Chapter 1

1. ① 可行解的集合　② 最优的准则　③ 寻找的解法.

Mathematical Optimization :

$$\text{minimize} \quad f_0(x)$$
$$\text{subject to} \quad f_i(x) \leq b_i, \quad i = 1, \dots, m$$

Optimization Variable : the vector $x = (x_1, \dots, x_n)$

Objective function 最优准则 : $f_0 : \mathbb{R}^n \to \mathbb{R}$

Inequality Constraints 不等式约束 : $f_i : \mathbb{R}^n \to \mathbb{R}$

可行解集　feasible set

Optimal : $x^* \iff \forall z \in \{ f_i(z) \leq b_i, \ i = 1, \dots, m \}$
$f_0(z) \geq f_0(x^*)$

2. Linear Program : if the objective function f_i and constraint functions are all linear, i.e., $f_i(\alpha x + \beta y) = \alpha f_i(x) + \beta f_i(y)$
for all $x, y \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$

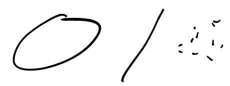
Non-linear Program : if one of the above conditions is not linear

Convex Program : If the objective and constraint functions are convex, which means $f_i(\alpha x+\beta y) \le \alpha f_i(x) + \beta f_i(y)$ for all $x, y \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$ with $\alpha+\beta=1$, $\alpha \ge 0$, $\beta \ge 0$

The convex program is more general than linear program

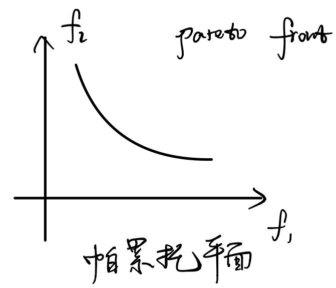
光滑/非光滑 : 但不是个本质上的差别，而 convex/non-convex 则是个本质上的差别。

↳ 每个点都是可微的.

连续/离散 : 针对 可行域 

↳ 一般为 non-convex

单目标/多目标 : multiple objective functions



Convex Optimization 研究求解较难的问题，并非研究例如求 $f_0(x)$ 有难度的问题，例如 $f_0(x)$ 代表一个人的头发有多少.

3. History

17世纪 Newton Raphson

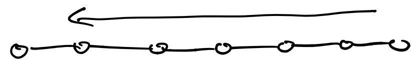
$$f(x)=0 \Rightarrow \min f^2(x)$$

19世纪 Gauss Siedel Jacobi

$$\begin{cases} f_1(x)=0 \\ \vdots \\ f_N(x)=0 \end{cases} \Rightarrow \min \sum_{i=1}^{N} f_i^2(x)$$

18世纪 Lagrange

1940 Bellman Dynamic Programming



从后往前去优化.

1944 Von Neumann Game Theory

↘ 1950 Nash

1939 Kantorovich Linear Programming

↘ 1947 Dantzig 单纯形法.

1979 Khachiyan Poly.

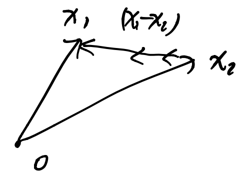
1984 Karmarker 内点法

1990s Non linear

Chapter 2 Convex Sets

1. Affine Sets 仿射集 (凸集的特例)

$x_1 \neq x_2$, $x_1, x_2 \in \mathbb{R}^n$, $\theta \in \mathbb{R}$



直线 line: $y = \theta x_1 + (1-\theta) x_2$

$\quad = x_2 + \theta(x_1 - x_2)$

线段 line segment: Add $\theta \in [0,1]$

仿射集 Affine Set: 若 $\forall x_1, x_2 \in C$, 则连接 x_1 与 x_2 的直线也在集合 C 内.

If for any $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, we have $\theta x_1 + (1-\theta) x_2 \in C$

(e.g., line is affine set, line segment is not an affine set.)

More generalized: If C is an affine set,

$x_1, \dots, x_k \in C$, $\theta_1 + \dots + \theta_k = 1$, $\in \mathbb{R}$.

仿射组合: $\theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k$, then $\theta_1 x_1 + \dots + \theta_k x_k \in C$

proof: $k=3$, $x_1, x_2, x_3 \in C$, $\theta_1 + \theta_2 + \theta_3 = 1$, $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$

$$\frac{\theta_1}{\theta_1+\theta_2} x_1 + \frac{\theta_2}{\theta_1+\theta_2} x_2 \in C$$

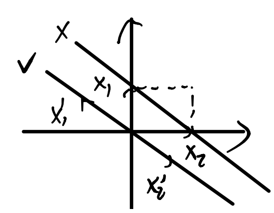
$$(\theta_1+\theta_2)\left[\frac{\theta_1}{\theta_1+\theta_2} x_1 + \frac{\theta_2}{\theta_1+\theta_2} x_2\right] + (1-\theta_1-\theta_2) x_3 \in C$$

$$\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 \in C$$

∎

对于一般的 affine set, $\forall x_1, x_2 \in C$, $\alpha x_1 + \beta x_2 \in C$ 仅当 $\alpha+\beta=1$ 时成立, 对于哪些 affine set, $\alpha, \beta \in \mathbb{R}$ 时, 该条件也成立呢?

(引出子空间)



If C is an affine set and $x_0 \in C$, then the set

$$V = C - x_0 = \{ x - x_0 \mid x \in C \}$$ is a subspace

Proof: in the book.

相当于把任意的 affine set 拓展成为一个性质更好的 set.

$\begin{cases} 0 \in V \\ \text{对加法 数乘 封闭} \end{cases}$

线性方程组的解是一个 Affine set.

Affine Hull 仿射包：任意集合 C，构造尽可能小的仿射集.

$$\text{aff } C = \{ \theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \ldots, x_k \in C,\ \theta_1 + \cdots + \theta_k = 1 \}$$

2. Convex Set 凸集 （任意仿射集也是凸集）

凸集：当任意两点之间的线段仍在 C 内.