# node *centrality*

introduction to *network analysis* (*ina*)

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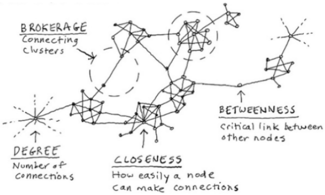
# centrality *measures*

which *nodes* are most *important*?

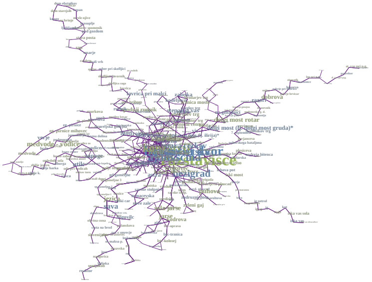
- *node centrality measures* for (*un*)*directed* networks
  - *clustering coefficients* [WS98, SV05, dNMB05]
  - *distance-based* measures [Fre77, FBW91, New05]
  - *spectral analysis* measures [Kat53, Bon87, BP98]
  - *fragment-based* measures [MSOI+02, Prž07, EK15]




- *link analysis algorithms* primarily for *directed* networks

# networkology *LPP*

- partial *LPP public bus transport network**
- $n = 416$ bus stops with $\langle k \rangle = 5.62$ connections
- *giant component* 95.4% nodes (6 components)
- *"small-world"* with $\langle C \rangle = 0.09$ and $\langle d \rangle = 14.26$
- *"scale-free"* with $\gamma = 2.62$ for cutoff $k_{min} = 5$



*reduced to largest connected component

important *nodes* are *strongly* *embedded*

- for *undirected* $G$ *clustering coefficient* $C$ [WS98] of $i$ is
  - $t_i$ is number of *linked neighbors* or *triangles* of $i$

$$C_i = \frac{2t_i}{k_i(k_i-1)} \qquad C_i = 0 \text{ for } k_i \leq 1$$

- $\omega$-*corrected clustering coefficient* $C^\omega$ [SV05] of $i$ is
  - $\omega_i$ is *maximum possible* $t_i$ with *respect to* $\{k\}$

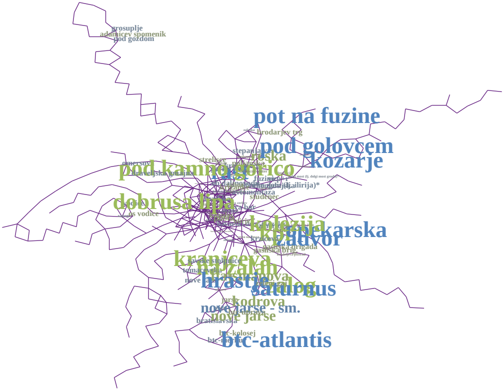
$$C_i^\omega = \frac{t_i}{\omega_i} \qquad C_i^\omega = 0 \text{ for } \omega_i = 0$$

- $\mu$-*corrected clustering coefficient* $C^\mu$ [Bat19] of $i$ is
  - $\mu$ is *maximum* number of *triangles* over *links*

$$C_i^\mu = \frac{2t_i}{k_i\mu} \qquad C_i^\mu = 0 \text{ for } k_i = 0$$

# networkology *clustering*

- *clustering coefficient* $C$ in partial LPP network†
- *highest* $C_i = 1.0$ nodes are *Na Žalah etc.* with $k_i = 2$

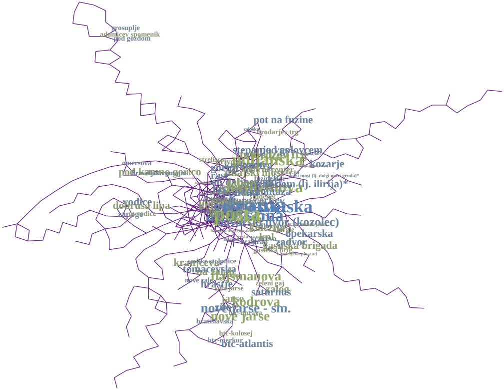



† reduced to simple undirected graph

# networkology *$\mu$-clustering*

— *$\mu$-corrected clustering $C^\mu$* in partial LPP network[‡]

— *highest* $C_i^\mu = 0.44$ node is *Drama* with $k_i = 10$




‡reduced to simple undirected graph

# centrality *closeness*

important *nodes* are *close to other* nodes

— for (*un*)*directed* $G$ *closeness centrality* $\ell^{-1}$ [New10] of $i$ is

  – $d_{ij}$ is (*un*)*directed distance* between $i$ and $j$
  – $d_{ij} = \infty$ for nodes in *different components*

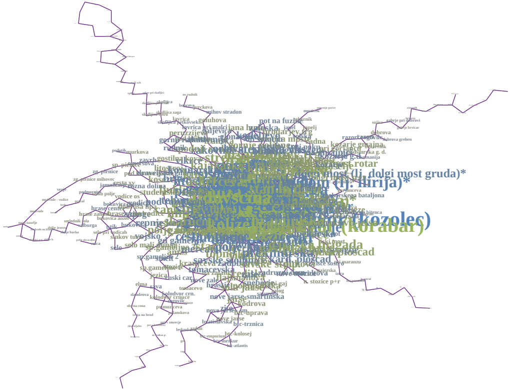
$$\ell_i^{-1} = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{d_{ij}}$$

— $\ell^{-1}$ spans *small range* in *small-world* networks

# networkology *closeness*

— *closeness centrality* $\ell^{-1}$ in partial LPP network[§]

— *highest* $\ell_i^{-1} = 0.208$ node is *Gosposvetska* with $k_i = 14$




§ reduced to simple undirected graph

# centrality *betweenness*

important *nodes* are *bridges between other* nodes

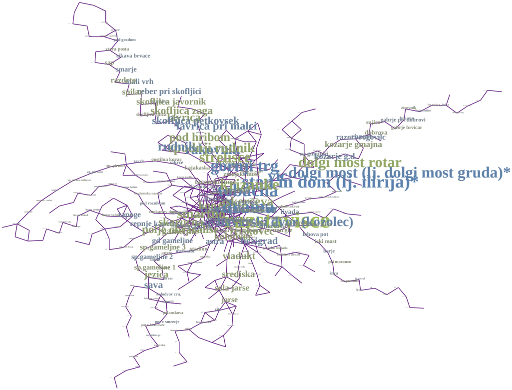
- for (*un*)*directed* $G$ *betweenness centrality* $\sigma$ [Fre77] of $i$ is
  - $g_{st}$ is number of *shortest paths between* $s$ and $t$
  - $g_{st}^i$ is number of *such shortest paths through* $i$

$$\sigma_i = \frac{1}{n^2}\sum_{st}\frac{g_{st}^i}{g_{st}}$$

- $\sigma$ considers *only shortest paths* [FBW91, New05]
- $\sigma$ mixes *local centers* with *global bridges* [JMK+16]

# networkology *betweenness*

- *betweenness centrality* $\sigma$ in partial LPP network¶
- *highest* $\sigma_i = 0.235$ node is *Razstavišče* with $k_i = 11$




¶ reduced to simple undirected graph

# centrality *degrees*

important *nodes* are *linked by many* nodes

— for *undirected* $G$ *degree centrality* $d$ of $i$ is

$$d_i = \frac{1}{n-1}\sum_{j\neq i} A_{ij} = \frac{k_i}{n-1}$$

— in *directed* $G$ *in-degree centrality* $d^{in}$ of $i$ is

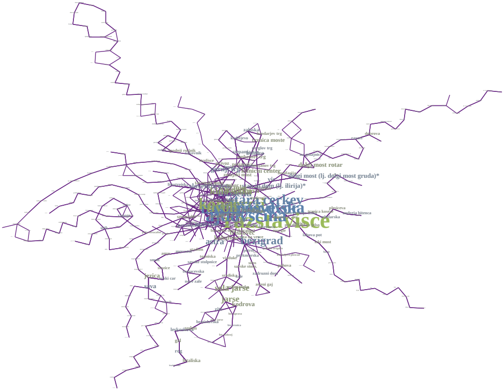
$$d_i^{in} = \frac{1}{n-1}\sum_{j\neq i} A_{ij} = \frac{k_i^{in}}{n-1}$$

— in *directed* $G$ *out-degree centrality* $d^{out}$ of $i$ is

$$d_i^{out} = \frac{1}{n-1}\sum_{j\neq i} A_{ji} = \frac{k_i^{out}}{n-1}$$

# networkology *degrees*

- *degree centrality* $d$ in partial LPP network
- *highest* $d_i = 0.099$ node is *Razstavišče* with $k_i = 41$
- *highest* $d_i^{in}$ node is *Razstavišče* with $k_i^{in} = 20$ and $k_i^{out} = 21$



# centrality *eigenvector*

important *nodes* are *linked by important* nodes

— for (*un*)*directed* $G$ *eigenvector centrality* $e$ [Bon87] of $i$ is

- $v$ and $\lambda$ are *eigenvectors* and *eigenvalues* of $A$
- $e$ is *proportional* to *leading eigenvector* $v_1$

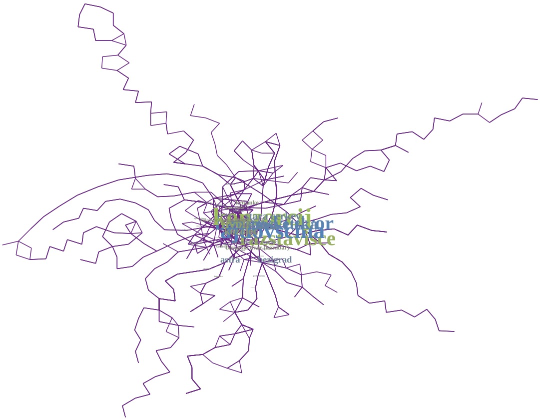
$$e(t) = A^t e(0) = A^t \sum_i C_i v_i = \sum_i C_i \lambda_i^t v_i = \lambda_1^t \sum_i C_i \left[\frac{\lambda_i}{\lambda_1}\right]^t v_i \rightarrow C_1 \lambda_1^t v_1$$

$$e_i = \lambda_1^{-1} \sum_j A_{ij} e_j$$

— in *directed* $G$ $e = 0$ for $k^{in} = 0$ *nodes etc.*

# networkology *eigenvector*

- *eigenvector centrality* $e$ in partial LPP network
- *highest* $e_i = 0.082$ node is *Konzorcij* with $k_i = 30$



# centrality *Katz*

*nodes get* small amount of *importance for free*

- for (*un*)*directed* $G$ *Katz centrality* $z$ [Kat53] of $i$ is
  - $\alpha$ and $\beta_i$ are some *positive constants*

$$z_i = \alpha \sum_j A_{ij} z_j + \beta_i$$

- for *convenience* $\beta_i = 1$ whereas $\alpha < \lambda_1^{-1}$
  - $\lambda_1$ is *leading eigenvalue* of $A$

# centrality *PageRank*

*nodes distribute equal* amount of *importance*

— for (*un*)*directed* $G$ *PageRank centrality* $p$ [BP98] of $i$ is

  – $\alpha$ and $\beta_i$ are some *positive constants*

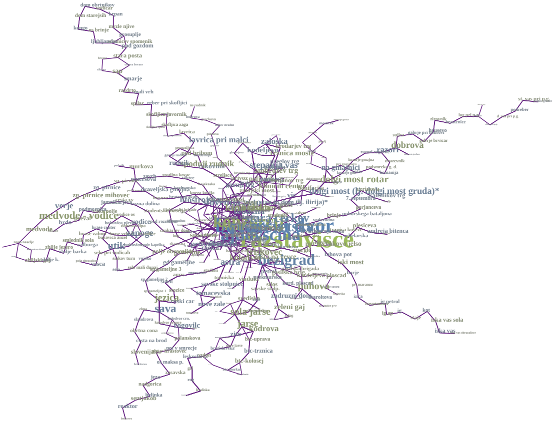
$$p_i = \alpha \sum_j A_{ij} \frac{p_j}{k_j^{out}} + \beta_i$$

— for *convenience* $\beta_i = \frac{1-\alpha}{n}$ whereas $\alpha = 0.85$

see PageRank algorithm NetLogo demo

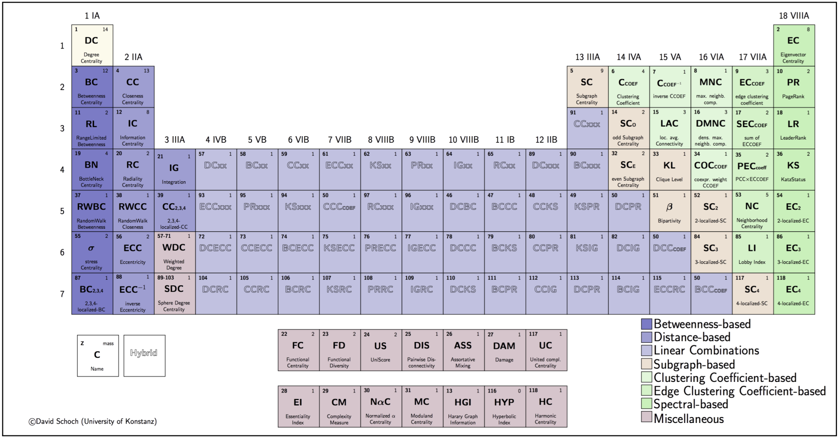
# networkology *PageRank*

- *PageRank centrality* $p$ in partial LPP network
- *highest* $p_i = 0.011$ node is *Razstavišče* with $k_i = 41$



# centrality *overview*

which *nodes* are most *important*?



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