Project 2
Fourth-Order Runge-Kutta Method

Lowell Monis

Mathematics 451 - Numerical Analysis I
Di Liu, Ph.D. (Instructor)
Jason Curtachio (Grader)

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Prompt: Implement a Python code for the fourth-order Runge-Kutta method. Verify the convergence rate using equation $x'(t) = -x(x^2 - 1)$

One proceeds with constructing a function that implements the fourth-order Runge-Kutta method, or RK_4. A plot is also generated to demonstrate the solution.

Setup One can use Algorithm §5.2.[1]

To proceed, one can use the following modules:

1. `numpy`: NumPy's vectorization property can be put to good use to perform advanced mathematical computations.
2. `matplotlib`: To demonstrate a good fit of the solution, one needs to generate plots. This module is imported to that end.
3. `scipy`: To compare the results from our code with the actual solution, this module can be imported.

The function is also defined here.

$$\frac{dx}{dt} = f(t, x) = -x(x^2 - 1) = x - x^3$$

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

def f(t, x):
    return x - (x**3)
```

Assumptions We will be assuming that the initial value of this problem is:

$$x(0) = 0.5$$

We will also assume the values of the start, end, and number of points.

- $a = 0$
- $b = 5$
- $N = 20$

From this, we can compute the value of step size using:

$$h = \frac{b - a}{N} = \frac{5 - 0}{20} = \frac{1}{4}$$

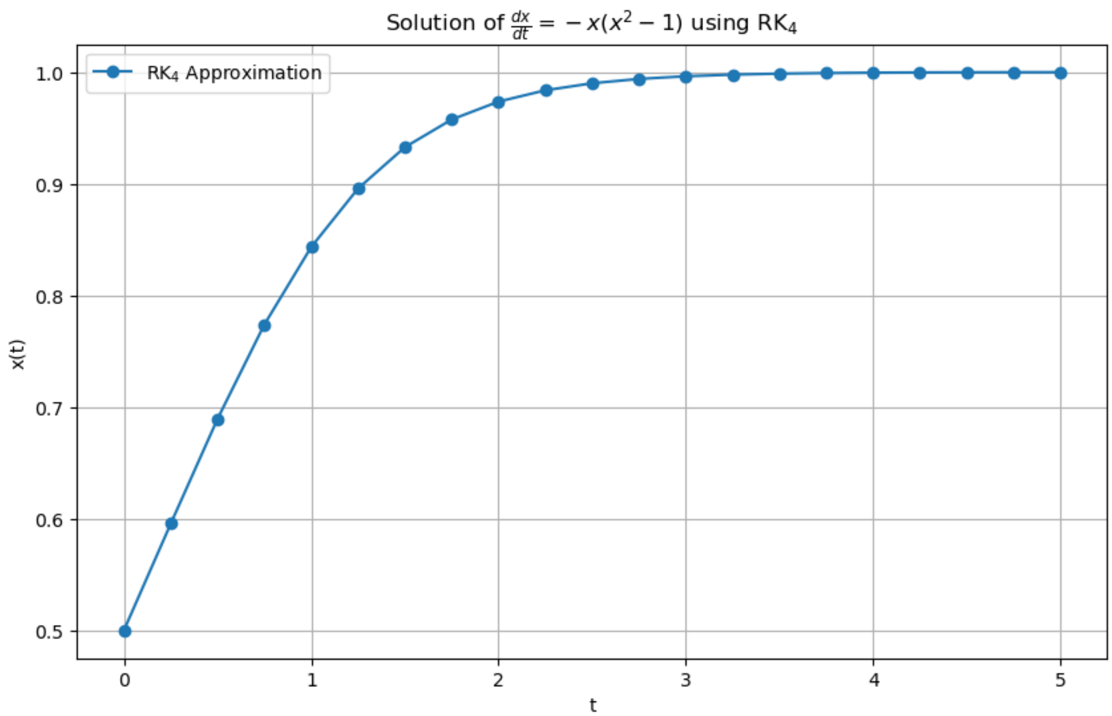
1. Richard L. Burden and J. Douglas Faires, *Numerical Analysis*, 9th ed. (Cengage Learning, 2010).

Defining a function for RK$_4$ One first defines a function to implement fourth-order Runge-Kutta, and then use it to solve the problem for an assumed value of N, and thus h.

[2]:
```
def RK4(f, a, b, N, x0):
    h = (b - a) / N                         # step size
    t = np.linspace(a, b, N + 1)            # time steps
    sols = np.zeros(N + 1)                  # creating placeholder for solutions
    sols[0] = x0                            # updating solutions with initial value

    # looping through each solution values to complete RK4
    for i in range(N):
        t_i = t[i]
        x = sols[i]

        # computing slopes
        k1 = h * f(t, x)
        k2 = h * f(t + (h / 2), x + (k1 / 2))
        k3 = h * f(t + (h / 2), x + (k2 / 2))
        k4 = h * f(t + h, x + k3)

        sols[i + 1] = x + (k1 + (2 * k2) + (2 * k3) + k4) / 6

    return t, sols
```

Solving the given differential equation One now utilizes the function created above to solve $\frac{dx}{dt} = f(t, x) = -x(x^2 - 1) = x - x^3$. One then plots the computed values.

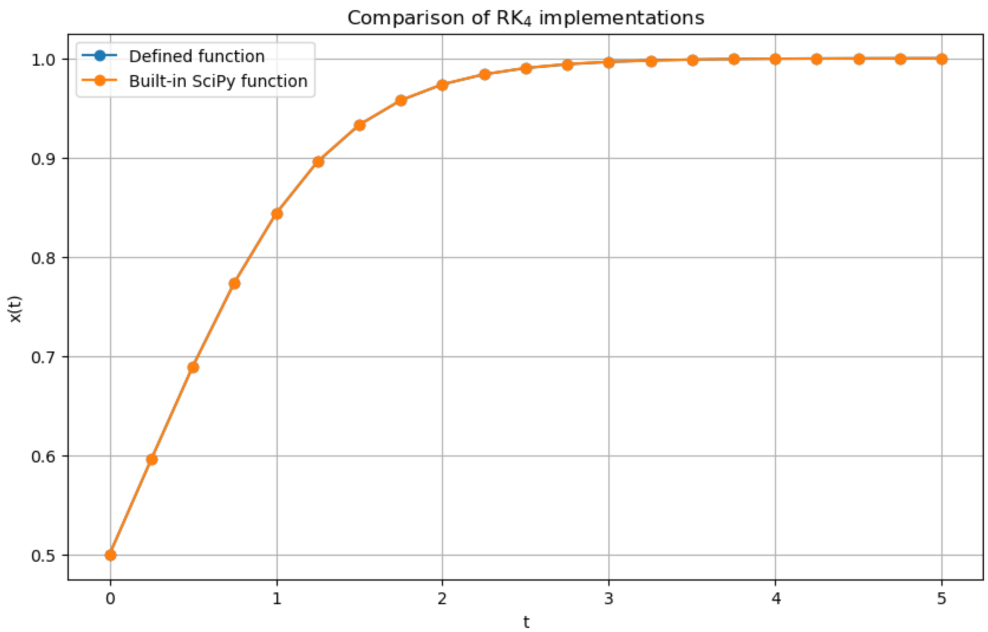
[3]:
```
a = 0      # start
b = 5      # end
x0 = 0.5   # initial value
N = 20     # Number of steps

t_values, x_values = RK4(f, a, b, N, x0)
```



Verifying the accuracy of the function defined One can now use Python's `SciPy` module to verify the accuracy of the function created to implement Runge-Kutta order of four. The method used in `scipy.integrate.solve_ivp()` should be `RK45`.

[4]:
```
t_eval = np.linspace(a, b, N + 1)
solution = solve_ivp(f, [a, b], [x0], method='RK45', t_eval=t_eval)
t_values = solution.t
x_scipy = solution.y[0]
```

Finally, one can plot the solutions to visually confirm the accuracy of the implementation. One can observe a perfect overlap.



Solutions for $\frac{dx}{dt} = f(t,x) = -x(x^2-1) = x - x^3$

t	x
0.00	0.50
0.25	0.60
0.50	0.69
0.75	0.77
1.00	0.84
1.25	0.90
1.50	0.93
1.75	0.96
2.00	0.97
2.25	0.98
2.50	0.99
2.75	0.99
3.00	1.00
3.25	1.00
3.50	1.00
3.75	1.00
4.00	1.00
4.25	1.00
4.50	1.00
4.75	1.00
5.00	1.00

Verification of Convergence Rate In order to verify the convergence rate, we will be decreasing the value of the step size. First, we find the solutions using the Runge-Kutta method for these decreasing values of step sizes. Thus, we input infinitesimally increasing N values.

[5]:
```
N_values = range(20, 10001, 10)

solutions = [RK4(f, a, b, N, x0) for N in N_values]
```

To compute the convergence rate, one needs to find all the errors for a decreasing value of step-size. To do this, one has to compute the analytical solutions.

The analytical solution was found to be:

$$x = \frac{1}{1+3e^{-2t}}$$

[6]:
```
def analytical(t):
    return np.sqrt(1/((3*np.exp(-2*t))+1))
```

We then compute the error. In order to simplify the calculation of the convergence rate, we use the following formula that uses the norm:

$$\text{Error } = \left|\left|\sum_{i=0}^{N} \bar{x}_i\right|\right|$$

To incorporate this, we use `numpy.linalg.norm()`

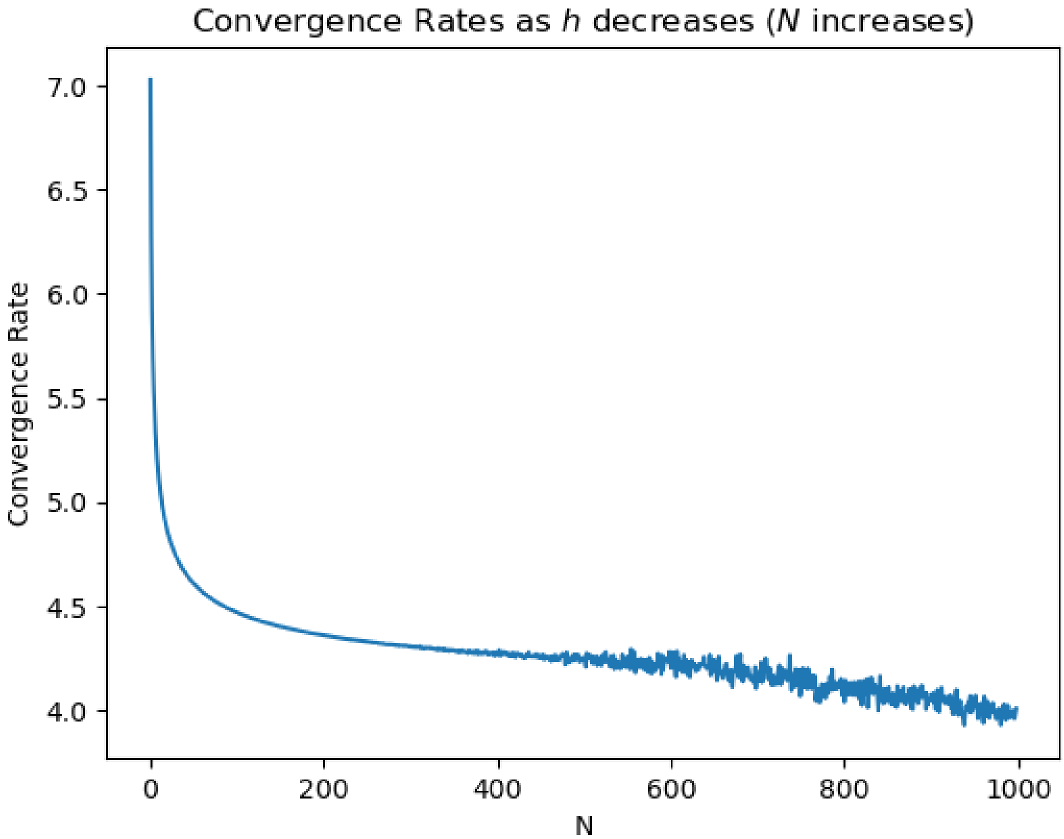
[87]:
```
errors = []

for solution in solutions:
    t, x = solution
    error = np.linalg.norm(x-analytical(t))
    errors.append(error)
```

Upon finding these errors, one can now compute the convergence rate. Since the order of the method was four, the rate should be $O(h^4)$.

Computing the convergence rate:

$$p_i = \frac{\log E_i}{\log h_i}$$

[8]:
```
h_values_log = np.log((b-a)/np.array(N_values))

errors_log = np.log(errors)

rate = errors_log/h_values_log
```



Since we used the norm for the error values, you can observe the values of convergence rates as we approach 4 fluctuates between 3.8 and 4.2.

Conclusion Upon computing the convergence rate for each value of N, we find that as the step size h decreases (hence increasing the accuracy of Runge-Kutta methods), the convergence rate converges to 4, showing that the convergence rate of the equation is of the order 4, or $O(h^4)$. Hence, verified. $\square$

References

Burden, Richard L., and J. Douglas Faires. *Numerical Analysis*. 9th ed. Cengage Learning, 2010.