configuration graph model

introduction to *network analysis* (*ina*)

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configuration *model*

— random graphs *Poisson distribution* $p_k \simeq \frac{\langle k\rangle^k e^{-\langle k\rangle}}{k!}$ [ER59]

— real networks *power-law degree distribution* $p_k \sim k^{-\gamma}$ [BA99]

— *configuration model* random graph for *arbitrary* $\{k\}$ [NSW01]

assume *undirected* G from now on

Mark Newman

Steven Strogatz

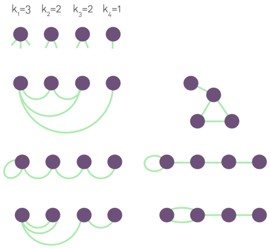
Duncan Watts

configuration $G(\{k\})$ *model*

- $G(\{k\})$ *configuration model* [NSW01]
- *randomly link m stub pairs* between n nodes
- *computationally convenient* and *analytically tractable*

graphical $k_1, k_2 \ldots k_n$ $\qquad m = \frac{1}{2}\sum_i k_i$

input sequence $\{k\}$
output graph G
1: $G \leftarrow n$ nodes with $\{k\}$ stubs
2: **while** G has node stubs **do**
3: link random node stub pair
4: **return** G



configuration *probability*

— *probability of self-loop* p_i on i

$$p_i = m\frac{\binom{k_i}{2}}{\binom{2m}{2}} \approx \frac{k_i(k_i-1)}{4m}$$

— *probability of link* p_{ij} between i and j

$$p_{ij} = m\frac{k_i k_j}{\binom{2m}{2}} = k_i\frac{k_j}{2m-1} \approx \frac{k_i k_j}{2m}$$

— thus *number of multilinks* and *self-loops* is

$$\left[\frac{\langle k^2\rangle - \langle k\rangle}{\sqrt{2}\langle k\rangle}\right]^2 \qquad \sum_i p_i = \sum_i \frac{k_i(k_i-1)}{2n\langle k\rangle} = \frac{\langle k^2\rangle - \langle k\rangle}{2\langle k\rangle}$$

configuration *neighbors*

- *neighbor degree distribution* p_k is *not* p_k
  - n_k is *number of degree-k* nodes thus $n_k = np_k$

$$\{neighbor\ p_k\} = n_k \frac{k}{2m-1} \approx \frac{kp_k}{\langle k \rangle}$$

- *average neighbor degree* $\langle k \rangle$ is *not* $\langle k \rangle$

$$\frac{\langle k^2 \rangle}{\langle k \rangle} - \langle k \rangle = \frac{\langle k^2 \rangle - \langle k \rangle^2}{\langle k \rangle} = \frac{\sigma_k^2}{\langle k \rangle} > 0$$

$$\langle neighbor\ k \rangle \approx \sum_k k \frac{kp_k}{\langle k \rangle} = \frac{\langle k^2 \rangle}{\langle k \rangle} > \langle k \rangle$$

- $\frac{\langle k^2 \rangle}{\langle k \rangle} = \frac{\langle k \rangle^2 + \langle k \rangle}{\langle k \rangle} = \langle k \rangle + 1$ even for *random graph* [ER59]

network *neighbors*

— *friendship paradox* $\langle neighbor\ k \rangle > \langle k \rangle$ [Fel91] in real networks

— $\langle neighbor\ k \rangle$ *well estimated* by $\frac{\langle k^2 \rangle}{\langle k \rangle}$ whereas $\langle k \rangle \lll \frac{\langle k^2 \rangle}{\langle k \rangle}$

network	n	$\langle k \rangle \lll$	$\langle$**neighbor** $k\rangle$	$\approx \frac{\langle k^2 \rangle}{\langle k \rangle}$
Southern women [DGG41]	32	5.56	7.57	7.02
Karate club [Zac77]	34	4.59	9.61	7.77
American football [GN02]	115	10.71	10.78	10.79
Java dependencies [ŠB11]	1368	16.20	207.52	140.53
Facebook circles [ML12]	4039	43.69	105.55	106.57
Physics collaboration [New01]	36 458	9.42	21.65	27.88
Enron e-mails [LLDM09]	36 692	20.04	472.86	280.16
Internet map [HJJ$^+$03]	75 885	9.42	1853.73	1461.54
Actors collaboration [BA99]	382 219	78.69	282.72	417.69
Physics citation [ŠFB14]	438 943	21.56	78.38	77.72
Patent citation [HJT01]	3 774 768	8.75	17.15	21.33
Facebook snowball [Fer12]	8 217 272	3.06	308.52	157.06

configuration *clustering*

— (*neighbor*) *excess degree distribution* q_k defined as

excess degree is "remaining" neighbor degree or neighbor degree-1

$$q_k = \frac{(k+1)p_{k+1}}{\langle k \rangle}$$

— then *network clustering coefficient* C [NSW01] is

$$\sum_{k_i k_j} q_{k_i} q_{k_j} \frac{k_i k_j}{2m} = \frac{1}{2m} \left[\sum_k k q_k\right]^2 = \frac{1}{2m\langle k \rangle^2} \left[\sum_k k(k+1)p_{k+1}\right]^2 = \frac{1}{n\langle k \rangle^3} \left[\sum_k (k-1)k p_k\right]^2$$

$$C = \sum_{k_i k_j} q_{k_i} q_{k_j} p_{ij} \approx \frac{\left[\langle k^2 \rangle - \langle k \rangle\right]^2}{n\langle k \rangle^3}$$

network *clustering*

— *average clustering coefficient* $\langle C \rangle$ [WS98] of real networks

— *neither* $G(n,p)$ [ER59] *nor* $G(\{k\})$ [NSW01] *explain* $\langle C \rangle \gg 0$

network	n	$\langle C \rangle$	$\gg \frac{[\langle k^2 \rangle - \langle k \rangle]^2}{n \langle k \rangle^3}$	$\gg \frac{\langle k \rangle}{n-1}$
Southern women [DGG41]	32	0.000	0.204	0.179
Karate club [Zac77]	34	0.571	0.294	0.139
American football [GN02]	115	0.403	0.078	0.094
Java dependencies [ŠB11]	1368	0.497	0.879	0.012
Facebook circles [ML12]	4039	0.606	0.063	0.011
Physics collaboration [New01]	36 458	0.657	0.002	0.000
Enron e-mails [LLDM09]	36 692	0.497	0.106	0.001
Internet map [HJJ+03]	75 885	0.160	2.985	0.000
Actors collaboration [BA99]	382 219	0.780	0.006	0.000
Physics citation [ŠFB14]	438 943	0.227	0.001	0.000
Patent citation [HJT01]	3 774 768	0.076	0.000	0.000
Facebook snowball [Fer12]	8 217 272	0.019	0.001	0.000

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