

node *centrality*

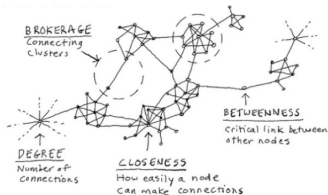
introduction to *network analysis* (*ina*)

Lovro Šubelj  
University of Ljubljana  
spring 2022/23

# centrality *measures*

which *nodes* are most *important*?

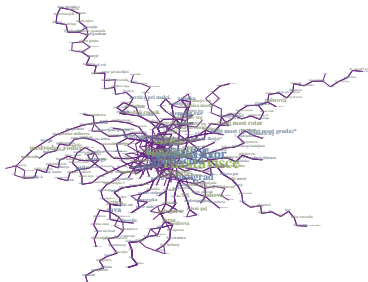
- *node centrality measures* for (*un*)*directed* networks
  - *clustering coefficients* [WS98, SV05, dNMB05]
  - *distance-based* measures [Fre77, FBW91, New05]
  - *spectral analysis* measures [Kat53, Bon87, BP98]
  - *fragment-based* measures [MSOI<sup>+</sup>02, Prž07, EK15]



- *link analysis algorithms* primarily for *directed* networks

# networkology *LPP*

- partial *LPP public bus transport network*\*
- $n = 416$  bus stops with  $\langle k \rangle = 5.62$  connections
- *giant component* 95.4% nodes (6 components)
- “small-world” with  $\langle C \rangle = 0.09$  and  $\langle d \rangle = 14.26$
- “scale-free” with  $\gamma = 2.62$  for cutoff  $k_{min} = 5$



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\* reduced to largest connected component

# centrality *clustering*

important *nodes* are *strongly embedded*

- for *undirected*  $G$  *clustering coefficient*  $C$  [WS98] of  $i$  is
  - $t_i$  is number of *linked neighbors* or *triangles* of  $i$

$$C_i = \frac{2t_i}{k_i(k_i-1)} \quad C_i = 0 \text{ for } k_i \leq 1$$

- $\omega$ -*corrected clustering coefficient*  $C^\omega$  [SV05] of  $i$  is
  - $\omega_i$  is *maximum possible*  $t_i$  with *respect to*  $\{k\}$

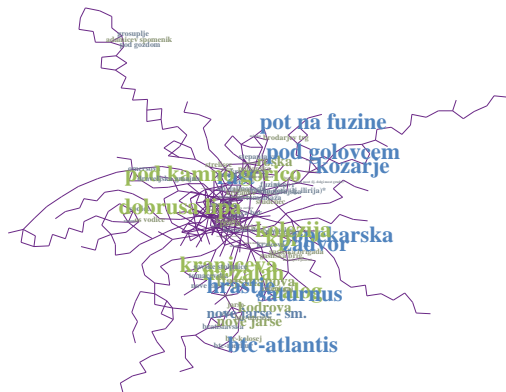
$$C_i^\omega = \frac{t_i}{\omega_i} \quad C_i^\omega = 0 \text{ for } \omega_i = 0$$

- $\mu$ -*corrected clustering coefficient*  $C^\mu$  [Bat19] of  $i$  is
  - $\mu$  is *maximum* number of *triangles* over *links*

$$C_i^\mu = \frac{2t_i}{k_i\mu} \quad C_i^\mu = 0 \text{ for } k_i = 0$$

# networkology *clustering*

- *clustering coefficient*  $C$  in partial LPP network<sup>†</sup>
- *highest*  $C_i = 1.0$  nodes are *Na Žalah etc.* with  $k_i = 2$



<sup>†</sup> reduced to simple undirected graph



## centrality *closeness*

important *nodes* are *close to other* nodes

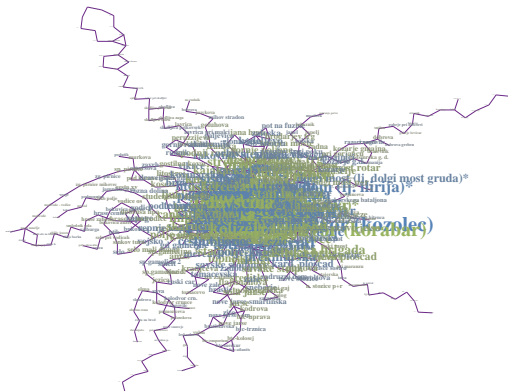
- for (*un*)*directed*  $G$  *closeness centrality*  $\ell^{-1}$  [New10] of  $i$  is
  - $d_{ij}$  is (*un*)*directed distance* between  $i$  and  $j$
  - $d_{ij} = \infty$  for nodes in *different components*

$$\ell_i^{-1} = \frac{1}{n-1} \sum_{j \neq i} \frac{1}{d_{ij}}$$

- $\ell^{-1}$  spans *small range* in *small-world* networks

# networkology *closeness*

- *closeness centrality*  $\ell^{-1}$  in partial LPP network<sup>§</sup>
- *highest*  $\ell_i^{-1} = 0.208$  node is *Gospodsvetska* with  $k_i = 14$



<sup>§</sup> reduced to simple undirected graph



# centrality *betweenness*

important *nodes* are *bridges between other* nodes

- for (*un*)*directed*  $G$  *betweenness centrality*  $\sigma$  [Fre77] of  $i$  is
  - $g_{st}$  is number of *shortest paths between*  $s$  and  $t$
  - $g_{st}^i$  is number of *such shortest paths through*  $i$

$$\sigma_i = \frac{1}{n^2} \sum_{st} \frac{g_{st}^i}{g_{st}}$$

- $\sigma$  considers *only shortest paths* [FBW91, New05]
- $\sigma$  mixes *local centers* with *global bridges* [JMK<sup>+</sup>16]



# centrality *degrees*

important *nodes* are *linked by many* nodes

- for *undirected*  $G$  *degree centrality*  $d$  of  $i$  is

$$d_i = \frac{1}{n-1} \sum_{j \neq i} A_{ij} = \frac{k_i}{n-1}$$

- in *directed*  $G$  *in-degree centrality*  $d^{in}$  of  $i$  is

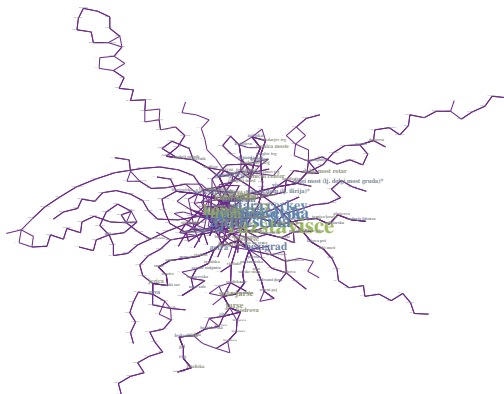
$$d_i^{in} = \frac{1}{n-1} \sum_{j \neq i} A_{ij} = \frac{k_i^{in}}{n-1}$$

- in *directed*  $G$  *out-degree centrality*  $d^{out}$  of  $i$  is

$$d_i^{out} = \frac{1}{n-1} \sum_{j \neq i} A_{ji} = \frac{k_i^{out}}{n-1}$$

## networkology *degrees*

- *degree centrality*  $d$  in partial LPP network
- *highest*  $d_i = 0.099$  node is *Razstavišče* with  $k_i = 41$
- *highest*  $d_i^{in}$  node is *Razstavišče* with  $k_i^{in} = 20$  and  $k_i^{out} = 21$



## centrality *eigenvector*

important *nodes* are *linked by important* nodes

- for (*un*)*directed*  $G$  *eigenvector centrality*  $e$  [Bon87] of  $i$  is
  - $v$  and  $\lambda$  are *eigenvectors* and *eigenvalues* of  $A$
  - $e$  is *proportional* to *leading eigenvector*  $v_1$

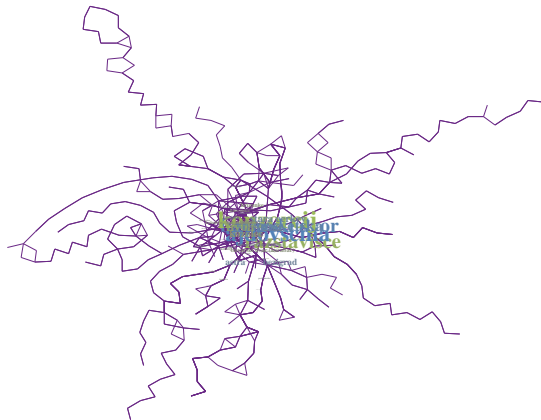
$$e(t) = A^t e(0) = A^t \sum_i C_i v_i = \sum_i C_i \lambda_i^t v_i = \lambda_1^t \sum_i C_i \left[ \frac{\lambda_i}{\lambda_1} \right]^t v_i \rightarrow C_1 \lambda_1^t v_1$$

$$e_i = \lambda_1^{-1} \sum_j A_{ij} e_j$$

- in *directed*  $G$   $e = 0$  for  $k^{in} = 0$  *nodes etc.*

## networkology *eigenvector*

- *eigenvector centrality*  $e$  in partial LPP network
- *highest*  $e_i = 0.082$  node is *Konzorcij* with  $k_i = 30$



## centrality *Katz*

*nodes* get small amount of *importance* for free

- for (un)directed  $G$  *Katz centrality*  $z$  [Kat53] of  $i$  is
  - $\alpha$  and  $\beta_i$  are some *positive constants*

$$z_i = \alpha \sum_j A_{ij} z_j + \beta_i$$

- for *convenience*  $\beta_i = 1$  whereas  $\alpha < \lambda_1^{-1}$ 
  - $\lambda_1$  is *leading eigenvalue* of  $A$

## centrality *PageRank*

*nodes distribute equal* amount of *importance*

- for (un)directed  $G$  *PageRank centrality*  $p$  [BP98] of  $i$  is
  - $\alpha$  and  $\beta_i$  are some *positive constants*

$$p_i = \alpha \sum_j A_{ij} \frac{p_j}{k_j^{\text{out}}} + \beta_i$$

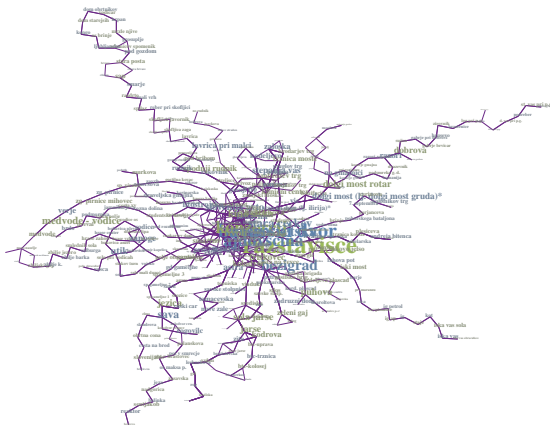
- for *convenience*  $\beta_i = \frac{1-\alpha}{n}$  whereas  $\alpha = 0.85$

see PageRank algorithm *NetLogo* demo



# networkology *PageRank*

- *PageRank centrality*  $p$  in partial LPP network
- *highest*  $p_i = 0.011$  node is *Razstavišče* with  $k_i = 41$



# centrality *overview*

which *nodes* are most *important*?

1 IA												18 VIIIA											
1	1	34											2	8									
	DC												EC										
	Degree Centrality												Eigenvector Centrality										
2	3	12	4	13											5	2							
	BC		CC												SC								
	Betweenness Centrality		Closeness Centrality												Subgraph Centrality								
3	11	2	12	8											6	1							
	RL		IC												CCoef								
	Range-Linked Betweenness		Information Centrality												Clustering Coefficient								
4	19	4	20	2											7	1							
	BN		RC												CCoef <sup>-1</sup>								
	BetweenBlock Centrality		Radiality Centrality												Inverse Coefficient								
5	37	1	38	1											8	1							
	RWBC		RWCC												MNC								
	RandomWalk Betweenness		RandomWalk Closeness												max. neigh. comp.								
6	56	2	56	2											9	3							
	$\sigma$		ECC												DMNC								
	sigma Centrality		Eccentricity												edge clustering coefficient								
7	87	1	88	1											10	1							
	BC <sub>2,3,4</sub>		ECC <sup>-1</sup>												SEC <sub>Coef</sub>								
	2,3,4-localized-BC		Inverse Eccentricity												LR								
															LeaderRank								
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