$X: (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$ → 6-algebra

$\mathcal{F}$ ↓ 6-field

6-algebra

$\mathcal{B}$ ↓ Borel sets

contains all intervals $[a,b]$

⇓

Can prove

(a,b) $[a,b]$, $[a,b)$ $\{a\} \subset \mathcal{B}$

$\{a\} \Rightarrow \bigcap_{n=1}^{\infty} (a-\frac{1}{n}, a]$

$X^{-1}(B) \xleftarrow{X^{-1}} B$

$\subset \Omega$

$X^{-1}(B) = \{\omega : X(\omega) \in B\}$

$\in \mathcal{F}$ measurable Condition

$P_X(B) := P(X^{-1}(B))$

$P_X(\mathbb{R}) = 1$

$(-\infty, +\infty)$

$P_X(\phi) = 0.$

$$F_X(x) = \Pr\{X \le x\} = P\left(X^{-1}(-\infty, x)\right)$$

$$= P_X\left\{(-\infty, x]\right\}$$

$$= \begin{cases} \sum_{u \le x} p_X(u) & \text{discrete} \\ \int_{-\infty}^{x} f_X(u)\, du & \text{continuous} \end{cases}$$

dominating measure μ

q is μ-density of $Q = P_X$

if $Q(A) = \int_A q\, d\mu = \int q\, I_A\, d\mu$

$$F_X(x) = \int_{(-\infty, x]} p(u)\, d\mu(u)$$

counting measure $\to$ sum

p is μ-density

Lesbegue measure $\to$ L.I

$X_1, X_2, \ldots, X_n, \ldots$ as $n \to \infty$

$(X_n, \mathcal{F}_n, P_n)$

modes of convergence

L^p convergence

0 Complete Convergence

1 Almost Sure Convergence $X_n \xrightarrow{a.s.} X$

$\Downarrow$

$(X_n, \mathcal{F}, P)$

2 Convergence in Probability $X_n \xrightarrow{P} X$

$\Downarrow$

$(X_n, \mathcal{F}_n, P_n)$ 3 Convergence in distribution $X_n \xrightarrow{d} X$

Lecture 2

For $\varepsilon > 0$, $\exists\ n_\varepsilon(\omega)$ s.t.

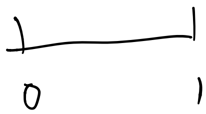
$$P\left\{\omega : |X_n(\omega) - X(\omega)| < \varepsilon,\ \forall n \geq n_\varepsilon(\omega)\right\} = 1$$

$$X_n \xrightarrow{P} X_n$$

For $\varepsilon > 0$,

$$\underset{P_n(\varepsilon)}{\underset{\parallel}{P(|X_n - X| > \varepsilon)}} \longrightarrow 0, \quad \text{as } n \to \infty$$

$$P(|X_n - X| \leq \varepsilon) \to 0, \quad \text{as } n \to \infty$$



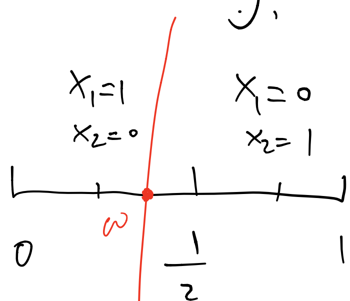
$\Omega = [0,1)$

$\mathcal{F} = \mathcal{B}(\Omega)$ Borel sets

$P = \lambda$: Lebesgue Measure.

$P(A) = \int_A 1\,dx$

eg. $P([0,\tfrac{1}{2}]) = \tfrac{1}{2}$, etc.



$X_3 = 1, 0, 0, 0$

$X_4 = 0, 1, 0, 0$

$\vdots$

$X \equiv 0$

$$P\{|X_n - X| > \varepsilon\}$$

$\varepsilon < \frac{1}{2}$

$$P\{|X_n - X| > \varepsilon\} = P\{X_n = 1\} \longrightarrow 0$$

as $n \to \infty$

$$X_n \xrightarrow{P} 0$$

$$\cancel{X_n \xrightarrow{a.s.} 0}$$

n	$X_n(\omega)$
1	1
2	0
3	0
4	1
	⋮
	1
	0
	0
	1
	0
	0
	⋮
	1
	0
	⋮
	1
	0
	1
	⋮

with Prob 1, $X_n(\omega)$ does not converge.

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$$

$$P\Big\{ \underbrace{\omega : |X_n(\omega) - X(\omega)| \leq \varepsilon, \ \forall\, n \geq n_\varepsilon(\omega)}_{A} \Big\} = 1$$

For $k > 0$, $\{\omega : |X_n$