

Notes of the Introduction To Algorithms

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August 12, 2016

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Part I

Foundations

Chapter 1

The Role of Algorithms in Computing

1.1 Algorithms

Exercies

1.1-1 Give a real-world example that requires sorting or a real-world example that requires computing a convex hull.

Answer: One example that requires sorting is that teachers will sort our scores after the exam.

1.1-2 Other than speed, what other measures of efficiency might one use in a real-world setting ?

Answer: cost, space, manpower, material resources. In different cases, each can be the key of measures of efficiency.

Reference: <https://www.quora.com/Other-than-speed-what-other-measures-of-efficiency-might-one-use-in-a-real-world-setting>

1.1-3 Select a data structure that you have seen previously, and discuss its strengths and limitations.

Answer: Array

strengths: access directly

limitations: costs lot when insert or delete

1.1-4 How are the [shortest-path](#) and [traveling-salesman](#) problems given [similar](#)? How they are [different](#)?

Answer:

1.1-5 Come up with a real-world problem in which only the best solution will do. Then come up with one in which a solution that is "approximately" the best is good enough.

Answer:

1.2 Algorithms as a technology

Chapter 2

Getting Started

Chapter 3

Growth of Functions

Chapter 4

Divide-and-conquer

Chapter 5

Probabilistic Analysis and Randomized Algorithms

Part II

Sorting and Order Statistics

Chapter 6

Heapsort

Part III

Data Structures

Part IV

Advanced Design and Analysis Techniques

Part V

Advanced Data Structures

Part VI

Graph Algorithms

Chapter 7

Minimum Spanning Tree

7.1 Notes

- (i) There may be more than one MST in a forest.
- (ii) The number of all the edges in the MST is equal to $V - 1$.

7.2 Growing a minimum spanning tree

7.2.1 Definition

A

A is a subset of some minimum spanning tree.

Safe edge

Safe edge is an edge that can be added to A and A is also a subset of some minimum spanning tree.

7.2.2 Generic-MST

GENERIC-MST(G, w)

```

1  $A = \emptyset$ 
2 while A does not form a spanning tree
3   find an edge  $(u, v)$  that is safe edge for A
4    $A = A \cup \{(u, v)\}$ 
5 return A
```

Initialization: After line 1, the set A trivially satisfies the loop invariant.

Maintenance: The loop in lines 2-4 maintains the invariant by adding only safe edges.

Termination: All edges added to A are in a minimum spanning tree, and so the set A returned in line 5 must be a minimum spanning tree.

7.2.3 Theorem 1.

Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function ω defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a light edge crossing $(S, V - S)$. Then, edge (u, v) is **safe** for A . **Namely, $A \cup (u, v)$ is also included in some minimum spanning tree for G .**

Proof Let T be a minimum spanning tree that includes A , and **assume that T does not contain the light edge (u, v)** , since if it does, the edge is obviously **safe** for A . We shall construct another minimum spanning tree T' that includes $A \cup (u, v)$ by using cut-and-paste technique, thereby showing that (u, v) is a **safe** edge for A .

The edge (u, v) forms a **cycle** with the edges on the simple path p from u to v in T . Since u and v are on opposite sides of the cut $(S, V - S)$, at least one edge in T lies on the simple path p and also crosses the cut. Let (x, y) be any such edge. The edge (x, y) is not in A , because the cut respects A . Since (x, y) is on the unique simple path from u to v in T , removing (x, y) breaks T into two components. Adding (u, v) reconnects them to form a new spanning tree $T' = T - \{(x, y)\} \cup \{(u, v)\}$.

We next show that T' is a minimum spanning tree. Since (u, v) is a light edge crossing $(S, V - S)$ and (x, y) also crosses this cut, $w(u, v) \leq w(x, y)$. Therefore, $w(T') = w(T) - w(x, y) + w(u, v) \leq w(T)$.

When $w(T') = w(T)$, we know that T' is also a minimum spanning tree, so the edge (u, v) is **safe** for A .

When $w(T') < w(T)$, since we let T be a minimum spanning tree and **assume** that T does not contain the light edge (u, v) . Therefore, the **assume** is false, so T must contain the light edge (u, v) , and the edge (u, v) is **safe** for A .

7.2.4 Exercises

23.1-1

Let (u, v) be a minimum-weight edge in a connected graph G . Show that (u, v) belongs to some minimum spanning tree of G .

Solution

Let E_u be all the edges that connected to the point u .

- a. If there is only one edge connected to the point u , the edge belongs to **all** the minimum spanning tree of G .
- b. If there is more than one edge connected to the point u , we assume that (u, v) is not in any minimum spanning trees of G . There must be one edge (u, x) $x \neq v$ that is in some minimum spanning tree of G , since $w(u, v) < w(u, x)$, therefore, the edge (u, x) can not be in some minimum spanning tree of G . So there is conflict and the assume is false. So, the (u, v) belongs to some minimum spanning tree of G .

23.1-2

Professor Sabatier conjectures the following converse of Theorem 1. in Minimum Spanning Tree. Let $G = (V, E)$ be a connected, undirected graph with a real-valued weight function w defined on E . Let A be a subset of E that is included in some minimum spanning tree for G , let $(S, V - S)$ be any cut of G that respects A , and let (u, v) be a safe edge for A crossing $(S, V - S)$. Then, (u, v) is a light edge for the cut. Show that the professor's conjecture is incorrect by giving a counterexample.

Solution

- a. Here is a special case, the point v of (u, v) only has one edge, and $w(u, v)$ is the largest, let (x, y) be any other edge that crosses the cut, obviously, (u, v) is not a light edge for the cut.
- b. Here is a generic case, assume that there is a light edge (u', v') crossing the cut and the edge has no common point with (u, v) , so $w(u', v') < w(u, v)$. After combine A with (u', v') , there is another cut cut' that crossing (u, v) , and it is a light edge for cut' . The previous case shows that (u, v) is not a light edge for any cut but some cut when (u, v) is a safe edge for A .

23.1-3

Show that if an edge (u, v) is contained in some minimum spanning tree, then it is a light edge crossing some cut of the graph.

Solution

Let T be the minimum spanning tree that contains the edge (u, v) , if we remove the edge from T , and the other edges are A , obviously there is some cut that crosses the edge (u, v) which respects A . Then we are going to show that the edge (u, v) is a light edge crossing these cut.

If there is only one edge crossing the cut, obviously the edge (u, v) is a light edge crossing the cut.

If there is more than one edge crossing the cut, let (x, y) be any edges crossing the cut other than (u, v) . Assume that $w(x, y) < w(u, v)$, there will another minimum spanning tree T' and $w(T') = w(T) - \{(u, v)\} + \{(x, y)\} < w(T)$ which is impossible since the T is a minimum spanning tree. So the assume is contradiction and $w(x, y) \geq w(u, v)$, so the edge (u, v) is a light edge crossing some cut of the graph.

23.1-4

Give a simple example of a connected graph such that the set of edges $\{(u, v): \text{there exists a cut } (S, V - S) \text{ such that } (u, v) \text{ is a light edge crossing } (S, V - S)\}$ does not form a minimum spanning tree.

Solution

There is a quadrangle: $V = A, B, C, D, E = (A, B), (A, C), (B, C), (B, D), (C, D)$, $w(A, B) = w(A, C) = w(B, C) = 1, w(B, D) = w(C, D) = 2$. Obviously, $(A, B), (A, C)$ and (B, C) are lights edges crossing some cut. So the tree edges can join the set. And they construct a circle, so the set can not form a minimum spanning tree.

I think if we add **respect** to the set, then the set will form a minimum spanning tree. Such as, $\{(u, v): \text{there exists a cut } (S, V - S) \text{ which respects this set such that } (u, v) \text{ is a light edge crossing } (S, V - S)\}$, and the set will form a minimum spanning tree.

23.1-5

Let e be a maximum-weight edge on some cycle of connected graph $G = (V, E)$. Prove that there is a minimum spanning tree of $G' = (V, E - e)$ that is also a minimum spanning tree of G . That is, there is a minimum spanning tree of G that does not include e .

Solution

Assume that there is a minimum spanning tree T of G including the edge e . Firstly, we construct a tree T' same as T , and remove the edge e from T' . There is another edge e' on the same cycle of G with e , and T' does not have a cycle after add e' to T' . Since $w(e) \geq w(e')$, so $w(T) \geq w(T')$. Therefore, there is a minimum spanning tree T' that does not include e .

23.1-6

Show that a graph has a unique minimum spanning tree if, for every cut of the graph, there is a unique light edge crossing the cut. Show that the converse is not true by giving a counterexample.

Solution**1. Proof**

Let T be the minimum spanning tree that is constructed by the unique light edges crossing each cut. Assume that T' is another minimum spanning tree which is different from T . We are going to show that the assume is contradiction that T' can not be a minimum spanning tree.

Let x be the vertex which has different edges in T and T' . Let edge (x, y_1) be the edge in T but not in T' . Let edge (x, y_2) be the edge in T' but not in T .

Now we are going to show that $w(x, y_1) < w(x, y_2)$. If we add the edge (x, y_2) into the T , there will be an cycle including the edge (x, y_1) and (x, y_2) . Since there is a unique light edge for each cut, so $w(x, y_1) \leq w(x, y_2)$. Assume $w(x, y_1) > w(x, y_2)$, so we will get a better minimum spanning tree after replace (x, y_1) with (x, y_2) . Since T is a minimum spanning tree, so there can not be a better minimum spanning tree. So the assume that $w(x, y_1) > w(x, y_2)$ is contradiction. So $w(x, y_1) < w(x, y_2)$.

If we add the edge (x, y_1) into the T' , there will an cycle including the edge (x, y_1) and (x, y_2) . Since $w(x, y_1) < w(x, y_2)$, we can get a better minimum spanning tree if we replace (x, y_2) with (x, y_1) . So the T' is not a minimum spanning tree. Therefore, the assume that T' is another minimum spanning tree which is different from T is contradiction.

2. Counterexample

$G = (V, E)$ has three vertex: A, B, C and two edges $(A, B), (A, C)$ which $w(A, B) =$

$w(A, C)$. There is a unique minimum spanning tree. However, the cut of $\{A\}, \{B, C\}$ does not have a unique light spanning tree.

23.1-7

Argue that if all edge weights of a graph are positive, then any subset of edges that connects all vertices and has minimum total weight must be a tree. Give an example to show that the same conclusion does not follow if we allow some weights to be nonpositive.

Solution

Firstly, we prove that the subset is a graph. Secondly, we prove that the subset does not contain a cycle.

- 1) Since the subset of edges connect all vertices, the subset must be a graph.
- 2) Assume there is a cycle in the subset, since all the weights are positive, if we remove one edge in the cycle, we will get a lesser total weight. However, the subset has minimum total weight, so the assume is contradiction. So, there is no cycle in the subset.

So the subset must be a tree.

Counterexample

$G = (V, E), V = \{A, B, C, D\}, E = \{(A, B), (B, C), (C, A), (A, D), (B, D), (C, D)\}, w(A, B) = w(B, C), w(C, A) = -1, w(A, D) = 1, w(B, D) = 2, w(C, D) = 3$, the minimum total weight is $w(A, B) + w(B, C) + w(C, A) + w(A, D) = -2$ but it has a cycle.

the tree is a minimum spanning tree ?????

23.1-8

Let T be a minimum spanning tree of a graph G , and let L be the sorted list of the edge weights of T . Show that for any other minimum spanning tree T' of G , the list L is also the sorted list of edge weights of T' .

Solution-1

We are going to replace different edges in T' with the same weight edges in T . If finally T' is the same as T , then we are done.

- 1 **while** find a vertex u in T' which only in two different edges in T' and T
- 2 Let (u, x) in T' but not in T
- 2 Let (u, y) in T but not in T'
- 2 There is a cut $(S, V - S)$ which S includes x, y and excludes u . Since both the T and T' are minimum spanning tree, so $w(u, x) = w(u, y)$, so we can replace (u, x) with (u, y) .
- 3 The final T' is the same as T , since each edge replaced in T' has the same weight as in T , so we are done.

Solution-2

Reference

Let list $A = a_1, a_2, \dots, a_{(i-1)}, a_i, a_{(i+1)}, \dots, a_n$ be the sorted weights of T in ascending order.

Let list $B = b_1, b_2, \dots, b_{(i-1)}, b_i, b_{(i+1)}, \dots, b_n$ be the sorted weights of T in ascending order.

Assume there is a difference weight between A and B which is the i th, so $a_i \neq b_i$. We are going to show that the assume is contradiction. We are going to prove when the $a_i > b_i$, and it is also applied to $a_i < b_i$.

(1) If b_i in the list A , then there is a_j in the list A and $a_j = b_i$. Since $a_x = b_x$ when x is from 1 to $(i-1)$, $j \geq i$, so $b_i = a_j \geq a_i$, so $b_i \geq a_i$, so the assume is contradiction.

(2) If b_i does not in the list A , there will a cycle in T when we add the edge of b_i . And b_i is not less than any other weights in the cycle. There is must a edge in the cycle which does not exist in T' . Let the edge be a_x and $b_i \geq a_x$. Since $b_i \geq a_x \geq a_i$, the assume is contradiction.

Therefore, the assume is contradiction, so there is not a difference weight between A and B .

23.1-9

Let T be a minimum spanning tree of a graph $G = (V, E)$, and let V' be a subset of V . Let T' be the subgraph of T , and let G' be the subgraph of G induced by V' . Show that if T' is connected, then T' is a minimum spanning tree of G' .

Solution

Firstly, we will prove that **when T' is connected, if we replace T' with another tree T'_1 that connects all the V' in G' , the T of G will be T_1 , and T_1 is also a tree.** Assume that there is cycle in T_1 after replace T' with T'_1 . Let A, B, A', B' in the cycle, and A, B is in V not in V' , A', B' is in V' , and A' is the first vertex that A connects T' , B' is the first vertex that B connects T' . Since A, B is in the cycle, there must be several edges that connect A and B . Also, there must be several edges connect A' and B' . For all the trees in G' which connects all the V' , A' connects B' , so does the T' , so there is also a cycle in T which is a minimum spanning tree. So the assume is contradiction.

Secondly, assume that there is a lesser weight tree than T' in G' , then replace T' with it, we will get a lesser weight tree in G . Obviously, the assume is contradiction. So there is not a lesser weight tree in G' than T' . So T' is a minimum spanning tree of G' .

TODO

TODO: Prim -> Kruskal

TODO: Kruskal -> Prim

Part VII

Selected Topics

Part VIII

Appendix: Mathematical Background

