# node *centrality*

introduction to *network analysis* (*ina*)

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centrality *measures*

which *nodes* are most *important*?

— *node centrality measures* for (*un*)*directed* networks
  - *clustering coefficients* [WS98, SV05, dNMB05]
  - *distance-based* measures [Fre77, FBW91, New05]
  - *spectral analysis* measures [Kat53, Bon87, BP98]
  - *fragment-based* measures [MSOI$^+$02, Prž07, EK15]
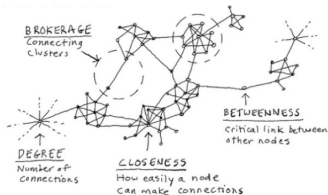
— *link analysis algorithms* primarily for *directed* networks

— partial *LPP* public bus transport network*

— $n = 416$ bus stops with $\langle k \rangle = 5.62$ connections

— *giant component* 95.4% nodes (6 components)

— "small-world" with $\langle C \rangle = 0.09$ and $\langle d \rangle = 14.26$

— "scale-free" with $\gamma = 2.62$ for cutoff $k_{min} = 5$
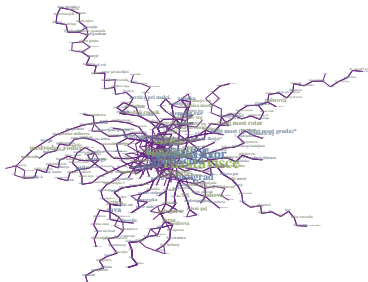
*reduced to largest connected component

centrality *clustering*

important *nodes* are *strongly embedded*

— for *undirected* G *clustering coefficient* C [WS98] of i is

    – t_i is number of *linked neighbors* or *triangles* of i

$$C_i = \frac{2t_i}{k_i(k_i-1)} \qquad C_i = 0 \text{ for } k_i \leq 1$$

— ω-*corrected clustering coefficient* C^ω [SV05] of i is

   – ω_i is *maximum possible* t_i with *respect to* $\{k\}$

$$C_i^\omega = \frac{t_i}{\omega_i} \qquad C_i^\omega = 0 \text{ for } \omega_i = 0$$

— μ-*corrected clustering coefficient* C^μ [Bat19] of i is

    – μ is *maximum* number of *triangles* over *links*

$$C_i^\mu = \frac{2t_i}{k_i\mu} \qquad C_i^\mu = 0 \text{ for } k_i = 0$$

networkology *clustering*

— *clustering coefficient C* in partial LPP network[†]

— *highest $C_i = 1.0$* nodes are *Na Žalah etc.* with $k_i = 2$

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networkology μ-clustering

— *μ-corrected clustering C^{μ} in partial LPP network*[‡]

— *highest $C_i^{\mu} = 0.44$* node is *Drama* with $k_i = 10$
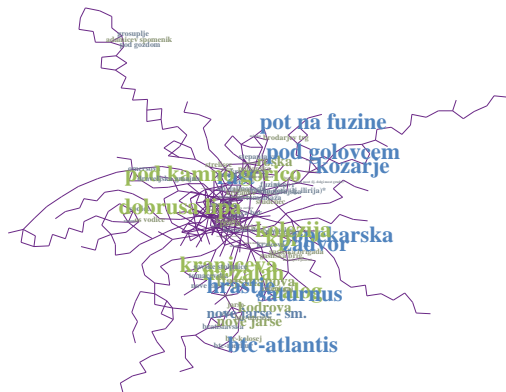
[‡]reduced to simple undirected graph

# centrality *closeness*

important *nodes* are *close to other* nodes

— for (*un*)*directed G closeness centrality* ℓ^{-1} [New10] of i is
  – d_{ij} is (*un*)*directed distance* between i and j
  – $d_{ij} = \infty$ for nodes in *different components*

$$\ell_i^{-1} = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{d_{ij}}$$

— ℓ^{-1} spans *small range* in *small-world* networks

networkology *closeness*

— *closeness centrality ℓ^{-1}* in partial LPP network[§]

— *highest $\ell_i^{-1} = 0.208$* node is *Gosposvetska* with $k_i = 14$
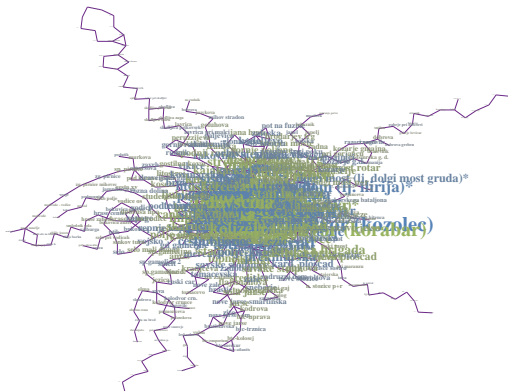
[§]reduced to simple undirected graph

centrality *betweenness*

important *nodes* are *bridges between other* nodes

— for (*un*)*directed G betweenness centrality* σ [Fre77] of *i* is
  – g_{st} is number of *shortest paths between s* and *t*
  – g_{st}^{i} is number of *such shortest paths through i*

$$\sigma_i = \frac{1}{n^2} \sum_{st} \frac{g_{st}^i}{g_{st}}$$

— σ considers *only shortest paths* [FBW91, New05]
— σ mixes *local centers* with *global bridges* [JMK+16]

— *betweenness centrality* σ in partial LPP network[¶]

— *highest* $\sigma_i = 0.235$ node is *Razstavišče* with $k_i = 11$

[¶] reduced to simple undirected graph

centrality *degrees*

important *nodes* are *linked by many* nodes

— for *undirected G degree centrality d* of *i* is
$$d_i = \frac{1}{n-1} \sum_{j \neq i} A_{ij} = \frac{k_i}{n-1}$$

— in *directed G in-degree centrality d^{in}* of *i* is
$$d_i^{in} = \frac{1}{n-1} \sum_{j \neq i} A_{ij} = \frac{k_i^{in}}{n-1}$$

— in *directed G out-degree centrality d^{out}* of *i* is
$$d_i^{out} = \frac{1}{n-1} \sum_{j \neq i} A_{ji} = \frac{k_i^{out}}{n-1}$$

networkology *degrees*

- — *degree centrality d* in partial LPP network
- — *highest $d_i = 0.099$* node is *Razstavišče* with $k_i = 41$
- — *highest d_i^{in}* node is *Razstavišče* with $k_i^{in} = 20$ and $k_i^{out} = 21$
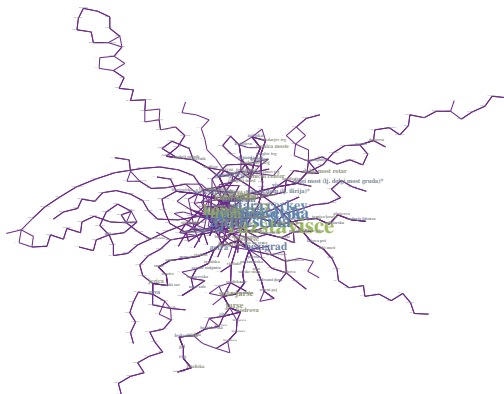
centrality *eigenvector*

important *nodes* are *linked by important* nodes

— for (*un*)*directed G eigenvector centrality e* [Bon87] of i is

  – v and λ are *eigenvectors* and *eigenvalues* of A

  – e is *proportional* to *leading eigenvector* v_1

$$e(t) = A^t e(0) = A^t \sum_i C_i v_i = \sum_i C_i \lambda_i^t v_i = \lambda_1^t \sum_i C_i \left[\frac{\lambda_i}{\lambda_1} \right]^t v_i \to C_1 \lambda_1^t v_1$$

$$e_i = \lambda_1^{-1} \sum_j A_{ij} e_j$$

— in *directed G* $e = 0$ for $k^{in} = 0$ *nodes etc.*

networkology *eigenvector*

— *eigenvector centrality* e in partial LPP network
— *highest* $e_i = 0.082$ node is *Konzorcij* with $k_i = 30$
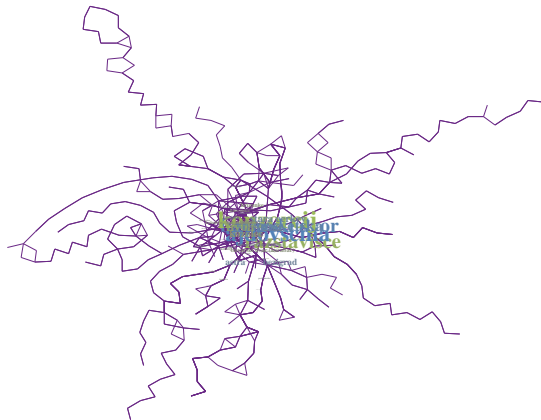
centrality *Katz*

nodes get small amount of *importance for free*

— for (*un*)*directed G Katz centrality z* [Kat53] of *i* is
  – α and β_i are some *positive constants*

$$z_i = \alpha \sum_j A_{ij} z_j + \beta_i$$

— for *convenience* $\beta_i = 1$ whereas $\alpha < \lambda_1^{-1}$
  – λ_1 is *leading eigenvalue* of *A*

centrality *PageRank*

nodes distribute *equal* amount of *importance*

— for (*un*)*directed G PageRank centrality p* [BP98] of *i* is
  – α and β_i are some *positive constants*

$$p_i = \alpha \sum_j A_{ij} \frac{p_j}{k_j^{out}} + \beta_i$$

— for *convenience* $\beta_i = \frac{1-\alpha}{n}$ whereas $\alpha = 0.85$

see PageRank algorithm NetLogo demo

networkology *PageRank*

— *PageRank centrality p* in partial LPP network
— *highest $p_i = 0.011$* node is *Razstavišče* with $k_i = 41$

which *nodes* are most *important*?

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Legend:
- Betweenness-based
- Distance-based
- Linear Combinations
- Subgraph-based
- Clustering Coefficient-based
- Edge Clustering Coefficient-based
- Spectral-based
- Miscellaneous

centrality *references*

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