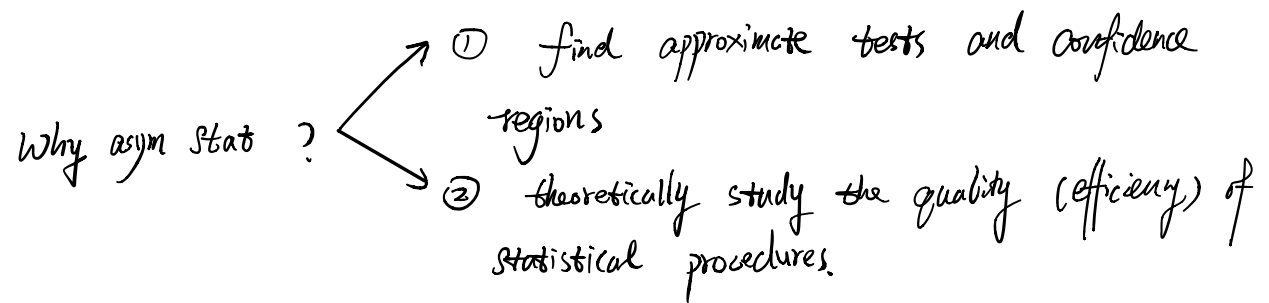


Chapter 1

Introduction



Chapter 2 Stochastic Convergence

2.1 Basic Theory

1. A **random vector** in \mathbb{R}^k is a vector $X = (X_1, \dots, X_k)$ of real random variables.
2. The **distribution function** of X is the map

$$x \longrightarrow P(X \leq x).$$

(more formally it is a Borel measurable map from some probability space in \mathbb{R}^k)

3. A **sequence of random vectors** X_n is said to **converge in distribution** to a random vector X if

$$P(X_n \leq x) \longrightarrow P(X \leq x)$$

for every x at which the limit distribution function

$$\lim_{n \rightarrow \infty} P(X_n \leq x) = P(X \leq x)$$

$x \longrightarrow P(X \leq x)$ is continuous

(Alternative names are **weak convergence**, **converge in law**)

(As the last name suggests, the convergence only depends on the induced laws of vectors and not on the probability spaces on which they are defined)

Notation: $X_n \rightsquigarrow X$; if $X \sim L$ or $X \sim N(0,1)$,
 $X_n \rightsquigarrow L$; $X_n \rightsquigarrow N(0,1)$

4. Let $d(x, y)$ be a distance function on \mathbb{R}^k that generates the usual topology. For instance, the Euclidean distance

$$d(x, y) = \|x - y\| = \left(\sum_{i=1}^k (x_i - y_i)^2 \right)^{\frac{1}{2}}$$

5. A sequence of random variables X_n is said to **converge in probability** to X if for all $\varepsilon > 0$,

$$P(d(X_n, X) > \varepsilon) \rightarrow 0, \quad n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} P(\|X_n - X\| > \varepsilon) = 0$$

Denoted by

$$X_n \xrightarrow{P} X$$

$$d(X_n, X) \xrightarrow{P} 0$$

6. A sequence X_n is said to **converge almost surely** to X if $d(X_n, X) \rightarrow 0$ with probability one with $n \rightarrow \infty$:

$$P(\lim_{n \rightarrow \infty} d(X_n, X) = 0) = 1$$

$$P(\lim_{n \rightarrow \infty} \|X_n - X\| = 0) = 1$$

$$P(\lim_{n \rightarrow \infty} X_n = X) = 1$$

Denoted by $X_n \xrightarrow{a.s.} X$