core-periphery structure

introduction to *network analysis* (*ina*)

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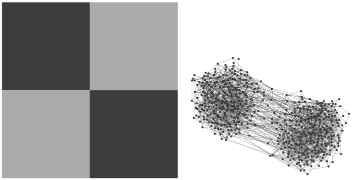
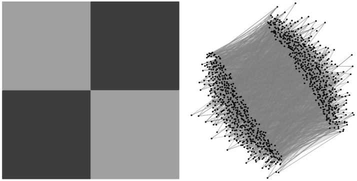
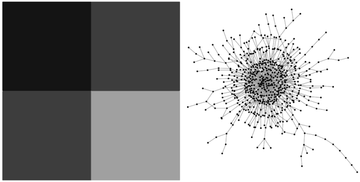
core-periphery *block model*



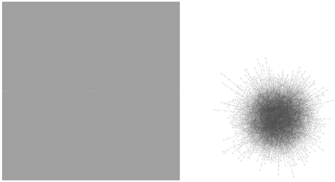
community block model [GN02]



disassortative (bipartite) block model [NL07]



core-periphery block model [Sei83]

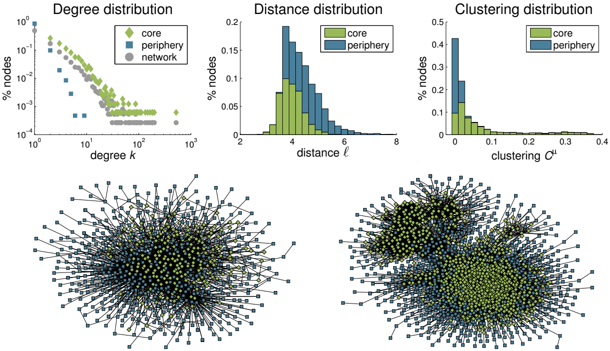


random block model [ER59]

* origin of core-periphery structure in international relations

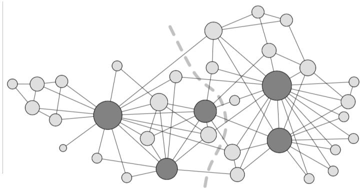
core-periphery *structure*

- *core/periphery nodes* have *higher/lower degrees k*
- *core/periphery nodes* are on *shorter/longer distances* ℓ
- *core/periphery nodes* have *higher/lower clustering C*

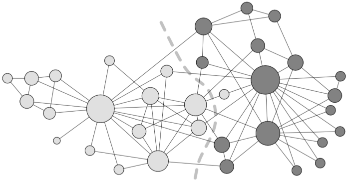


core-periphery *stochastic*

- $G(\{C_1, C_2\}, \{p_{11}, p_{12}, p_{22}\})$ *stochastic block model* [HLL83]
  - n_i is *size* of *cluster* C_i & p_{ij} is *link density* between C_i and C_j
- *density-based core-periphery* structure when $p_{11} \gg p_{12} \gg p_{22}$
- *lookalike core-periph.* when $n_1 p_{11} \gg 1$, $n_1 p_{12} \ll 1$, $n_2 p_{22} \approx 1$



non-corrected block model $p_{11} > p_{12} > p_{22}$



degree-corrected block model $p_{11} \approx p_{22} > p_{12}$

core-periphery *discrete/continuos*

- *discrete core-periphery division* $\delta \in \{0, 1\}$ [BE00]
  - $\delta_i = 1$ for *core nodes* i & $\delta_i = 0$ for *peripheral nodes* i

$$\rho_{\{0,1\}} = \sum_{ij} A_{ij}\Delta_{ij} \qquad \Delta_{ij} = \begin{cases} 1 & \text{if } \delta_i = \delta_j = 1 \\ 0 & \text{if } \delta_i = \delta_j = 0 \\ \in [0, 1] & \text{if } \delta_i - \delta_j \neq 0 \end{cases}$$

- *continuos core-periphery centrality* $\delta \in [0, 1]$
  - $\delta_i \approx 1$ for *core nodes* i & $\delta_i \approx 0$ for *peripheral nodes* i

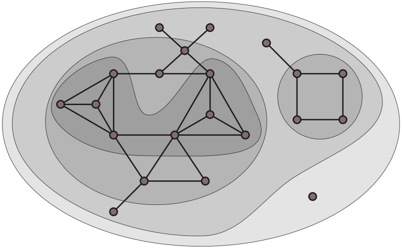
$$\rho_{[0,1]} = \sum_{ij} A_{ij}\delta_i\delta_j$$

$$\Delta^1 = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \quad \Delta^\alpha = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & \alpha & \alpha & \alpha \\ 1 & 0 & 1 & \alpha & \alpha & \alpha \\ 1 & 1 & 0 & \alpha & \alpha & \alpha \\ \hline \alpha & \alpha & \alpha & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & 0 & 0 & 0 \\ \alpha & \alpha & \alpha & 0 & 0 & 0 \end{array}\right] \quad \delta = \left[\begin{array}{c} 1 \\ 0.8 \\ 0.7 \\ \hline 0.4 \\ 0.2 \\ 0.1 \end{array}\right]$$

core-periphery *k-cores*

- *k-cores* are *subgraphs of nodes* with $\geq k$ *neighbors* [Sei83]

  remove nodes with degree $< k$ until no such node remains [BZ11]
- *k-shells* are *nodes of k-cores* that are *not in* $k+1$*-cores*
- *k-cores* are *nested* while *k-shells* form *decomposition*



1-cores are connected components w/o isolates & k-cores can be disconnected

core-periphery k^*-core

— Holme's k^*-core maximizes closeness centrality ℓ^{-1} [Hol05]

- d_{ij} is distance between i and j & ℓ_i is farness centrality of i
- ℓ_C^{-1} is closeness centrality of cluster C & n_c is size of C

$$\ell_i = \frac{1}{n-1}\sum_{j\neq i} d_{ij} \qquad \ell_C^{-1} = \left(\frac{1}{n_c}\sum_{i\in C}\ell_i\right)^{-1}$$

— Holme's core-periphery coefficient c_{cp} for k^*-core

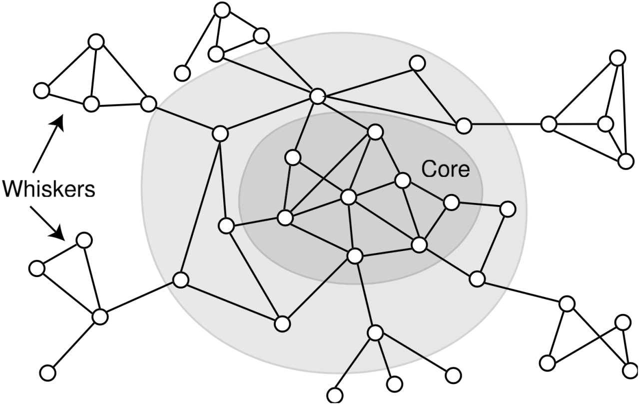
- N is set of nodes & N_k are nodes in k-core
- $\langle\ldots\rangle_{G'}$ is expectation in random graph G'

$$c_{cp} = \ell_{N_{k^*}}^{-1}/\ell_N^{-1} - \left\langle \ell_{N'_{k^*}}^{-1}/\ell_{N'}^{-1}\right\rangle_{G'}$$

core-periphery *coefficient*

Network		N	M	c_{cp}
Geographical networks	Interstate highways	935	1315	0.231(1)
	Pipelines	2999	3079	0.180(2)
	Streets, Stockholm	3325	5100	0.255(1)
	Streets, Göteborg	1258	1516	0.040(3)
	Airport	449	2795	0.0523(3)
	Internet	1968(66)	4051(121)	0.045(2)
One-mode projections of affiliation networks	arXiv	48561	287570	–0.08(3)
	Board of directors	6193	43074	–0.037(2)
	Ajou University students	7285(128)	75898(6566)	–0.08(1)
Acquaintance networks	High School friendship	571(43)	1078(85)	0.006(7)
	Prisoners	58	83	–0.043(2)
	Social scientists	34	265(35)	-0.002(4)
Electronic communication	e-mail, Ebel *et al.*	39592	57703	–0.229(4)
	e-mail, Eckmann *et al.*	3186	31856	–0.091(2)
	Internet community, nioki.com	49801	239265	–0.014(2)
	Internet community, pussokram.com	28295	115335	–0.183(5)
Reference networks	WWW, nd.edu	325729	1090108	–0.027(3)
	HEP citations	27400	352021	–0.10(1)
Software dependencies	GNU / Linux	504	793	–0.155(1)
Food webs	Little Rock Lake	92	960	0.005(6)
	Ythan Estuary	134	593	–0.020(1)
Neural network	*C. elegans*	280	1973	0.040(6)
Biochemical networks	*Drosophila* protein	2915	4121	–0.035(2)
	S. cervisiae protein	3898	7283	–0.249(1)
	S. cervisiae genetic	1503	5043	–0.0646(7)
	Metabolic networks	427(27)	1257(88)	–0.002(6)
	Whole cellular networks	623(32)	1752(103)	–0.004(6)

core-periphery *nestedness*



nested cores & whiskers communities [LLDM09, YL13]

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core-periphery *references*

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