Advanced network algorithms, random graph models

You are given four networks in Pajek format (edge list and LNA formats are also available).

- A simple toy network for testing (tiny)
- The famous Zachary karate club network (small)
- iMDB actors collaboration network (medium)
- A part of Google web graph (large)

I. Number and size of connected components (ii)

1. **(answer)** Study the following algorithm for computing (weakly) connected components $\{C\}$ by any order link traversal. Does the algorithm implement breadth-first or depth-first search? Why? What is the time complexity of the algorithm?

```
input   graph G, nodes N
output  network components {C}
    {C} ← empty list
    while not N empty do
        {C}.add(component(G, N, N.next()))
    return {C}
```

```
input   graph G, nodes N, node i
output  weak component C
    C ← empty list
    S ← empty stack
    N.remove(S.push(i))
    while not S empty do
        C.add(i ← S.pop())
        for neighbors j ∈ Γ_i do
            if N.remove(j) then
                S.push(j)
    return C
```

2. **(code)** Implement the algorithm and compute number of (weakly) connected components s and size of the largest (weakly) connected component S of all four networks. Are the results expected or are they surprising?

3. **(answer)** How could you further improve the algorithm to *only* compute s and S?

II. Average node distance and network diameter

1. **(answer)** Study the following algorithm for computing distances between the nodes $\{d\}$ by level order link traversal. Does the algorithm implement breadth-first or depth-first search? Why? What is the time complexity of the algorithm?

```
input   graph G
output  network distances {D}
  {D} ← empty list
  for nodes i ∈ N do
      {D}.add(distances(G, i))
  return {D}

input   graph G, node i
output  directed distances D
  ...
  for successors j ∈ Γ_i^out do
      ...
```

```
input   graph G, node i
output  undirected distances D
  D ← empty array
  Q ← empty queue
  D[Q.add(i)] ← 0
  while not Q empty do
      i ← Q.remove()
      for neighbors j ∈ Γ_i do
          if D[j] undefined then
              D[j] ← D[i] + 1
              Q.add(j)
  return D
```

2. **(code)** Implement the algorithm and compute average distance between the nodes $\langle d \rangle$ and maximum distance or diameter $d_{\max}$ of *smaller* networks. Are the results expected or are they surprising?

3. **(answer)** How is the algorithm different from the famous Dijkstra's algorithm? In which case you would necessarily have to use the Dijkstra's algorithm?

4. **(answer)** How could you speed up the algorithm to *only* approximate $\langle d \rangle$ and $d_{\max}$?

III. Average node clustering coefficient

1. **(answer)** Study the following algorithm for computing node clustering coefficients $\{C\}$ by link triad counting. Why does the algorithm count triads over the links and not over the nodes? What is the time complexity of the algorithm?

```
input   graph G
output  average clustering ⟨C⟩
  ⟨C⟩ ← 0
  for nodes i ∈ N do
      ⟨C⟩ ← clustering(G, i)/n
  return ⟨C⟩

input   graph G, node i
output  node clustering C
  if k_i ≤ 1 then
      return 0
  return triads(G, i) · 2/(k_i^2 − k_i)
```

```
input   graph G, node i
output  node triads t
  t ← 0
  for neighbors j ∈ Γ_i do
      if |Γ_i| ≤ |Γ_j| then
          t ← t + triads(G, i, j)/2
      else
          t ← t + triads(G, j, i)/2
  return t

input   graph G, link i, j
output  link triads t
  t ← 0
  for neighbors k ∈ Γ_i do
      if k ∈ Γ_j then
          t ← t + 1
  return t
```

2. **(code)** Implement the algorithm and compute average node clustering coefficient $\langle C \rangle$ of all four

networks. Are the results expected or are they surprising?

3. **(answer)** What kind of network representation is required by the algorithm?

IV. Erdös-Rényi random graphs and link indexing

1. **(answer)** Study the following two algorithms for generating Erdös-Rényi random graphs $G(n,m)$ with and without link indexing $\binom{i}{2}+j, i>j$. What is the main difference between the algorithms? What is the time complexity of the algorithms?

```
input   nodes n, links m
output  simple random G
  H ← empty set
  G ← n isolated nodes
  while not G has m links do
      h ← {0, ..., (n^2 − n)/2 − 1}.random()
      if H.add(h) then
          i ← 1 + ⌊−0.5 + √(0.25 + 2h)⌋
          add link between i and h − (i^2 − i)/2
  return G
```

```
input   nodes n, links m
output  random multi G
  G ← n isolated nodes
  while not G has m links do
      i, j ← {0, ..., n − 1}.random()
      if i ≠ j then
          add link between i and j
  return G
```

2. **(code)** Implement one of the algorithms and generate Erdös-Rényi random graphs corresponding to all four networks and compute graphs' S, $\langle d\rangle$ and $\langle C\rangle$. Are the results expected or are they surprising?

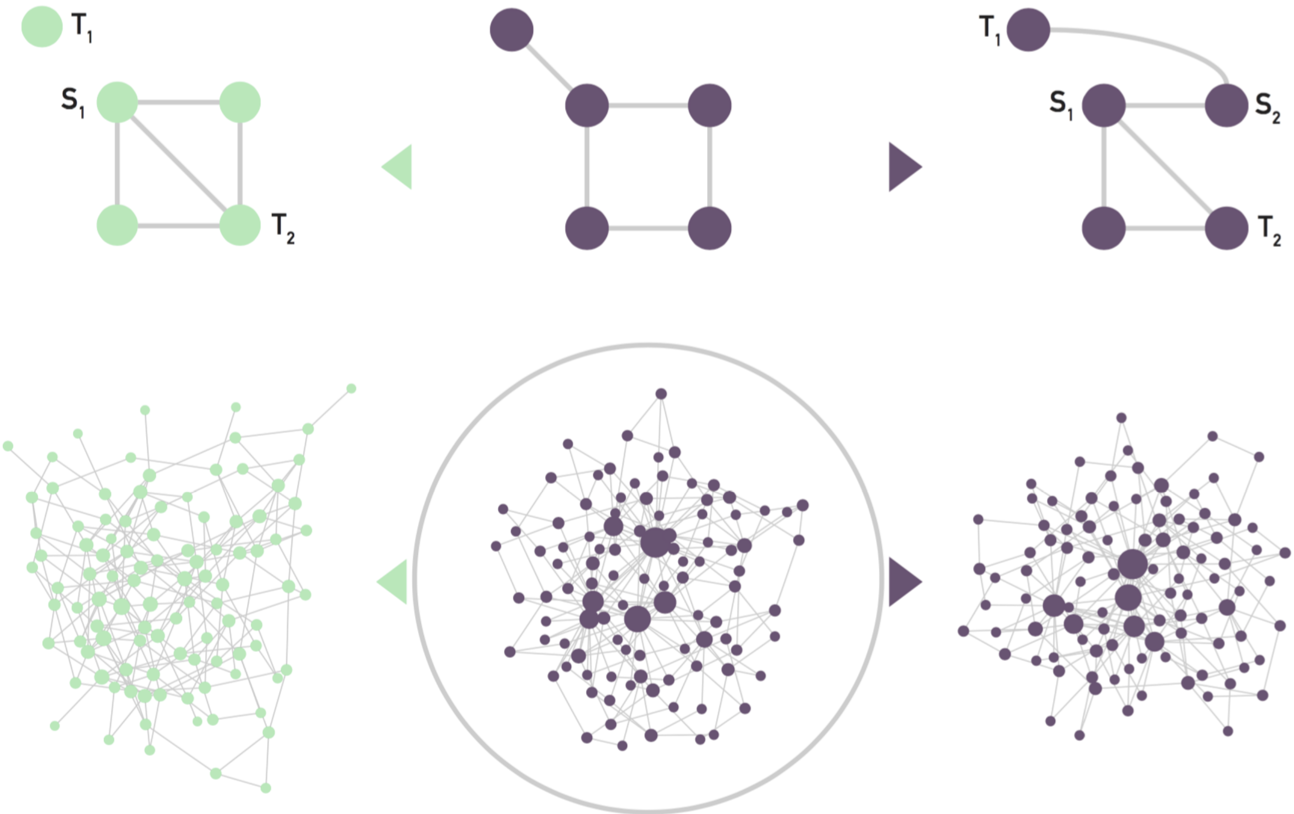
V. Configuration model graphs and link rewiring

1. **(answer)** Study the following two algorithms for generating configuration model graphs $G(\{k\})$ with link rewiring and stub matching. What is the main difference between the algorithms? What is the time complexity of the algorithms?

```
input   simple links L
output  configuration simple G
  H ← empty set
  for links {i, j} ∈ L do
      H.add(h_ij)
  while not links L rewired do
      {i, j}, {s, t} ← L.random()    ▷ removes links
      if H.contains(h_it or h_sj) or i = t or s = j then
          L.add({i, j})
          L.add({s, t})
      else
          L.add({i, t}) H.add(h_it) H.remove(h_ij)
          L.add({s, j}) H.add(h_sj) H.remove(h_st)
  return G on links L
```

```
input   nodes n, degrees {k}
output  configuration pseudo G
  Q ← empty queue
  G ← n isolated nodes
  for nodes i ∈ N do
      for k_i times do
          Q.add(i)
  while not Q empty do
      i, j ← Q.random()    ▷ removes nodes
      add link between i and j
  return G
```

2. **(code)** Implement one of the algorithms and generate configuration model graphs corresponding to all four networks and compute graphs' S, $\langle d\rangle$ and $\langle C\rangle$. Are the results expected or are they surprising?



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