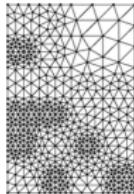


# network *blockmodeling*

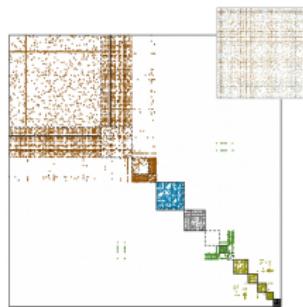
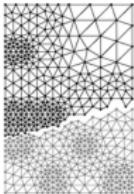
introduction to *network analysis* (*ina*)

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spring 2024/25

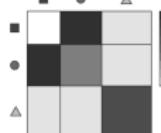
# blockmodeling overview



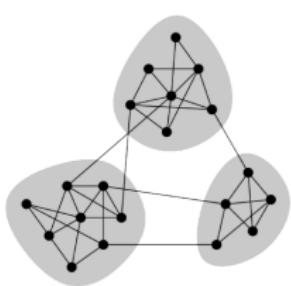
graph partitioning [KL70, Fie73]



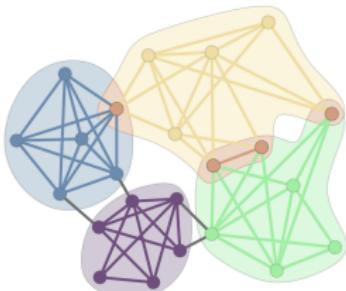
blockmodeling [LW71, WR83]



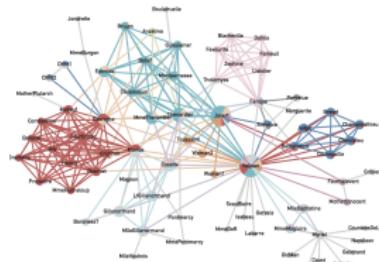
stochastic block model [Pei15]



communities [GN02]



overlapping communities [PDFV05]

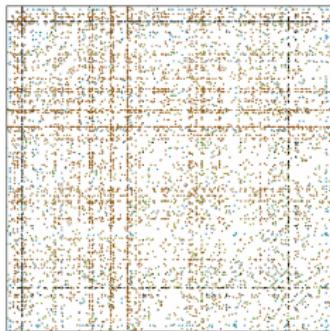


link communities [EL09, ABL10]

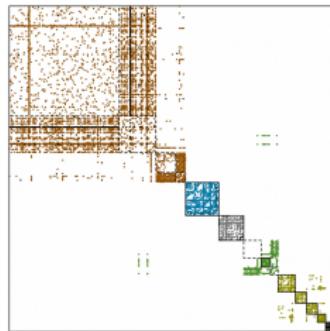
\* assortative & disassortative equivalence blockmodeling

# blockmodeling *equivalence*

- standard equivalence blockmodeling [DBF05]
  - define *node similarity* as (*structural*) equivalence
$$\sigma_{ij} \sim |\Gamma_i \cap \Gamma_j|$$
- 1. *blockmodeling* by (*hierarchical*) clustering  $\mathcal{O}(n^2)$
- 2. return *block model* at desired *clustering resolution*



javax adjacency matrix



javax block model



javax.swing, javax.management, javax.naming, javax.print, javax.xml, javax.lang etc.

# blockmodeling *structural*

*similar* nodes have *same* neighbors

- standard structural equivalence [LW71] of  $i$  and  $j$  is

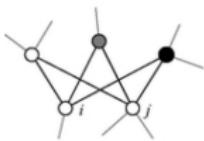
$$\sigma_{ij} = \sum_x A_{ix} A_{xj} = |\Gamma_i \cap \Gamma_j|$$

- Salton structural equivalence [SM83] of  $i$  and  $j$  is

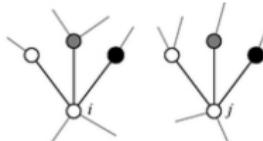
–  $\theta_{ij}$  is angle between neighborhoods  $A_i$  and  $A_j$

$$\sigma_{ij} = \cos \theta_{ij} = \frac{\sum_x A_{ix} A_{xj}}{\sqrt{\sum_x A_{ix}^2} \sqrt{\sum_x A_{xj}^2}} = \frac{|\Gamma_i \cap \Gamma_j|}{\sqrt{k_i k_j}}$$

- Leicht structural equivalence [LHN06] of  $i$  and  $j$  is  $\sigma_{ij} = \frac{|\Gamma_i \cap \Gamma_j|}{k_i k_j / n}$



structural

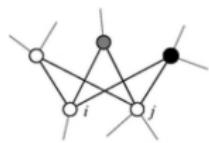


regular equivalence

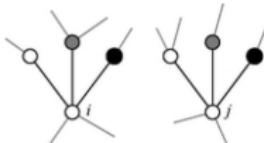
# blockmodeling *regular*

*similar* nodes have *equivalent* *neighbors*

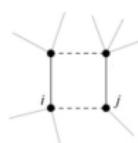
- *standard regular equivalence* [WR83] of *i* and *j* is
  - $\alpha < \lambda^{-1}$  is *positive constant* and  $\lambda$  *leading eigenvalue* of  $A$
  - $\sigma_{ij} = \alpha \sum_{xy} A_{ix} A_{jy} \sigma_{xy} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sum_{y \in \Gamma_j} \sigma_{xy} + \delta_{ij}$
  - $\sigma = \alpha A \sigma A + I$  and thus  $\sigma^{(0)} = 0$ ,  $\sigma^{(1)} = I$ ,  $\sigma^{(2)} = \alpha A^2 + I$ ,  $\sigma^{(3)} = \alpha^2 A^4 + \alpha A^2 + I$  etc.
- *Katz regular equivalence* [Kat53] of *i* and *j* is
  - $\sigma_{ij} = \alpha \sum_x A_{ix} \sigma_{xj} + \delta_{ij} = \alpha \sum_{x \in \Gamma_i} \sigma_{xj} + \delta_{ij}$
  - $\sigma = \alpha A \sigma + I$  and thus  $\sigma^{(0)} = 0$ ,  $\sigma^{(1)} = I$ ,  $\sigma^{(2)} = \alpha A + I$ ,  $\sigma^{(3)} = \alpha^2 A^2 + \alpha A + I$  etc.



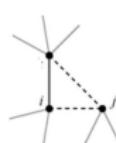
structural



regular equivalence

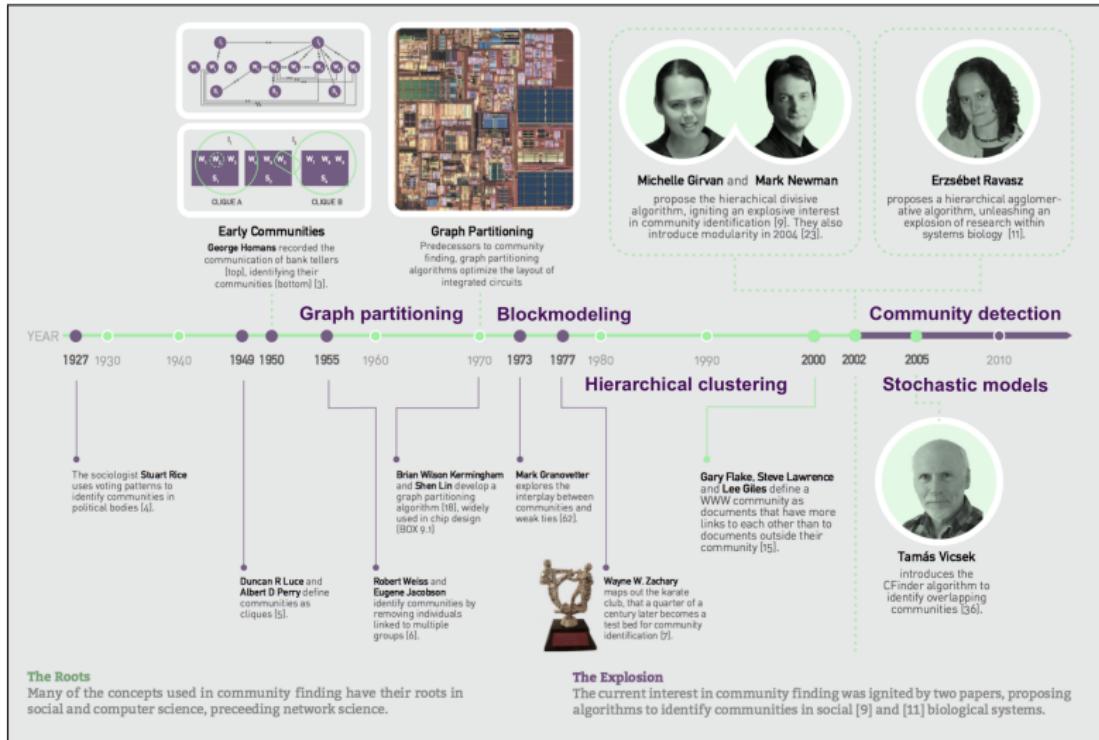


standard



Katz

# blockmodeling *history*



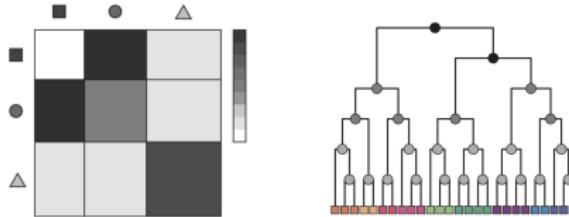
# *stochastic* models

introduction to *network analysis* (*ina*)

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spring 2024/25

# stochastic *models*

- random graph model  $G(n, m)$  for network links  $m$  [ER59]
- configuration model  $G(\{k\})$  for node degrees  $\{k\}$  [NSW01]
- exponential  $p^*$ -model  $G(n, \{\langle x \rangle\})$  for any expectations  $\{\langle x \rangle\}$
- stochastic block model  $G(\{C\})$  for node clusters  $\{C\}$  [HLL83]
- hierarchical model  $G(H)$  for node hierarchy  $H$  [CMN08]

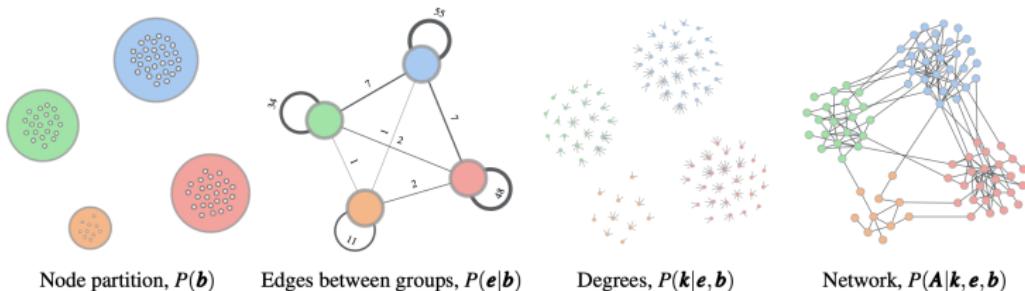


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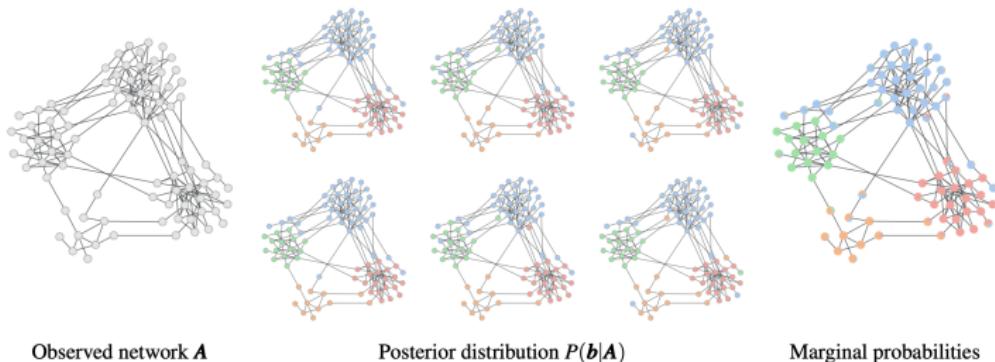
\* assortative & disassortative stochastic block models

# stochastic *process*

(a) Generative process



(b) Inference procedure



# stochastic $G(\{C\})$ model

- $G(\{C\}, \{p\})$  stochastic block model [HLL83]
- link between  $i$  and  $j$  placed with probability  $p_{c_i c_j}$

—  $m_{c_i c_j}$  is number of links between  $C_i$  and  $C_j$

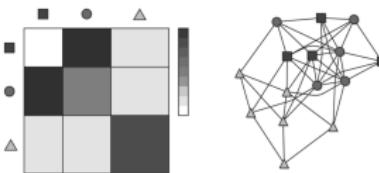
—  $M_{c_i c_j}$  is maximum  $m_{c_i c_j}$  hence  $n_i n_j$  or  $\binom{n_i}{2}$

$$P(A|\{C\}, \{p\}) = \prod_{i \leq j} p_{c_i c_j}^{A_{ij}} (1 - p_{c_i c_j})^{1 - A_{ij}} = \prod_{c_i \leq c_j} p_{c_i c_j}^{m_{c_i c_j}} (1 - p_{c_i c_j})^{M_{c_i c_j} - m_{c_i c_j}}$$

- maximum likelihood  $G(\{C\})$  block model

—  $\frac{m_{c_i c_j}}{M_{c_i c_j}}$  is maximum likelihood estimate for  $p_{c_i c_j}$

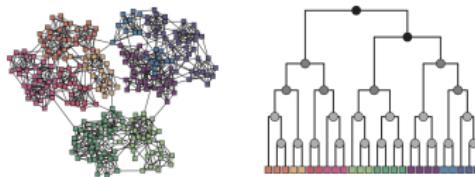
$$\mathcal{L}(A|\{C\}) = \log P(A|\{C\}) = \sum_{c_i \leq c_j} m_{c_i c_j} \log \frac{m_{c_i c_j}}{M_{c_i c_j} - m_{c_i c_j}} + M_{c_i c_j} \log \frac{M_{c_i c_j} - m_{c_i c_j}}{M_{c_i c_j}}$$



see graph-tool implementation

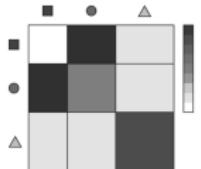
# stochastic $G(H)$ model

- $G(H, \{p\})$  hierarchical model [CMN08]
- link between  $i$  and  $j$  placed with probability  $p_{r_{ij}}$ 
  - $r$  is root with subtrees  $L_r, R_r$  and  $r_{ij}$  lowest root of  $i$  and  $j$
  - $m_r$  is number of links between  $L_r, R_r$  and  $M_r$  is  $|L_r||R_r|$
- $P(A|H, \{p\}) = \prod_{i \leq j} p_{r_{ij}}^{A_{ij}} (1 - p_{r_{ij}})^{1 - A_{ij}} = \prod_r p_r^{m_r} (1 - p_r)^{M_r - m_r}$
- maximum likelihood  $G(H)$  hierarchical model
  - $\frac{m_r}{M_r}$  is maximum likelihood estimate for  $p_r$

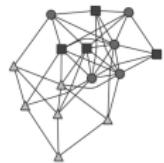


see randomgraphs implementation

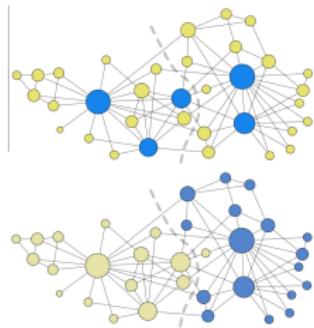
# stochastic overview



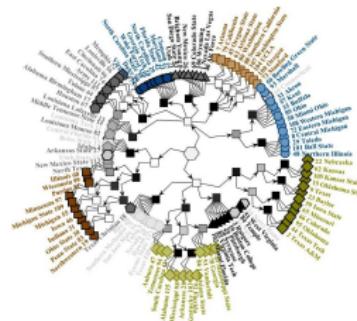
stochastic block model [HLL83]



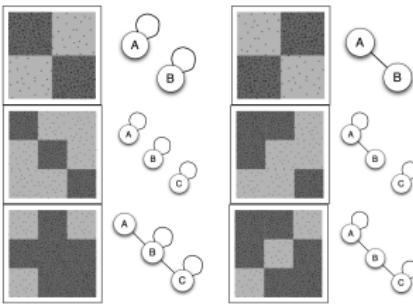
degree-corrected SBM [KN11]



nested SBM [Pei15]



hierarchical models [CMN08, ŠB14]



role models [RW07, NL07, GSPA07]

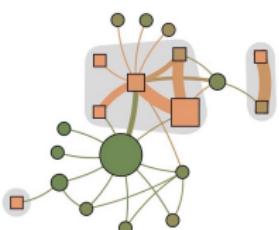


image graphs [ŠB12]

†

overlapping & corrected models also known as mixture & mixed membership models

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