

# *small-world* networks

introduction to *network analysis* (*ina*)

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*connected*  
*the power of six degrees*

documentary on small-world and scale-free networks



[WS98]



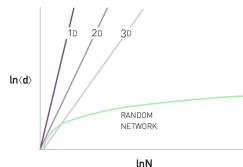
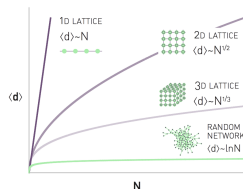
[BA99]



[AJB00]

# small-world *phenomenon*

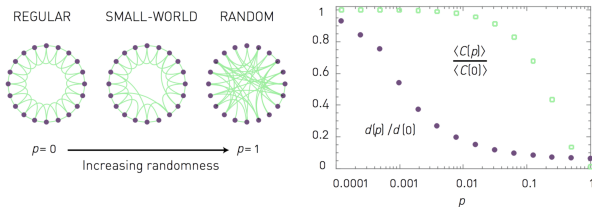
- for *regular lattices*
  - *high clustering*  $\langle C \rangle \gg 0$
  - long distances  $\langle d \rangle \simeq n^{1/D}$
- in *random graphs* [ER59]
  - low clustering  $\langle C \rangle = \frac{\langle k \rangle}{n-1}$
  - *short distances*  $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$
- real *small-world networks* [WS98]
  - *high clustering*  $\langle C \rangle \gg 0$
  - *short distances*  $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$



- $\langle d \rangle = 4.74$  for *Facebook* friendships [BBR<sup>+</sup>12] while  $\frac{\ln n}{\ln \langle k \rangle} = 3.98$
- $\langle C \rangle = 0.61$  for *Facebook* social circles [NL12] while  $\rho < 10^{-6}$

# small-world *model*

- $G(n, k, p)$  *small-world* model [WS98]
- *randomly rewire*  $pnk/2$  *links* of *regular lattice*
- *conceptually interesting* but *practically inapplicable*
  - for *some*  $p$  *small-world* with  $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$  and  $\langle C \rangle \gg 0$
  - for  $p = 1$  *random graph* with  $\langle d \rangle \simeq \frac{\ln n}{\ln \langle k \rangle}$
  - for  $p = 0$  *regular lattice* with  $C = \frac{3(k-2)}{4(k-1)}$



see small-world model [NetLogo](#) demo

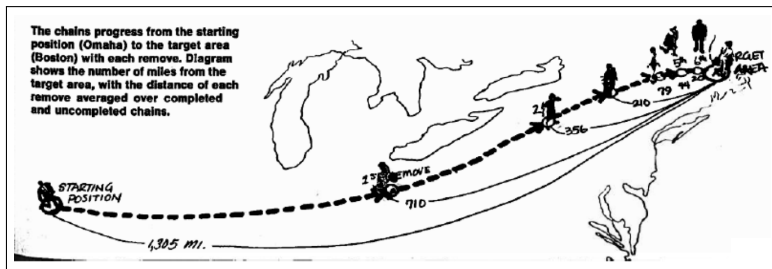
## small-world *networks*

- *clustering coefficient*  $\langle C \rangle$  in *real small-world* networks
- *average distance*  $\langle d \rangle$  in *real small-world* networks

network	$n$	$\langle C \rangle$	$\gg \frac{\langle k \rangle}{n-1}$	$\langle d \rangle$	$\approx \frac{\ln n}{\ln \langle k \rangle}$
southern women [DGG41]	32	0.000	0.179	2.31	2.02
karate club [Zac77]	34	0.571	0.139	2.41	2.31
American football [GN02]	115	0.403	0.094	2.51	2.00
Java dependencies [ŠB11]	1368	0.497	0.012	2.21	2.59
Facebook circles [ML12]	4039	0.606	0.011	3.69	2.20
physics collaboration [New01]	36 458	0.657	0.000	5.50	4.68
Enron e-mails [LLDM09]	36 692	0.497	0.001	3.39	3.51
Internet map [HJJ <sup>+</sup> 03]	75 885	0.160	0.000	5.83	5.01
actors collaboration [BA99]	382 219	0.780	0.000	$\approx 3.6$	2.94
physics citation [ŠFB14]	438 943	0.227	0.000	$\approx 5.0$	4.23
patent citation [HJT01]	3 774 768	0.076	0.000	$\approx 8.1$	6.98
Facebook snowball [Fer12]	8 217 272	0.019	0.000	$\approx 6.8$	14.23

## small-world *experiments*

- 6 *degrees of separation* in *letter* passing as  $\langle d \rangle = 6.2$  [Mil67]
- 4/7 *degrees of separation* in *e-mail* communication [DMW03]
- 4 *degrees of separation* on *Facebook* as  $\langle d \rangle = 4.74$  [BBR<sup>+</sup>12]

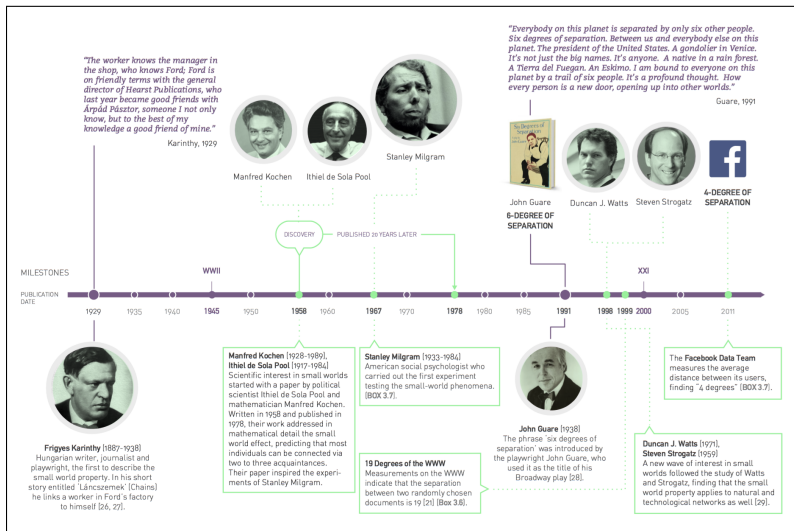


- the *strength* (*weakness*) of *weak* (*strong*) *ties* [Gra73]

## small-world *navigation*

does existence of short paths imply  
*navigable small-world* by *decentralized search*? [Kle00]

# small-world *history*





# small-world *references*



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