$X: (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathcal{B})$ σ-algebra

$\mathcal{F}$: σ-field

σ-algebra

$\mathcal{B}$: Borel sets

contains all intervals $[a,b]$

$\Downarrow$

Can prove

(a,b) $[a,b]$, $[a,b)$ $\{a\} \subset \mathcal{B}$

$$\{a\} \Rightarrow \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a\right]$$

$$X^{-1}(B) \xleftarrow{X^{-1}} B$$

$\subset \Omega$

$$X^{-1}(B) = \{\omega : X(\omega) \in B\}$$

$\in \mathcal{F}$ measurable condition

$$P_X(B) := P(X^{-1}(B))$$

$$P_X(\mathbb{R}) = 1$$

$(-\infty, +\infty)$

$$P_X(\phi) = 0.$$

$$F_X(x) = Pr\{X \le x\} = P(X^{-1}(-\infty, x])$$

$$= P_X\{(-\infty, x]\}$$

$$= \begin{cases} \sum_{u \le x} P_X(u) & \text{discrete} \\ \int_{-\infty}^{x} f_X(u)\, du & \text{continuous} \end{cases}$$

dominating measure μ

q is μ-density of $Q = P_X$

if $Q(A) = \int_A q \, d\mu = \int q \, I_A \, d\mu$

$$F_X(x) = \int_{(-\infty, x]} p(u) \, d\mu(u)$$

counting measure $\rightarrow$ Sum

p is μ-density

Lesbegue measure $\rightarrow$ L.I

$X_1, X_2, \dots, X_n, \dots$ as $n \to \infty$

$(X_n, \mathcal{F}_n, P_n)$

modes of convergence

L^p convergence

⓪ Complete Convergence

① Almost Sure Convergence $X_n \xrightarrow{a.s.} X$

$\Downarrow$

$(X_n, \mathcal{F}, P)$

② Convergence in Probability $X_n \xrightarrow{P} X$

$\Downarrow$

$(X_n, \mathcal{F}_n, P_n)$ ③ Convergence in distribution $X_n \xrightarrow{d} X$

Lecture 2

For $\varepsilon > 0$, $\exists\, n_\varepsilon(\omega)$ s.t.

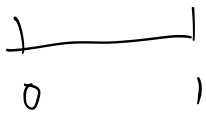
$$P\{\omega : |X_n(\omega) - X(\omega)| < \varepsilon, \ \forall n \geq n_\varepsilon(\omega)\} = 1$$

$$X_n \xrightarrow{P} X_n$$

For $\varepsilon > 0$,

$$\underset{\substack{\parallel \\ P_n(\varepsilon)}}{P(|X_n - X| > \varepsilon)} \longrightarrow 0 \quad , \text{ as } n \to \infty$$

$$P(|X_n - X| \leq \varepsilon) \to 0, \quad \text{as } n \to \infty$$



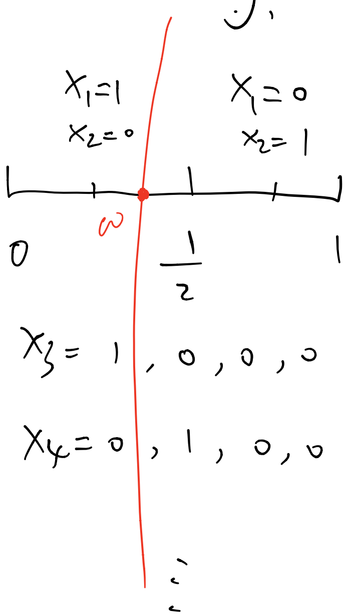
$$\Omega = [0,1)$$

$$\mathcal{F} = \mathcal{B}(\Omega) \quad \text{Borel sets}$$

$$P = \lambda : \text{Lebesgue Measure.}$$

$$P(A) = \int_A 1\, dx$$

e.g. $P([0,\tfrac{1}{2}]) = \tfrac{1}{2}$, etc.



$$X \equiv 0$$

$$P\{|X_n - X| > \varepsilon\}$$

$$\varepsilon < \tfrac{1}{2}$$

$$P\{|X_n - X| > \varepsilon\}$$

$$= P\{X_n = 1\} \longrightarrow 0$$

as $n \to \infty$

$$X_n \xrightarrow{P} 0$$

$$\cancel{X_n \xrightarrow{a.s.} 0}$$

n	$X_n(\omega)$
1	1
2	0
3	0
4	1
	$\vdots$
	1
	0
	0
	1
	0
	0
	$\vdots$
	1
	0
	$\vdots$
	1
	0
	1
	$\vdots$

with Prob 1, $X_n(\omega)$ does not converge.

$$X_n \xrightarrow{a.s.} X \implies X_n \xrightarrow{P} X$$

$$P\underbrace{\left\{ \omega : |X_n(\omega) - X(\omega)| \leq \varepsilon,\ \forall\, n \geq n_\varepsilon(\omega) \right\}}_{A} = 1$$

For $k > 0$, $\{ \omega : | X_n$

$P(|X_n - X| > \varepsilon \ , \ i.o.) = 0$

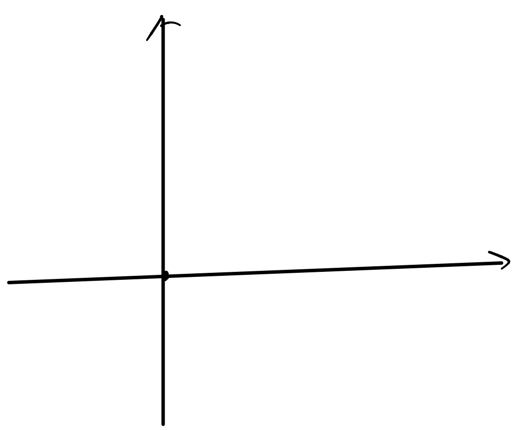
$P(\lim_{n\to\infty} X_n =$

$P(\frac{1}{n} \leq 0) = 1$

X_n

$P(X_n \leq x)$ | $P(X_n \leq 0) = 0$, $n \to 0$

$P(X \leq x)$ | $P(X \leq 0) =$



X_n

Lecture Jan 27

L_p Convergence

X on $(\Omega, \mathcal{F}, P)$

$\infty > p \geq 1$

$\|X\|_p = \left(E|X|^p \right)^{1/p}$

$p = 1$ abs

$p = 2$ sd when x centerized.

$$\begin{cases} \|X+Y\|_p \leq \|X\|_p + \|Y\|_p \\ \|cX\|_p = |c| \, \|X\|_p, \quad c \in \mathbb{R} \\ \|X\|_p = 0, \quad X = 0, \text{ a.e.}, \quad P(X \neq 0) = 0 \end{cases}$$

X_n, X should be on $(\Omega, \mathcal{F}, P)$

$X_p \xrightarrow{L_p} X$

iff $\|X_n - X\|_p \to 0, \quad n \to \infty$

(for one specific p)

Consequence: $E(|X_n|^p),\ E(|X|^p) < \infty$

$$X_n \xrightarrow{L_p} X \implies E|X_n|^p \to E|X|^p$$

$$E(X_n^p) \to E(X^p)$$

whenever the moments are defined.

$$X^p = \exp\{p \log x\}$$

$$\big| E|X_n| - E|X| \big|$$

or

$$| E(X_n) - E(X) | \leq E|X_n - X|$$

$1 \leq r \leq s$

$$\|X\|_r \leq \|X\|_s$$

Thus L_2 convergence $\implies$ L_1 convergence

$$X_n \xrightarrow{L_p} X \implies E|X_n|^s \to E|X|^s$$

for all $1 \leq s \leq p$

$$0 \le \Delta_n = |X_n - X|$$

$$\{E\Delta_n^r\}^{\frac{1}{r}} \le \{E\Delta^s\}^{\frac{1}{s}}$$

$$\Longleftrightarrow \quad E\Delta_n^r \le \{E\Delta^s\}^{\frac{r}{s}}$$

$$\Delta_n^r = y$$

$$\Delta_n^s = y^{s/r} \qquad s/r = t \geq 1$$

$$Ey \le \{E\,y^t\}^{\frac{1}{t}}$$

$$(Ey)^t \le E(y^t) \qquad \text{always true.}$$

$$f(EX) \le Ef(X) \quad \text{iff} \quad \text{is convex}$$

L_p Convergence $\Rightarrow$ Convergen in probability

$$\varepsilon > 0, \quad P_r\{|X_n - X| > \varepsilon\} \le \frac{E|X_n - X|^p}{\varepsilon^p}$$

$$P_r\{|Y| > t\} \le \frac{E|Y|}{t} \qquad p = 1$$

$$E|Y| = E[|Y| I(|Y| \le t)] + E[|Y| I(|Y| > t)]$$

$$\ge \qquad \times \qquad + \; t \, P(|Y| > t)$$

$$P(|Y| > t) \le \frac{E|Y|}{t}$$