1. If A_n is a sequence of subsets of Ω, we let

$$\limsup A_n = \lim_{m\to\infty} \bigcup_{n=m}^{\infty} A_n = \{\omega \text{ that are in infinitely many } A_n\}$$

$$= \bigcap_{n\geq 1} \bigcup_{n=\infty}^{\infty} A_n$$

$$\liminf A_n = \lim_{m\to\infty} \bigcap_{n=m}^{\infty} A_n = \{\omega \text{ that are in all but finitely many } A_n\}$$

$$= \bigcup_{n\geq 1} \bigcap_{n=m}^{\infty} A_n$$

2. Borel-Cantelli Lemma

If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n, \text{ i.o.}) = 0$

$P(\limsup_{n\to\infty} A_n) = 0$

That is let $A_1, A_2, \dots$, be a sequence of events in some probability space, if the sum of the probabilities of the event A_n is finite, then the probability that infinitely many of them occur is 0.

The theorem states that,

if the sum of the probabilities of the event A_n is finite, then the set of all outcomes that are repeated infinitely many times must occur with probability zero.

Example: Suppose X_n is a sequence of random variables with

$$P(X_n=0) = \frac{1}{n^2}$$

$$A_1 = \{\omega : X_1 = 0\} \qquad P(A_1) = 1$$

$$A_2 = \{\omega : X_2 = 0\} \qquad P(A_2) = \frac{1}{4}$$

$$\cdots \qquad\qquad \cdots$$

$$A_n = \{\omega : X_n = 0\} \qquad P(A_n) = \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} P(X_n=0) = 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{\infty^2} = \frac{\pi^2}{6} < \infty$,

the probability of $X_n=0$ occurring for infinitely many n is 0. Almost surely (i.e. with probability 1), X_n is nonzero for all but finitely many n.