Homework 4 – NE 255 – Lorenzo Vergari

1a - Calculate the eigenvalues and corresponding eigenvectors of the matrix:

$$A = \begin{pmatrix} 0 & 3 \\ 4 & 4 \end{pmatrix}$$

Eigenvalues:

$$0 = \det(\boldsymbol{A} - \lambda \boldsymbol{I}) \rightarrow 0 = \det\begin{bmatrix} -\lambda & 3 \\ 4 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) - 12$$

$$0 = -4\lambda + \lambda^2 - 12 \rightarrow \boldsymbol{\lambda_1 = 6, \lambda_2 = -2}$$

Eigenvectors:

$$\boldsymbol{A}x_1 = \lambda_1 x_1 \rightarrow (\boldsymbol{A} - \lambda_1 \boldsymbol{I})x_1 = 0 \rightarrow \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} = 0$$

$$-6x_{11} + 3x_{12} = 0 \rightarrow \boldsymbol{x_1} = \boldsymbol{C_1}\begin{bmatrix} \mathbf{1} \\ \mathbf{-2} \end{bmatrix}$$

$$\boldsymbol{A}x_2 = \lambda_2 x_2 \rightarrow (\boldsymbol{A} - \lambda_2 \boldsymbol{I})x_2 = 0 \rightarrow \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} = 0$$

$$2x_{21} + 3x_{22} = 0 \rightarrow \boldsymbol{x_2} = \boldsymbol{C_2}\begin{bmatrix} \mathbf{3} \\ \mathbf{2} \end{bmatrix}$$

1b –
What is the spectral radius of the above matrix?

Spectral radius:

$$\rho(\boldsymbol{A}) = \max|\lambda| = 6$$

Why do we care about the spectral radius of a matrix?

The spectral radius influences convergence behavior of a matrix in a problem $\boldsymbol{A}x = b$
The smaller the spectral radius, the faster the convergence.

What is the largest possible spectral radius for an iteration matrix in nuclear engineering?

Given the problem $\boldsymbol{A}x = b$, $\boldsymbol{A}$ is convergent if and only if $\rho(\boldsymbol{A}) < 1$. So, the largest possible spectral radius is 1.

Calculate the spectral radii for the following systems:

$\Sigma_s[cm^{-1}]$	$\Sigma_t[cm^{-1}]$
0.3	1.5
0.8	1.0
5.0	10.0

Spectral radius is $\rho = \Sigma_s/\Sigma_t$, hence for the three cases we have $\rho_1 = 0.2$; $\rho_2 = 0.8$; $\rho_3 = 0.5$

Which of these systems will converge most quickly? Which most slowly?

The system that converges most quickly is that with the smallest spectral radius (the first one). The system that converges most slowly is that with the largest spectral radius (the second one)

1c - What is the physical interpretation of the k eigenvalue?

The k eigenvalue represents the multiplication factor, i.e. the ratio of the number of neutrons of the n+1 th generation to the neutrons of the n-th. A value k>1 represents a super critical system, k=1 indicates a critical system and k<1 a subcritical system.

2 - Write a Monte Carlo code to estimate π. Plot the result, relative error, and absolute error for:

- **100**
- **1000**
- **10000 histories**

Use a value of $\pi = 3.14159$ for error comparison

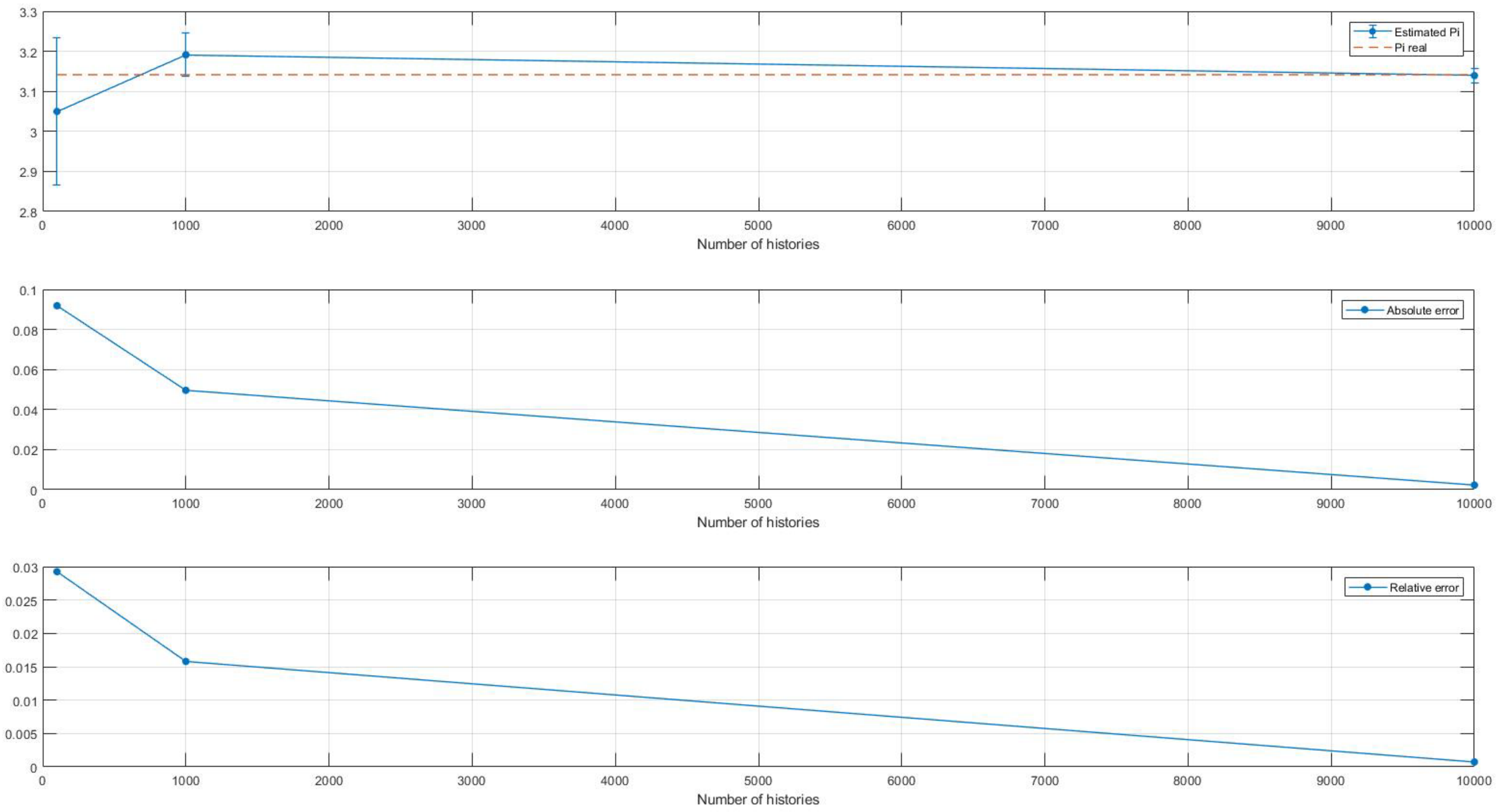
Matlab code and higher resolution plots are provided in the repository.
A plot follows (error bars at 2 standard deviations).

For the represented case, the estimated values for Pi were the following:

```
>> Pi_est

Pi_est =

   3.049718750288954   3.191206537737192   3.139302603780134
```


Estimated Pi
Pi real
Number of histories
Absolute error
Number of histories
Relative error
Number of histories

3.

3a. Why can we run Monte Carlo particle histories in parallel?

We can run Monte Carlo histories in parallel because we assume that all particles are independent one from another, so that we can sample independent random number.

3b. Consider a particle undergoing a collision in a material with the following properties:

$$\Sigma_s = 200 cm^{-1}$$
$$\Sigma_f = 530 cm^{-1}$$
$$\Sigma_a = 150 cm^{-1}$$

What is the probability that the particle undergoes scattering? Fission?

We compute the probability of a given interaction x as the ratio of its cross section to the sum of the cross sections of all possible interactions. Therefore, in this case we have:

$$\Sigma_t = \Sigma_s + \Sigma_f + \Sigma_a = 880\ cm^{-1}$$

$$\boldsymbol{P}(x = Scattering) = \frac{\Sigma_s}{\Sigma_t} = \frac{200}{880} = 22.7\%$$

$$\boldsymbol{P}(x = Fission) = \frac{\Sigma_f}{\Sigma_t} = \frac{530}{880} = 60.2\%$$

The material's scattering cross section is broken down as follows:

$$\Sigma_s^{elastic} = 180 cm^{-1}$$
$$\Sigma_s^{inelastic} = 20 cm^{-1}$$

What is the probability that a given *scattering* collision is inelastic scattering? What is the probability that a given collision is elastic scattering?

The probability that a given <u>scattering</u> collision is of inelastic type is defined as follows:

$$\boldsymbol{P}(x = Inelastic\ Sc. \,|\, x = Scattering) = \frac{\mathbf{P}(x = Inelastic\ Sc.)}{P(x = Scattering)} = \frac{\Sigma_s^{inel}}{\Sigma_s} = \frac{20}{200} = 10\%$$

See appendix for a detailed proof.

The probability that a given collision is elastic scattering is:

$$\boldsymbol{P}(x = Elastic\ Sc.) = \frac{\Sigma_s^{el}}{\Sigma_t} = \frac{180}{880} = 20.5\%$$

Appendix:

Consider an arbitrary distance L. The probability of exactly one interaction within this distance L can be modeled with a Poisson:

$$P(x = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

In our case, *k=1,* and λ can be expressed as the product of a probability of interaction per unit length Σ_t and the distance L. Moreover, by definition, $\Sigma_t = \Sigma_s + \Sigma_a + \Sigma_f$. Therefore, we can write:

$$P(interaction = 1) = \lambda e^{-\lambda} = \Sigma_t L * e^{-\Sigma_t L}$$

Since we have exactly one interaction, it can be a scattering event OR a fission event OR an absorption event (mutually exclusive). Therefore, using Poisson:

$$P(interaction = S) = P(1\ scattering, 0\ absorptions, 0\ fissions)$$
$$= \lambda_s e^{-\lambda_s} * e^{-\lambda_a} * e^{-\lambda_f} = \Sigma_s L * e^{-\Sigma_t L}$$

Therefore, we can compute the conditional probability using Bayes theorem as:

$$P(int. = S|int.takes\ place) = \frac{P(int.takes\ place|int. = S) * P(int. = S)}{P(int\ takes\ place)}$$
$$= \frac{1 * \Sigma_s L * e^{-\Sigma_t L}}{\Sigma_a L * e^{-\Sigma_t L}} = \frac{\Sigma_s}{\Sigma_t}$$

The same reasoning can be used in part 2, replacing the total cross section with the total scattering cross section and the scattering cross section with the inelastic scattering cross section.