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6.5 Reductions

- *introduction*
- *designing algorithms*
- *establishing lower bounds*
- *classifying problems*

Overview: introduction to advanced topics

Main topics. [next 3 lectures]

- Reduction: design algorithms, establish lower bounds, classify problems.
- Linear programming: the ultimate practical problem-solving model.
- Intractability: problems beyond our reach.

Shifting gears.

- From individual problems to problem-solving models.
- From linear/quadratic to polynomial/exponential scale.
- From details of implementation to conceptual framework.

Goals.

- Place algorithms we've studied in a larger context.
- Introduce you to important and essential ideas.
- Inspire you to learn more about algorithms!

Algorithms

Robert Sedgewick | Kevin Wayne

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Bird's-eye view

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, …
linearithmic	$N \log N$	sorting, convex hull, closest pair, farthest pair, …
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Bird's-eye view

Desiderata. Classify problems according to computational requirements.

Desiderata'.

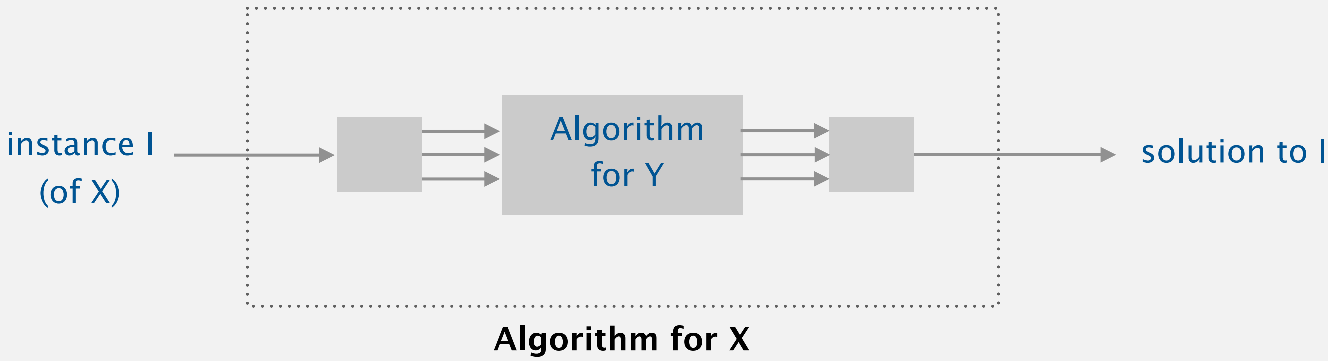
Suppose we could (could not) solve problem X efficiently.

What else could (could not) we solve efficiently?

" Give me a lever long enough and a fulcrum on which to place it, and I shall move the world. " — *Archimedes*

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



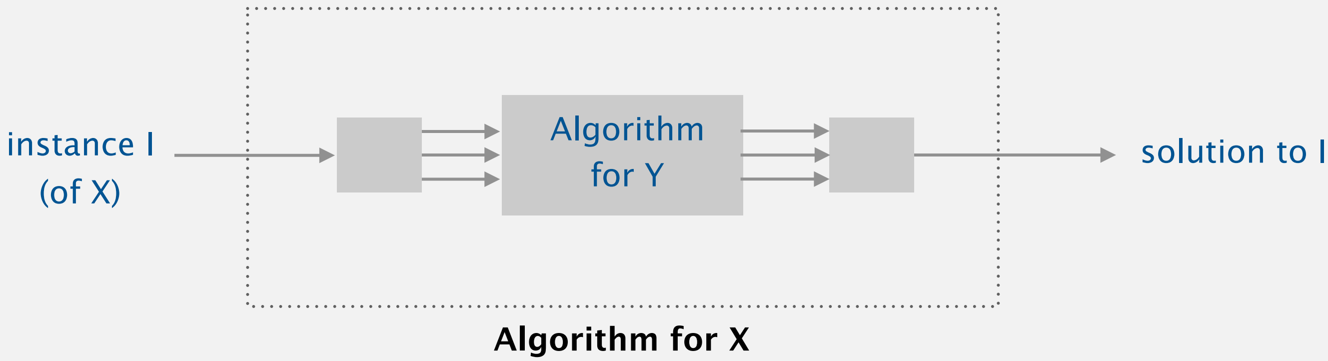
Cost of solving X = total cost of solving Y + cost of reduction.

perhaps many calls to Y
on problems of different sizes

preprocessing and postprocessing

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 1. [finding the median reduces to sorting]

To find the median of N items:

- Sort N items.
- Return item in the middle.

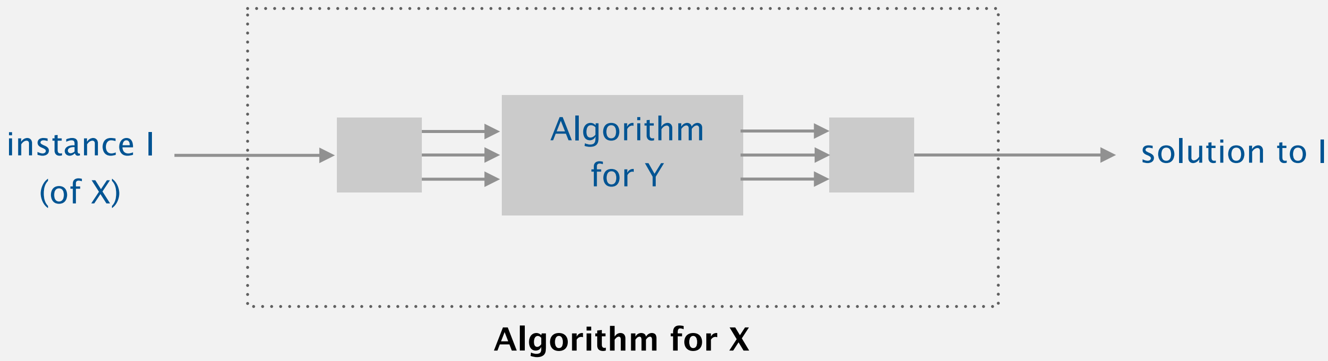
cost of sorting

cost of reduction

Cost of solving finding the median. $N \log N + 1$.

Reduction

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.



Ex 2. [element distinctness reduces to sorting]

To solve element distinctness on N items:

- Sort N items.
- Check adjacent pairs for equality.

cost of sorting

cost of reduction

Cost of solving element distinctness. $N \log N + N$.

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Reduction: design algorithms

Def. Problem X reduces to problem Y if you can use an algorithm that solves Y to help solve X.

Design algorithm. Given algorithm for Y, can also solve X.

Ex.

- 3-collinear reduces to sorting. [assignment]
- Finding the median reduces to sorting.
- Element distinctness reduces to sorting.
- CPM reduces to topological sort. [shortest paths lecture]
- Arbitrage reduces to shortest paths. [shortest paths lecture]
- Burrows-Wheeler transform reduces to suffix sort. [assignment]
- ...

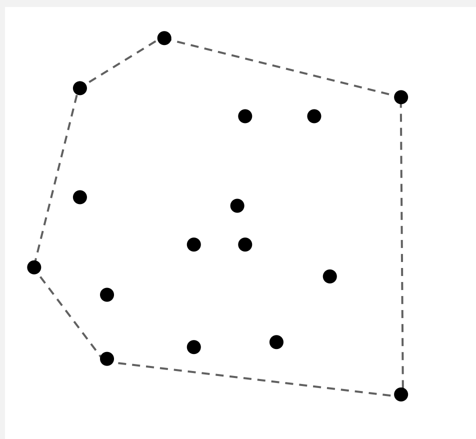
Mentality. Since I know how to solve Y, can I use that algorithm to solve X?

programmer's version: I have code for Y. Can I use it for X?

Convex hull reduces to sorting

Sorting. Given N distinct integers, rearrange them in ascending order.

Convex hull. Given N points in the plane, identify the extreme points of the convex hull (in counterclockwise order).



convex hull

1251432
2861534
3988818
4190745
8111033
13546464
89885444
43434213
34435312

sorting

Proposition. Convex hull reduces to sorting.

Pf. Graham scan algorithm (see next slide).

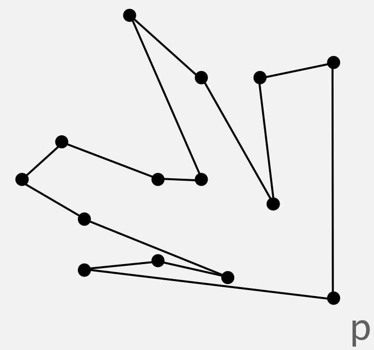
cost of sorting   cost of reduction

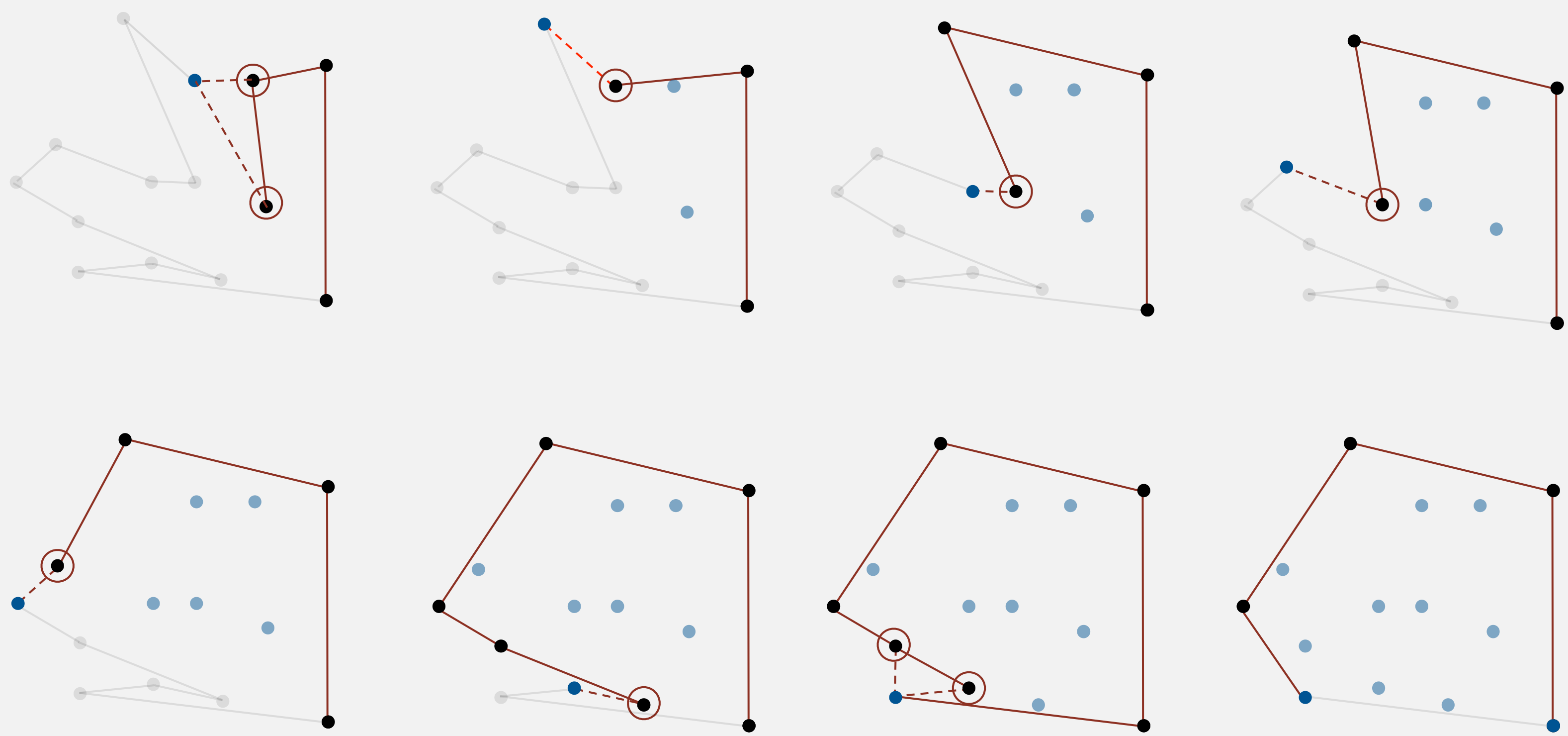
Cost of convex hull. $N \log N + N$.

Graham scan algorithm

Graham scan.

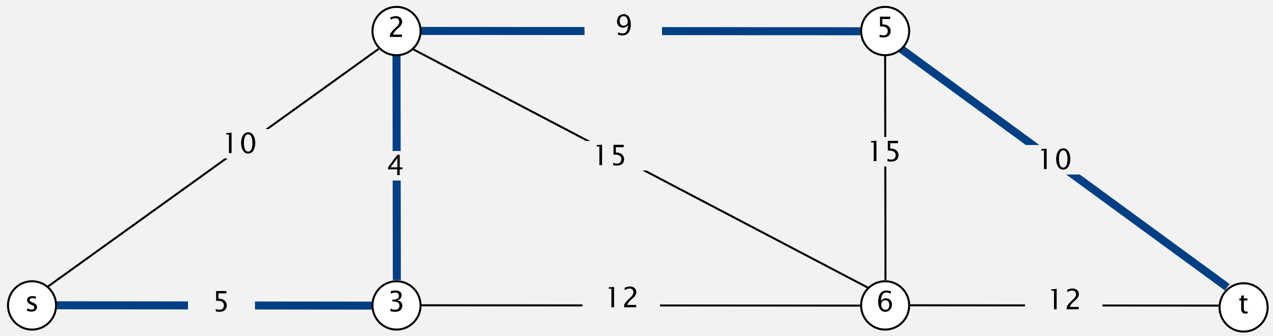
- Choose point p with smallest (or largest) y-coordinate.
- Sort points by polar angle with p to get simple polygon.
- Consider points in order, and discard those that would create a clockwise turn.



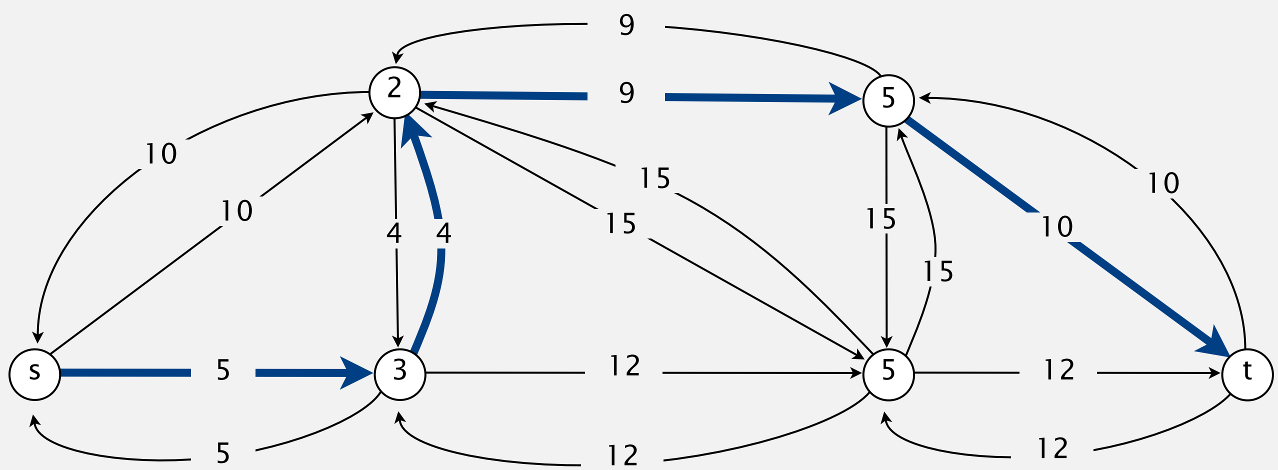


Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.

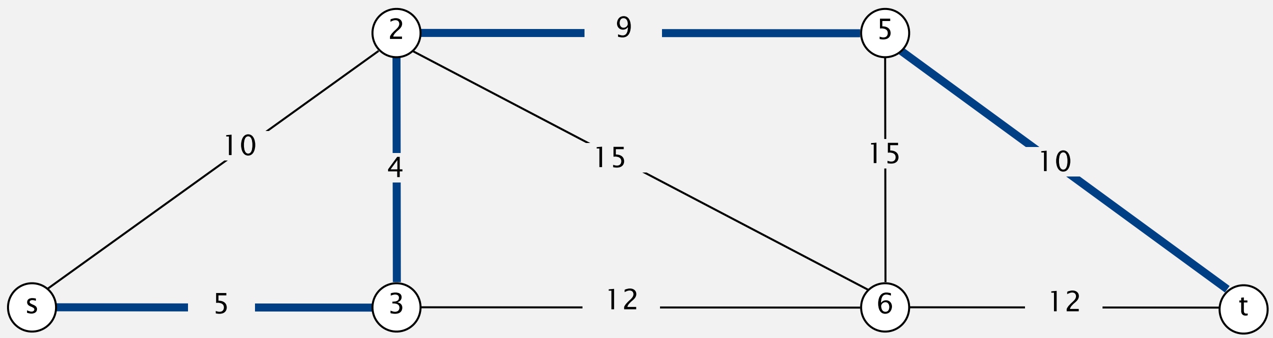


Pf. Replace each undirected edge by two directed edges.



Shortest paths on edge-weighted graphs and digraphs

Proposition. Undirected shortest paths (with nonnegative weights) reduces to directed shortest path.



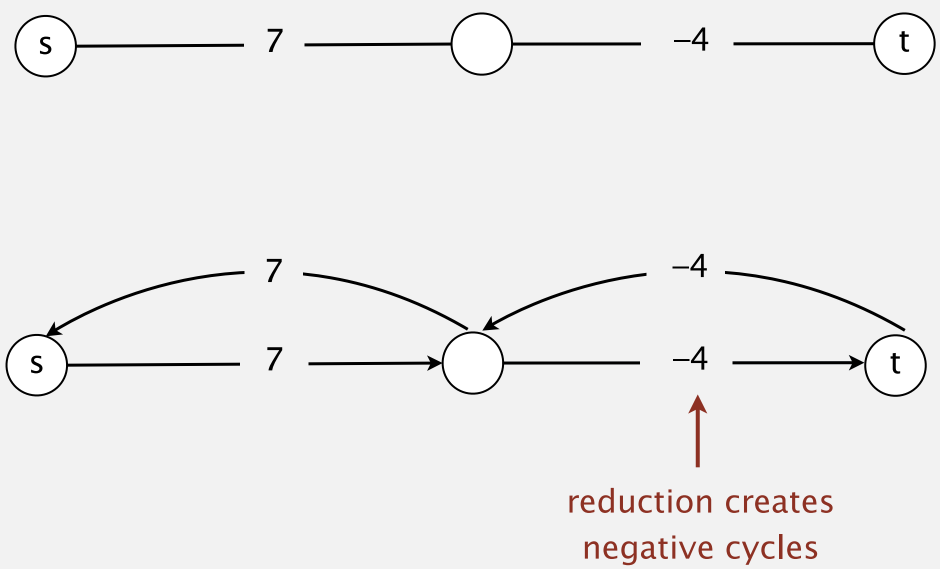
cost of shortest paths in digraph

cost of reduction

Cost of undirected shortest paths. $E \log V + E$.

Shortest paths with negative weights

Caveat. Reduction is invalid for edge-weighted graphs with negative weights (even if no negative cycles).

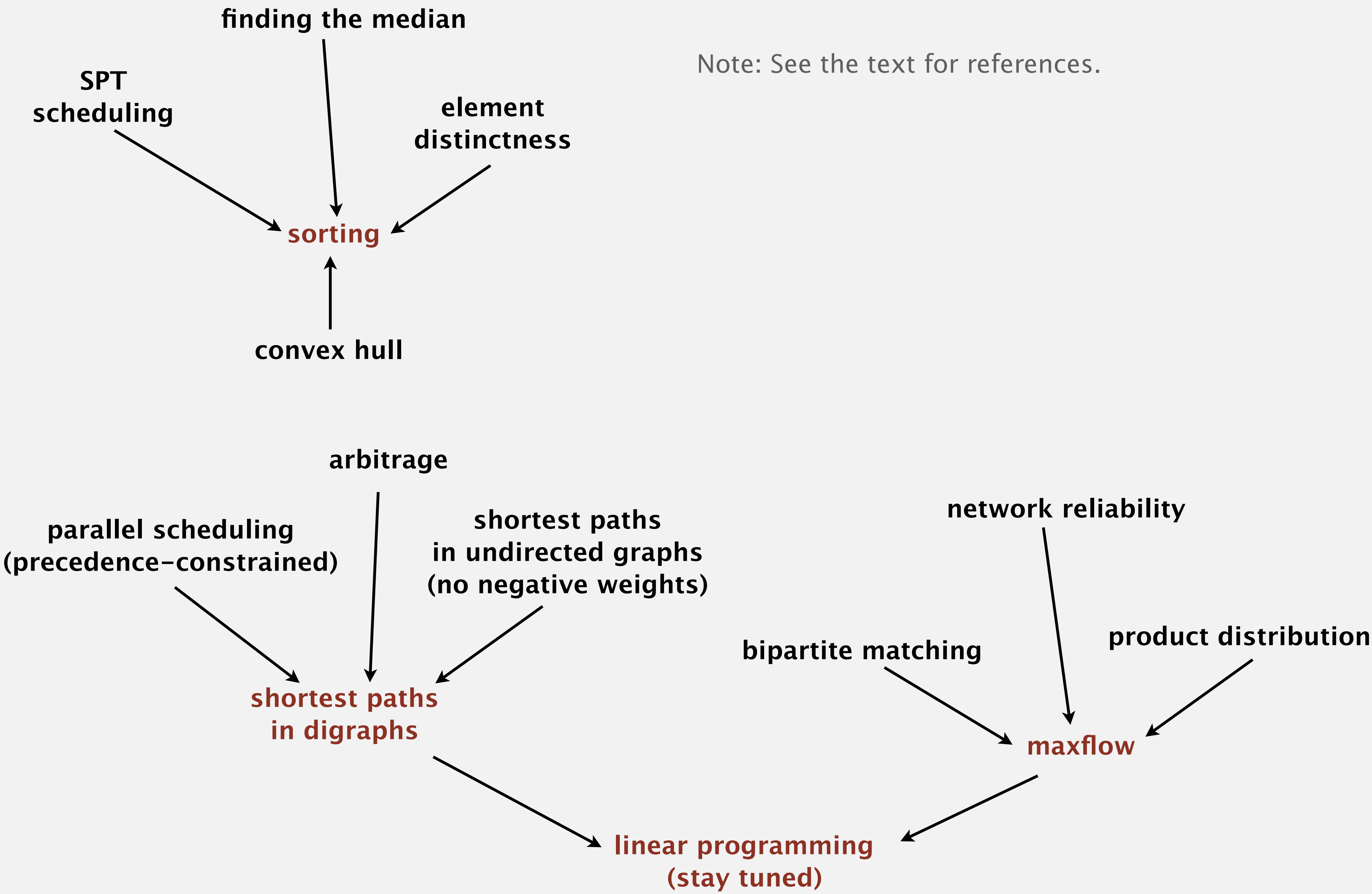


Remark. Can still solve shortest-paths problem in undirected graphs (if no negative cycles), but need more sophisticated techniques.

reduces to weighted
non-bipartite matching (!)

Linear-time reductions involving familiar problems

Note: See the text for references.



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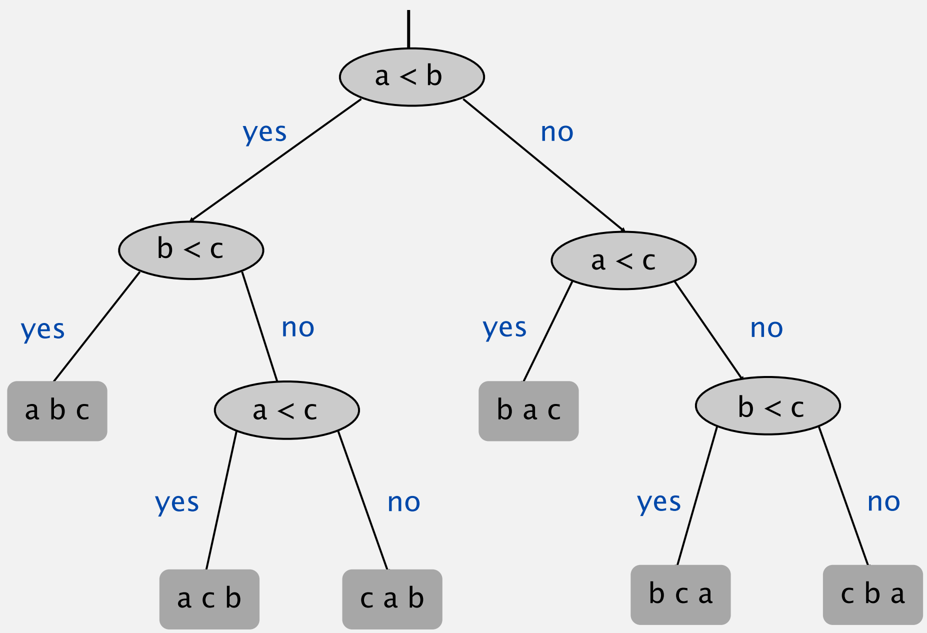
Bird's-eye view

Goal. Prove that a problem requires a certain number of steps.

Ex. In decision tree model, any compare-based sorting algorithm requires $\Omega(N \log N)$ compares in the worst case.



argument must apply to all conceivable algorithms

Bad news. Very difficult to establish lower bounds from scratch.

Good news. Spread $\Omega(N \log N)$ lower bound to Y by reducing sorting to Y.

assuming cost of reduction is not too high

Linear-time reductions

Def. Problem X linear-time reduces to problem Y if X can be solved with:

- Linear number of standard computational steps.
- Constant number of calls to Y.

Ex. Almost all of the reductions we've seen so far. [Which ones weren't?]

Establish lower bound:

- If X takes $\Omega(N \log N)$ steps, then so does Y.
- If X takes $\Omega(N^2)$ steps, then so does Y.

Mentality.

- If I could easily solve Y, then I could easily solve X.
- I can't easily solve X.
- Therefore, I can't easily solve Y.

Lower bound for convex hull

Proposition. In quadratic decision tree model, any algorithm for sorting N integers requires $\Omega(N \log N)$ steps.

allows linear or quadratic tests:

$\underline{x_i} < \underline{x_j}$ or $(x_j - x_i)\,(x_k - x_i) - (x_j)\,(\underline{x_j} - x_i) < 0$

Proposition. Sorting linear-time reduces to convex hull.

Pf. [see next slide]

lower-bound mentality:
if I can solve convex hull
efficiently, I can sort efficiently

1251432
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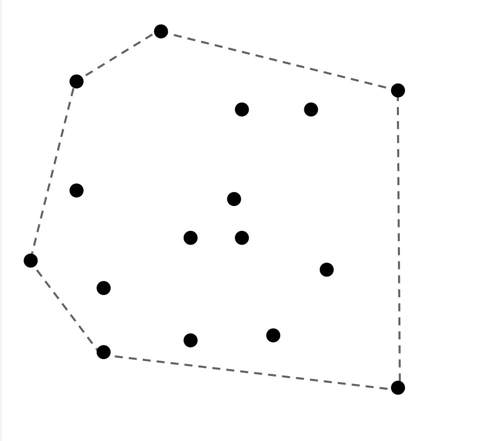
sorting



convex hull

linear or
quadratic tests

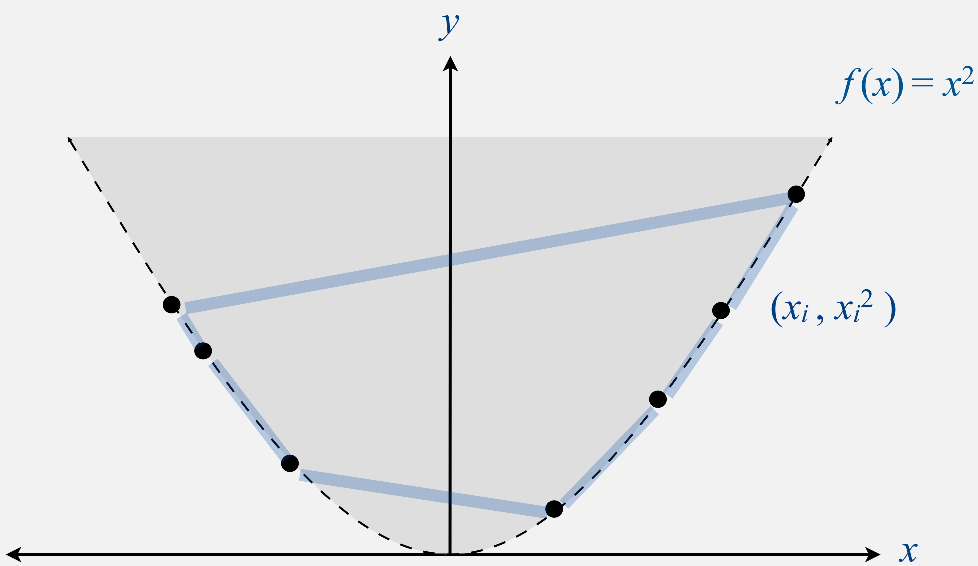
Implication. Any ccw-based convex hull algorithm requires $\Omega(N \log N)$ ops.

Sorting linear-time reduces to convex hull

Proposition. Sorting linear-time reduces to convex hull.

- Sorting instance: $x_1, x_2, \ldots, x_N$.
- Convex hull instance: $(x_1, x_1^2), (x_2, x_2^2), \ldots, (x_N, x_N^2)$.

lower-bound mentality: if I can solve convex hull efficiently, I can sort efficiently



Pf.

- Region $\{ x : x^2 \geq x \}$ is convex $\Rightarrow$ all points are on hull.
- Starting at point with most negative x, counterclockwise order of hull points yields integers in ascending order.

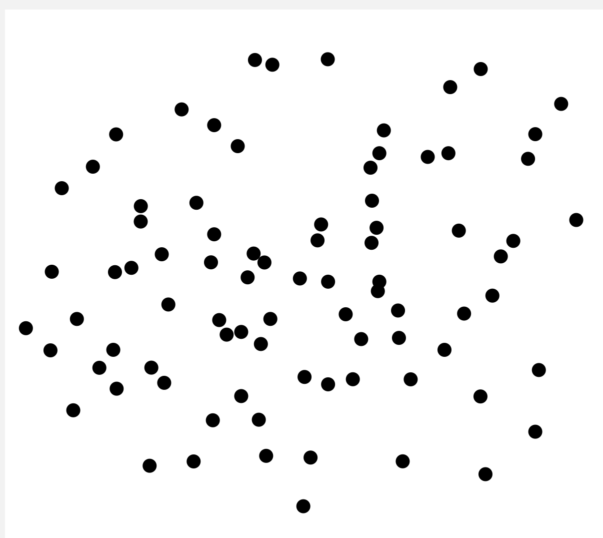
Establishing lower bounds: summary

Establishing lower bounds through reduction is an important tool in guiding algorithm design efforts.

Q. How to convince yourself no linear-time convex hull algorithm exists?

A1. [hard way] Long futile search for a linear-time algorithm.

A2. [easy way] Linear-time reduction from sorting.



convex hull

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Classifying problems: summary

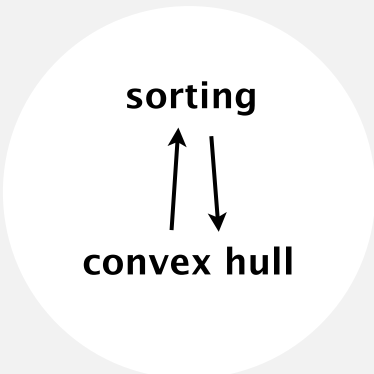
Desiderata. Problem with algorithm that matches lower bound.

Ex. Sorting and convex hull have complexity $N \log N$.

Desiderata'. Prove that two problems X and Y have the same complexity.

- First, show that problem X linear-time reduces to Y.
- Second, show that Y linear-time reduces to X.
- Conclude that X and Y have the same complexity.

even if we don't know what it is!



Caveat

SORT. Given N distinct integers, rearrange them in ascending order.

CONVEX HULL. Given N points in the plane, identify the extreme points of the convex hull (in counterclockwise order).

Proposition. SORT linear-time reduces to CONVEX HULL.

Proposition. CONVEX HULL linear-time reduces to SORT.

Conclusion. SORT and CONVEX HULL have the same complexity.

A possible real-world scenario.

- System designer specs the APIs for project.
- Alice implements `sort()` using `convexHull()`. well, maybe not so realistic
- Bob implements `convexHull()` using `sort()`.
- Infinite reduction loop!
- Who's fault?

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

								1	1	0	1	0	1	0	1
							×	0	1	1	1	1	1	0	1
								1	1	0	1	0	1	0	1
							0	0	0	0	0	0	0	0	
						1	1	0	1	0	1	0	1		
					1	1	0	1	0	1	0	1			
				1	1	0	1	0	1	0	1				
			1	1	0	1	0	1	0	1					
		1	1	0	1	0	1	0	1						
	0	0	0	0	0	0	0	0							
0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	1

Integer arithmetic reductions

Integer multiplication. Given two N-bit integers, compute their product.

Brute force. N^2 bit operations.

problem	arithmetic	order of growth
integer multiplication	$a \times b$	M(N)
integer division	a / b, $a \bmod b$	M(N)
integer square	a^2	M(N)
integer square root	$\lfloor \sqrt{a} \rfloor$	M(N)

integer arithmetic problems with the same complexity as integer multiplication

Q. Is brute-force algorithm optimal?

History of complexity of integer multiplication

year	algorithm	order of growth
?	brute force	N^2
1962	Karatsuba-Ofman	$N^{1.585}$
1963	Toom-3, Toom-4	$N^{1.465}$, $N^{1.404}$
1966	Toom-Cook	$N^{1 + \varepsilon}$
1971	Schönhage–Strassen	$N \log N \log \log N$
2007	Fürer	$N \log N \, 2^{\log^* N}$
?	?	N

number of bit operations to multiply two N-bit integers

used in Maple, Mathematica, gcc, cryptography, ...

Remark. GNU Multiple Precision Library uses one of five different algorithm depending on size of operands.

GMP
«Arithmetic without limitations»

Linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

$$
\begin{array}{rc}
 & \\
\text{row } i &
\end{array}
\begin{bmatrix} 0.1 & 0.2 & 0.8 & 0.1 \\ 0.5 & 0.3 & 0.9 & 0.6 \\ 0.1 & 0.0 & 0.7 & 0.4 \\ 0.0 & 0.3 & 0.3 & 0.1 \end{bmatrix}
\times
\begin{array}{c}
\begin{array}{cccc} & & \text{column } j & \end{array} \\
\begin{bmatrix} 0.4 & 0.3 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.0 & 0.6 \\ 0.0 & 0.0 & 0.4 & 0.5 \\ 0.8 & 0.4 & 0.1 & 0.9 \end{bmatrix}
\end{array}
=
\begin{array}{c}
\begin{array}{cccc} & & j & \end{array} \\
\begin{array}{c} \\ i \\ \\ \\ \end{array}
\begin{bmatrix} 0.16 & 0.11 & 0.34 & 0.62 \\ 0.74 & 0.45 & 0.47 & 1.22 \\ 0.36 & 0.19 & 0.33 & 0.72 \\ 0.14 & 0.10 & 0.13 & 0.42 \end{bmatrix}
\end{array}
$$

$0.5 \cdot 0.1 + 0.3 \cdot 0.0 + 0.9 \cdot 0.4 + 0.6 \cdot 0.1 = 0.47$

Linear algebra reductions

Matrix multiplication. Given two N-by-N matrices, compute their product.

Brute force. N^3 flops.

problem	linear algebra	order of growth
matrix multiplication	$A \times B$	MM(N)
matrix inversion	A^{-1}	MM(N)
determinant	$\vert A \vert$	MM(N)
system of linear equations	$Ax = b$	MM(N)
LU decomposition	$A = L U$	MM(N)
least squares	$\min \Vert Ax - b \Vert_2$	MM(N)

numerical linear algebra problems with the same complexity as matrix multiplication

Q. Is brute-force algorithm optimal?

History of complexity of matrix multiplication

year	algorithm	order of growth
?	brute force	N^3
1969	Strassen	$N^{2.808}$
1978	Pan	$N^{2.796}$
1979	Bini	$N^{2.780}$
1981	Schönhage	$N^{2.522}$
1982	Romani	$N^{2.517}$
1982	Coppersmith-Winograd	$N^{2.496}$
1986	Strassen	$N^{2.479}$
1989	Coppersmith-Winograd	$N^{2.376}$
2010	Strother	$N^{2.3737}$
2011	Williams	$N^{2.3727}$
?	?	$N^{2+\varepsilon}$

number of floating-point operations to multiply two N-by-N matrices

Birds-eye view: review

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median, Burrows-Wheeler transform, ...
linearithmic	$N \log N$	sorting, convex hull, closest pair, farthest pair, ...
quadratic	N^2	?
⋮	⋮	⋮
exponential	c^N	?

Frustrating news. Huge number of problems have defied classification.

Birds-eye view: revised

Desiderata. Classify problems according to computational requirements.

complexity	order of growth	examples
linear	N	min, max, median,
linearithmic	$N \log N$	sorting, convex hull,
$M(N)$	?	integer multiplication, division, square root, ...
$MM(N)$	?	matrix multiplication, $Ax = b$, least square, determinant, ...
⋮	⋮	⋮
NP-complete	probably not N^b	SAT, IND-SET, ILP, ...

STAY TUNED!

Good news. Can put many problems into equivalence classes.

Complexity zoo

Complexity class. Set of problems sharing some computational property.

http://qwiki.stanford.edu/index.php/Complexity_Zoo

Bad news. Lots of complexity classes.

Summary

Reductions are important in theory to:

- Design algorithms.
- Establish lower bounds.
- Classify problems according to their computational requirements.

Reductions are important in practice to:

- Design algorithms.
- Design reusable software modules.
  - stacks, queues, priority queues, symbol tables, sets, graphs
  - sorting, regular expressions, Delaunay triangulation
  - MST, shortest path, maxflow, linear programming
- Determine difficulty of your problem and choose the right tool.

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