Notes of the Introduction To Algorithms

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Part I

Foundations

Chapter 1

The Role of Algorithms in Computing

1.1 Algorithms

Exercies

1.1-1 Give a real-world example that requires sorting or a real-world example that requires computing a convex hull.

Answer: One example that requires sorting is that teachers will sort our scores after the exam.

1.1-2 Other than speed, what other measures of efficiency might one use in a real-world setting ?

Answer: cost, space, manpower, material resources. In different cases, each can be the key of meausres of efficiency.

Reference: https://www.quora.com/Other-than-speed-what-other-measures-of-efficiency-might-one-use-in-a-real-world-setting

1.1-3 Select a data structure that you have seen previously, and discuss its strengths and limitations.

Answer: Array

strengths: access directly

limitations: costs lot when insert or delete

1.1-4 How are the shortest-path and traveling-salesman problems given similar? How they are different?

Answer:

1.1-5 Come up with a real-world problem in which only the best solution will do. Then come up with one in which a solution that is "approximately" the best is good enough.

Answer:

1.2 Algorithms as a technology

Chapter 2

Getting Started

Chapter 3

Growth of Functions

Chapter 4

Divide-and-conquer

Chapter 5

Probabilistic Analysis and Randomized Algorithms

Part II

Sorting and Order Statistics

Chapter 6

Heapsort

Part III

Data Structures

Part IV

Advanced Design and Analysis Techniques

Part V

Advanced Data Structures

Part VI

Graph Algorithms

Chapter 7

Minimum Spanning Tree

7.1 Notes

(i) There maybe more than one MST in a forest.

(ii) The number of all the edges in the MST is equal to $\boldsymbol{V} - 1$.

7.2 Growing a minimum spanning tree

7.2.1 Definition

A

A is a subset of some minimum spanning tree.

Safe edge

Safe edge is a edge that add to A and A is also a subset of some minimum spanning tree.

7.2.2 Generic-MST

GENERIC-MST(G, w)
1 $A = \varnothing$
2 **while** A does not form a spanning tree
3 find an edge (u, v) that is **safe edge** for A
4 $A = A \cup \{(u, v)\}$
5 return A

Initialization: After line 1, the set A trivially satisfies the loop invariant.
Maintenance: The loop in lines 2-4 maintains the invariant by adding only safe edges.
Termination: All edges added to A are in a minimum spanning tree, and so the set A returned in line 5 must be a minimum spanning tree.

7.2.3 Theorem 1.

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function ω defined on E. Let A be a subset of E that is inclued in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A, and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is **safe** for A. Namely, $A \cup (u, v)$ is also included in some minimum spanning tree for G.

Proof Let T be a minimum spanning tree that includes A, and assume that T does not contain the light edge (u, v), since if it does, the edge is obviously **safe** for A. We shall construct another minimum spanning tree $T^{'}$ that includes $A \cup (u, v)$ by using cut-and-paste technique, thereby showing that (u, v) is a **safe** edge for A.

The edge (u, v) forms a cycle with the edges on the simple path p from u to v in T. Since u and v are on opposite sides of the cut $(S, V - S)$, at least one edge in T lies on the simple path p and also crosses the cut. Let (x, y) be any such edge. The edge (x, y) is not in A, because the cut respects A. Since (x, y) is on the unique simple path from u to v in T, removing (x, y) breaks T into two components. Adding (u, v) reconnects them to form a new spanning tree $T^{'} = T - \{(x, y)\} \cup \{(u, v)\}$.

We next show that $T^{'}$ is a minimum spanning tree. Since (u, v) is a light edge crossing $(S, V - S)$ and (x, y) also crosses this cut, $w(u, v) \leq w(x, y)$. Therefore, $w(T^{'}) = w(T) - w(x, y) + w(u, v) \leq w(T)$.

When $w(T^{'}) == w(T)$, we know that $T^{'}$ is also a minimum spanning tree, so the edge (u, v) is **safe** for A.

When $w(T^{'}) < w(T)$, since we let T be a minimum spanning tree and **assume** that T does not contain the light edge (u, v). Therefore, the **assume** is false, so T must contain the light edge (u, v), and the edge (u, v) is **safe** for A.

7.2.4 Exercises

23.1-1

Let (u, v) be a minimum-weight edge in a connected graph G. Show that (u, v) belongs to some minimum spanning tree of G.

Solution

Let E_u be all the edges that connected to the point u.

a. If there is only one edge connected to the point u, the edge belongs to **all** the minimum spanning tree of G.

b. If there is more than one edge connected to the point u, we assume that (u, v) is not in any minimum spanning trees of G. There must be one edge $(u, x)x! = v$ that is in some minimum spanning tree of G, since $w(u, v) < w(u, x)$, therefore, the edge (u, x) can not be in some minimum spanning tree of G. So there is conflict and the assume is false. So, the (u, v) belongs to some minimum spanning tree of G.

23.1-2

Professor Sabatier conjectures the following converse of Theorem 1. in Minimum Spanning Tree. Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E. Let A be a subset of E that is included in some minimum spanning tree for G, let $(S, V - S)$ be any cut of G that respects A,and let (u, v) be a safe edge for A crossing $(S, V - S)$. Then, (u, v) is a light edge for the cut. Show that the professorfis conjecture is incorrect by giving a counterexample.

Solution

a. Here is a special case, the point v of (u, v) only has one edge, and $w(u, v)$ is the largest, let (x, y) be any other edge that crosses the cut, obviously, (u, v) is not a light edge for the cut.

b. Here is a generic case, assume that there is a light edge $(u^{'}, v^{'})$ crossing the cut and the edge has no common point with (u, v), so $w(u^{'}, v^{'}) < w(u, v)$. After combine A with $(u^{'}, v^{'})$, there is another $cut^{'}$ that crossing (u, v), and it is a light edge for $cut^{'}$. The previous case shows that (u, v) is not a light edge for any cut but some cut when (u, v) is a safe edge for A.

23.1-3

Show that if an edge (u, v) is contained in some minimum spanning tree, then it is a light edge crossing some cut of the graph.

Solution

Let T be the minimum spanning tree that contains the edge (u, v), if we remove the edge from T, and the other edges are A, obviously there is some cut that crosses the edge (u, v) which respects A. Then we are going to show that the edge (u, v) is a light edge crossing these cut.

If there is only one edge crossing the cut, obviously the edge (u, v) is a light edge crossing the cut.

If there is more than one edge crossing the cut, let (x, y) be any edges crossing the cut other than (u, v). Assume that $w(x, y) < w(u, v)$, there will another minimum spanning tree $T^{'}$ and $w(T^{'}) = w(T) - \{(u, v)\} + \{(x, y)\} < w(T)$ which is impossible since the T is a minimum spanning tree. So the assume is contradiction and $w(x, y) \geq w(u, v)$, so the edge (u, v) is a light edge crossing some cut of the graph.

23.1-4

Give a simple example of a connected graph such that the set of edges $\{(u, v)$: there exists a cut $(S, V - S)$ such that (u, v) is a light edge crossing $(S, V - S)\}$ does not form a minimum spanning tree.

Solution

There is a quadrangle: $V = A, B, C, D, E = (A, B), (A, C), (B, C), (B, D), (C, D)$ $, w(A, B) = w(A, C) = w(B, C) = 1, w(B, D) = w(C, D) = 2$. Obviously, $(A, B), (A, C) and (B, C)$ are lights edges crossing some cut. So the tree edges can join the set. And they construct a circle, so the set can not form a minimum spanning tree.

I think if we add respect to the set, then the set will form a minimum spanning tee. Such as, $\{(u, v)$: there exists a cut $(S, V - S)$ which repects this set such that (u, v) is a light edge crossing $(S, V - S)\}$, and the set will form a minimum spanning tree.

23.1-5

Let e be a maximum-weight edge on some cycle of connected graph $G = (V, E)$. Prove that there is a minimum spanning tree of $G^{'} = (V, E - e)$ that is also a minimum spanning tree of G. That is, there is a minimum spanning tree of G that does not include e.

Solution

Assume that there is a minimum spanning tree T of G including the edge e. Firstly, we construct a tree $T^{'}$ same as T, and remove the edge e from $T^{'}$. There is another edge $e^{'}$ on the same cycle of G with e, and $T^{'}$ does not have a cycle after add $e^{'}$ to $T^{'}$. Since $w(e) \geq w(e^{'})$, so $w(T) \geq w(T^{'})$. Therefore, there is a minimum spanning tree $T^{'}$ that does not include e.

TODO: Double check the previous problem.

TODO: Prim -¿ Kruskal TODO: Kruskal -¿ Prim

Part VII

Selected Topics

Part VIII

Appendix: Mathematical Background