network *clustering*

introduction to *network analysis* (*ina*)

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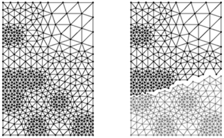
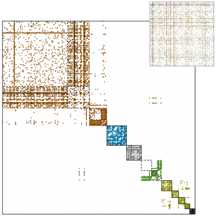
spring 2020/21

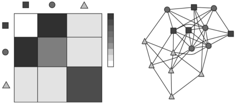
clustering *overview*



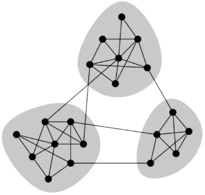
graph partitioning [KL70, Fie73]



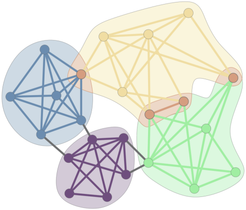
blockmodeling [LW71, WR83]



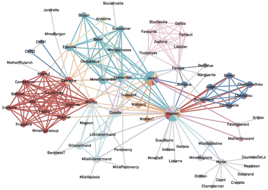
stochastic block models [Pei15]



communities [GN02]

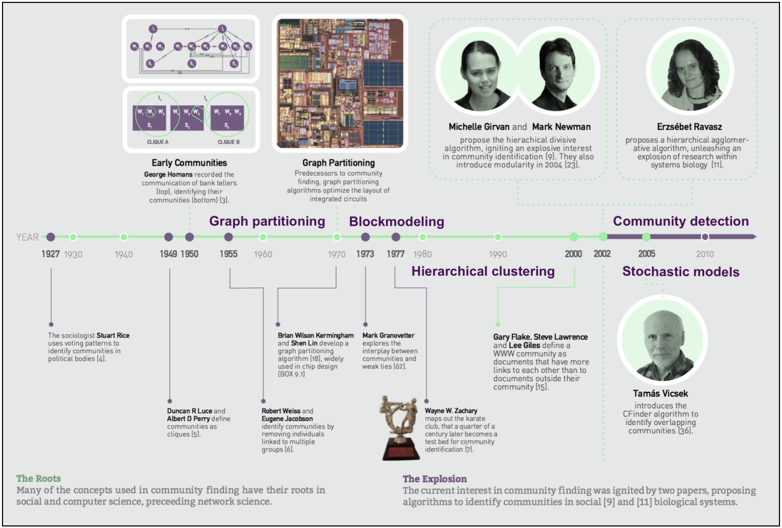


overlapping communities [PDFV05]



link communities [EL09, ABL10]

clustering *history*



graph *partitioning*

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partitioning *bisection*

— Kernighan-Lin *graph bisection* [KL70]

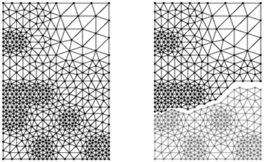
– define *bisection quality* as *cut size*

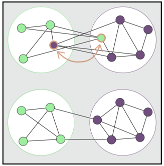
$$R = \frac{1}{2}\sum_{ij} A_{ij}(1-\delta_{c_i c_j}) \qquad \forall i : c_i = \pm 1$$

1. *swap nodes* by *minimizing cut size* $\mathcal{O}(cn^2m)$

$$\Delta R_{ij} = k_i^{ext} - k_i^{in} + k_j^{ext} - k_j^{in} - 2A_{ij}$$

2. repeat 1. *until* $\min(n_1, n_2)$ *nodes swapped*
3. return *bisection minimizing cut size*





*example mesh bisection with cut size equal to 40

partitioning *spectral*

— Fiedler *graph bisection* [Fie73]

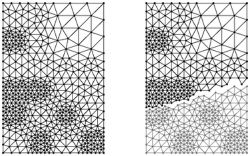
– define *bisection quality* as *cut size*

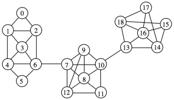
$$R = \frac{1}{4}\sum_{ij} A_{ij}(1 - s_i s_j) \qquad \forall i : s_i = \delta_{c_i c_1} - \delta_{c_i c_2}$$

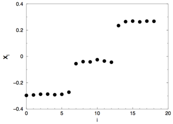
– formulate *eigenvector problem* of *graph Laplacian*

$$R = \frac{1}{4}\sum_i k_i s_i^2 - \frac{1}{4}\sum_{ij} A_{ij} s_i s_j = \frac{1}{4}\sum_{ij}(k_i\delta_{ij} - A_{ij}) s_i s_j = \frac{1}{4} s^T L s \simeq \frac{1}{4} v^T L v = \frac{n_1 n_2}{n}\lambda$$

1. find *eigenvector* v_2 with *algebraic connectivity* λ_2 $\mathcal{O}(nm)$
2. assign n_1 *nodes* with *largest*/*smallest* v_2 to C_1
3. return *bisection minimizing cut size*







see graclus and metis implementations

† example mesh bisection with cut size equal to 46

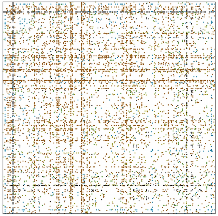
partitioning *blockmodeling*

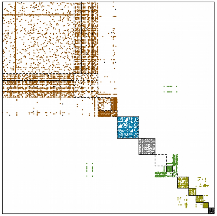
— standard *equivalence blockmodeling* [DBF05]

– define *node similarity* as *(structural) equivalence*

$$s_{ij} \sim |\Gamma_i \cap \Gamma_j|$$

1. *blockmodeling* by *(hierarchical) clustering* $\mathcal{O}(n^2)$
2. return *block model* at desired *clustering resolution*





see catrege implementation

‡ `javax.swing`, `javax.management`, `javax.naming`, `javax.print`, `javax.xml`, `javax.lang` etc.

community detection

introduction to *network analysis* (*ina*)

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community *agglomerative*

— Ravasz *hierarchical clustering* [RSM+02]

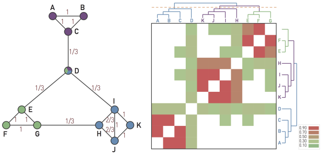
- define *node similarity* as *topological overlap*

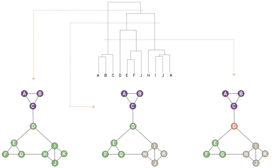
$$s_{ij} = \frac{|\Gamma_i \cap \Gamma_j| + A_{ij}}{\min(k_i, k_j)}$$

- define *cluster similarity* as *average linkage*

$$S_{ij} = \frac{1}{n_i n_j} \sum_{xy} s_{xy} \delta_{c_x c_i} \delta_{c_y c_j}$$

1. bottom-up *agglomerative hierarchical clustering* $\mathcal{O}(n^2)$
2. cut *cluster dendrogram* at desired *clustering resolution*





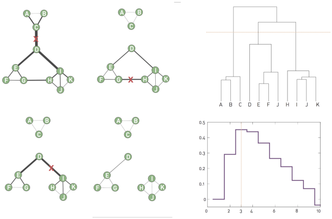
community *divisive*

- Girvan-Newman *hierarchical clustering* [GN02]
  - define *node dissimilarity* as *link betweenness*

$$\sigma_{ij} = \sum_{st \notin \{i,j\}} \frac{g_{st}^{ij}}{g_{st}}$$

1. top-down *divisive hierarchical clustering* $\mathcal{O}(nm^2)$
2. cut *cluster dendrogram* at *maximum modularity*

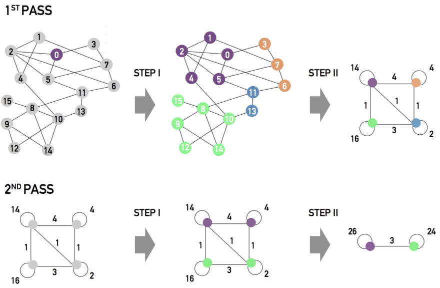
$$Q = \frac{1}{2m} \sum_{ij} (A_{ij} - \frac{k_i k_j}{2m}) \delta_{c_i c_j}$$



community *modularity*

- Louvain *modularity optimization* [BGLL08]
  1. set *node community* by *modularity optimization* $\mathcal{O}(cm)$
  2. *aggregate community nodes into supernodes* and repeat 1.
  3. return *community structure maximizing modularity*

$$Q = \frac{1}{2m}\sum_{ij}(A_{ij} - \frac{k_i k_j}{2m})\delta_{c_i c_j}$$



see louvain/leidenalg implementation

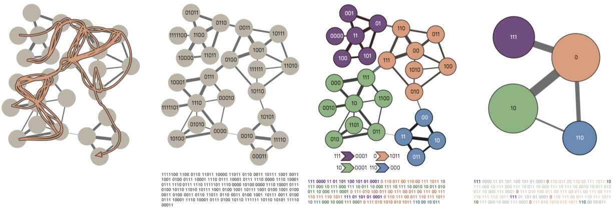
community *map equation*

details

— Infomap *map equation compression* [RB08]

1. set *node community* by *optimal coding* $\mathcal{O}(m \log m)$
2. *compress community nodes into supernodes* and repeat 1.
3. return *community structure maximizing map equation*

$$\mathcal{L} = \sum_i p_{i\rightsquigarrow} H(\widetilde{\mathcal{C}}) + \sum_i p_{i\hookleftarrow} H(\mathcal{C}_i)$$



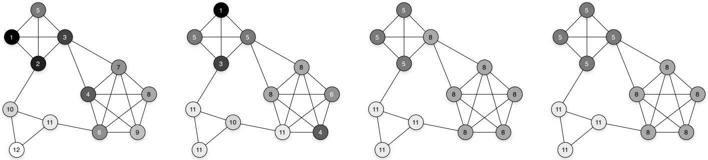
see mapequation implementation

community *propagation*

— Raghavan *label propagation* [RAK07, ŠB11]

1. set *node community* by *neighbors frequency* $\mathcal{O}(cm)$
2. *randomly shuffle nodes* and repeat 1. *until convergence*
3. return *community structure connected components*

$$\forall i : c_i = \arg\max_c \sum_j A_{ij}\delta_{c_j c}$$



see balanced implementation

community *percolation*

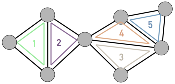
— Palla *clique percolation* [PDFV05, KKKS08]

1. find *k-node cliques* by *sequential enumeration* $\mathcal{O}(n_k)$
2. *merge clique nodes into supernodes* and *link adjacent*

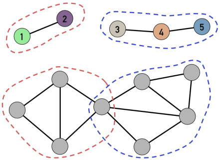
   adjacent k-node cliques share $k-1$ nodes

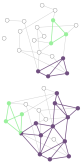
3. return *clique structure connected components*

   clique percolation at $(kn-n)^{\frac{1}{1-k}}$



	1	2	3	4	5
1	0	1	0	0	0
2	1	0	0	0	0
3	0	0	0	1	0
4	0	0	1	0	1
5	0	0	0	1	0





see kclique implementation

community *links*

— Ahn *link clustering* [EL09, ABL10]

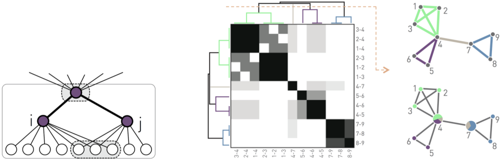
- define *link similarity* as *neighbors index*

$$\forall ij \in \Gamma_x : s_{ij}^x = \frac{|\Gamma_i^+ \cap \Gamma_j^+|}{|\Gamma_i^+ \cup \Gamma_j^+|}$$

- define *cluster similarity* as *single linkage*

$$S_{ij} = \max_{xy \in \Gamma_z}(s_{xy}^z \delta_{c_{xz}c_i} \delta_{c_{yz}c_j})$$

1. bottom-up *agglomerative hierarchical clustering* $\mathcal{O}(m^2)$
2. cut *cluster dendrogram* at desired *clustering resolution*



see linkcomm implementation

community *measures*

— *degree* K, *expansion* E and *Flake* F [FLG00, RCC+04] of $\{C\}$

$$K = \frac{1}{n}\sum_{ij} A_{ij}\delta_{c_i c_j} = \langle k\rangle - E \qquad F = \frac{|\{i : \sum_j A_{ij}\delta_{c_i c_j} < k_i/2\}|}{n}$$

— *normalized mutual information* NMI [DDGDA05] of $\{C\},\{D\}$

- p_c & p_{cd} are *standard* & *joint distributions* of $\{C\},\{D\}$
- $H(\mathcal{C})$ & $H(\mathcal{C}|\mathcal{D})$ are *standard* & *conditional entropies*
- MI & VI are *mutual* & *variation* of *information*

$$NMI = \frac{2MI(\mathcal{C},\mathcal{D})}{H(\mathcal{C})+H(\mathcal{D})} = \frac{2H(\mathcal{C})-2H(\mathcal{C}|\mathcal{D})}{H(\mathcal{C})+H(\mathcal{D})} = \frac{2H(\mathcal{C})+2\sum_{CD} p_{cd}\log\frac{p_{cd}}{p_d}}{-\sum_C p_c \log p_c + H(\mathcal{D})}$$

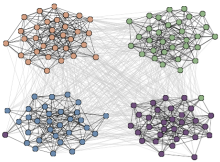
— *normalized variation of information* NVI [Mei07, KLN08]

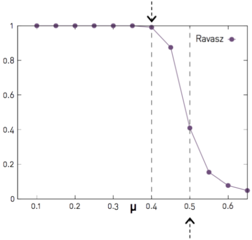
$$NVI = \frac{VI(\mathcal{C},\mathcal{D})}{\log n} = \frac{H(\mathcal{C}|\mathcal{D})+H(\mathcal{D}|\mathcal{C})}{\log n}$$

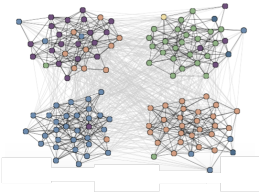
community *benchmarks*

- Girvan-Newman *synthetic graphs* [GN02]
- *planted partition* controlled by *mixing parameter* μ

$n = 128$ $\quad$ $\langle k \rangle = \langle k^{int} \rangle + \langle k^{ext} \rangle = 16$ $\quad$ $\mu = \frac{\langle k^{ext} \rangle}{\langle k \rangle}$







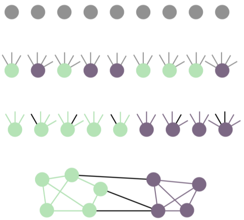
community *benchmarks*

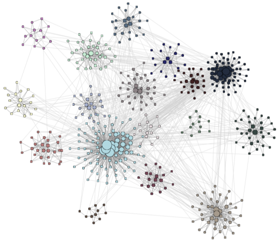
— Lanchichinetti *synthetic graphs* [LFR08]

— *power-law distributions* $p_k \sim k^{-\gamma_k}$ & $p_s \sim s^{-\gamma_s}$

— *planted communities* controlled by *mixing parameter* μ

$n = 1000,\ n_c \in [10, 50]$ $\gamma_k \in [2, 3],\ \gamma_s \in [1, 2]$ $\mu = \frac{\langle k^{ext} \rangle}{\langle k \rangle}$





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