

# graphology & *networkology*

advanced topics in *network science* (*ants*)

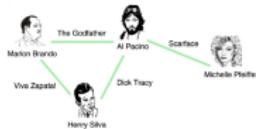
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# terminology *graphs & networks*

- *network science* perspective
  - *network* is some *real-world system*
  - *graph* is *representation of network*
- *graph theory* perspective
  - *graph* is formal *mathematical object*
  - *network* is *graph with real data*
- *social science* perspective
- but Web graph, Internet map



network



another network



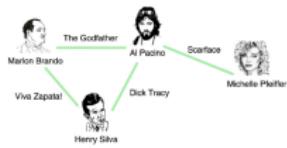
graph

# terminology *nodes & links*

- *network science* terminology
  - *nodes* and *links*
- *graph theory* terminology
  - *vertices* and *edges/relations*
- *social science* terminology
  - *agents/brokers/units* and *ties*



nodes & links



agents & ties



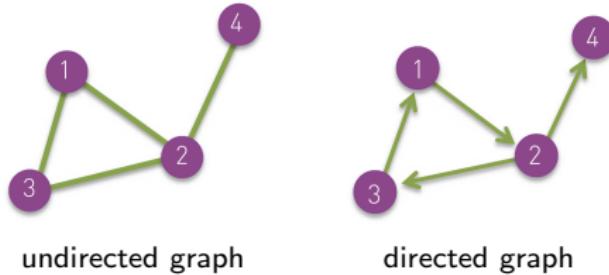
vertices & edges

# terminology *classes*

- *social* networks
  - nodes are *people or animals*, links are relations or interactions
  - Facebook, offline, online, affiliation, author/actor collaboration
- *information* networks
  - nodes are information sources, links resemble *information flow*
  - Web, Twitter, citation, communication, peer-to-peer
- *technological* networks
  - human-made infrastructure with *technological constraints*
  - Internet, telephone, transportation, power grid, software
- *biological* networks
  - interaction between genes, cells, neurons in *living beings*
  - gene regulatory, metabolic, protein interaction, neural
- *ecological, lexical, financial, sports* etc. networks

# graphology *graphs & digraphs*

- graph  $G$  is defined by
  - set of nodes  $N = \{1, 2, \dots, n\}$
  - set of links  $L$  where  $m = |L|$
- if  $G$  is *undirected* then  $L \subseteq \{\{i, j\} \mid i, j \in N\}$
- if  $G$  is *directed* then  $L \subseteq \{(i, j) \mid i, j \in N\}$



## graphology *adjacency*

- *adjacency matrix*  $A$  is  $n \times n$  matrix defined as
  - $A_{ij} = 1$  if there is link *from j to i*
  - $A_{ij} = 0$  if  $i = j$  or *otherwise*
- if  $G$  is *undirected* then  $A_{ij} = A_{ji}$  and  $\sum_{ij} A_{ij} = 2m$
- if  $G$  is *directed* then  $A_{ij} \neq A_{ji}$  and  $\sum_{ij} A_{ij} = m$

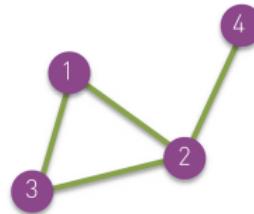
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

undirected graph

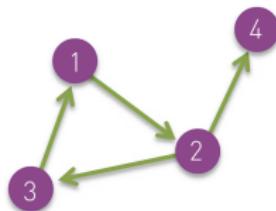
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

directed graph

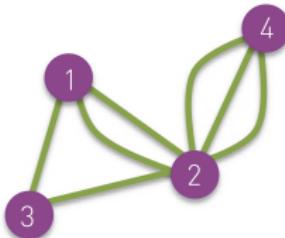
# graphology *multigraphs*



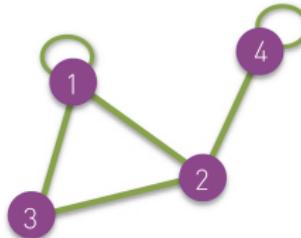
*simple undirected*  
 $A_{ij} = A_{ji} \in \{0, 1\}$



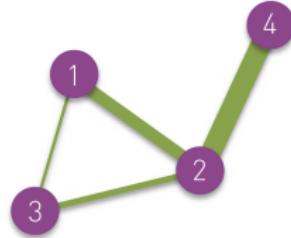
*simple directed*  
 $A_{ij} \neq A_{ji} \in \{0, 1\}$



*multigraph*  $A_{ij} \in \mathbb{N}_0$



*self-loops*  $A_{ii} = 2$



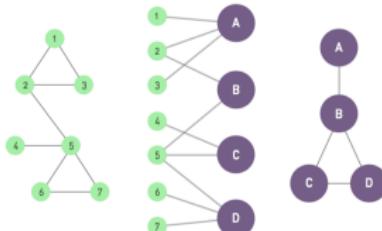
*weighted*  $W_{ij} \in \mathbb{R}_{\geq 0}$

# graphology *multipartite*

- undirected *bipartite graph*  $G_B$  is defined by
  - sets of nodes  $N_1 = \{1, 2 \dots n_1\}$  and  $N_2 = \{1, 2 \dots n_2\}$
  - set of  $m$  links  $\subseteq N_1 \times N_2$
- *incidence matrix*  $B$  is  $n_2 \times n_1$  matrix defined as
  - $B_{ij} = 1$  if there is link *between*  $j$  *and*  $i$
  - $B_{ij} = 0$  *otherwise*
- (*one-mode*) *projections* are multigraphs with

$$A = B^T B - D_1$$

$$A = BB^T - D_2$$

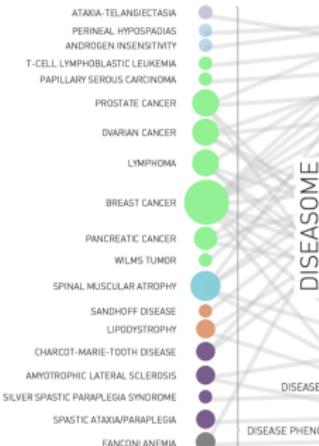


bipartite graph & projections

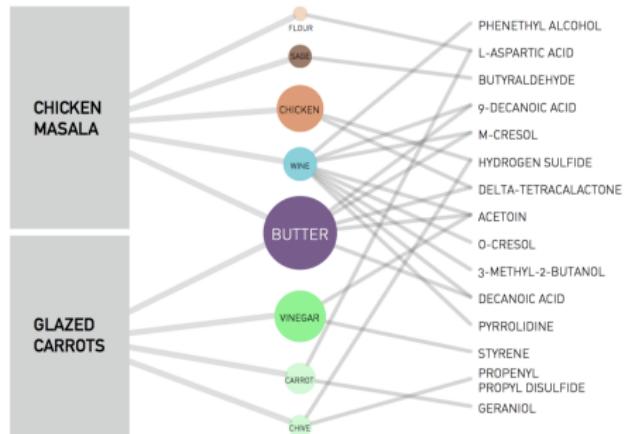
$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

incidence matrix

# networkkology *multi-mode*



DISEASOME



bipartite or two-mode network

tripartite graph or three-mode network

## graphology *degrees*

- for *undirected*  $G$  degree  $k_i$  of  $i$  is number of *incident links*

$$k_i = \sum_j A_{ij} = \sum_j A_{ji}$$

- for *directed*  $G$  degree  $k_i = k_i^{in} + k_i^{out}$

- *in-degree*  $k_i^{in}$  of  $i$  is number of *incoming links*

$$k_i^{in} = \sum_j A_{ij}$$

- *out-degree*  $k_i^{out}$  of  $i$  is number of *outgoing links*

$$k_i^{out} = \sum_j A_{ji}$$

- thus (*network*) *average degrees*  $\langle k \rangle$  and  $\langle k^{\cdot} \rangle$  are

$$\langle k \rangle = 2m/n \quad \langle k^{\cdot} \rangle = m/n$$

# networkology *degrees*

- average degrees  $\langle k \rangle$  of real networks [Bar16]
- mostly  $\langle k \rangle \leq 10$  despite very different  $n$

NETWORK	NODES	LINKS	DIRECTED UNDIRECTED	N	L	$\langle k \rangle$
Internet	Routers	Internet connections	Undirected	192,244	609,066	6.34
WWW	Webpages	Links	Directed	325,729	1,497,134	4.60
Power Grid	Power plants, transformers	Cables	Undirected	4,941	6,594	2.67
Mobile Phone Calls	Subscribers	Calls	Directed	36,595	91,826	2.51
Email	Email addresses	Emails	Directed	57,194	103,731	1.81
Science Collaboration	Scientists	Co-authorship	Undirected	23,133	93,439	8.08
Actor Network	Actors	Co-acting	Undirected	702,388	29,397,908	83.71
Citation Network	Paper	Citations	Directed	449,673	4,689,479	10.43
E. Coli Metabolism	Metabolites	Chemical reactions	Directed	1,039	5,802	5.58
Protein Interactions	Proteins	Binding interactions	Undirected	2,018	2,930	2.90

- $\langle k \rangle = 190.5$  for Facebook friendships [BBR<sup>+</sup>12]

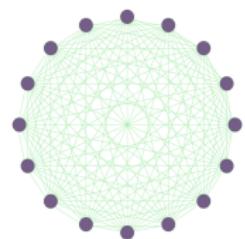
# graphology *density*

- for *undirected*  $G$  *density*  $\rho$  is defined as

$$\rho = \frac{2m}{n(n-1)} = \frac{\langle k \rangle}{n-1}$$

- for *directed*  $G$  *density*  $\rho^*$  is defined as

$$\rho^* = \frac{m}{n(n-1)} = \frac{\langle k^* \rangle}{n-1}$$



*complete*  $m = \binom{n}{2}$

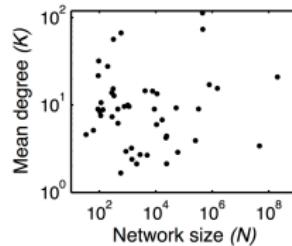
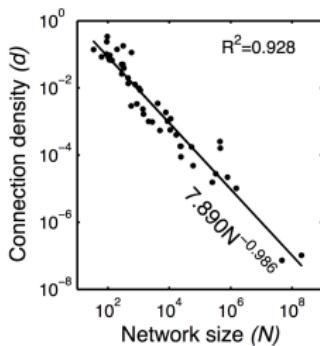


*tree*  $m = n - 1$

- $G$  is *dense* if  $\rho \rightarrow \text{const.}$  as  $n \rightarrow \infty$  thus  $\langle k \rangle = \mathcal{O}(n)$
- $G$  is *sparse* if  $\rho \rightarrow 0$  as  $n \rightarrow \infty$  thus  $\langle k \rangle \neq \mathcal{O}(n)$

# networkology *density*

- *density*  $\rho$  and *degree*  $\langle k \rangle$  of real networks [LJT<sup>+</sup>11]
- real networks are *sparse*  $\rho \approx \mathcal{O}(n^{-1})$  and  $\langle k \rangle \ll n$



- $\rho \approx \frac{138 \cdot 10^9}{721^2 \cdot 10^{12}} < 10^{-6}$  for Facebook friendships [BBR<sup>+</sup>12]
- $A$  of real networks is *almost all zeros*  $m \approx \mathcal{O}(n)$

# graphology *degree distribution*

— for *undirected G* *degree distribution*  $p_k$  is defined as

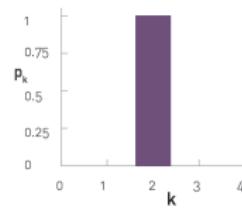
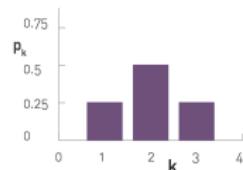
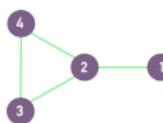
—  $n_k$  is number of *degree-k* nodes

$$p_k = n_k / n \quad \sum_k p_k = 1 \quad \langle k \rangle = \sum_k k p_k$$

— for *directed G* *in-/out-degree distributions*  $p_k^{in}$  and  $p_k^{out}$

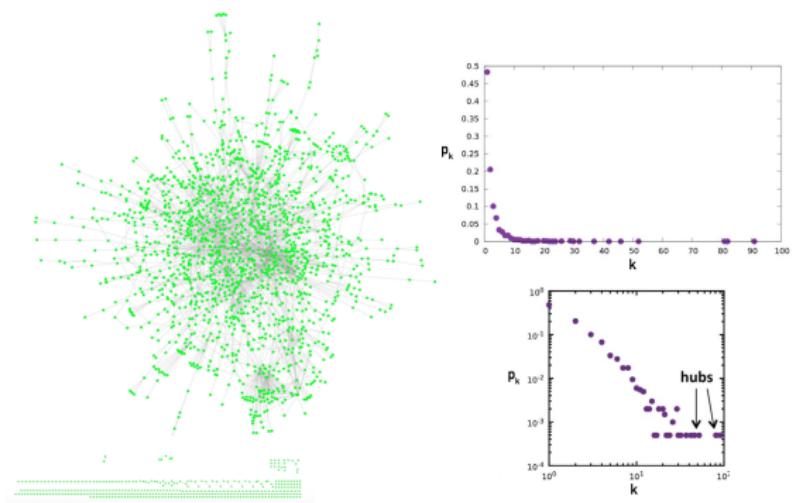
—  $n_k^{in}$  and  $n_k^{out}$  is number of *in-/out-degree-k* nodes

$$p_k^{in} = n_k^{in} / n \quad p_k^{out} = n_k^{out} / n \quad \langle k \cdot \rangle = \sum_k k p_k$$



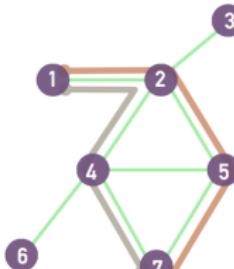
# networkology *degree distribution*

- *heavy-tail distribution*  $p_k$  of protein network [Bar16]
- nodes with *very high*  $k \gg \langle k \rangle$  are called *hubs*

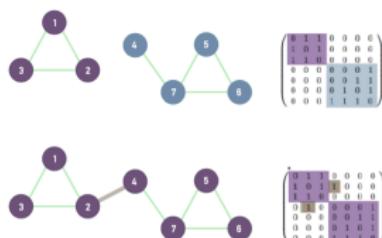


# pathology *connectivity*

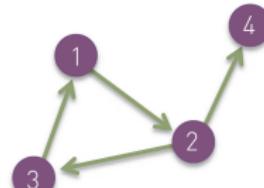
- for *undirected G* path  $P_{ij}$  is seq. of *links between i and j*
  - (*connected*) *component* is *maximal subset* thus  $\forall i, j : \exists P_{ij}$
  - *giant component* contains *nontrivial fraction* of nodes
  - *connected G* has *exactly one* connected component
- for *directed G* path  $\vec{P}_{ij}$  is seq. of *directed links from i to j*
  - *weak/strong connectivity* defined through  $P$  and  $\vec{P}$



paths  $P_{17}$



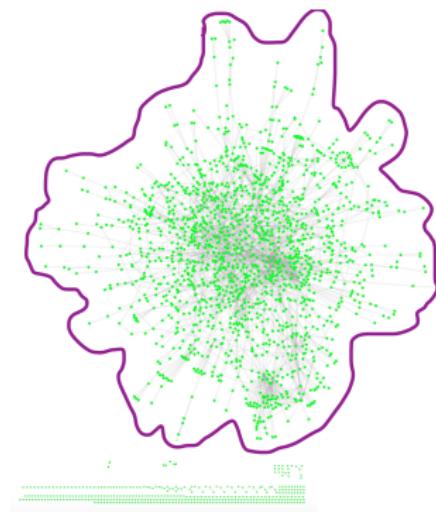
(dis)connected with bridge



WCC with two SCC

## networkology *connectivity*

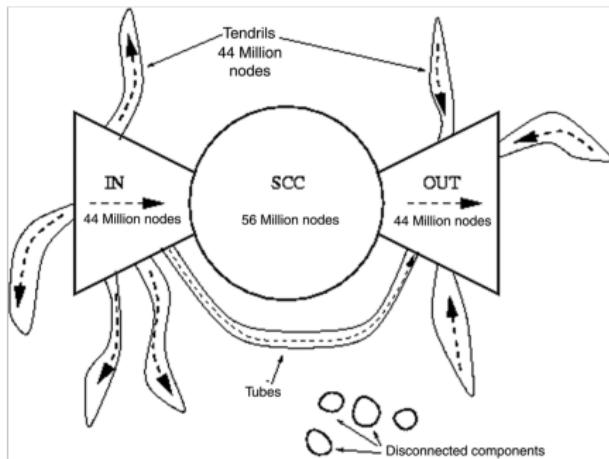
- *giant/largest component* of protein network [Bar16]



- *giant* > 99,7% for *Facebook* friendships [BBR<sup>+</sup>12]
- could real network have *two giant components*?

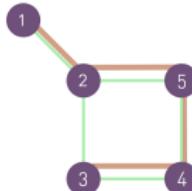
## networkology *diconnectivity*

- any *directed*  $G$  is *DAG* on its *SCC*-s
- *SCC* of  $i$  is *intersection of IN & OUT* of  $i$
- *bow-tie structure* of Web graph [BKM<sup>+</sup>00]

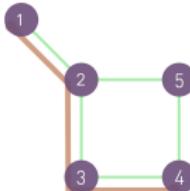


# pathology *distances*

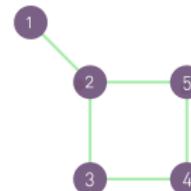
- *length* of path  $P$  or  $\overrightarrow{P}$  is number of *links/hops*
- *geodesic path*  $G_{ij}$  or  $\overrightarrow{G_{ij}}$  is any *shortest*  $P_{ij}$  or  $\overrightarrow{P_{ij}}$
- *distance*  $d_{ij}$  between  $i$  and  $j$  is *length* of  $G_{ij}$  or  $\overrightarrow{G_{ij}}$
- (network) *diameter*  $d_{\max}$  or  $D$  is *maximum*  $d_{ij}$
- (network) *average distance*  $\langle d \rangle$  or  $\ell^{-1}$  is defined as
  - $d_{ij} = 0$  or  $d_{ij} = \infty$  for  $i$  and  $j$  in different components
$$\langle d \rangle = \frac{1}{n(n-1)} \sum_{i \neq j} d_{ij} \quad \ell^{-1} = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{1}{d_{ij}}$$



$$P_{13} \neq G_{13}$$



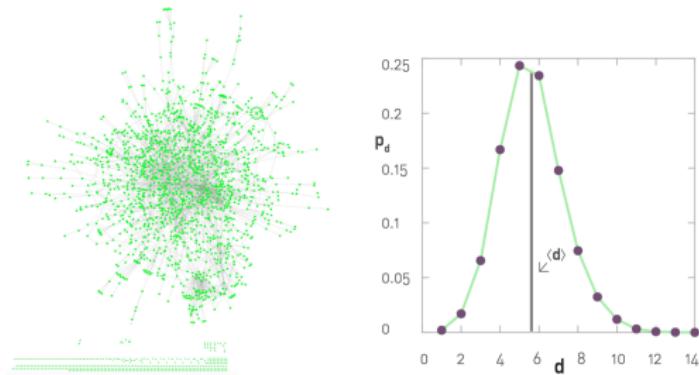
$$d_{14} = 3$$



$$\langle d \rangle = 1.6$$

## networkology *distances*

- *distance distribution*  $p_d$  of protein network [Bar16]
- most nodes are on *similar distances*  $d \approx \langle d \rangle$



- $\langle d \rangle = 4.74$  for *Facebook* friendships [BBR<sup>+</sup>12]
- real networks have *surprisingly small*  $\langle d \rangle \ll n$

# graphology *clustering*

- for *undirected G node clustering coefficient*  $C_i$  of  $i$  is
  - $t_i$  is number of *linked neighbors* or *triangles* of  $i$

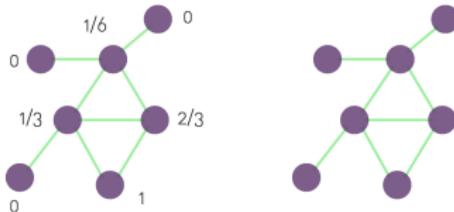
$$C_i = \frac{2t_i}{k_i(k_i-1)} \quad C_i = 0 \text{ for } k_i \leq 1$$

- *average clustering coefficient*  $\langle C \rangle$  [WS98] is defined as

$$\langle C \rangle = \frac{1}{n} \sum_i C_i$$

- *network clustering coefficient*  $C$  [NSW01] is defined as

$$C = \frac{3 \times \text{number of triangles or closed triads}}{\text{number of triples of nodes or connected triads}}$$

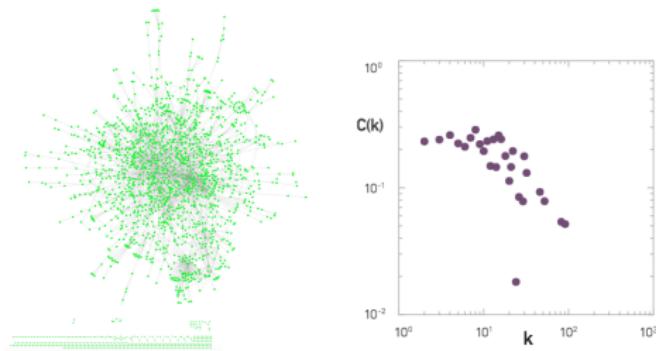


$$\langle C \rangle = \frac{13}{7 \cdot 6} = 0.31$$

$$C = \frac{3 \cdot 2}{16} = 0.38$$

# networkology *clustering*

- clustering  $C_i(k)$  of protein network [Bar16]
- hubs *much lower*  $C_i$  than nodes with  $k \approx \langle k \rangle$



- $\langle C \rangle = 0.61$  for *Facebook* social circles [ML12]
- real (social) networks have *significant*  $\langle C \rangle \gg 0$

# networkkology *references*

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