

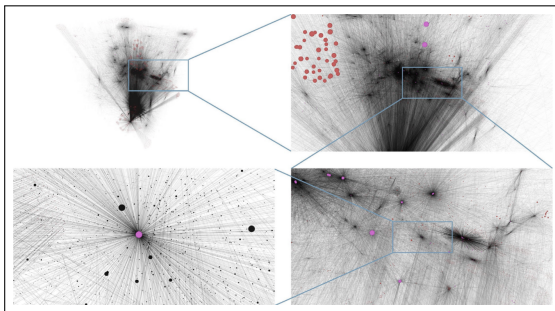
scale-free networks

introduction to *network analysis* (*ina*)

Lovro Šubelj
University of Ljubljana
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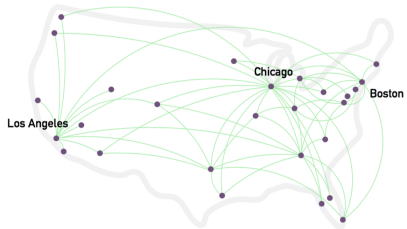
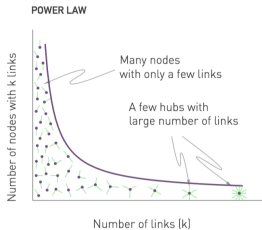
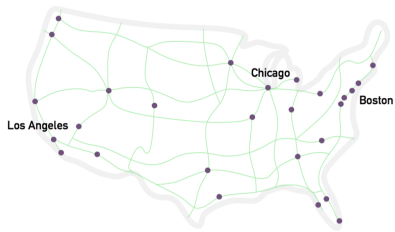
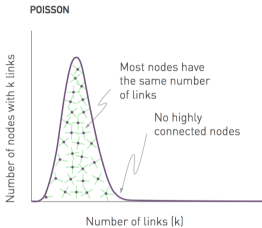
scale-free *property*

- *random graphs* = *Poisson degree distribution* p_k [ER59]
- *real networks* contain *highly linked hubs* [Pri65, FFF99]
- *scale-free networks* \sim *power-law degree distribution* p_k [BA99]



see zooming into *World Wide Web* demo

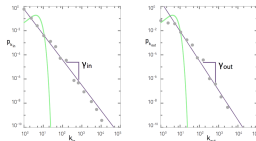
scale-free *structure*



scale-free *power-law*

- *power-law degree distribution* p_k with *exponent* $\gamma > 1$

$$p_k \sim k^{-\gamma}$$
$$\log p_k \sim -\gamma \log k$$



- *theoretically correct discrete power-law* p_k for $k \geq 1$

$$\sum_{k=1}^{\infty} p_k = C \sum_{k=1}^{\infty} k^{-\gamma} = C \zeta(\gamma) = 1$$
$$p_k = C k^{-\gamma} = \frac{k^{-\gamma}}{\zeta(\gamma)}$$

- *analytically convenient continuous power-law* $p(k)$ for $k \geq k_{\min}$

$$\int_{k_{\min}}^{\infty} p(k) dk = C \int_{k_{\min}}^{\infty} k^{-\gamma} dk = C \left. \frac{k^{-\gamma+1}}{-\gamma+1} \right|_{k_{\min}}^{\infty} = C \frac{k_{\min}^{-\gamma+1}}{\gamma-1} = 1$$
$$p(k) = C k^{-\gamma} = (\gamma - 1) k_{\min}^{\gamma-1} k^{-\gamma}$$

scale-free *hubs*

— for *small* $k \ll \langle k \rangle$ *power-law above Poisson*

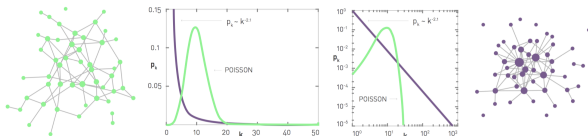
many *small degree nodes* in *scale-free network*

— for *average* $k \approx \langle k \rangle$ *power-law below Poisson*

most *nodes similar degree* in *random graph*

— for *large* $k \gg \langle k \rangle$ *power-law above Poisson*

existence of hubs in *scale-free network*



— *random graph* with $n \approx 10^{12}$ and $\langle k \rangle = 4.6$ then $n_{k \geq 100} \approx 10^{-82}$

— *scale-free network* with $n \approx 10^{12}$ and $\gamma = 2.1$ then $n_{k \geq 100} \approx 4 \cdot 10^9$

scale-free *cutoff*

- maximum degree k_{\max} by upper natural cutoff of $p(k)$
- for random graph with exponential $p(k) = \lambda e^{\lambda k_{\min}} e^{-\lambda k}$

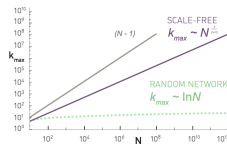
$$\int_{k_{\min}}^{\infty} p(k) dk = \lambda e^{\lambda k_{\min}} \left. \frac{e^{-\lambda k}}{-\lambda} \right|_{k_{\min}}^{\infty} = e^{\lambda k_{\min}} e^{-\lambda k_{\max}} = n^{-1}$$

$$k_{\max} = k_{\min} + \frac{\ln n}{\lambda}$$

- for scale-free network with power-law $p(k) = (\gamma - 1) k_{\min}^{\gamma-1} k^{-\gamma}$

$$\int_{k_{\min}}^{\infty} p(k) dk = (\gamma - 1) k_{\min}^{\gamma-1} \left. \frac{k^{-\gamma+1}}{-\gamma+1} \right|_{k_{\min}}^{\infty} = k_{\min}^{\gamma-1} k_{\max}^{-\gamma+1} = n^{-1}$$

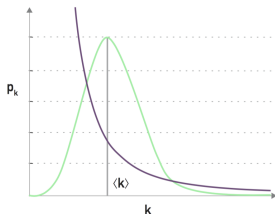
$$k_{\max} = k_{\min} n^{\frac{1}{\gamma-1}}$$



- random graph with $n \approx 3 \cdot 10^5$ and $\lambda = 1$ then $k_{\max} \approx 14$
- scale-free network with $n \approx 3 \cdot 10^5$ and $\gamma = 2.1$ then $k_{\max} \approx 10^5$

scale-free *moments*

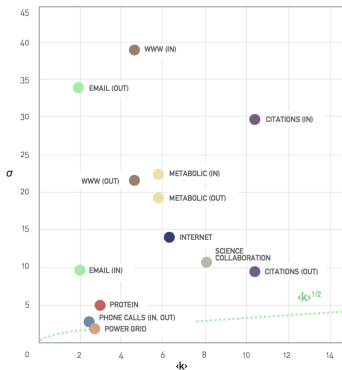
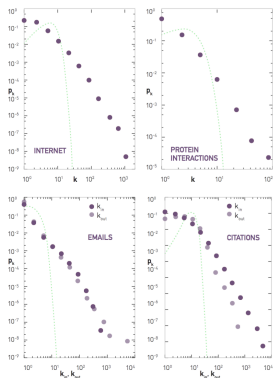
- x -th moment $\langle k^x \rangle$ of power-law $p_k \sim k^{-\gamma}$
 - $\langle k^2 \rangle = \sigma_k^2 + \langle k \rangle^2$ determines *spread* and $\langle k^3 \rangle$ determines *skewness*
- $\langle k^x \rangle = \sum_{k=1}^{\infty} k^x p_k \approx \int_{k_{\min}}^{k_{\max}} k^x p(k) dk \sim \frac{k_{\max}^{x-\gamma+1} - k_{\min}^{x-\gamma+1}}{x-\gamma+1}$
- *moments* $x \leq \gamma - 1$ *finite* whereas *moments* $x > \gamma - 1$ *diverge*



- *scale-free networks* $\gamma < 3$ *lack scale* as $k = \langle k \rangle \pm \infty$
- *random graphs have scale* as $k = \langle k \rangle \pm \sqrt{\langle k \rangle}$

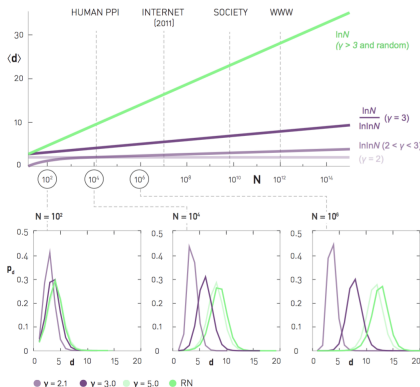
scale-free *networks*

- *heavy-tail* p_k of real networks [Bar16]
- *spread* $\sigma_k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$ in real networks

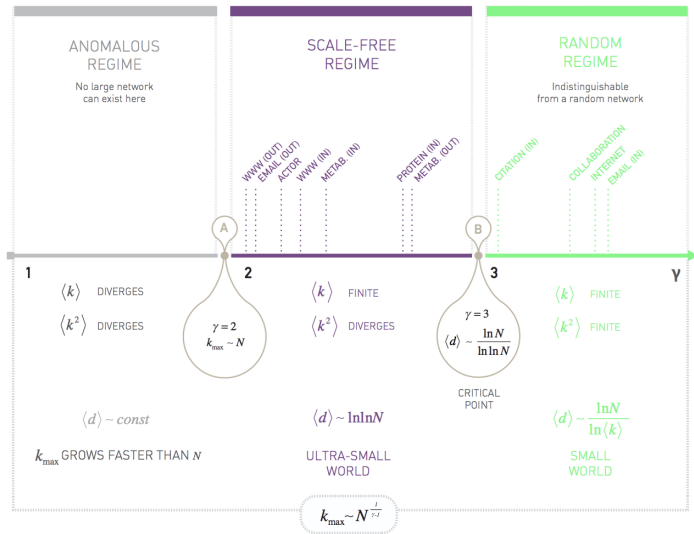


scale-free “small-world”

- *random graphs* are “small-world” as $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$
- *scale-free networks* $\gamma > 3$ are “small-world” as $\langle d \rangle \sim \ln n$
- *scale-free networks* $\gamma < 3$ “ultra-small-world” as $\langle d \rangle \sim \ln \ln n$



scale-free *exponent*



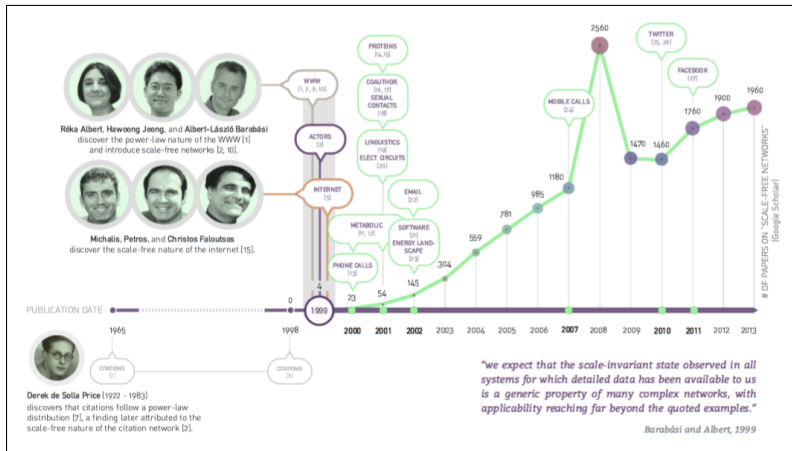
no graphical $\{k\}$ for $\gamma < 2$

$n = (k_{max}/k_{min})^{\gamma-1}$ nonexistent for $\gamma \gg 3$

scale-free *distributions*

NAME	$p_x/p(x)$	$\langle x \rangle$	$\langle x^2 \rangle$
Poisson (discrete)	$e^{-\mu} \mu^x / x!$	μ	$\mu(1 + \mu)$
Exponential (discrete)	$(1 - e^{-\lambda})e^{-\lambda x}$	$1/(e^\lambda - 1)$	$(e^\lambda + 1)/(e^\lambda - 1)^2$
Exponential (continuous)	$\lambda e^{-\lambda x}$	$1/\lambda$	$2/\lambda^2$
Power law (discrete)	$x^{-\alpha} / \zeta(\alpha)$	$\begin{cases} \zeta(\alpha - 2)/\zeta(\alpha), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \zeta(\alpha - 1)/\zeta(\alpha), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law (continuous)	$\alpha x^{-\alpha}$	$\begin{cases} \alpha/(\alpha - 1), & \text{if } \alpha > 2 \\ \infty, & \text{if } \alpha \leq 1 \end{cases}$	$\begin{cases} \alpha/(\alpha - 2), & \text{if } \alpha > 1 \\ \infty, & \text{if } \alpha \leq 2 \end{cases}$
Power law with cutoff (continuous)	$\frac{\lambda^{1-\alpha}}{\Gamma(1-\alpha)} x^{-\alpha} e^{-\lambda x}$	$\lambda^{-1} \frac{\Gamma(2-\alpha)}{\Gamma(1-\alpha)}$	$\lambda^{-2} \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}$
Stretched exponential (continuous)	$\beta \lambda^\beta x^{\beta-1} e^{-(\lambda x)^\beta}$	$\lambda^{-1} \Gamma(1 + \beta^{-1})$	$\lambda^{-2} \Gamma(1 + 2\beta^{-1})$
Log-normal (continuous)	$\frac{1}{x\sqrt{2\pi\sigma^2}} e^{-(\ln x - \mu)^2 / (2\sigma^2)}$	$e^{\mu + \sigma^2 / 2}$	$e^{2(\mu + \sigma^2)}$
Normal (continuous)	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2 / (2\sigma^2)}$	μ	$\mu^2 + \sigma^2$

scale-free *history*



scale-free *references*



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scale-free *references*



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