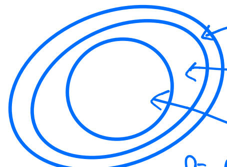
1. If A_n is a sequence of subsets of Ω, we let

只要在 infinitely many A_n 中的任一就可.

$$\limsup A_n = \lim_{m\to\infty} \bigcup_{n=m}^{\infty} A_n = \{\omega \text{ that are in infinitely many } A_n\}$$

$$= \bigcap_{m\geq 1} \bigcup_{n=m}^{\infty} A_n$$



$\bigcup_{n=1}^{\infty} A_n$ ω in $\forall$ all

$\bigcup_{n=2}^{\infty} A_n$ ω in $\forall$ all / A_1

$\bigcup_{n=3}^{\infty} A_n$ ω in $\forall$ all / $A_1 + A_2$

...

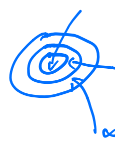
decreasing

final set

$$\liminf A_n = \lim_{m\to\infty} \bigcap_{n=m}^{\infty} A_n = \{\omega \text{ that are in all but finitely many } A_n\}$$

$$= \bigcup_{m\geq 1} \bigcap_{n=m}^{\infty} A_n$$

要在除去 finitely many A_n 后的每个 A_n 中.



$\bigcap_{n=1}^{\infty} A_n$ all

$\bigcap_{n=2}^{\infty} A_n$ all / A_1

$\bigcap_{n=3}^{\infty} A_n$ all / $A_1 + A_2$

...

increasing

final set $\Rightarrow$ all / $\underline{A_1 + A_2 + \cdots A_n}$ 有限多个 A_n

2. Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n, \text{ i.o.}) = 0$

$P(\limsup_{n\to\infty} A_n) = 0$

That is let $A_1, A_2, \ldots,$ be a sequence of events in some probability space, if the sum of the probabilities of the event A_n is finite, then the probability that infinitely many of them occur is 0.

The theorem states that,

if the sum of the probabilities of the event A_n is finite, then the set of all outcomes that are repeated infinitely many times must occur with probability zero.

Example: Suppose X_n is a sequence of random variables with

$$P(X_n = 0) = \frac{1}{n^2}$$

$$A_1 = \{\omega : X_1 = 0\} \qquad P(A_1) = 1$$

$$A_2 = \{\omega : X_2 = 0\} \qquad P(A_2) = \frac{1}{4}$$

$$\cdots \qquad\qquad \cdots$$

$$A_n = \{\omega : X_n = 0\} \qquad P(A_n) = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} P(X_n = 0) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{\infty^2} = \frac{\pi^2}{6} < \infty$,

the probability of $X_n = 0$ occurring for infinitely many n is 0. Almost surely (i.e. with probability 1), X_n is nonzero for all but finitely n.