

Erdős-Rényi *random graph*

introduction to *network analysis* (*ina*)

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graph *models*

- *graph model* is *ensemble* of random graphs
- *algorithm* for random graphs of given parameters
 - *baseline* for *network structure* statistics
 - for *reasoning* about *network evolution*
 - for *generating* new *large graphs*
- *random graph* refers to *Erdős-Rényi model* [ER59]

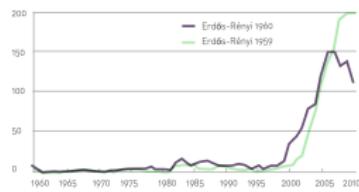
assume *undirected* G from now on



Pál Erdős



Alfréd Rényi



Erdős-Rényi model

graph $G(n, m)$ model

- $G(n, m)$ random graph model [ER59]
- randomly place m links between $\binom{n}{2}$ node pairs
- computationally convenient but analytically hard

$$n, m \text{ given} \quad \langle k \rangle = 2m/n$$

input parameters n, m

output graph G

- 1: $G \leftarrow n$ isolated nodes
- 2: while not G has m links do
- 3: add link btw random node pair
- 4: return G

graph $G(n, p)$ model

- $G(n, p)$ random graph model [SR51]
- place links between $\binom{n}{2}$ node pairs with probability p
- computationally hard but analytically convenient
 - n, p given
 - $m, \langle k \rangle$ unknown

input parameters n, p

output graph G

- 1: $G \leftarrow n$ isolated nodes
- 2: for all $\binom{n}{2}$ node pairs in G do
- 3: add link with probability p
- 4: return G

graph *density & degree*

- number of links m follows binomial distribution $B(\binom{n}{2}, p)$
 $x \sim B(n, p)$ then $p_x = \binom{n}{x} p^x (1-p)^{n-x}$ and $\langle x \rangle = np$

$$\langle m \rangle = \sum_{m=0}^{\binom{n}{2}} mP(m) = \sum_{m=0}^{\binom{n}{2}} m \binom{\binom{n}{2}}{m} p^m (1-p)^{\binom{n}{2}-m} = \binom{n}{2} p$$

- then density $\rho = p$ and average degree $\langle k \rangle = (n - 1)p$



graph *degree distribution*

- *degree distribution* p_k is *binomial distribution* $B(n - 1, p)$

$x \sim B(n, p)$ then $p_x = \binom{n}{x} p^x (1 - p)^{n-x}$ and $\langle x \rangle = np$

$$p_k = \binom{n-1}{k} p^k (1 - p)^{n-1-k}$$

- p_k approximately *Poisson distribution* $\text{Pois}(\langle k \rangle)$ for $n \gg \langle k \rangle$

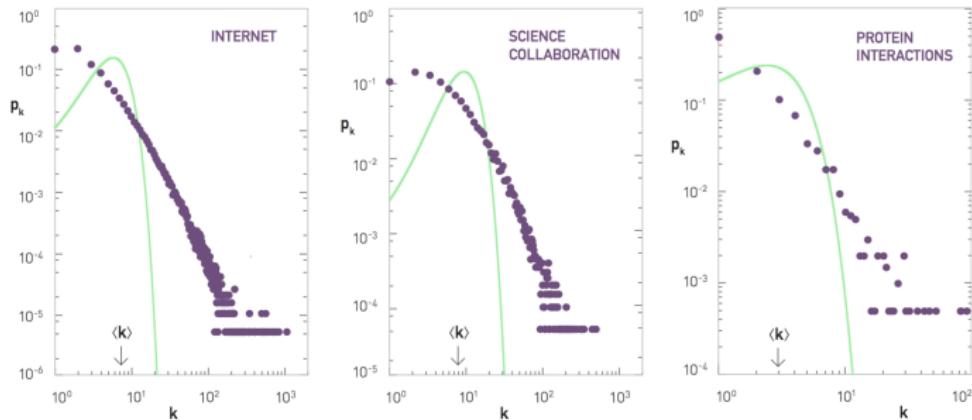
$x \sim \text{Pois}(\lambda)$ then $p_x = \frac{\lambda^x e^{-\lambda}}{x!}$ and $\langle x \rangle = \lambda$

$$\ln [(1 - p)^{n-1-k}] = (n - 1 - k) \ln \left(1 - \frac{\langle k \rangle}{n-1}\right) \simeq -(n - 1 - k) \frac{\langle k \rangle}{n-1} \simeq -\langle k \rangle$$

$$p_k \simeq \frac{(n-1)^k}{k!} \left(\frac{\langle k \rangle}{n-1}\right)^k e^{-\langle k \rangle} = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}$$

network *degree distribution*

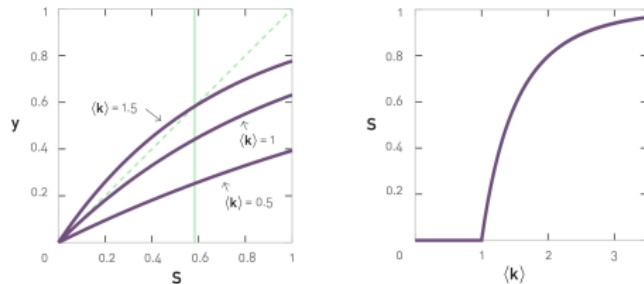
- scale-free $p_k \sim k^{-\gamma}$ of real networks [Bar16]
- real networks are *not random graphs* [ER59]
- random graphs *lack hubs* with $k \gg \langle k \rangle$



graph connectivity

- fraction of nodes in giant component S for $n \gg \langle k \rangle$

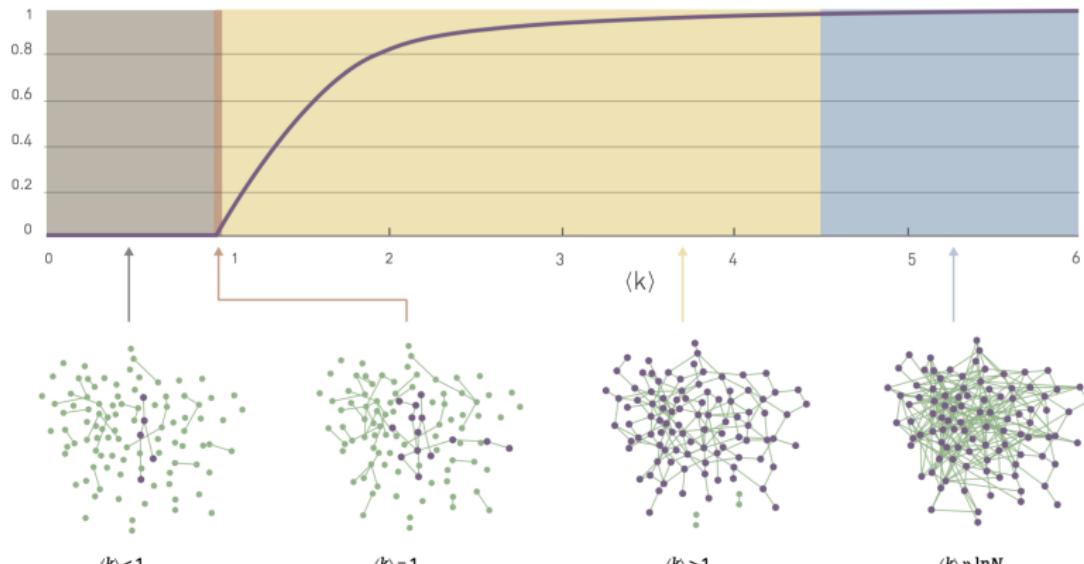
$$\ln(1 - S) = (n - 1) \ln(1 - pS) \simeq -(n - 1)pS = -(n - 1) \frac{\langle k \rangle}{n - 1} S = -\langle k \rangle S$$
$$1 - S = (1 - p + p(1 - S))^{n-1} \quad S = 1 - e^{-\langle k \rangle S}$$



- emergence of giant component or phase transition at $\langle k \rangle = 1$

$$\left. \frac{d}{dS} (1 - e^{-\langle k \rangle S}) \right|_{S=0} = \left. \langle k \rangle e^{-\langle k \rangle S} \right|_{S=0} = \langle k \rangle > 1$$

graph evolution



subcritical $n_S \sim \ln n$

critical point $n_S \sim n^{2/3}$

supercritical $n_S \sim n \frac{\langle k \rangle - 1}{n - 1}$

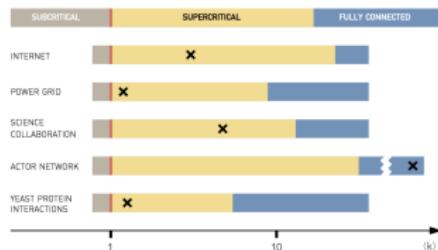
fully connected $n_S \approx n$

see random graph evolution NetLogo demo

network *connectivity*

- *connectivity* of real networks [Bar16]
- networks *supercritical* with $1 < \langle k \rangle < \ln n$

NETWORK	N	L	$\langle k \rangle$	$\ln N$
Internet	192,244	609,066	6.34	12.17
Power Grid	4,941	6,594	2.67	8.51
Science Collaboration	23,133	94,439	8.08	10.05
Actor Network	702,388	29,397,908	83.71	13.46
Protein Interactions	2,018	2,930	2.90	7.61



- Facebook friendships [BBR+12] *connected* $S > 0.997$

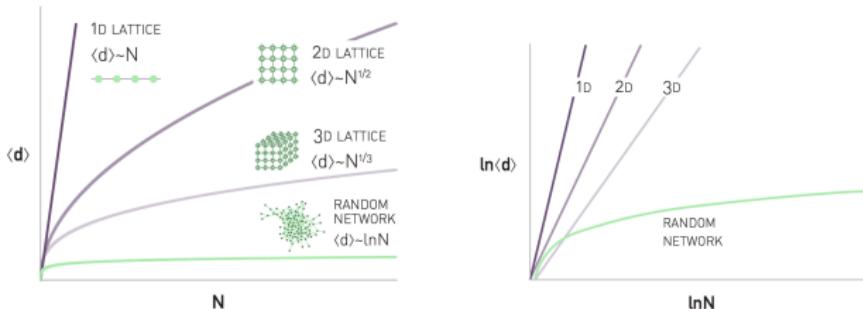
graph *diameter & distance*

- diameter d_{max} and average distance $\langle d \rangle$ for $n \gg \langle k \rangle$

$$1 + \langle k \rangle + \langle k \rangle^2 + \cdots + \langle k \rangle^{d_{max}} = \frac{\langle k \rangle^{d_{max}+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^{d_{max}} \simeq n$$

$$d_{max} \simeq \frac{\ln n}{\ln \langle k \rangle} \quad \langle d \rangle \approx \frac{\ln n}{\ln \langle k \rangle}$$

- $\langle d \rangle = 4.74$ for Facebook [BBR⁺12] while $\frac{\ln n}{\ln \langle k \rangle} = 3.98$
- random graphs *short distances* opposed to *lattices*



network *diameter & distance*

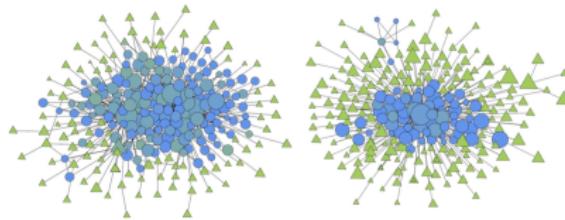
- *diameter* d_{max} and *distance* $\langle d \rangle$ of real networks [Bar16]
- $\langle d \rangle$ well estimated by $\frac{\ln n}{\ln \langle k \rangle}$ whereas $d_{max} \gg \frac{\ln n}{\ln \langle k \rangle}$

NETWORK	N	L	$\langle k \rangle$	$\langle d \rangle$	d_{max}	$\frac{\ln N}{\ln \langle k \rangle}$
Internet	192,244	609,066	6.34	6.98	26	6.58
WWW	325,729	1,497,134	4.60	11.27	93	8.31
Power Grid	4,941	6,594	2.67	18.99	46	8.66
Mobile Phone Calls	36,595	91,826	2.51	11.72	39	11.42
Email	57,194	103,731	1.81	5.88	18	18.4
Science Collaboration	23,133	93,439	8.08	5.35	15	4.81
Actor Network	702,388	29,397,908	83.71	3.91	14	3.04
Citation Network	449,673	4,707,958	10.43	11.21	42	5.55
E. Coli Metabolism	1,039	5,802	5.58	2.98	8	4.04
Protein Interactions	2,018	2,930	2.90	5.61	14	7.14

graph *clustering*

- *clustering coefficients* $\langle C \rangle$ [WS98] and C [NSW01]

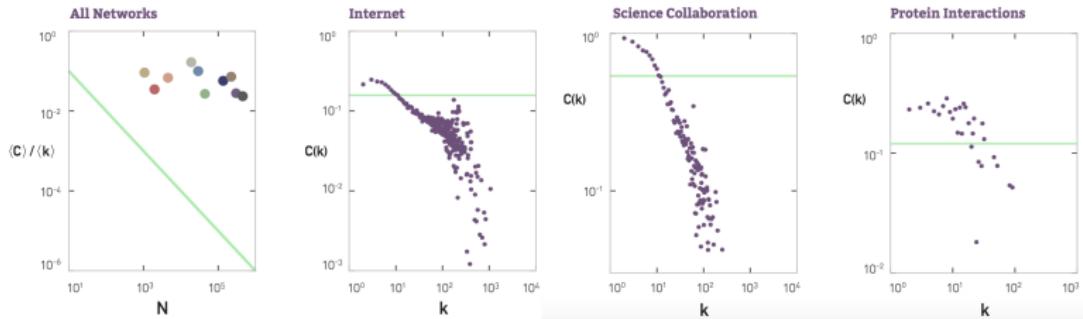
$$C = \langle C \rangle = \langle C_i \rangle = \frac{\langle t_i \rangle}{\binom{k_i}{2}} = \frac{p \binom{k_i}{2}}{\binom{k_i}{2}} = p$$



- $\langle C \rangle = 0.61$ for *Facebook* social circles [NL12] while $p < 10^{-6}$
- random graphs *lack clustering* for $n \gg \langle k \rangle$ opposed to *lattices*

network *clustering*

- clustering $\langle C \rangle$ and $C(k)$ of real networks [Bar16]
- C is under-/overestimated for low-/high- k nodes
- random graphs substantially underestimate $\langle C \rangle$



graph *references*

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