COL726 Homework 2

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1. The full SVD gives a decomposition of the form $A = U\Sigma V^T$ where both U and V are orthogonal and Σ is diagonal.
   Also, the direction of minimum or maximum distance to the surface of a hyperellipse from the origin is along the principal semiaxes.

   (a) Using $A = U\Sigma V^T$ and the fact that SVD exists for every matrix, we can see that the unit sphere in $\mathbb{R}^n$ under a linear transformation by any $m \times n$ matrix A is a hyperellipse. V^T preserves the unit sphere since it is orthogonal. The diagonal matrix Σ stretches the n orthogonal axes by weights $\sigma_1, \cdots, \sigma_n$ to form a hyperellipse. The final matrix U rotates or reflects this hyperellipse since U is orthogonal.
   Also,
   $$\min_{x \subset S_k} \frac{||Ax||_2}{||x||_2} \equiv \min_{x \subset S_k, ||x||_2 = 1} ||Ax||_2$$
   $||x||_2 = 1$ is a unit sphere and Ax is a hyperellipse. We want to find the vector x which has the minimum $||Ax||_2$ given $||x||_2 = 1$. Clearly it is along one of the semiaxes of the hyperellipse. But since $x \subset S_k$, Ax will lie on the hyperellipse whose basis is spanned by $u_1, u_2, \cdots, u_k$ and the minimum distance is the shortest semiaxes lying in the range of $u_1, u_2, \cdots, u_k$ which is
   $$min(||\sigma_1 u_1||_2, ||\sigma_2 u_2||_2, \cdots, ||\sigma_k u_k||_2) = min(\sigma_1, \sigma_2, \cdots, \sigma_k) = \sigma_k$$

   (b) Any k-dimensional subspace $S \subset \mathbb{R}^n$ will have k linearly independent vectors as its basis. These k vectors can be formed by choosing atleast k vectors from $u_1, u_2, \cdots, u_n$ since these form a basis for $\mathbb{R}^n$ as they are orthogonal. Now, atleast 1 of these vectors will be chosen from $u_k, u_{k+1}, \cdots, u_n$ because the rest are only $k - 1$ vectors. Using the result from part (a), we will have
   $$\sigma_n \leq \min_{x \subset S, ||x||_2 = 1} ||Ax||_2 \leq \sigma_k$$
   So, no matter what subspace S you choose, there will exist atleast one vector x with $||x||_2 = 1$ in that subspace such that
   $$||Ax||_2 \leq \sigma_k$$
   where the equality is satisfied when the subspace S has basis $u_1, u_2, \cdots, u_k$.

2. Let us denote $\{a_1, a_2, \cdots\}$ by A and $\{b_1, b_2, \cdots\}$ by B. Also, since the subspaces S and T are complementary, any vector $v \subset \mathbb{R}^m$ can be written as sum of 2 vectors - one lying in S and another in T.

   (a) First, we'll prove that $A \cup B$ is a linearly independent set using contradiction. Suppose $A \cup B$ is not linearly independent. Then there exists a vector $k = a_i$ or b_i lying completely in S or T such that

   $$k = \sum_{i, a_i \neq k} \alpha_i a_i + \sum_{j, b_j \neq k} \beta_j b_j$$

   where atleast one of both α_i and β_j are non zero otherwise k will lie entirely in $range(A)$ or $range(B)$ and this is not possible since both A and B are linearly independent sets on account of being basis. Now, $\sum_i \alpha_i a_i$ is a vector lying in S and $\sum_j \beta_j b_j$ is a vector lying in T. Therefore, their sum lies in $\mathbb{R}^m$ with components both in S and T which is a contradiction to the fact that k lies completely in S or T. Hence, $A \cup B$ is linearly independent.

   Now, we'll prove that $A \cup B$ forms a basis for $\mathbb{R}^m$. Any vector in S can be written as a linear combination of a_i's and any vector in T can be written as a linear combination of b_j's. Now, since any vector v in $\mathbb{R}^m$ can be written as sum of vectors in S and T, we observe that v can be written as a linear combination of a_i's and b_j's. Therefore $A \cup B$ forms a basis for $\mathbb{R}^m$.

   (b) We want to find a projector P such that P applied to any vector v gives a projection lying in S. Also, any vector that lies in T will have null projection in S since S and T are complementary to each other. Therefore, we have:

   $$Pa_i = a_i \quad \forall i = 1, 2, \cdots \tag{1}$$

   $$Pb_i = \vec{0} \quad \forall i = 1, 2, \cdots \tag{2}$$

   Using (1) and (2), we can write

   $$PA = A \tag{3}$$

   $$PB = 0 \tag{4}$$

   where A and B are the matrices formed by using $a_1, a_2, \cdots$ as columns of A and similarly for B. Using (3) and (4), we can write

   $$P[A|B] = A \tag{5}$$

   where $[A|B]$ is the matrix formed by stacking matrices A and B side by side. So the columns of $[A|B]$ are $A \cup B$ which we proved in last part that they are linearly independent and form a basis for $\mathbb{R}^m$ and since these columns also lie in m dimensional space, $[A|B]$ is a square matrix which has full rank and thus, it is invertible. Using (5), we can write

   $$P = A([A|B])^{-1}$$

3. We have the C-inner product between any 2 vectors u and v as

$$f(u,v) = u^T C v$$

   (a) We want to find the C-orthogonal projector matrix P that gives a projection in the direction of x. Therefore, for any vector v, we have

$$Pv = \lambda x \tag{6}$$

$$f(v, v - Pv) = f(x, v - Pv) = 0 \tag{7}$$

   Using (7), we have

$$x^T C(v - \lambda x) = 0 \Rightarrow \lambda = \frac{x^T C v}{x^T C x}$$

   Using (6), we have

$$Pv = \lambda x = (\frac{x^T C v}{x^T C x})x = (\frac{x x^T C}{x^T C x})v$$

   Therefore,

$$P = \frac{x x^T C}{x^T C x}$$

   (b) We want to compute a factorization of A similar to reduced QR factorization i.e. $A = \hat{X}\hat{R}$ where the columns of $\hat{X}$ are C-orthonormal. Therefore all the inner products and norms in Gram Schmidt algorithm will be replaced by C-inner products and C-norms. Given below is the pseudocode for classical Gram Schmidt algorithm.

      i. for $j = 1$ to n
      ii. $v_j = a_j$
      iii. for $i = 1$ to $j - 1$
      iv. $r_{ij} = q_i^T C a_j$
      v. $v_j = v_j - r_{ij} q_i$
      vi. $r_{jj} = ||v_j||_2 = \sqrt{v_j^T C v_j}$
      vii. $q_j = v_j / r_{jj}$

4. We have

$$F = I - 2\frac{vv^T}{v^T v},\ v = x - x' \text{ and } ||x||_2 = ||x'||_2$$

We can write $P = \frac{vv^T}{v^T v}$ where P is the orthogonal projector in the direction of v. Therefore, $P = P^T$ and $P^2 = P$. So,

$$F^T F = (I - 2P)^T (I - 2P) = (I - 2P^T)(I - 2P) = (I - 2P)(I - 2P) = I - 4P + 4P^2 = I - 4P + 4P = I$$

Now, we have

$$Fx = (I - 2\frac{vv^T}{v^T v})x = (I - 2\frac{vv^T}{v^T v})x - x' + x' = x - 2(\frac{vv^T}{v^T v})x - x' + x' = v - 2(\frac{vv^T}{v^T v})x + x' \tag{8}$$

Now,

$$v - 2(\frac{vv^T}{v^T v})x = v - \frac{2}{v^T v}(vv^T)x = v - \frac{2}{v^T v}v(v^T x) \tag{9}$$

Also $||x||_2 = ||x'||_2$ implies $x^T x = x'^T x'$, therefore we also have,

$$v^T v = (x - x')^T (x - x') = (x^T - x'^T)(x - x') = x^T x - x^T x' - x'^T x + x'^T x' = 2x^T x - 2x^T x' = 2x^T (x - x') = 2x^T v \tag{10}$$

Putting (5) in (4), we get

$$v - 2(\frac{vv^T}{v^T v})x = v - \frac{2}{v^T v}v(v^T x) = v - \frac{2}{2x^T v}(v^T x)v = v - \frac{2}{2x^T v}(x^T v)v = v - v = \vec{0} \tag{11}$$

Putting (6) in (3), we get

$$Fx = v - 2(\frac{vv^T}{v^T v})x + x' = \vec{0} + x' = x'$$

5. If we write A as $[a_1 \ a_2 \ \cdots \ a_n]$ and B as $[b_1 \ b_2 \ \cdots \ b_n]$ where the respective a_i's and b_i's are column vectors. The property $||b_i||_2 = ||a_i||_2$ is equivalent to $b_i^T b_i = a_i^T a_i$. Combined with $b_i^T b_j = a_i^T a_j \ \forall i, j$, we can write

$$A^T A = B^T B$$

Therefore, we want to find a matrix B such that the above property is satisfied.

Now, the reduced SVD of A is given by $A = \hat{U}\Sigma\hat{V}^T$. $A^T A$ is given by

$$A^T A = (\hat{U}\Sigma\hat{V}^T)^T(\hat{U}\Sigma\hat{V}^T) = \hat{V}\Sigma^T\hat{U}^T\hat{U}\Sigma\hat{V}^T = \hat{V}\Sigma\hat{U}^T\hat{U}\Sigma\hat{V}^T$$

Now, the columns of $\hat{U}$ are orthonormal. Therefore $\hat{U}^T\hat{U} = I$. Hence,

$$A^T A = \hat{V}\Sigma\Sigma\hat{V}^T = (\Sigma\hat{V}^T)^T(\Sigma\hat{V}^T)$$

Therefore,

$$B = \Sigma\hat{V}^T$$

We can clearly observe that A is $m \times n$ and B is $n \times n$. Hence the given B satisfies the required properties and the columns of B are the required vectors.

6. Low rank approximation

   (a) The `split` and `join` functions are simple using simple flatten and reshape functions in python.

   (b) We have the full SVD as $A = U\Sigma V^T$. The `compress` function basically chooses the first r columns of U and V and the first r diagonal elements of Σ. Therefore if we denote U as $[U_{keep}|U_{discard}]$ where U_{keep} is the matrix formed by the r columns that we are keeping and similarly for Σ and V, we can write

$$A = U\Sigma V^T = [U_{keep}|U_{discard}] \begin{bmatrix} \Sigma_{keep} & 0 \\ 0 & \Sigma_{discard} \end{bmatrix} ([V_{keep}]|V_{discard}])^T$$

   Hence,

$$A = [U_{keep}\Sigma_{keep}|U_{discard}\Sigma_{discard}] \begin{bmatrix} V_{keep}^T \\ V_{discard}^T \end{bmatrix} = (U_k\Sigma_k V_k^T) + (U_d\Sigma_d V_d^T)$$

   where k and d are short for $keep$ and $discard$. Hence the r-rank approximation of A is $U_k\Sigma_k V_k^T$

   (c) The error can be written as

$$\frac{||A - A_{approx}||_2}{||A||_2} = \frac{||U_d\Sigma_d V_d^T||_2}{||U\Sigma V^T||_2}$$

   Now, because U and V are orthogonal and U_d and V_d have orthonormal columns, we have the error as

$$error = \frac{||U_d\Sigma_d V_d^T||_2}{||U\Sigma V^T||_2} = \frac{||\Sigma_d V_d^T||_2}{||\Sigma V^T||_2} = \frac{||\Sigma_d||_2}{||\Sigma||_2} = \frac{\sqrt{\sigma_{r+1}^2 + \cdots + \sigma_n^2}}{\sqrt{\sigma_1^2 + \cdots + \sigma_n^2}}$$

   where σ_i's are the singular values of A.