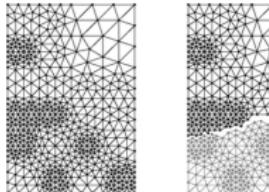


# network *blockmodeling*

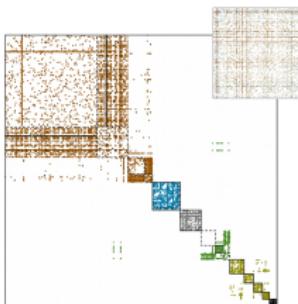
introduction to *network science in Python* (*NetPy*)

Lovro Šubelj  
University of Ljubljana  
3rd Dec 2022

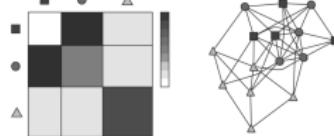
# blockmodeling overview



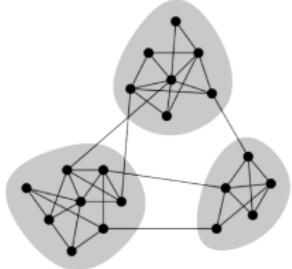
graph partitioning [KL70, Fie73]



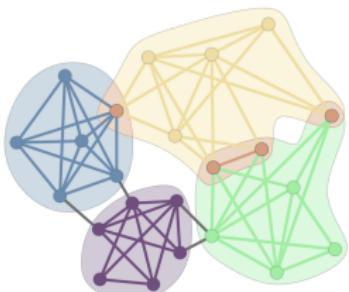
blockmodeling [LW71, WR83]



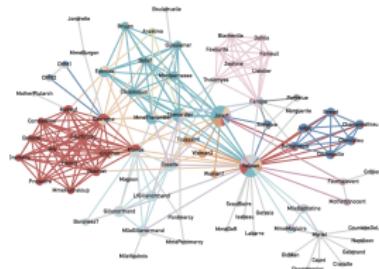
stochastic block models [Pei15]



communities [GN02]



overlapping communities [PDFV05]

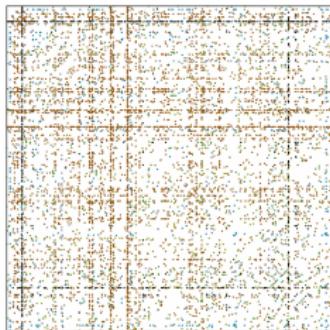


link communities [EL09, ABL10]

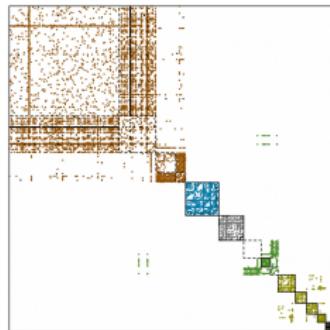
\* assortative & disassortative equivalence blockmodeling

# blockmodeling *equivalence*

- standard equivalence blockmodeling [DBF05]
  - define *node similarity* as (*structural*) equivalence
$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$
- 1. *blockmodeling* by (*hierarchical*) clustering  $\mathcal{O}(n^2)$
- 2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model



javax.swing, javax.management, javax.naming, javax.print, javax.xml, javax.lang etc.

# blockmodeling *structural*

*similar* nodes have *same* neighbors

- standard structural equivalence [LW71] of  $i$  and  $j$  is

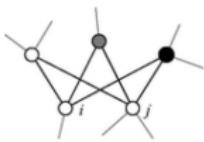
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

- Salton structural equivalence [SM83] of  $i$  and  $j$  is

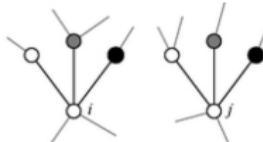
- $\theta_{ij}$  is angle between neighborhoods  $A_i$  and  $A_j$

$$\sigma_{ij} = \cos \theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2} \sqrt{\sum_x A_{xj}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

- Leicht structural equivalence [LHN06] of  $i$  and  $j$  is  $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural



regular equivalence

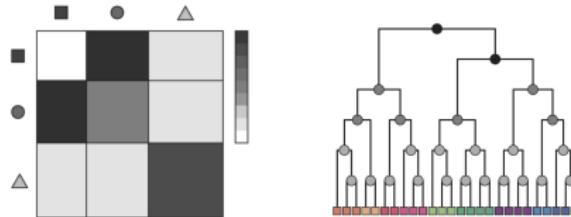
# *stochastic* models

introduction to *network science in Python* (*NetPy*)

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3rd Dec 2022

# stochastic *models*

- random graph model  $G(n, m)$  for network links  $m$  [ER59]
- configuration model  $G(\{k\})$  for node degrees  $\{k\}$  [NSW01]
- exponential  $p^*$ -model  $G(n, \{\langle x \rangle\})$  for any expectations  $\{\langle x \rangle\}$
- stochastic block model  $G(\{C\})$  for node clusters  $\{C\}$  [HLL83]
- hierarchical model  $G(H)$  for node hierarchy  $H$  [CMN08]



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\* assortative & disassortative stochastic block models

## stochastic $G(\{C\})$ model

- $G(\{C\}, \{p\})$  stochastic block model [HLL83]
- link between  $i$  and  $j$  placed with probability  $p_{c_i c_j}$

—  $m_{c_i c_j}$  is number of links between  $C_i$  and  $C_j$

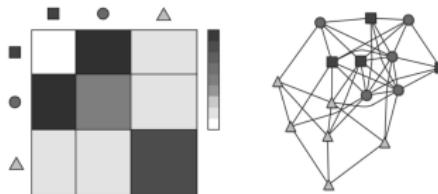
—  $M_{c_i c_j}$  is maximum  $m_{c_i c_j}$  hence  $n_i n_j$  or  $\binom{n_i}{2}$

$$P(G|\{C\}, \{p\}) = \prod_{i \leq j} p_{c_i c_j}^{A_{ij}} (1 - p_{c_i c_j})^{1 - A_{ij}} = \prod_{c_i \leq c_j} p_{c_i c_j}^{m_{c_i c_j}} (1 - p_{c_i c_j})^{M_{c_i c_j} - m_{c_i c_j}}$$

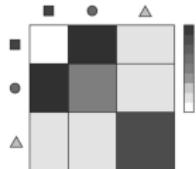
- maximum likelihood  $G(\{C\})$  block model

—  $\frac{m_{c_i c_j}}{M_{c_i c_j}}$  is maximum likelihood estimate for  $p_{c_i c_j}$

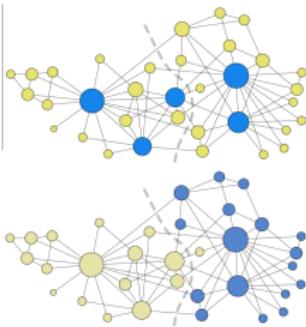
$$\mathcal{L}(G|\{C\}) = \log P(G|\{C\}) = \sum_{c_i \leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j} - m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j} - m_{c_i c_j}}{M_{c_i c_j}}$$



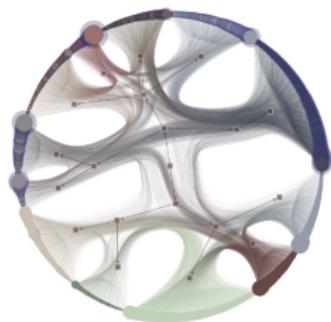
# stochastic overview



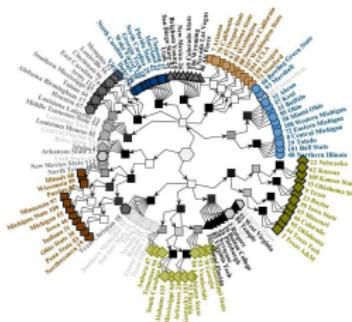
stochastic block models [HLL83]



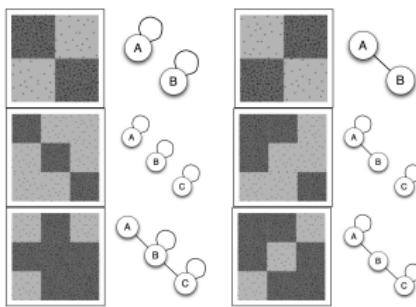
corrected block models [KN11]



principled block models [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

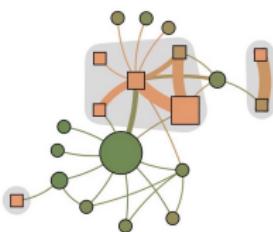


image graphs [ŠB12]

†

overlapping & corrected models also known as mixture & mixed membership models

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