lambda_T^2 = 1 \qquad \rightarrow \qquad \lambda_T = \frac{1}{\sqrt{\lambda}} \tag{2.56}$$

The deformation gradient and Cauchy-Green deformation tensor then take the form:

$$\boldsymbol{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1/\sqrt{(\lambda)} & 0 \\ 0 & 0 & 1/\sqrt{(\lambda)} \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & 1/\lambda & 0 \\ 0 & 0 & 1/\lambda \end{bmatrix} \tag{2.57}$$

We can then compute the nominal stress using (2.50), to find, simiarly to the previous case that:

$$P = K(\lambda - 1) \tag{2.58}$$

The expression of the Cauchy stress is however different from the previous situation since $J = 1$. In this case, equation (2.45) degenerates to:

$$P = P\lambda = K\lambda(\lambda - 1) \tag{2.59}$$

In the case, we find that the stress increases nonlinearly with deformation. This is explained by reduction of cross-sectional area of the bar with stretch, which results in an additional increase in stress.