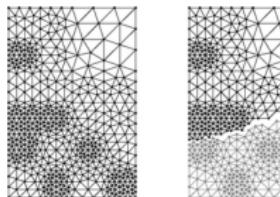


network *blockmodeling*

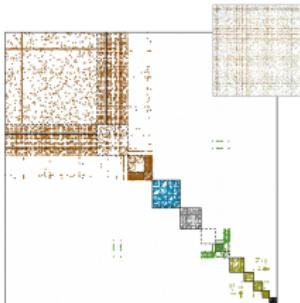
introduction to *network analysis in Python* (*NetPy*)

Lovro Šubelj
University of Ljubljana
19th Sep 2019

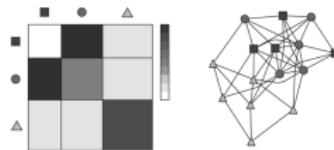
blockmodeling overview



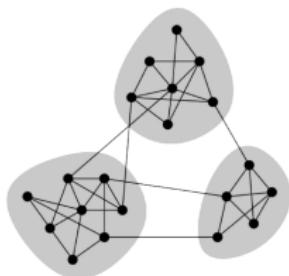
graph partitioning [KL70, Fie73]



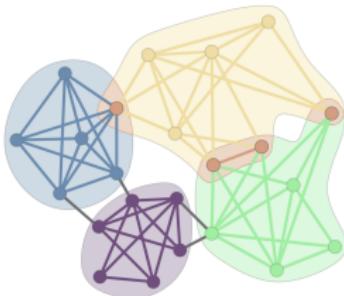
blockmodeling [LW71, WR83]



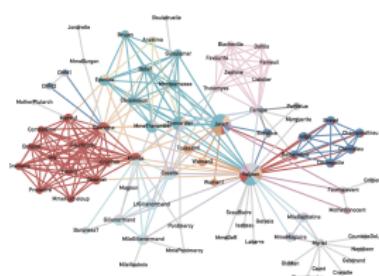
stochastic block models [Pei15]



communities [GN02]



overlapping communities [PDFV05]

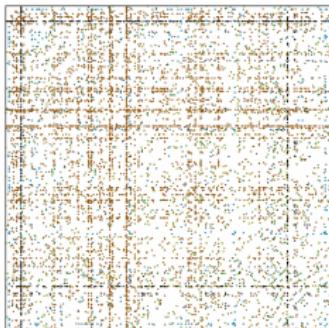


link communities [EL09, ABL10]

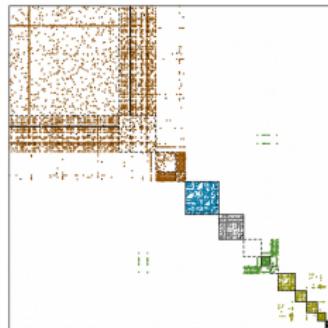
* assortative & disassortative equivalence blockmodeling

blockmodeling *equivalence*

- standard equivalence blockmodeling [DBF05]
 - define *node similarity* as (*structural*) equivalence
$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$
 - 1. *blockmodeling* by (*hierarchical*) *clustering* $\mathcal{O}(n^2)$
 - 2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model



javax.swing, javax.management, javax.naming, javax.print, javax.xml, javax.lang etc.

blockmodeling *structural*

similar nodes have *same neighbors*

- *standard structural equivalence* [LW71] of i and j is

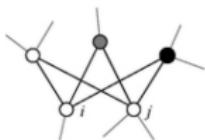
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

- *Salton structural equivalence* [SM83] of i and j is

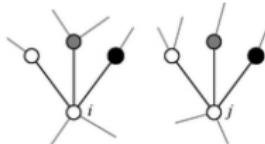
- θ_{ij} is *angle* between neighborhoods A_i and A_j

$$\sigma_{ij} = \cos \theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2} \sqrt{\sum_x A_{jx}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

- *Leicht structural equivalence* [LHN06] of i and j is $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural

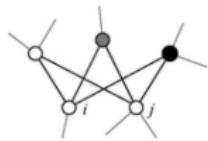


regular equivalence

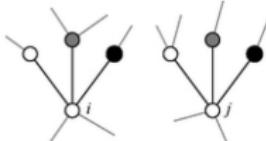
blockmodeling *regular*

similar nodes have *equivalent* *neighbors*

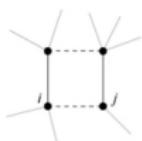
- *standard regular equivalence* [WR83] of *i* and *j* is
 - $\alpha < \lambda^{-1}$ is *positive constant* and λ *leading eigenvalue* of A
$$\sigma_{ij} = \alpha \sum_{xy} A_{ix} A_{jy} \sigma_{xy} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sum_{y \in \Gamma_j} \sigma_{xy} + \delta_{ij}$$
$$\sigma = \alpha A \sigma A + I \text{ and thus } \sigma^{(0)} = 0, \sigma^{(1)} = I, \sigma^{(2)} = \alpha A^2 + I, \sigma^{(3)} = \alpha^2 A^4 + \alpha A^2 + I \text{ etc.}$$
- *Katz regular equivalence* [Kat53] of *i* and *j* is
 - $\sigma_{ij} = \alpha \sum_x A_{ix} \sigma_{xj} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sigma_{xj} + \delta_{ij}$
$$\sigma = \alpha A \sigma + I \text{ and thus } \sigma^{(0)} = 0, \sigma^{(1)} = I, \sigma^{(2)} = \alpha A + I, \sigma^{(3)} = \alpha^2 A^2 + \alpha A + I \text{ etc.}$$



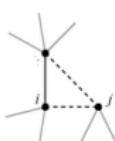
structural



regular equivalence

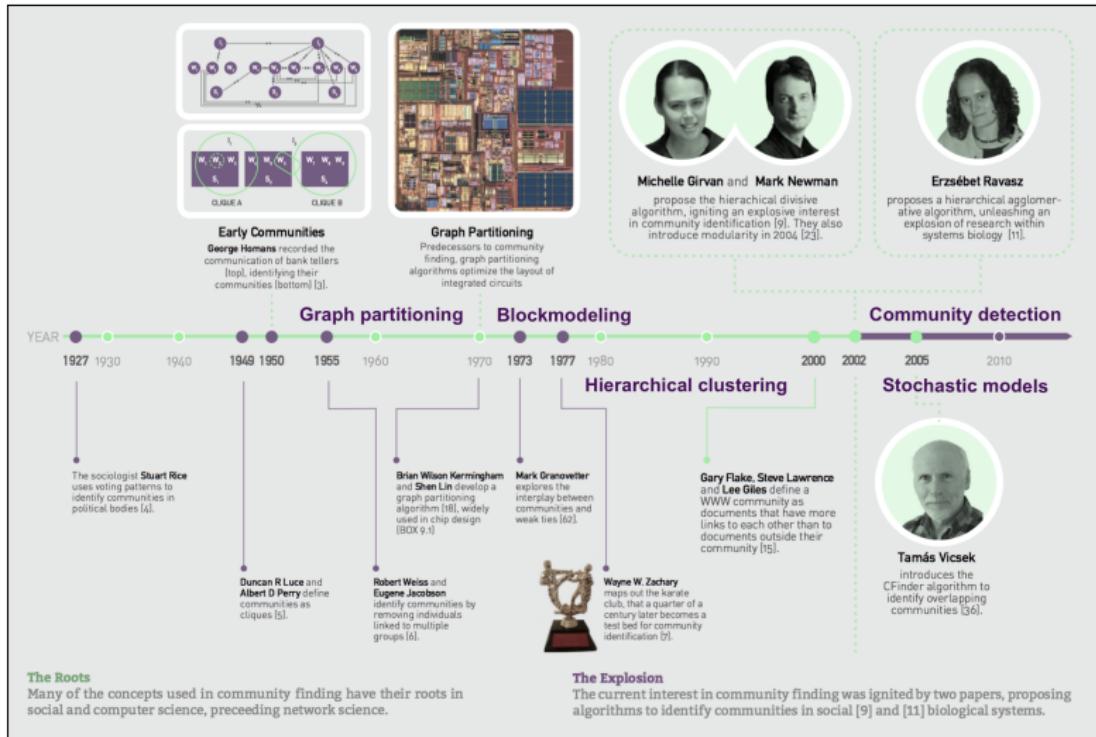


standard



Katz

blockmodeling *history*



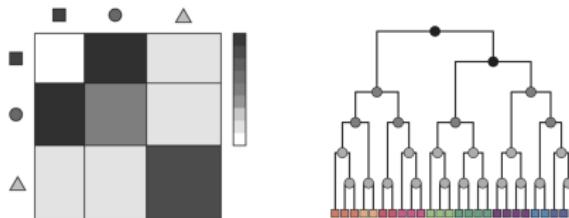
stochastic models

introduction to *network analysis in Python* (*NetPy*)

Lovro Šubelj
University of Ljubljana
19th Sep 2019

stochastic *models*

- random graph model $G(n, m)$ for network links m [ER59]
- configuration model $G(\{k\})$ for node degrees $\{k\}$ [NSW01]
- exponential p^* -model $G(n, \{\langle x \rangle\})$ for any expectations $\{\langle x \rangle\}$
- stochastic block model $G(\{C\})$ for node clusters $\{C\}$ [HLL83]
- hierarchical model $G(H)$ for node hierarchy H [CMN08]



* assortative & disassortative stochastic block models

stochastic $G(\{C\})$ model

- $G(\{C\}, \{p\})$ stochastic block model [HLL83]
- link between i and j placed with probability $p_{c_i c_j}$

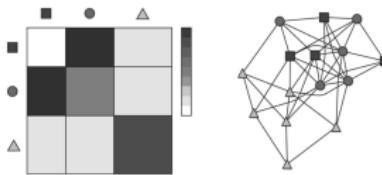
- $m_{c_i c_j}$ is number of links between C_i and C_j
- $M_{c_i c_j}$ is maximum $m_{c_i c_j}$ hence $n_i n_j$ or $\binom{n_i}{2}$

$$P(G|\{C\}, \{p\}) = \prod_{i \leq j} p_{c_i c_j}^{A_{ij}} (1 - p_{c_i c_j})^{1 - A_{ij}} = \prod_{c_i \leq c_j} p_{c_i c_j}^{m_{c_i c_j}} (1 - p_{c_i c_j})^{M_{c_i c_j} - m_{c_i c_j}}$$

- maximum likelihood $G(\{C\})$ block model

- $\frac{m_{c_i c_j}}{M_{c_i c_j}}$ is maximum likelihood estimate for $p_{c_i c_j}$

$$\mathcal{L}(G|\{C\}) = \log P(G|\{C\}) = \sum_{c_i \leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j} - m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j} - m_{c_i c_j}}{M_{c_i c_j}}$$

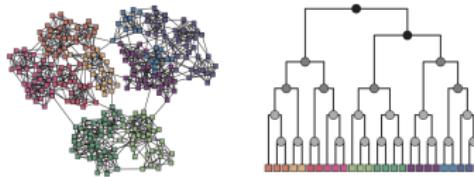


see `graph-tool` implementation

stochastic $G(H)$ model

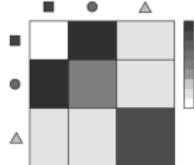
- $G(H, \{p\})$ hierarchical model [CMN08]
- link between i and j placed with probability $p_{r_{ij}}$
 - r is root with subtrees L_r, R_r and r_{ij} lowest root of i and j
 - m_r is number of links between L_r, R_r and M_r is $|L_r||R_r|$
- maximum likelihood $G(H)$ hierarchical model
 - $\frac{m_r}{M_r}$ is maximum likelihood estimate for p_r

$$\mathcal{L}(G|H) = \log P(G|H) = \sum_r m_r \log \frac{m_r}{M_r - m_r} + M_r \log \frac{M_r - m_r}{M_r}$$

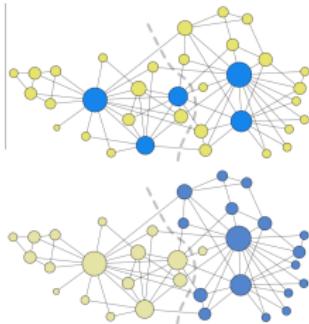


see `randomgraphs` implementation

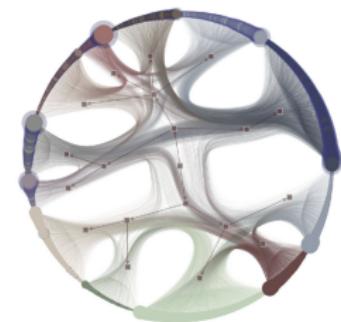
stochastic overview



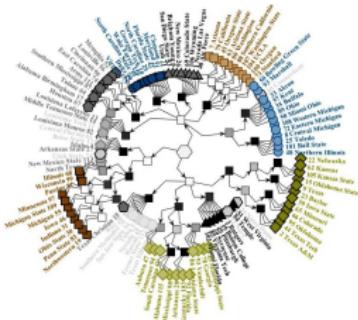
stochastic block models [HLL83]



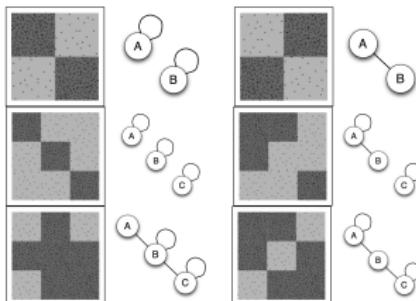
corrected block models [KN11]



principled block models [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

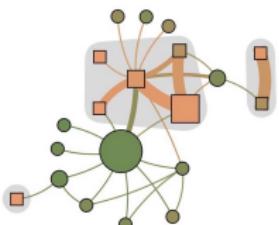


image graphs [ŠB12]

†

overlapping & corrected models also known as mixture & mixed membership models

blockmodeling *references*

-  Yong-Yeol Ahn, James P. Bagrow, and Sune Lehmann.
Link communities reveal multiscale complexity in networks.
Nature, 466(7307):761–764, 2010.
-  Aaron Clauset, Cristopher Moore, and M. E. J. Newman.
Hierarchical structure and the prediction of missing links in networks.
Nature, 453(7191):98–101, 2008.
-  Patrick Doreian, Vladimir Batagelj, and Anuska Ferligoj.
Generalized Blockmodeling.
Cambridge University Press, Cambridge, 2005.
-  T. S. Evans and R. Lambiotte.
Line graphs, link partitions and overlapping communities.
Phys. Rev. E, 80(1):016105, 2009.
-  P. Erdős and A. Rényi.
On random graphs I.
Publ. Math. Debrecen, 6:290–297, 1959.
-  M. Fiedler.
Algebraic connectivity of graphs.
Czech. Math. J., 23:298–305, 1973.
-  M. Girvan and M. E. J Newman.
Community structure in social and biological networks.
P. Natl. Acad. Sci. USA, 99(12):7821–7826, 2002.
-  Roger Guimerà, Marta Sales-Pardo, and Luis A. N. Amaral.
Classes of complex networks defined by role-to-role connectivity profiles.
Nat. Phys., 3(1):63–69, 2007.

blockmodeling *references*

-  Paul W. Holland, Kathryn Blackmond Laskey, and Samuel Leinhardt.
Stochastic blockmodels: First steps.
Soc. Networks, 5(2):109–137, 1983.
-  Leo Katz.
A new status index derived from sociometric analysis.
Psychometrika, 18(1):39–43, 1953.
-  Brian W. Kernighan and S. Lin.
An efficient heuristic procedure for partitioning graphs.
Bell Sys. Tech. J., 49(2):291–308, 1970.
-  Brian Karrer and M. E. J Newman.
Stochastic blockmodels and community structure in networks.
Phys. Rev. E, 83(1):016107, 2011.
-  E. A. Leicht, Petter Holme, and M. E. J. Newman.
Vertex similarity in networks.
Phys. Rev. E, 73(2):026120, 2006.
-  F. Lorrain and H. C. White.
Structural equivalence of individuals in social networks.
J. Math. Sociol., 1(1):49–80, 1971.
-  M. E. J Newman and E. A Leicht.
Mixture models and exploratory analysis in networks.
P. Natl. Acad. Sci. USA, 104(23):9564–9569, 2007.
-  M. E. J. Newman, S. H. Strogatz, and D. J. Watts.
Random graphs with arbitrary degree distributions and their applications.
Phys. Rev. E, 64(2):026118, 2001.

blockmodeling *references*

-  Gergely Palla, Imre Derényi, Illes Farkas, and Tamas Vicsek.
Uncovering the overlapping community structure of complex networks in nature and society.
Nature, 435(7043):814–818, 2005.
-  Tiago P. Peixoto.
Model selection and hypothesis testing for large-scale network models with overlapping groups.
Phys. Rev. X, 5(1):011033, 2015.
-  J. Reichardt and D. R. White.
Role models for complex networks.
Eur. Phys. J. B, 60(2):217–224, 2007.
-  Lovro Šubelj and Marko Bajec.
Ubiquitousness of link-density and link-pattern communities in real-world networks.
Eur. Phys. J. B, 85(1):32, 2012.
-  Lovro Šubelj and Marko Bajec.
Group detection in complex networks: An algorithm and comparison of the state of the art.
Physica A, 397:144–156, 2014.
-  G. Salton and M. J. McGill.
Introduction to Modern Information Retrieval.
McGraw-Hill, 1983.
-  D. R. White and K. P. Reitz.
Graph and semigroup homomorphisms on networks of relations.
Soc. Networks, 5(2):193–234, 1983.