network *blockmodeling*

introduction to *network science in Python* (*NetPy*)

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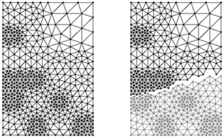
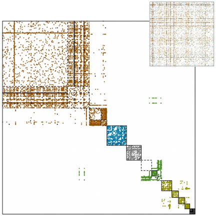
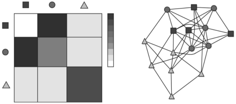
blockmodeling *overview*



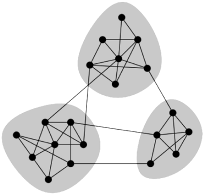
graph partitioning [KL70, Fie73]



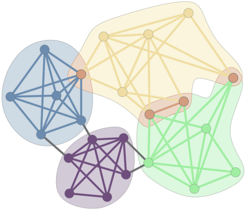
blockmodeling [LW71, WR83]



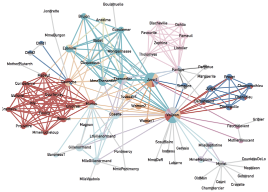
stochastic block models [Pei15]



communities [GN02]



overlapping communities [PDFV05]



link communities [EL09, ABL10]

* assortative & disassortative equivalence blockmodeling

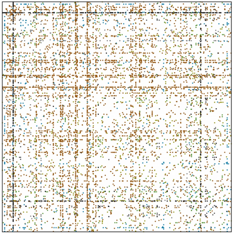
blockmodeling *equivalence*

— *standard equivalence blockmodeling* [DBF05]

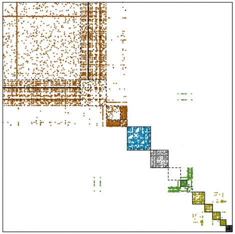
- define *node similarity* as *(structural) equivalence*

$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$

1. *blockmodeling* by *(hierarchical) clustering* $\mathcal{O}(n^2)$
2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model

‡ `javax.swing`, `javax.management`, `javax.naming`, `javax.print`, `javax.xml`, `javax.lang` etc.

blockmodeling *structural*

similar nodes have *same neighbors*

— *standard structural equivalence* [LW71] of i and j is

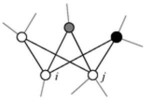
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

— *Salton structural equivalence* [SM83] of i and j is

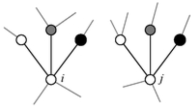
- θ_{ij} is *angle* between neighborhoods A_i and A_j

$$\sigma_{ij} = \cos\theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2}\sqrt{\sum_x A_{jx}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

— *Leicht structural equivalence* [LHN06] of i and j is $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural



regular equivalence

stochastic models

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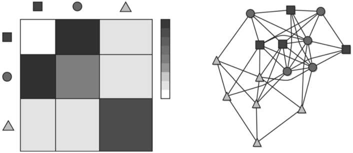
stochastic *models*

— *random graph model* $G(n, m)$ for *network links* m [ER59]

— *configuration model* $G(\{k\})$ for *node degrees* $\{k\}$ [NSW01]

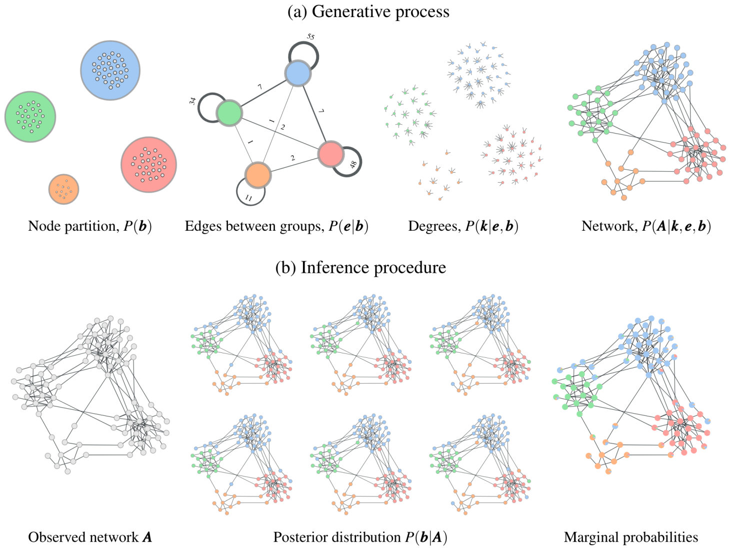
— *exponential* p^**-model* $G(n, \{\langle x\rangle\})$ for *any expectations* $\{\langle x\rangle\}$

— *stochastic block model* $G(\{C\})$ for *node clusters* $\{C\}$ [HLL83]



*assortative & disassortative stochastic block models

stochastic *process*



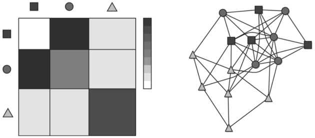
stochastic *SBM*

- $G(\{C\},\{p\})$ *stochastic block model* [HLL83]
- *link* between i and j *placed with probability* $p_{c_ic_j}$
  - $m_{c_ic_j}$ is *number of links* between C_i and C_j
  - $M_{c_ic_j}$ is *maximum* $m_{c_ic_j}$ hence n_in_j or $\binom{n_i}{2}$

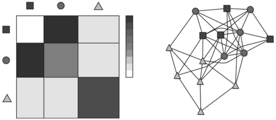
$$\mathrm{P}(G|\{C\},\{p\}) = \prod_{i\leq j} p_{c_ic_j}^{A_{ij}}(1-p_{c_ic_j})^{1-A_{ij}} = \prod_{c_i\leq c_j} p_{c_ic_j}^{m_{c_ic_j}}(1-p_{c_ic_j})^{M_{c_ic_j}-m_{c_ic_j}}$$

- *maximum likelihood* $G(\{C\})$ *block model*
  - $\frac{m_{c_ic_j}}{M_{c_ic_j}}$ is *maximum likelihood estimate* for $p_{c_ic_j}$

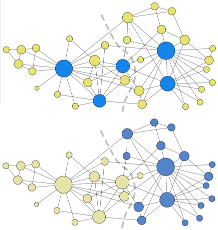
$$\mathcal{L}(G|\{C\}) = \log \mathrm{P}(G|\{C\}) = \sum_{c_i\leq c_j} m_{c_ic_j}\log\frac{m_{c_ic_j}}{M_{c_ic_j}-m_{c_ic_j}} + M_{c_ic_j}\log\frac{M_{c_ic_j}-m_{c_ic_j}}{M_{c_ic_j}}$$



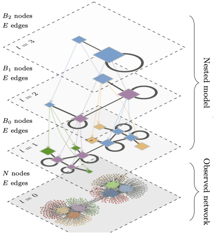
stochastic *overview*



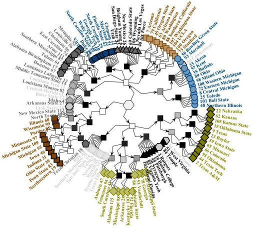
stochastic block model [HLL83]



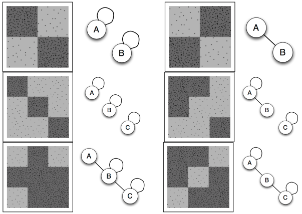
degree-corrected SBM [KN11]



nested SBM [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

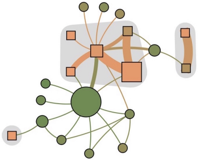


image graphs [ŠB12]

† overlapping & corrected models also known as mixture & mixed membership models

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