Homework 2

STT 465, Bayesian Statistical Methods

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Question 1

Suppose that if $\theta = 1$, then y has a normal distribution with mean 1 and standard deviation σ, and if $\theta = 2$, then y has a normal distribution with mean 2 and standard deviation σ. Also suppose that $P(\theta = 1) = 0.5 = P(\theta = 2)$.

(a) 5 points

For $\sigma = 2$, write the formula for the marginal probability density for y. Then draw the density curve of y.

Answer

For $\sigma = 2$, the marginal (mixture) density of y is

$$f_Y(y) \;=\; \tfrac{1}{2}\cdot\frac{1}{2\sqrt{2\pi}}\exp\!\left(-\frac{(y-1)^2}{8}\right) \;+\; \tfrac{1}{2}\cdot\frac{1}{2\sqrt{2\pi}}\exp\!\left(-\frac{(y-2)^2}{8}\right).$$

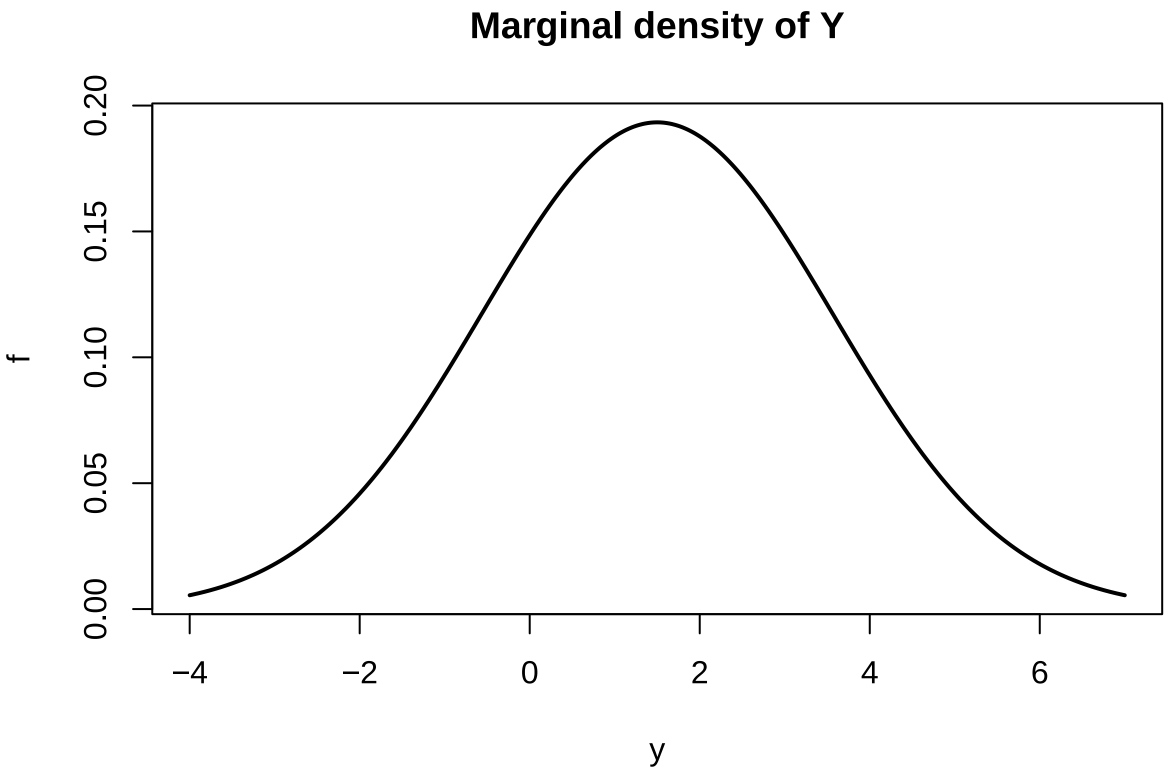
Equivalently,

$$f_Y(y) \;=\; \frac{1}{4}\left[\varphi\!\left(\tfrac{y-1}{2}\right) + \varphi\!\left(\tfrac{y-2}{2}\right)\right],$$

where $\varphi(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ is the standard normal density.

The density can be plotted as follows:

```r
f <- function(y) 0.5*dnorm(y, mean=1, sd=2) + 0.5*dnorm(y, mean=2, sd=2)
ys <- seq(-4, 7, length=1000)
plot(ys, f(ys), type="l", lwd=2, main="Marginal density of Y",
     xlab="y", ylab="f")
```



(b) 5 points

What is $P(\theta = 1|y = 1)$ for $\sigma = 2$.

Answer

By Bayes' rule, the posterior probability that $\theta = 1$ given $Y = y$ is

$$P(\theta = 1 \mid y) = \frac{P(\theta = 1) f(y \mid \theta = 1)}{P(\theta = 1) f(y \mid \theta = 1) + P(\theta = 2) f(y \mid \theta = 2)} = \frac{f(y \mid \theta = 1)}{f(y \mid \theta = 1) + f(y \mid \theta = 2)},$$

since $P(\theta = 1) = P(\theta = 2) = 1/2$.

For the normal likelihoods with common standard deviation σ,

$$f(y \mid \theta = i) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-i)^2}{2\sigma^2}\right), \qquad i = 1, 2.$$

Thus

$$P(\theta = 1 \mid y) = \frac{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)}{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) + \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right)}.$$

Now plug in $\sigma = 2$ and $y = 1$. For $\sigma = 2$,

$$f(1 \mid \theta = 1) = \frac{1}{2\sqrt{2\pi}}, \qquad f(1 \mid \theta = 2) = \frac{1}{2\sqrt{2\pi}} \exp\left(-\frac{1^2}{8}\right).$$

$$\sqrt{2\pi} \approx 2.50662827463, \qquad \frac{1}{2\sqrt{2\pi}} \approx 0.19947114020071635,$$

and

$$\exp\left(-\tfrac{1}{8}\right) \approx 0.8824969025845955,$$

so

$$f(1 \mid 2) \approx 0.19947114020071635 \times 0.8824969025845955 \approx 0.17603266338214976.$$

Therefore

$$P(\theta = 1 \mid y = 1) = \frac{0.19947114020071635}{0.19947114020071635 + 0.17603266338214976} \approx \boxed{0.5312093734}.$$

(c) 10 points

Describe how the posterior density of θ changes in shape as θ is increasing and as it is decreased.

Answer

We derive a compact analytic form for the posterior and then analyze how its shape depends on σ.

Starting from

$$P(\theta = 1 \mid y) = \frac{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)}{\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right) + \exp\left(-\frac{(y-2)^2}{2\sigma^2}\right)},$$

divide numerator and denominator by $\exp\left(-\frac{(y-1)^2}{2\sigma^2}\right)$ to obtain

$$P(\theta = 1 \mid y) = \frac{1}{1 + \exp\left(\dfrac{(y-1)^2 - (y-2)^2}{2\sigma^2}\right)}.$$

Compute the quadratic difference:

$$(y-1)^2 - (y-2)^2 = (y^2 - 2y + 1) - (y^2 - 4y + 4) = 2y - 3.$$

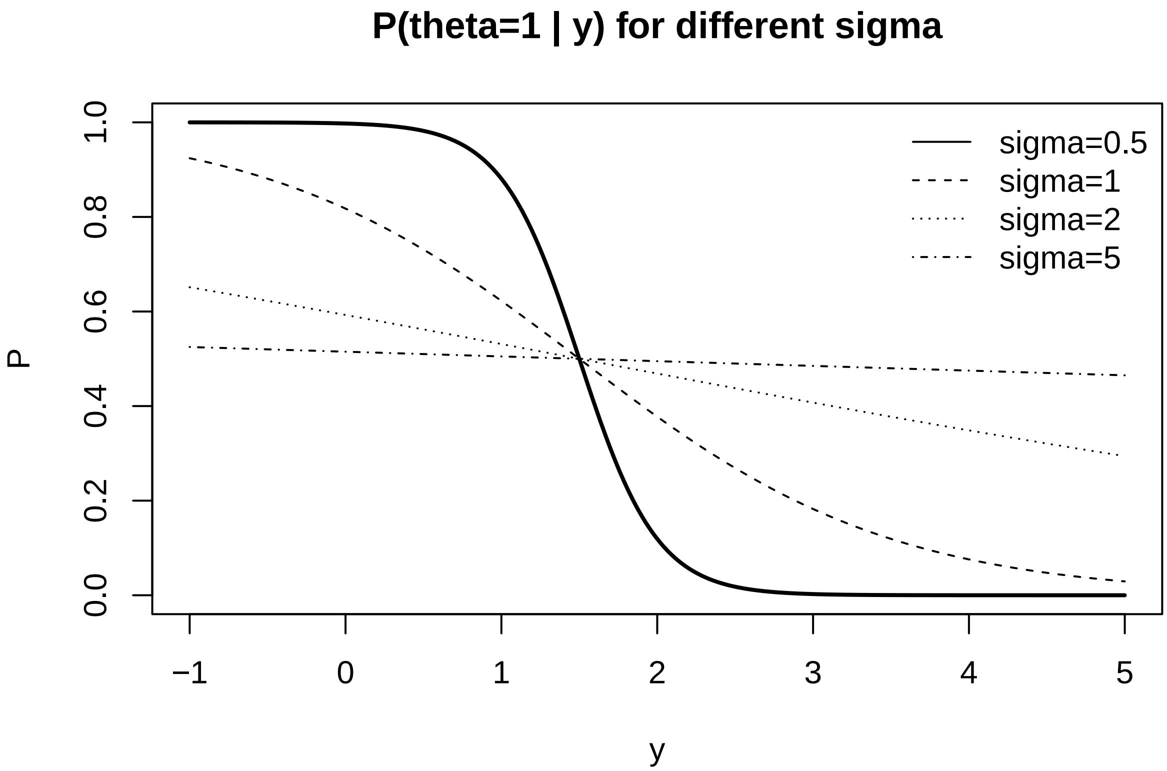
Hence the posterior simplifies to the logistic form

$$\boxed{P(\theta = 1 \mid y) = \frac{1}{1 + \exp\left(\frac{2y-3}{2\sigma^2}\right)}}.$$

The posterior probability is

$$P(\theta = 1 \mid y) = \frac{1}{1 + \exp\left(\frac{2y-3}{2\sigma^2}\right)}$$

```r
posterior <- function(y, sigma) {
  num <- exp(- (y-1)^2 / (2*sigma^2))
  den <- num + exp(- (y-2)^2 / (2*sigma^2))
  num/den
}
ys <- seq(-1, 5, length=1000)
plot(ys, posterior(ys, 0.5), type="l", lwd=2, ylim=c(0,1),
     main="P(theta=1 | y) for different sigma", xlab="y", ylab="P")
lines(ys, posterior(ys, 1), lty=2)
lines(ys, posterior(ys, 2), lty=3)
lines(ys, posterior(ys, 5), lty=4)
legend("topright", legend=c("sigma=0.5","sigma=1","sigma=2","sigma=5"),
       lty=1:4, bty="n")
```



Posterior and prior are closer at a higher value of σ.

Question 2

Logan (1983) reported the following joint density on (Y_1, Y_2) = (Father's Occupation, Son's Occupation):

Father's Occupation	Son's Occupation				
	farm	operatives	craftsmen	sales	professional
farm	0.018	0.035	0.031	0.008	0.018
operatives	0.002	0.112	0.064	0.032	0.069
craftsmen	0.001	0.066	0.094	0.032	0.084
sales	0.001	0.018	0.019	0.010	0.051
professional	0.001	0.029	0.032	0.043	0.130

Use the joint density to calculate the following:

(a) 5 points

The marginal density for a father's occupation.

Answer

The marginal distribution of the father's occupation is obtained by summing over the son's occupation:

$$p_{Y_1}(y_1) = \sum_{y_2} p(y_1, y_2).$$

$$
\begin{aligned}
p_{Y_1}(\text{farm}) &= 0.018 + 0.035 + 0.031 + 0.008 + 0.018 = 0.110, \\
p_{Y_1}(\text{operatives}) &= 0.002 + 0.112 + 0.064 + 0.032 + 0.069 = 0.279, \\
p_{Y_1}(\text{craftsmen}) &= 0.001 + 0.066 + 0.094 + 0.032 + 0.084 = 0.277, \\
p_{Y_1}(\text{sales}) &= 0.001 + 0.018 + 0.019 + 0.010 + 0.051 = 0.099, \\
p_{Y_1}(\text{professional}) &= 0.001 + 0.029 + 0.032 + 0.043 + 0.130 = 0.235.
\end{aligned}
$$

(b) 5 points

The marginal density for a son's occupation.

Answer

The marginal distribution of the son's occupation is obtained by summing over the father's occupation:

$$
p_{Y_2}(y_2) = \sum_{y_1} p(y_1, y_2).
$$

$$
\begin{aligned}
p_{Y_2}(\text{farm}) &= 0.018 + 0.002 + 0.001 + 0.001 + 0.001 = 0.023, \\
p_{Y_2}(\text{operatives}) &= 0.035 + 0.112 + 0.066 + 0.018 + 0.029 = 0.260, \\
p_{Y_2}(\text{craftsmen}) &= 0.031 + 0.064 + 0.094 + 0.019 + 0.032 = 0.240, \\
p_{Y_2}(\text{sales}) &= 0.008 + 0.032 + 0.032 + 0.010 + 0.043 = 0.125, \\
p_{Y_2}(\text{professional}) &= 0.018 + 0.069 + 0.084 + 0.051 + 0.130 = 0.352.
\end{aligned}
$$

(c) 5 points

The conditional density of a son's occupation given that the father's occupation is a craftsman.

Answer

We compute

$$
p_{Y_2|Y_1}(y_2 \mid Y_1 = \text{craftsman}) = \frac{p(Y_1 = \text{craftsman}, y_2)}{p_{Y_1}(\text{craftsman})},
$$

where

$$
p_{Y_1}(\text{craftsman}) = 0.277.
$$

Thus,

$$
\begin{aligned}
p_{Y_2|Y_1}(\text{farm} \mid \text{craftsman}) &= \tfrac{0.001}{0.277} \approx 0.00361, \\
p_{Y_2|Y_1}(\text{operatives} \mid \text{craftsman}) &= \tfrac{0.066}{0.277} \approx 0.23827, \\
p_{Y_2|Y_1}(\text{craftsmen} \mid \text{craftsman}) &= \tfrac{0.094}{0.277} \approx 0.33935, \\
p_{Y_2|Y_1}(\text{sales} \mid \text{craftsman}) &= \tfrac{0.032}{0.277} \approx 0.11552, \\
p_{Y_2|Y_1}(\text{professional} \mid \text{craftsman}) &= \tfrac{0.084}{0.277} \approx 0.30325.
\end{aligned}
$$

Question 3

Suppose that X and Y are random variables with joint density function:

$$
f_{X,Y}(x, y) = \begin{cases} \frac{1}{4}(x + y), & 0 \le x \le y \le 2 \\ 0, & \text{otherwise} \end{cases}
$$

(a) 5 points

Check that $f_{X,Y}$ is a joint density function.

Answer

Let

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(x+y), & 0 \le x \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

We check that the integral of $f_{X,Y}$ over its support equals 1:

$$\iint_{0\le x\le y\le 2} \frac{1}{4}(x+y)\,dx\,dy = \int_{y=0}^{2}\int_{x=0}^{y} \frac{1}{4}(x+y)\,dx\,dy.$$

Compute the inner integral:

$$\int_{x=0}^{y}(x+y)\,dx = \left[\frac{x^2}{2}+yx\right]_0^y = \frac{y^2}{2}+y^2 = \frac{3}{2}y^2.$$

Multiply by $\frac{1}{4}$ and integrate in y:

$$\int_0^2 \frac{1}{4}\cdot\frac{3}{2}y^2\,dy = \frac{3}{8}\int_0^2 y^2\,dy = \frac{3}{8}\cdot\frac{2^3}{3} = 1.$$

Hence the total probability is 1, so $f_{X,Y}$ is a valid joint density.

(b) 5 points

Calculate the probability $P\{Y < 2X\}$.

Answer

We want the probability of the set $\{(x,y) : 0 \le x \le y \le 2,\ y < 2x\}$.

For a given x the upper limit for y is $\min(2x, 2)$. Thus we split the x-integration:

$$P(Y<2X) = \int_{x=0}^{1}\int_{y=x}^{2x}\frac{1}{4}(x+y)\,dy\,dx + \int_{x=1}^{2}\int_{y=x}^{2}\frac{1}{4}(x+y)\,dy\,dx.$$

First term: for $0 \le x \le 1$,

$$\int_{y=x}^{2x}(x+y)\,dy = \left[xy+\tfrac{1}{2}y^2\right]_{y=x}^{2x} = (2x^2+2x^2)-(x^2+\tfrac{1}{2}x^2) = \frac{5}{2}x^2.$$

Multiplying by $\frac{1}{4}$ and integrating x from 0 to 1 gives

$$\int_0^1 \frac{1}{4}\cdot\frac{5}{2}x^2\,dx = \frac{5}{8}\int_0^1 x^2\,dx = \frac{5}{8}\cdot\frac{1}{3} = \frac{5}{24}.$$

Second term: for $1 \le x \le 2$,

$$\int_{y=x}^{2} (x+y)\,dy = \left[xy + \tfrac{1}{2}y^2\right]_{y=x}^{2} = (2x+2) - \frac{3}{2}x^2.$$

Multiply by $\frac{1}{4}$ to get the integrand in x:

$$\frac{1}{4}\left(2x+2-\tfrac{3}{2}x^2\right) = \frac{1}{2}x + \frac{1}{2} - \frac{3}{8}x^2.$$

Integrate this from $x = 1$ to 2:

$$\int_1^2 \left(\frac{1}{2}x + \frac{1}{2} - \frac{3}{8}x^2\right)dx = \left[\frac{1}{4}x^2 + \frac{1}{2}x - \frac{1}{8}x^3\right]_1^2 = \left(\frac{1}{4}\cdot 4 + \frac{1}{2}\cdot 2 - \frac{1}{8}\cdot 8\right) - \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right) = \frac{3}{8}.$$

Combine both parts:

$$P(Y < 2X) = \frac{5}{24} + \frac{3}{8} = \frac{5}{24} + \frac{9}{24} = \frac{14}{24} = \boxed{\frac{7}{12}}.$$

(c) 10 points

Find the marginal density function $f_Y(y)$ of Y.

Answer

For $0 \le y \le 2$,

$$f_Y(y) = \int_{x=0}^{y} \frac{1}{4}(x+y)\,dx = \frac{1}{4}\left[\frac{x^2}{2} + yx\right]_0^y = \frac{1}{4}\left(\frac{y^2}{2} + y^2\right) = \frac{3}{8}y^2.$$

Thus

$$\boxed{f_Y(y) = \frac{3}{8}y^2, \quad 0 \le y \le 2, \quad \text{and } f_Y(y) = 0 \text{ otherwise.}}$$

Question 4 (5 points)

Suppose $X \sim \text{Beta}(a,b)$. Derive $E(X^2)$.

Answer

Suppose $X \sim \text{Beta}(a, b)$ with density

$$f_X(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \qquad 0 < x < 1,$$

where $B(a, b) = \dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

We compute $E[X^2]$:

$$E[X^2] = \int_0^1 x^2 \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}\,dx = \frac{1}{B(a,b)} \int_0^1 x^{a+1}(1-x)^{b-1}\,dx.$$

The integral is a Beta function:

$$\int_0^1 x^{a+1}(1-x)^{b-1}\,dx = B(a+2, b).$$

Therefore

$$E[X^2] = \frac{B(a+2,b)}{B(a,b)} = \frac{\Gamma(a+2)\Gamma(b)/\Gamma(a+b+2)}{\Gamma(a)\Gamma(b)/\Gamma(a+b)} = \frac{\Gamma(a+2)\,\Gamma(a+b)}{\Gamma(a)\,\Gamma(a+b+2)}.$$

Use $\Gamma(a+2) = a(a+1)\Gamma(a)$ and $\Gamma(a+b+2) = (a+b)(a+b+1)\Gamma(a+b)$ to simplify:

$$E[X^2] = \frac{a(a+1)\Gamma(a)\,\Gamma(a+b)}{\Gamma(a)\,(a+b)(a+b+1)\Gamma(a+b)} = \frac{a(a+1)}{(a+b)(a+b+1)}.$$

Hence

$$\boxed{E[X^2] = \frac{a(a+1)}{(a+b)(a+b+1)}\,.}$$
