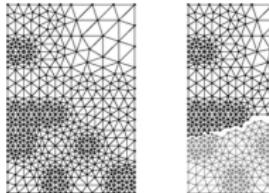


network *blockmodeling*

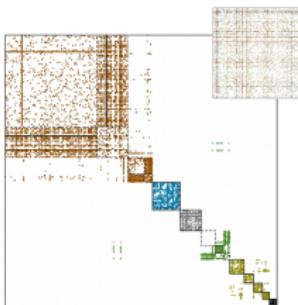
introduction to *network science in Python* (*NetPy*)

Lovro Šubelj
University of Ljubljana
4th July 2024

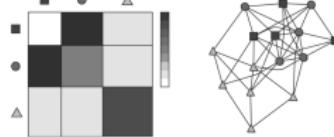
blockmodeling overview



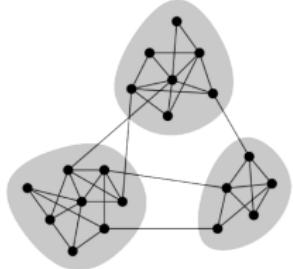
graph partitioning [KL70, Fie73]



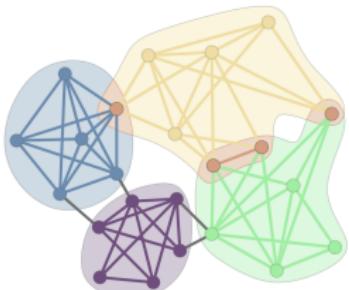
blockmodeling [LW71, WR83]



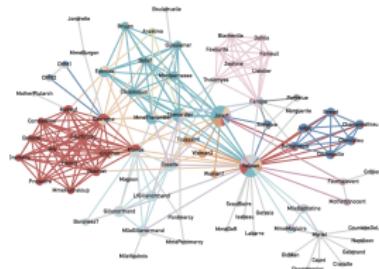
stochastic block models [Pei15]



communities [GN02]



overlapping communities [PDFV05]

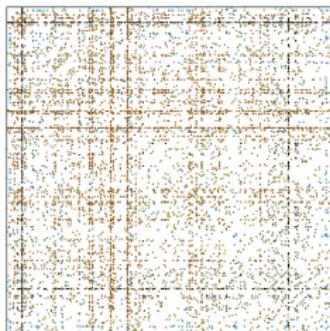


link communities [EL09, ABL10]

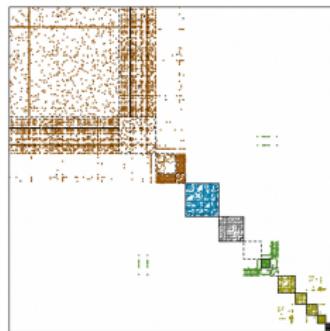
* assortative & disassortative equivalence blockmodeling

blockmodeling *equivalence*

- standard equivalence blockmodeling [DBF05]
 - define *node similarity* as (*structural*) equivalence
$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$
- 1. *blockmodeling* by (*hierarchical*) clustering $\mathcal{O}(n^2)$
- 2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model



javax.swing, javax.management, javax.naming, javax.print, javax.xml, javax.lang etc.

blockmodeling *structural*

similar nodes have *same* neighbors

- standard structural equivalence [LW71] of i and j is

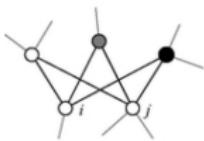
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

- Salton structural equivalence [SM83] of i and j is

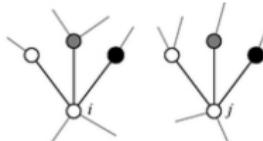
- θ_{ij} is angle between neighborhoods A_i and A_j

$$\sigma_{ij} = \cos \theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2} \sqrt{\sum_x A_{xj}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

- Leicht structural equivalence [LHN06] of i and j is $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural



regular equivalence

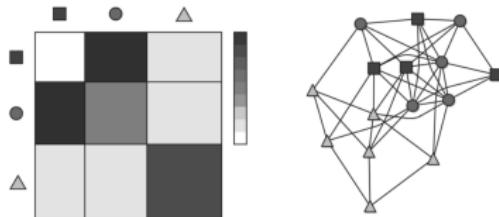
stochastic models

introduction to *network science in Python* (*NetPy*)

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stochastic *models*

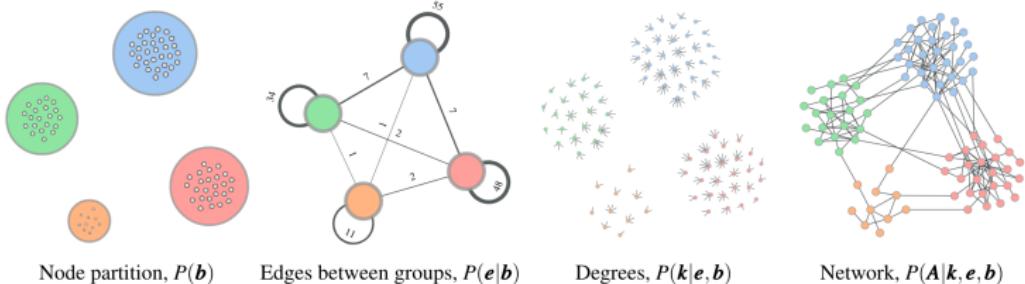
- random graph model $G(n, m)$ for network links m [ER59]
- configuration model $G(\{k\})$ for node degrees $\{k\}$ [NSW01]
- exponential p^* -model $G(n, \{\langle x \rangle\})$ for any expectations $\{\langle x \rangle\}$
- stochastic block model $G(\{C\})$ for node clusters $\{C\}$ [HLL83]



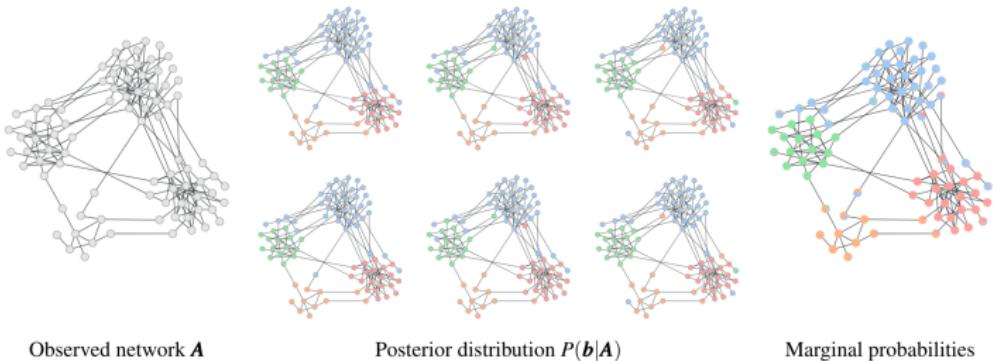
* assortative & disassortative stochastic block models

stochastic *process*

(a) Generative process



(b) Inference procedure



stochastic *SBM*

- $G(\{C\}, \{p\})$ stochastic block model [HLL83]
- link between i and j placed with probability $p_{c_i c_j}$

— $m_{c_i c_j}$ is number of links between C_i and C_j

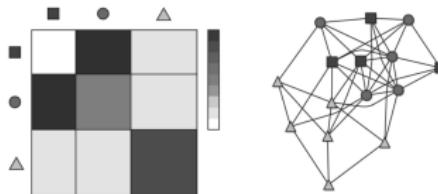
— $M_{c_i c_j}$ is maximum $m_{c_i c_j}$ hence $n_i n_j$ or $\binom{n_i}{2}$

$$P(G|\{C\}, \{p\}) = \prod_{i \leq j} p_{c_i c_j}^{A_{ij}} (1 - p_{c_i c_j})^{1 - A_{ij}} = \prod_{c_i \leq c_j} p_{c_i c_j}^{m_{c_i c_j}} (1 - p_{c_i c_j})^{M_{c_i c_j} - m_{c_i c_j}}$$

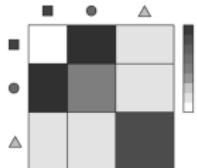
- maximum likelihood $G(\{C\})$ block model

— $\frac{m_{c_i c_j}}{M_{c_i c_j}}$ is maximum likelihood estimate for $p_{c_i c_j}$

$$\mathcal{L}(G|\{C\}) = \log P(G|\{C\}) = \sum_{c_i \leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j} - m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j} - m_{c_i c_j}}{M_{c_i c_j}}$$



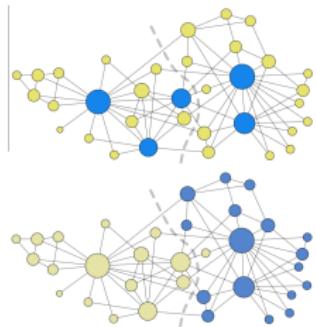
stochastic overview



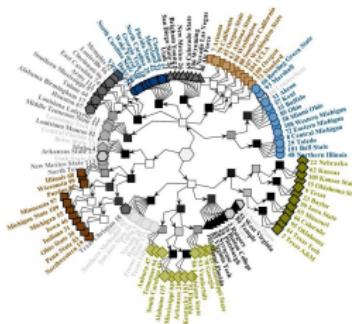
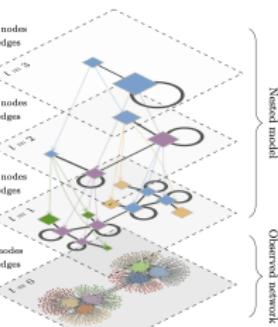
stochastic block model [HLL83]



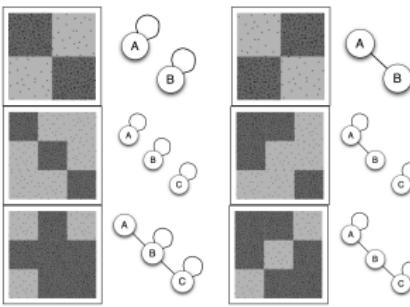
degree-corrected SBM [KN11]



nested SBM [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

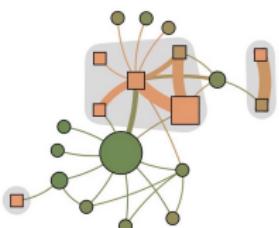


image graphs [ŠB12]

†

overlapping & corrected models also known as mixture & mixed membership models

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