Iteration method for evolutionary problems with non-local condition in time

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Useful references

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Abstract model (non-local problem)

Consider the following abstract differential equation in Banach space E on segment $[0,T] \subset \mathbb{R}$

$$u'(t) = Au(t), \quad 0 \leqslant t \leqslant T,$$

with additional condition

$$\int\limits_0^T u(t)\eta(t)dt = u_1.$$

Element $u_1 \in E$ is known.

It is required to recover the solution $u(t)$

and its initial state $u_0 = u(0)$.

Operator equation

Suppose that

1) A is a closed linear operator in E,
2) $A \to U(t) \in (C_0)$, $D(A)$ is dense in E, $u_1 \in D(A)$,
3) $\eta \in C^1[0, T]$, $\eta(0) > 0$.

Consider $\beta = \eta(0)$ and $f = -Au_1$.

The original problem is reduced to an operator equation

$$\beta u_0 - Bu_0 = f$$

with a bounded linear operator

$$B = \eta(T)U(T) - \int_0^T \eta'(t)U(t)\,dt.$$

Equivalence condition: $\exists\, A^{-1}$ (usually satisfied).

Neumann series

Consider the equation

$$\beta u_0 - B u_0 = f,$$

where $\beta = \eta(0)$, $f = -Au_1$.

Теорема («Differential equations», 1998)

Let $\|U(t)\| \leqslant e^{-\alpha t}$ *with* $t \geqslant 0$ *and constant* $\alpha > 0$.
Let $\eta(t) > 0$ *and* $\eta'(t) \leqslant 0$ *on* $[0, T]$.
Then $\|B\| < \beta$ *and solution of the operator equation can be represented by converging Neumann series*

$$u_0 = \sum_{k=0}^{\infty} \frac{1}{\beta^{k+1}} B^k f.$$

Possible generalizations

1. Instead of $\eta \in C^1[0,T]$, we can take $\eta \in BV[0,T]$ assuming that $\eta(0) = \eta(0+0)$ and $\eta(T) = \eta(T-0)$. Then

$$B = \eta(T)U(T) - \int_0^T U(t)\, d\eta(t)$$

with the vector Riemann–Stieltjes integral.

2. Instead of $\|B\|$, we evaluate the spectral radius $r(B)$. Condition $r(B) < \beta \equiv \eta(0)$ also implies convergence of the series

$$u_0 = \sum_{k=0}^{\infty} \frac{1}{\beta^{k+1}} B^k f.$$

Convergence condition – 1 (CC – 1)

Теорема (Carson transform)

Let $U(t)$ *be an analytical semigroup with* $t > 0$, *and* $\sigma(A) \subset (-\infty, -\alpha]$ *with some* $\alpha > 0$. *Then if*

$$0 < H_T(p) \equiv p \int\limits_0^T e^{-pt}\eta(t)\,dt < 2\eta(0), \qquad \forall p \geqslant \alpha,$$

then the Neumann series

$$u_0 = \sum_{k=0}^{\infty} \frac{1}{\beta^{k+1}} B^k f$$

converges under the norm of E.

The spectral theory of semigroups is used in the proof.
Э. Хилле, Р. Филлипс. Функциональный анализ и полугруппы. М.: ИЛ, 1962. Глава XVI.

Convergence condition – 2 (CC – 2)

Теорема (Lipschitz type restriction)

Let $U(t)$ *be an analytical semigroup with* $t > 0$,
and $\sigma(A) \subset (-\infty, -\alpha]$ *with some* $\alpha > 0$.
Let $|\eta(t) - \eta(0)| \leqslant Lt$ *with* $0 \leqslant t \leqslant T$ *and constant* $L \geqslant 0$.
Then if

$$L < \alpha\eta(0),$$

then CC–1, is satisfied, i.e.

$$0 < H_T(p) < 2\eta(0), \quad \forall p \geqslant \alpha,$$

and the Neumann series

$$u_0 = \sum_{k=0}^{\infty} \frac{1}{\beta^{k+1}} B^k f$$

converges under the norm of E.

Iteration method (program realization)

$$u_0 = \sum_{k=0}^{\infty} \frac{1}{\beta^{k+1}} B^k f.$$

Let $u_0^{(k)}$ be approximation of u_0 on the k-th iteration:

$$u_0^{(0)} = \frac{1}{\beta} f,$$

$$u_0^{(1)} = \frac{1}{\beta} \left(f + B u_0^{(0)} \right),$$

$$\dots$$

$$u_0^{(k)} = \frac{1}{\beta} \left(f + B u_0^{(k-1)} \right).$$

The operator B is constructively calculated.

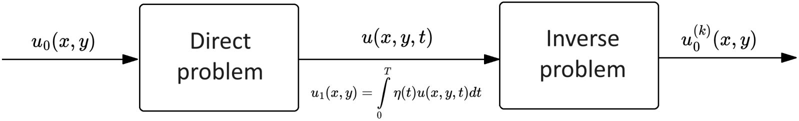
Specific model

In the rectangle $D = [0, l_1] \times [0, l_2]$
consider the two-dimensional heat conduction problem:

$$\begin{cases} u_t = a^2(u_{xx} + u_{yy}) - c(x,y)u, \quad (x,y) \in D, \quad t \in [0,T], \\ u|_{\partial D} = 0, \\ \int\limits_0^T u(x,y,t)\eta(t)\,dt = u_1(x,y). \end{cases}$$

$$u_0(x,y) \equiv u(x,y,0) =?$$

Method verification scheme



1. Setting an arbitrary initial condition $u_0(x,y)$.
2. Solving the direct problem, i.e. calculating $u(x,y,t)$.
3. Finding $u_1(x,y) = \int_0^T \eta(t)u(x,y,t)\,dt$ and solving the inverse problem by iteration method.
4. Comparing the resulting values $u_0^{(k)}$ with the original u_0.

Error estimation

$$d^{(k)} = \max_{(x,y)\in D} \left| u_0^{(k)}(x,y) - u_0(x,y) \right|,$$

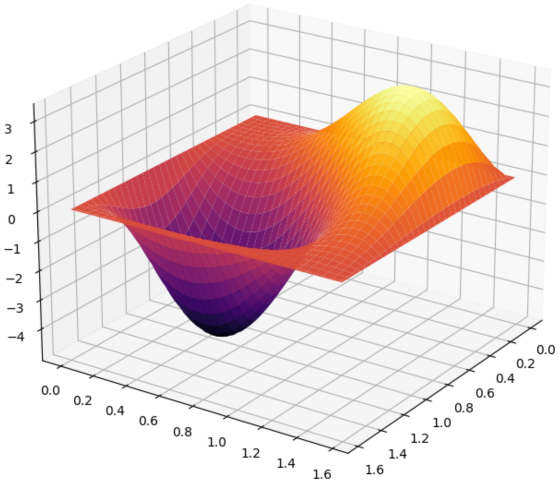
k — number of iterations.

Two sufficient conditions of convergence:

- CC−1 (Carson transform),
- CC−2 (Lipschitz type condition).

Test initial condition

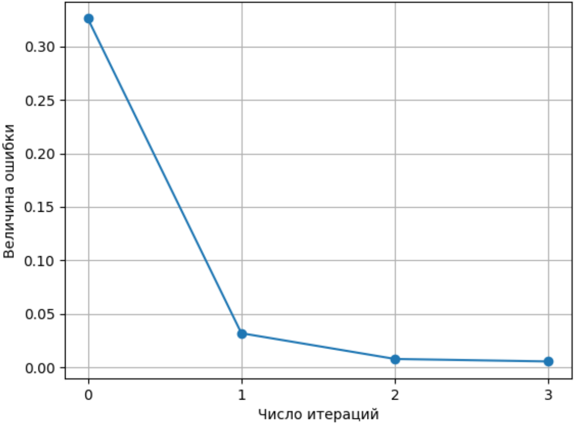
$$u_0(x,y) = 10xy^3\,(l_1 - x)^2\,(l_2 - y) - 50x^5 y\,(l_1 - x)^3\,(l_2 - y)^2,$$
$$0 \leqslant x \leqslant l_1, \quad 0 \leqslant y \leqslant l_2, \quad l_1 = l_2 = \frac{\pi}{2}\,.$$



Example 1 $l_1 = l_2 = \frac{\pi}{2},\ \ T = \frac{3\pi}{2},\ \ a^2 = \frac{1}{8},\ \ c(x,y) = x + y,\ \ \eta(t) = \cos t$

$\alpha \geqslant \alpha_0 = 1, \quad L \approx 0.725 < \alpha\eta(0) = \alpha$ — correct. $\Longrightarrow$ CC − 2 is satisfied.

The method converges, a small number of iterations is enough.



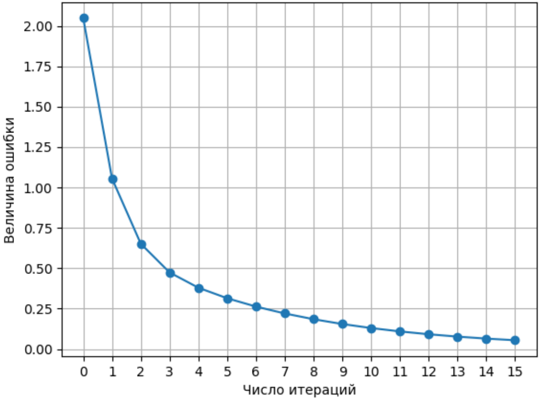
Example 2 $\quad l_1 = l_2 = \frac{\pi}{2}, \quad T = \frac{3\pi}{2}, \quad a^2 = \frac{1}{16}, \quad c(x,y) \equiv 0, \quad \eta(t) = \cos t$

$\alpha = 0.5, \quad L \approx 0.725 < \alpha\eta(0) = 0.5$ — incorrect. $\Longrightarrow$ CC – 2 is not satisfied.

But the more general CC – 1 is satisfied:

$$0 < H_T(p) \equiv p \int\limits_0^T e^{-pt}\,\eta(t)\,dt < 2\eta(0), \quad \forall p \geqslant \alpha.$$

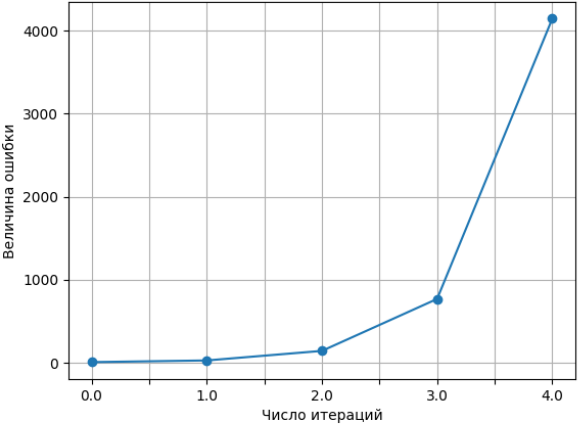
The method converges, but slowly — more iterations are required.



Example 3 $\quad l_1 = l_2 = \frac{\pi}{2}, \quad T = \frac{3\pi}{2}, \quad a^2 = \frac{1}{16}, \quad c(x,y) \equiv 0, \quad \eta(t) = e^t \cos t$

$\alpha = 0.5, \quad L \approx 9.574 < \alpha\eta(0) = 0.5$ — incorrect. $\implies$ CC−2 is not satisfied.

The sufficient CC−1 is also not satisfied. The method diverges.



Thank you for your attention!