Chapter 1 Introduction

Why asym stat ?

① find approximate tests and confidence regions

② theoretically study the quality (efficiency) of statistical procedures.

Chapter 2 Stochastic Convergence

2.1 Basic Theory

1. A random vector in $\mathbb{R}^k$ is a vector $X = (X_1, \dots, X_k)$ of real random variables.
2. The distribution function of X is the map

$$x \longrightarrow P(X \le x).$$

(more formally it is a Borel measuable map from some probability space in $\mathbb{R}^k$)

3. A sequence of random vectors X_n is said to converge in distribution to a random vector X if

$$P(X_n \le x) \longrightarrow P(X \le x)$$

for every x at which the limit distribution function

$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$

$$x \longrightarrow P(X \le x) \text{ is continuous}$$

(Alternative names are weak convergence, converge in law)

(As the last name suggests, the convergence only depends on the induced laws of vectors and not on the probability spaces on which they are defined)

Notation: $X_n \rightsquigarrow X$; if $X \sim L$ or $X \sim N(0,1)$, $X_n \rightsquigarrow L$; $X_n \rightsquigarrow N(0,1)$

4. Let $d(x,y)$ be a distance function on $\mathbb{R}^k$ that generates the usual topology. For instance, the Euclidean distance

$$d(x,y) = \|x-y\| = \left(\sum_{i=1}^{k} (x_i-y_i)^2\right)^{\frac{1}{2}}$$

5. A sequence of random variables X_n is said to converge in probability to X if for all $\varepsilon>0$,

$$P(d(X_n,X)>\varepsilon) \longrightarrow 0, \quad n\to\infty$$

$$\lim_{n\to\infty} P(\|X_n-X\|>\varepsilon) = 0$$

Denoted by

$$X_n \xrightarrow{P} X$$

$$d(X_n,X) \xrightarrow{P} 0$$

6. A sequence X_n is said to converge almost surely to X if $d(X_n,X)\to 0$ with probability one with $n\to\infty$:

$$P(\lim_{n\to\infty} d(X_n,X)=0)=1$$

$$P(\lim_{n\to\infty} \|X_n-X\|=0)=1$$

$$P(\lim_{n\to\infty} X_n = X) = 1$$

Denoted by $\quad X_n \xrightarrow{a.s.} X$