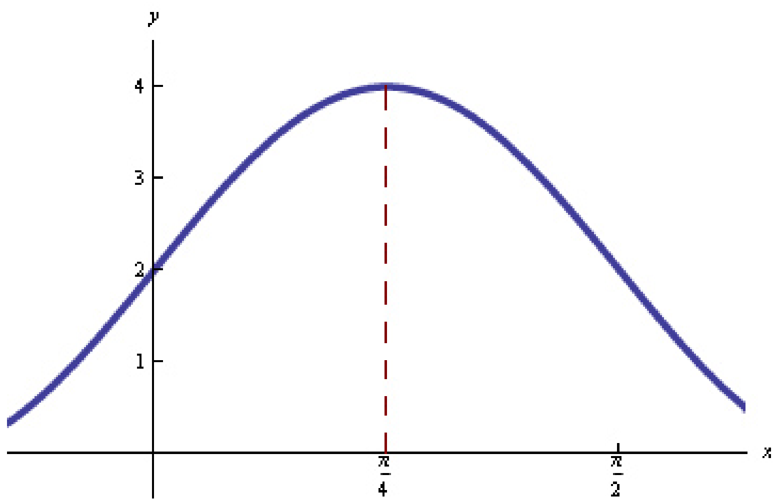
Chapter 3: Behavior of Functions, Extreme Values

- Maximum and Minimum Function Values
- Applications involving an Absolute Extremum on a Closed Interval
- Rolle's Theorem and the Mean Value Theorem
- Increasing and Decreasing Functions and the First Derivative Test

- Concavity, Points of Inflection and the Second Derivative Test
- Summary of Sketching Graphs of Functions
- Additional Applications of Absolute Extrema

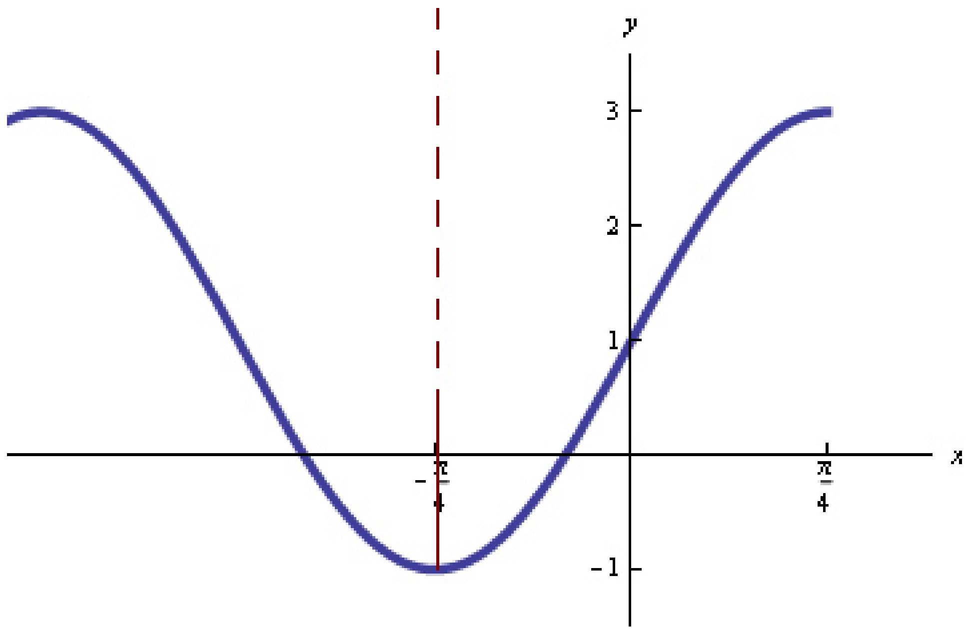
Maximum and Minimum Function Values

Definition 1. The function f has a relative maximum value at the number c if there exists an open interval containing c, on which f is defined such that $f(c) \geq f(x)$ for all x in this interval.



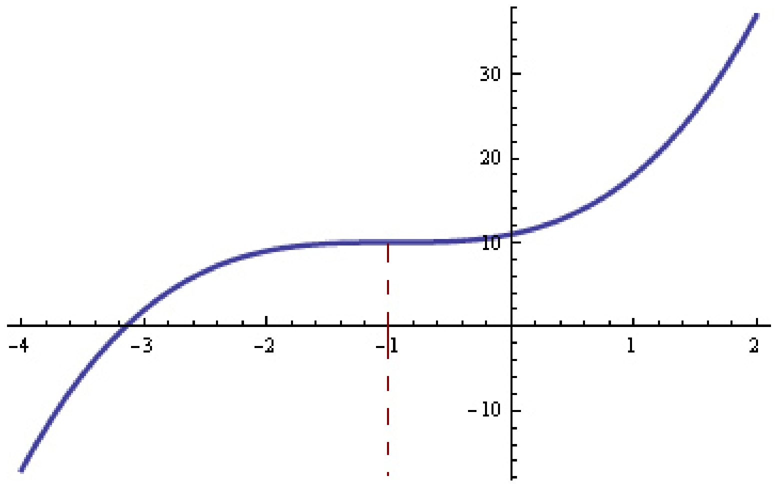
Relative Maximum at $x = \pi/4$

Definition 2. The function f has a relative minimum value at the number c if there exists an open interval containing c, on which f is defined such that $f(c) \leq f(x)$ for all x in this interval.



Relative Minimum at $x = -\pi/4$

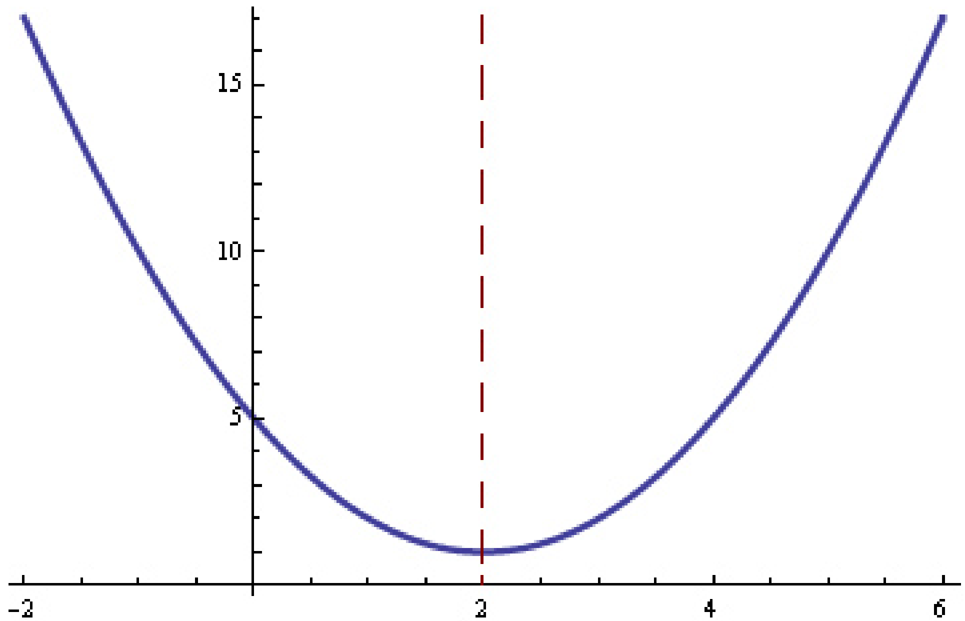
If a function has either a relative maximum or a relative minimum value at c, the function has a **relative extremum** at c.



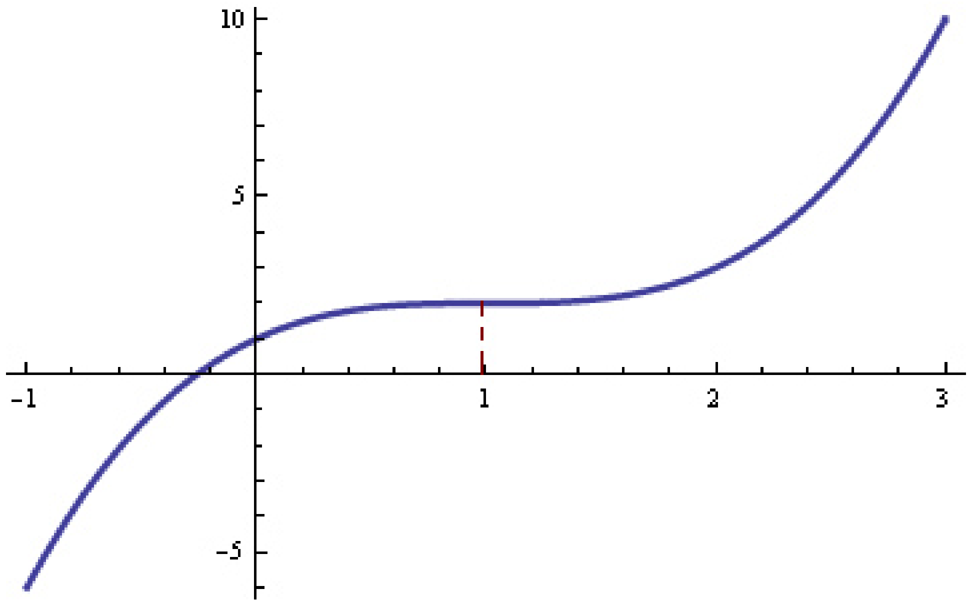
No Extremum at $x = -1$

Theorem 1. *If $f(x)$ exists for all values of x in the open interval (a, b) and f has a relative extremum at c, where $a < c < b$, then if $f'(c)$ exists, $f'(c) = 0$.*

Example 1. Let f be the function defined by $f(x) = x^2 - 4x + 5$. Then $f'(x) = 2x - 4$ and $f'(2) = 0$. Thus, f **may have** a relative extremum at $c = 2$. Upon checking we see that $f(2) = 1$ and $f(x) > 1$ for all $x < 2$ or $x > 2$.



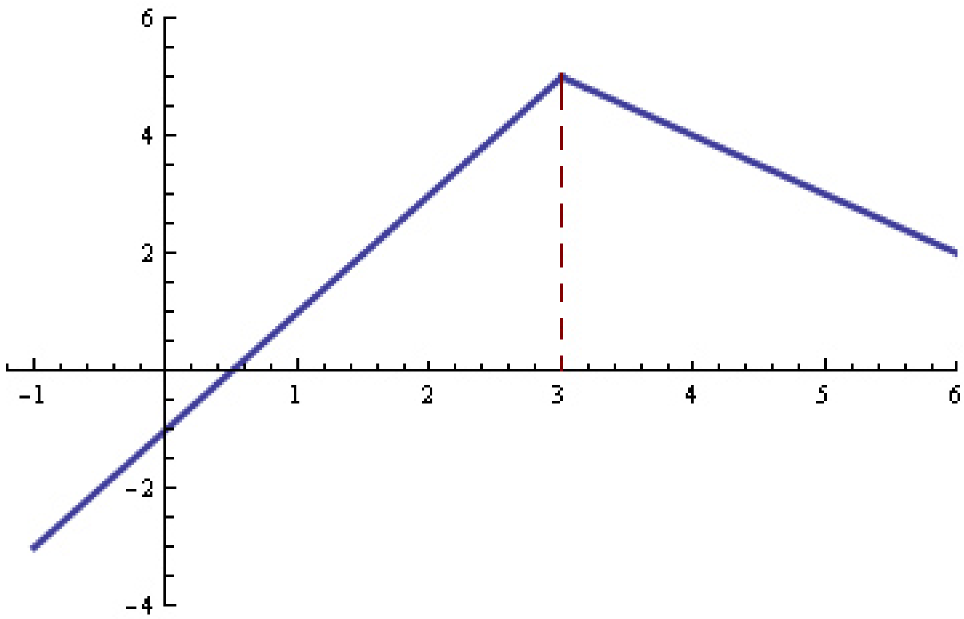
Example 2. Consider the function defined by $f(x) = (x-1)^3 + 2$ so that $f'(x) = 3(x-1)^2$ and $f'(1) = 0$ which implies that there is **a possibility** that f has a relative extremum at $c = 1$. Now, $f(1) = 2$; for $x < 1$ we have $f(x) < 2$ and for $x > 1$ we have $f(x) > 2$. This only shows that f has no relative extremum at $x = 1$ although the derivative at this point is 0.



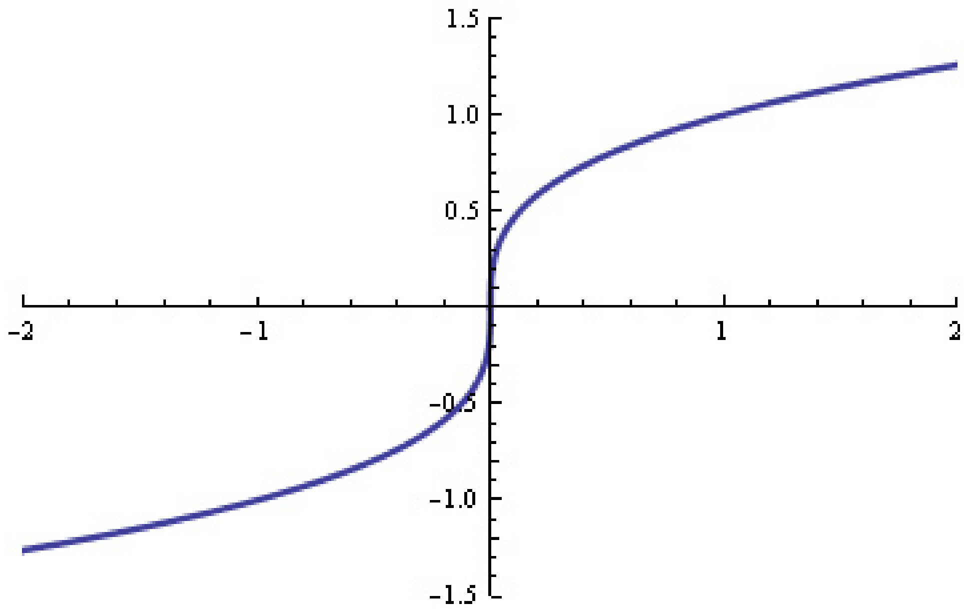
Example 3. Let f be defined by

$$f(x) = \begin{cases} 2x - 1, & \text{if } x \le 3 \\ 8 - x, & \text{if } x > 3. \end{cases}$$

Observe that $f'_-(3) = 2$ and $f'_+(3) = -1$ so that the derivative does not exist at $x = 3$. However, observe that $f(3) = 5 \le f(x)$ for all x so that the function has a maximum at $x = 3$.



Example 4. Let the function f be defined by $f(x) = x^{1/3}$ so that $\mathrm{dom}\ f = \mathbb{R}$. Now $f'(x) = \dfrac{1}{3x^{2/3}}$, $x \neq 0$ so that $f'(0)$ does not exist. The graph of this function shows that it has no relative extrema.



Definition 3. If c is a number in the domain of the function f, and if either $f'(c) = 0$ or $f'(c)$ does not exist, then c is a critical number of f.

Example 5. Find the critical numbers of the following functions.

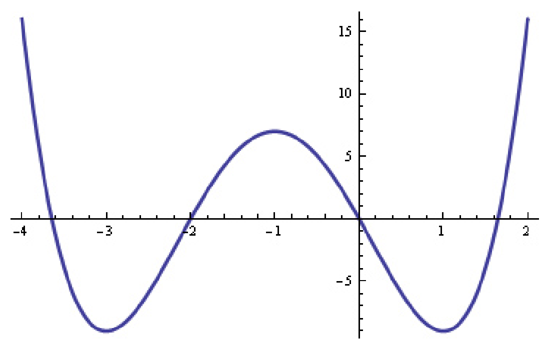
1. $f(x) = x^4 + 4x^3 - 2x^2 - 12x$

   Solution.

   $f'(x) = 4x^3+12x^2-4x-12 = 4(x^3+3x^2-x-3) = 4(x+3)(x-1)(x+1)$

   $0 = (x+3)(x-1)(x+1)$

   Therefore, the critical numbers are $-3, -1$ and 1.



$y = x^4 + 4x^3 - 2x^2 - 12x$

2. $f(x) = x^{4/3} + 4x^{1/3}$

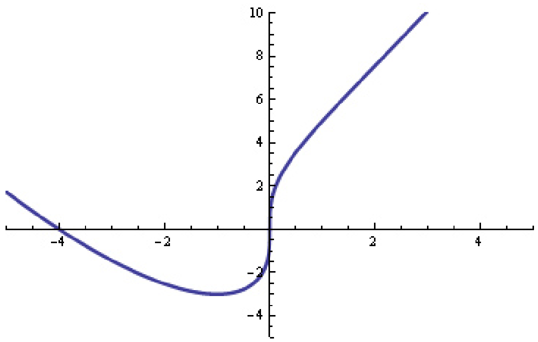
Solution.

$$f'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \frac{4(x+1)}{3x^{2/3}}$$

$$0 = \frac{4(x+1)}{3x^{2/3}} \Leftrightarrow x = -1$$

Also, $f'(0)$ does not exist.

Therefore, the critical numbers are -1 and 0.



$y = x^{4/3} + 4x^{1/3}$

3. $f(x) = \sin x \cos x$

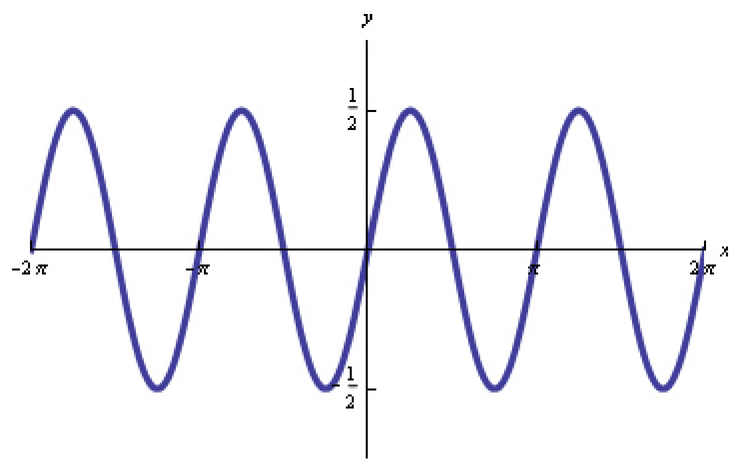
Solution.

$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = -\sin^2 x + \cos^2 x$

$= (\cos x + \sin x)(\cos x - \sin x)$

$0 = (\cos x + \sin x)(\cos x - \sin x) \Leftrightarrow \cos x = \pm \sin x$

Hence, $x = \frac{\pi}{4} + k\pi, k \in \mathbb{Z}$ which are the critical numbers.

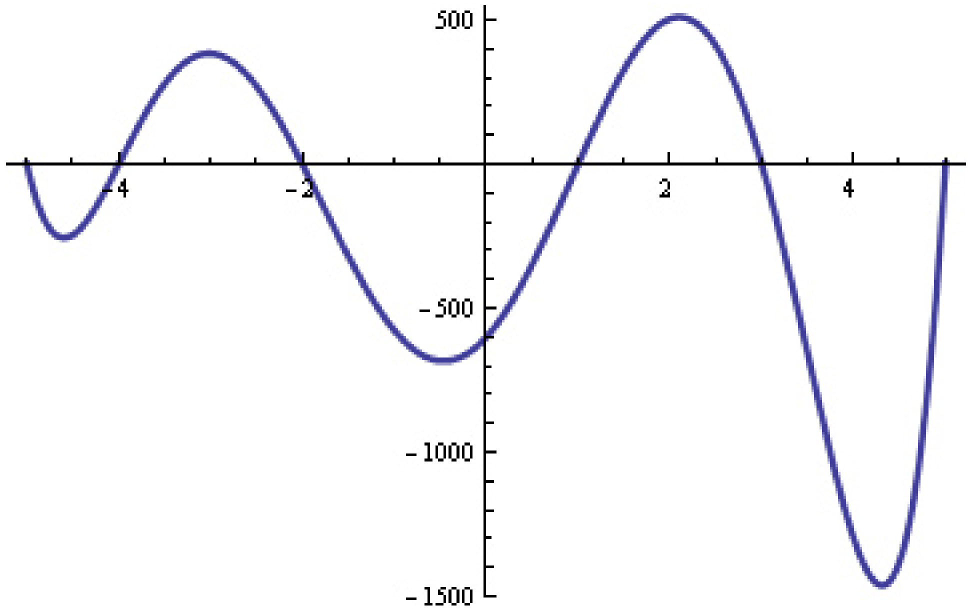


$y = \sin x \cos x$

Definition 4. The function f has an absolute maximum value on an interval if there is some number c in the interval such that $f(c) \geq f(x)$ for all x in the interval. The number $f(c)$ is called the absolute maximum value of f on the interval.

Definition 5. The function f has an absolute minimum value on an interval if there is some number c in the interval such that $f(c) \leq f(x)$ for all x in the interval. The number $f(c)$ is called the absolute minimum value of f on the interval.

An absolute extremum of a function on an interval is either an absolute maximum value or an absolute minimum value of the function on the interval.



$y = f(x)$ on the Closed Interval $[-5, 5]$

Absolute Maximum Value occurs at $x = 2$ and
Absolute Minimum Value occurs at $x = 4$

Theorem 2. `The Extreme Value Theorem.` *If the function f is continuous on the closed interval $[a, b]$, then f has an absolute maximum value and an absolute minimum value on $[a, b]$.*

REMARK: An absolute extremum of a function continuous on a closed interval must either be a relative extremum or a function value at an endpoint of the interval.

Because a necessary condition for a function to have a relative extremum at a number c is for c to be a critical number, the absolute extremum values of a continuous function f on a closed interval $[a, b]$ can be determined by the following procedures:

1. Find the function values at the critical numbers of f on (a, b).

2. Find the values of $f(a)$ and $f(b)$.

3. The largest of the values from steps 1 and 2 is the absolute maximum value and the smallest is the absolute minimum value.

Example 6. Find the absolute extrema of f on $[-2,3]$ if $f(x) = x^3 - 6x - 1$

Solution. Because f is continuous on $[-2,3]$,t he extreme-value theorem applies.To find the critical numbers of f we first find f':

$$f'(x) = 3x^2 - 6 = 3(x^2 - 2)$$

$f'(x)$ exists for all real numbers, and so the only critical numbers of f will be the values of x for which $f'(x) = 0$. Setting $f'(x) = 0$, we have

$$0 = 3(x^2 - 2)$$

from which we obtain

$$x = \pm\sqrt{2}$$

The critical numbers of f are $-\sqrt{2}$ and $\sqrt{2}$, and each of these numbers is in the given closed interval $[-2,3]$.We find the function values at the critical numbers and at the endpoints of the interval.

$$f(-2) = 3$$

$$f(-\sqrt{2}) = 4\sqrt{2} - 1 \approx 4.66$$

$$f(\sqrt{2}) = -1 - 4\sqrt{2} \approx -6.66$$

$$f(3) = 8$$

The absolute maximum value of f on $[-2, 3]$ is therefore 8 which occurs at the right endpoint 3, and the absolute minimum value of f on $[-2, 3]$ is $1 - 4\sqrt{2}$, which occurs at $\sqrt{2}$.

Example 7. Find the number in the interval $[\frac{1}{3}, 2]$ such that the sum of the number and its reciprocal is (**a**) a minimum and (**b**) a maximum.

Solution. Let $x =$ the number. Hence, $f(x) = x + \frac{1}{x}$ and

$$f'(x) = 1 - \tfrac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

$$0 = \frac{x^2 - 1}{x^2} \Rightarrow x = \pm 1$$

Also, $f'(0)$ does not exist. Hence, the critical numbers are $-1, 0$ and 1. Only 1 is in the interval $[\frac{1}{3}, 2]$. We find the function values at the critical number 1 and at the endpoints of the interval.

$$f(\tfrac{1}{3}) = \frac{10}{3}$$

$$f(1) = 2$$

$$f(2) = \frac{5}{2}$$

The absolute maximum value of f on $[\frac{1}{3}, 2]$ is therefore $\frac{10}{3}$ which occurs at the left endpoint $\frac{1}{3}$, and the absolute minimum value of f on $[\frac{1}{3}, 2]$ is 2, which occurs at 1.

Example 8. Find the number in the interval $[-1, 1]$ such that the difference of the number minus its square is (**a**) a maximum and (**b**) a minimum.

Solution. Let $x =$ the number. Hence, $f(x) = x - x^2$ and

$$f'(x) = 1 - 2x$$

$$0 = 1 - 2x \Rightarrow x = \tfrac{1}{2}$$

Hence, the critical number is $\frac{1}{2}$ which is in the interval $[-1, 1]$. We find the function values at the critical number $\frac{1}{2}$ and at the endpoints of the interval.

$$f(-1) = -2$$

$$f(\tfrac{1}{2}) = \frac{1}{4}$$

$$f(1) = 0$$

The absolute maximum value of f on $[-1, 1]$ is therefore $\frac{1}{4}$ which occurs at $\frac{1}{2}$, and the absolute minimum value of f on $[-1, 1]$ is -2, which occurs at the left endpoint -1.

Exercises. In Exercises 1 through 10, find the critical numbers of the given function.

1. $f(x) = x^3 + 7x^2 - 5x$

2. $f(x) = 2x^3 - 2x^2 - 16x + 1$

3. $f(x) = x^4 + 4x^3 - 2x^2 - 12x$

4. $f(x) = x^{7/3} + x^{4/3} - 3x^{1/3}$

5. $f(x) = x^{6/5} - 12x^{1/5}$

6. $f(x) = x^4 + 11x^3 + 34x^2 + 15x - 2$

7. $f(x) = (x^2 - 4)^{2/3}$

8. $f(x) = (x^3 - 3x^2 + 4)^{1/3}$

9. $f(x) = \dfrac{x}{x^2 - 9}$

10. $f(x) = \dfrac{x + 1}{x^2 - 5x + 4}$

In Exercises 11 through 20, find the absolute maximum value and the absolute minimum value of the given function on the indicated interval.

11. $f(x) = x^3 + 5x - 4; [-3, -1]$

12. $f(x) = x^3 + 3x^2 - 9x; [-4, 4]$

13. $f(x) = x^4 - 8x^2 + 16; [-4, 0]$

14. $f(x) = x^4 - 8x^2 + 16; [-1, 4]$

15. $f(x) = x^4 - 8x^2 + 16; [0, 3]$

16. $f(x) = x^4 - 8x^2 + 16; [-3, 2]$

17. $f(x) = \dfrac{x}{x+2}; [-1, 2]$

18. $f(x) = \dfrac{x+5}{x-3}; [-5, 2]$

19. $f(x) = (x+1)^{2/3}; [-2, 1]$

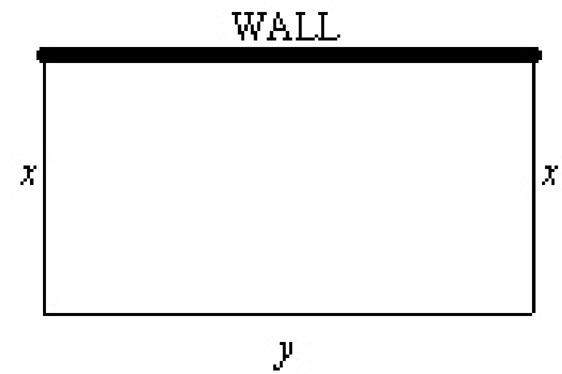
20. $f(x) = 1 - (x-3)^{2/3}; [-5, 4]$

Applications involving an Absolute Extremum on a Closed Interval

Each of the following is an application of the Extreme Value Theorem in which the solution is an absolute extremum of a function on a closed interval.

Example 9. A farmer has 200 yd of fence with which to construct three sides of a rectangular pen; an existing long, straight wall will form the fourth side. What dimensions will maximize the area of the pen?

We let x denote the length of each of the two sides of the pen perpendicular to the wall. We also let y denote the length of the side parallel to the wall.



Then the area of the rectangle is given by $A = xy$. Because all 200 yd of fence are to be used, we have $2x + y = 200$ or $y = 200 - 2x$ so that the area can be written in terms of the variable x alone; i.e.

$$A(x) = x(200 - 2x)$$

From Extreme Value Theorem (EVT), we need a closed interval. Here we consider the closed interval $[0, 100]$ and the values 0 and 100 correspond to "degenerate" pens of area 0. Applying what we have just discussed,

$$A'(x) = -4x + 200$$

The critical number is 50 which is in the interval $[0, 100]$. Since the area at the endpoints of the interval is 0, we get the results $x = 50$ and $y = 100$ so that the maximum area is 5000 yd^2.

Example 10. A piece of sheet metal is rectangular, 5 ft wide and 8 ft long. Congruent squares are to be cut from its four corners. The resulting piece of metal is to be folded and welded to form an open-topped box. How should this be done to get a box of largest possible volume?

Let x be the length of the edge of each corner square removed. To write the volume V as function of x, we have

$$V(x) = x(5-2x)(8-2x) = 4x^3 - 26x^2 + 40x.$$

Here, we shall take the closed interval $[0, 2.5]$ to obtain the result $x = 1$. The maximum volume is then equal to 18ft^3 and the resulting box will measure 6 ft by 3 ft by 1 ft.

Example 11. We need to design a cylindrical can with radius r and height h. The top and bottom must be made of copper, which will cost PhP $2/\text{in}^2$. The curved side is to be made of aluminum, which will cost PhP $1/\text{in}^2$. We seek the dimensions that will maximize the volume of the can. The only constraint is that the total cost of the can is to be PhP 300π.

The volume is given by $V = \pi r^2 h$, but we need to express it as a function of r or of h alone. Now we consider the restriction on the total cost so that

$$300\pi = 4\pi r^2 + 2\pi r h.$$

Using this equation, we can eliminate h

$$300\pi - 4\pi r^2 = 2\pi r h$$

$$h = \frac{2(75 - r^2)}{r}$$

and obtain

$$V(r) = 2\pi(75r - r^3).$$

Here, we consider the domain $[0, 5\sqrt{3}]$ since $V(0) = 0$ and $V(5\sqrt{3}) = 0$.

$$V'(r) = 2\pi(75 - 3r^2)$$

The critical number which is in the interval is 5. The absolute maximum is obtainable at $r = 5$ in.

Exercises.

1. Find the area of the largest rectangle having a perimeter of 200 ft.

2. Find the area of the largest isosceles triangle having a perimeter of 18 in.

3. A cardboard manufacturer wishes to make open boxes from rectangular pieces of cardboard with dimensions 10 in by 7 in by cutting equal squares from the four corners and turning up the sides. We wish to find the length of the side of the cut out square so that the box has the largest possible volume.

4. A rectangular plot of ground is to be enclosed by a fence and then divided down the middle by another fence. If the fence down the middle costs $\$1$ per running foot and the other fence costs $\$2.50$ per running foot, find the dimensions of the plot of largest possible area that can be enclosed with $\$480$ worth of fence.

5. Find two nonnegative numbers whose sum is 12 such that their product is an absolute maximum.

6. Find two nonnegative numbers whose sum is 12 such that the sum of their squares is an absolute minimum.

7. .Given the circle having the equation $x^2 + y^2 = 9$, find (a) the shortest distance from the Point $(4, 5)$ to a point on the circle, and (b) the longest distance from the point $(4, 5)$ to a point on the circle.

8. Find the area of the largest rectangle having two vertices on the x axis and two vertices on or above the x axis and on the parabola $y = 9 - x^2$.

Rolle's Theorem and the Mean Value Theorem

Theorem 3. *Rolle's Theorem.*
Let f be a function such that

(i) it is continuous on the closed interval $[a, b]$;

(ii) it is differentiable on the open interval (a, b);

(iii) $f(a) = 0$ and $f(b) = 0$.

Then there is a number c in the open interval (a, b) such that $f'(c) = 0$.

Proof. (Rolle's Theorem)
We consider two cases:

Case 1: $f(x) = 0$ for all $x \in [a, b]$

In this case, any number between a and b satisfies the property that $f'(x) = 0$.

Case 2: $f(x) \neq 0$ for some $x \in (a, b)$.

Since f is continuous on the closed interval $[a, b]$, by the extreme-value theorem, f has an absolute maximum value and an absolute minimum value on $[a, b]$. Furthermore, by (iii) we know that $f(a) = f(b) = 0$ and there exists some x in (a, b) for which $f(x) \neq 0$. Hence, f will either have a positive maximum value at some $c_1 \in (a, b)$ or a negative absolute minimum value at some $c_2 \in (a, b)$, or both. Thus for $c = c_1$ or $c = c_2$, there is an absolute extremum at an interior point of the interval $[a, b]$. Therefore, the absolute extremum $f(c)$ is also a relative extremum and since $f'(c)$ exists, by Theorem (1) $f'(c) = 0$. $\square$

Example 12. Given $f(x) = 4x^3 - 9x$ verify the conditions of the hypothesis of the Rolle's Theorem for the intervals $[-\frac{3}{2}, 0]$, $[0, -\frac{3}{2}]$ and $[-\frac{3}{2}, \frac{3}{2}]$. Then find a suitable c in each of these intervals for which $f'(c) = 0$

Solution.

$$f'(x) = 12x^2 - 9$$

Because $f'(x)$ exists for all values of x, f is differentiable on $(-\infty, +\infty)$ and therefore continuous on $(-\infty, +\infty)$. Conditions (i) and (ii) of Rolle's Theorem thus hold on any interval. To determine on which intervals condition (iii) holds, we find the values of x for which $f(x) = 0$. If $f(x) = 0$,

$$4x(x^2 - \frac{9}{4}) = 0$$

$$x = -\frac{3}{2}, x = 0, x = \frac{3}{2}$$

With $a = -\frac{3}{2}$ and $b = 0$, Rolle's Theorem holds on $[-\frac{3}{2}, 0]$. Similarly, Rolle's Theorem holds on $[0, \frac{3}{2}]$ and $[-\frac{3}{2}, \frac{3}{2}]$.

To find the suitable values for c, set $f'(x) = 0$ and get

$12x^2 - 9 = 0$

$x = -\frac{1}{2}\sqrt{3}, x = \frac{1}{2}\sqrt{3}$

Therefore, in the interval $[-\frac{3}{2}, 0]$ a suitable choice for c is $-\frac{1}{2}\sqrt{3}$. In the interval $[0, \frac{3}{2}]$, take $c = \frac{1}{2}\sqrt{3}$. In the interval $[-\frac{3}{2}, \frac{3}{2}]$ there are two possibilities for c: either $-\frac{1}{2}\sqrt{3}$ or $\frac{1}{2}\sqrt{3}$.

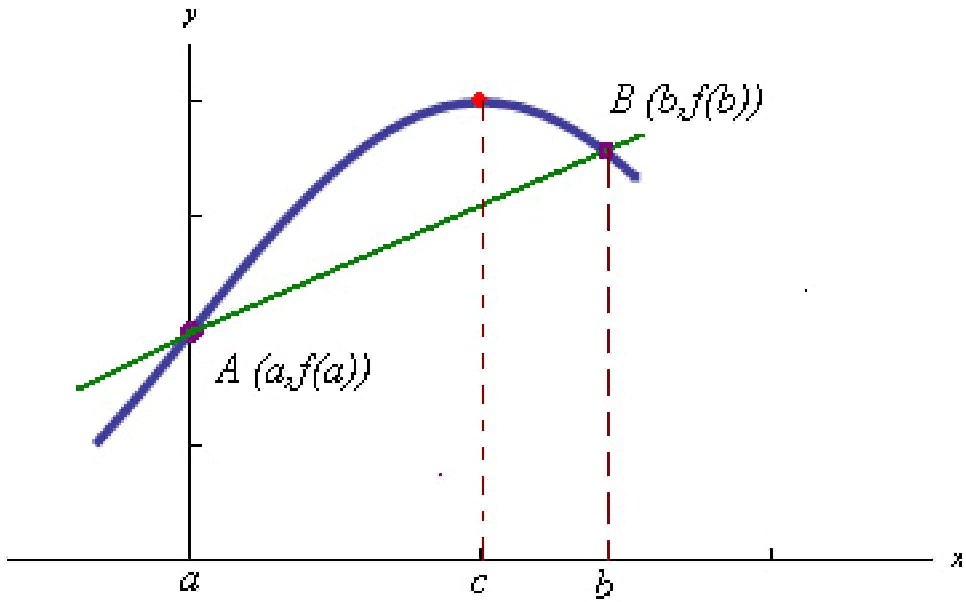
Theorem 4. `Mean-Value Theorem.`
Let f be a function such that

(i) it is continuous on the closed interval $[a, b]$;

(ii) it is differentiable on the open interval (a, b).

Then there is a number c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



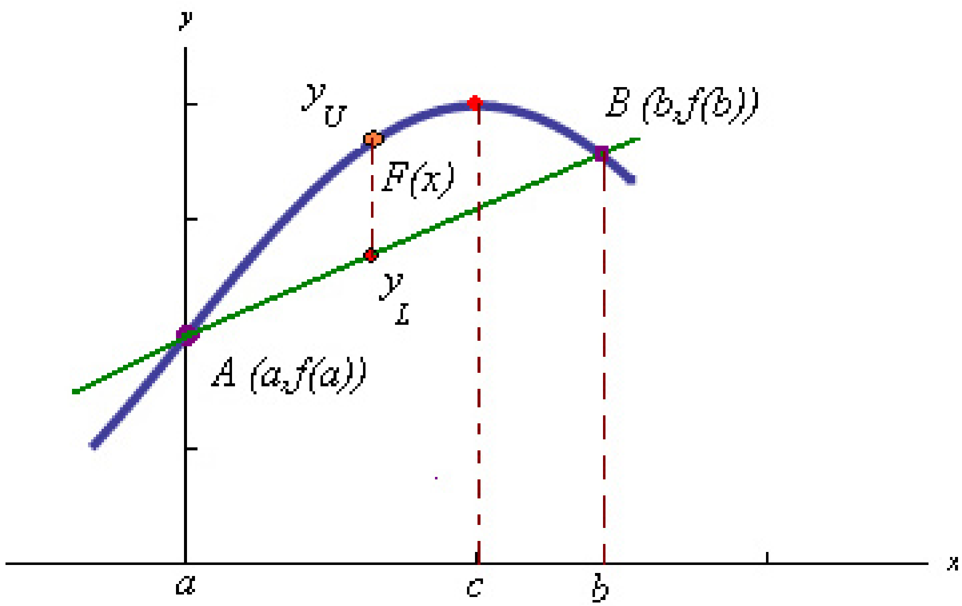
Proof. An equation of the line through A and B in the figure above is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\Leftrightarrow \quad y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Now if $F(x)$ measures the vertical distance between a point $(x, f(x))$ on the graph of the function f and the corresponding point on the secant through A and B, then

$$\begin{aligned} F(x) &= y_U - y_L \\ &= f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \end{aligned}$$



We show that this function satisfies the three conditions of the hypothesis of the Rolle's theorem.

The function F is continuous on the closed interval $[a, b]$ because it is the sum of f and linear function, both of which are continuous there. Therefore, condition (i) is satisfied. Condition (ii) is also satisfied because f is differentiable on (a, b). Now from (1), $F(a) = F(b) = 0$ so that (iii) is also satisfied.

The conclusion of Rolle's theorem states that there is a c in the open interval (a, b) such that $F'(c) = 0$. But

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Thus,

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Therefore there is a number $c \in (a, b)$ such that

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a}$$

$$\Leftrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

Example 13. Given $f(x) = x^3 - x^2 - 2x$ verify that the hypothesis of the MVT is satisfied for $a = 1$ and $b = 3$. Then find a number c in the open interval $(1, 3)$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}.$$

Solution. Because f is a polynomial function, f is continuous and differentiable everywhere. The hypothesis of the MVT is, therefore, satisfied for any a and b.

$$f'(x) = 3x^2 - 2x - 2$$

Because $f(1) = -2$ and $f(3) = 12$,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} = \frac{12 - (-2)}{2} = 7$$

We set $f'(c) = 7$ to obtain

$$3c^2 - 2c - 2 = 7$$

$$3c^2 - 2c - 9 = 0$$

$$c = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-9)}}{2(3)}$$

$$c = \frac{2 + \sqrt{112}}{6} \approx 2.10, c = \frac{2 - \sqrt{112}}{6} \approx -1.43$$

Because -1.43 is not in the open interval $(1, 3)$ the only possible value for c is 2.10

Example 14. Use the mean-value theorem to prove that if $x > 0$ then $\sin x < x$.

Solution. When $x > 1$, it is easy to see that $x \geq \sin x$ since $\sin x \leq 1$. Now for $0 < x \leq 1$ we let

$$f(x) = x - \sin x$$

so that

$$f'(x) = 1 - \cos x.$$

Because f is continuous and differentiable everywhere, we conclude from the mean-value theorem that there is some number c for which $0 < c < x \leq 1$, such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

or

$$f(x) = x(1 - \cos c) \quad 0 < c < 1.$$

On the right-hand side of this equation, both factors are positive. Thus,

$$0 < f(x) = x - \sin x$$

or

$$\sin x < x.$$

Theorem 5. *If f is a function such that $f'(x) = 0$ for all values of x in an interval I, then f is constant on I.*

Exercises.

In Exercises 1 through 4, verify that the three conditions of the hypthotesis of the Rolle's theorem are satisfied by the function on the indicated interval. Then find a suitable value for c that satisfies the conclusion of Rolle's theorem. Support your choice of c graphically by plotting in the same window the graph of f and the horizontal tangent line at $(c, f(c))$.

1. $f(x) = x^2 - 4x + 3; [1, 3]$
2. $f(x) = x^3 - 2x^2 - x + 2; [1, 2]$
3. $f(x) = \sin 2x; [0, \frac{1}{2}\pi]$
4. $f(x) = 3\cos^2 x; [\frac{1}{2}\pi, \frac{3}{2}\pi]$

In Exercises 5 through 10, do the following: (a) Plot the graph of the function on the indicated interval; (b) test the three conditions of the hypthotesis of the Rolle's theorem and determine which conditions are satisfied and which, if any, are not satisfied; (c) if the three conditions in part (b) are satisfied, determine a point at which there is a horizontal tangent line and support your answer graphically.

5. $f(x) = x^{4/3} - 3x^{1/3}; [0, 3]$

6. $f(x) = x^{3/4} - 2x^{1/4}; [0, 4]$

7. $f(x) = \dfrac{x^2 - x - 12}{x - 13}; [-3, 4]$

8. $f(x) = 1 - |x|; [-1, 1]$

9. $f(x) = \begin{cases} x^2 - 4 & \text{if } x < 1 \\ 5x - 8 & \text{if } 1 \leq x \end{cases}; [-2, \frac{8}{3}]$

10. $f(x) = \begin{cases} 3x + 6 & \text{if } x < 1 \\ x - 4 & \text{if } 1 \leq x \end{cases}; [-2, 4]$

In Exercises 11 through 20, verify that the hypthotesis of the mean-value theorem is satisfied for the function on the indicated interval $[a, b$. Then find a suitable choice for c that satisfies the conclusion of the mean-value theorem. Support your choice of c graphically by plotting in the same window the graph of f on the closed interval $[a, b$, the tangent line at $(c, f(c))$, and the secant line through the points $(a, f(a))$ and $(b, f(b))$ and observing that the tangent line and the secant line are parellel.

11. $f(x) = x^2 + 2x - 1; [0, 1]$
12. $f(x) = x^3 + x^2 - x; [-2, 1]$
13. $f(x) = x^{2/3}; [0, 1]$
14. $f(x) = \dfrac{x^2 + 4x}{x - 7}; [2, 6]$
15. $f(x) = \sqrt{1 + \cos x}; [-\frac{1}{2}\pi, \frac{1}{2}\pi]$
16. $f(x) = \sqrt{1 - \sin x}; [0, \frac{1}{2}\pi]$
17. $f(x) = x^2; [3, 5]$
18. $f(x) = x^2; [2, 4]$
19. $f(x) = \sin x; [0, \frac{1}{2}\pi]$
20. $f(x) = 2\cos x; [\frac{1}{3}\pi, \frac{2}{3}\pi]$

For each of the functions Exercises 21 through 24, there is a number c in the open interval (a, b) that satisfies the conclusion of the mean-value theorem. In each exercise, determine which part of the hypothesis of the mean-value theorem fails to hold. Sketch the graph of f and the line through the points $(a, f(a))$ and $(b, f(b))$.

21. $f(x) = \dfrac{4}{(x-3)^2}; a = 1, b = 6$

22. $f(x) = \dfrac{2x-1}{3x-4}; a = 1, b = 2$

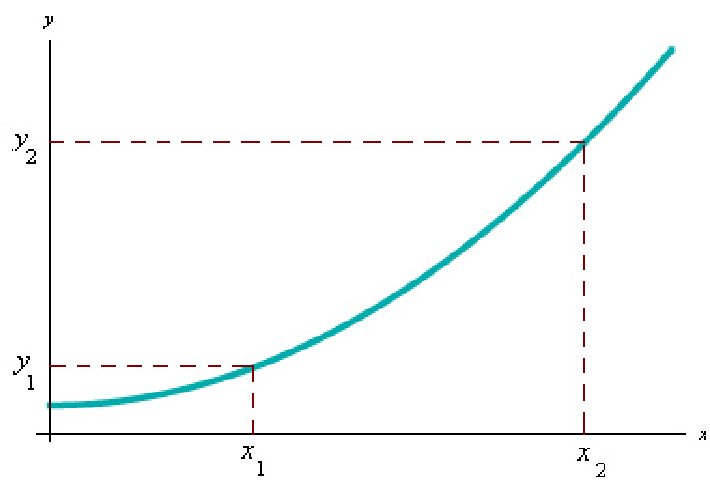
23. $f(x) = 3(x-4)^{2/3}; a = -4, b = 5$

24. $f(x) = \begin{cases} 2x+3 & \text{if } x < 3 \\ 15-2x & \text{if } 3 \le x \end{cases};$

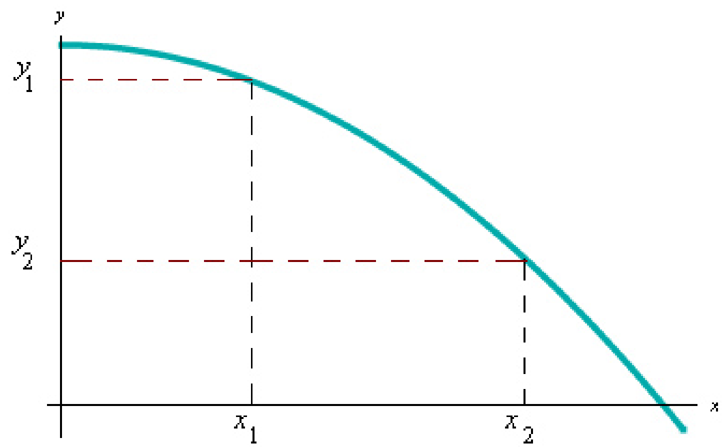
$a = -1, b = 5$

Increasing and Decreasing Functions and the First Derivative Test

Definition 6. A function f defined on an interval is increasing on that interval if and only if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ where x_1 and x_2 are any numbers in the interval.



Definition 7. A function f defined on an interval is decreasing on that interval if and only if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ where x_1 and x_2 are any numbers in the interval.



If a function is either increasing or decreasing on an interval, then it is said to be *monotonic* on the interval.

Theorem 6. *Let the function f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b):*

(i) if $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$;

(ii) if $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$;

Example 15. Determine the intervals on which the function $f(x) = x^2 - 4$ is increasing (decreasing).

Solution. The function f is continuous everywhere.

$$f'(x) = 2x$$

$f'(x)$ exists for al lvalues of x. Setting $f'(x) = 0$,we have

$$0 = 2x \Rightarrow x = 0$$

We consider the intervals $(-\infty, 0]$ and $[0, +\infty)$ and choose a number in each of the interval. We choose -1 and 1.

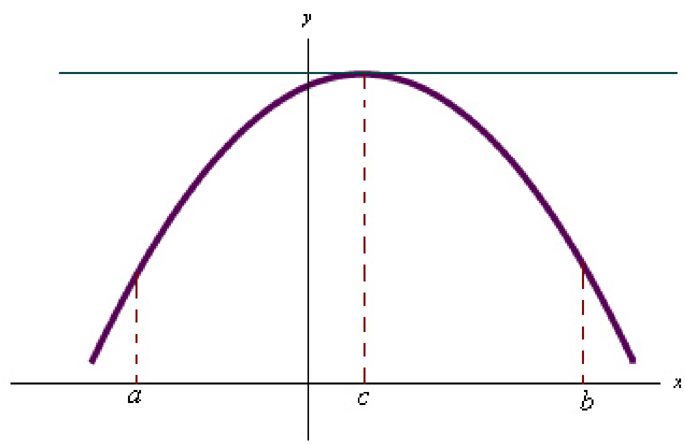
$$f'(-1) = -2 < 0$$

$$f'(1) = 2 > 0$$

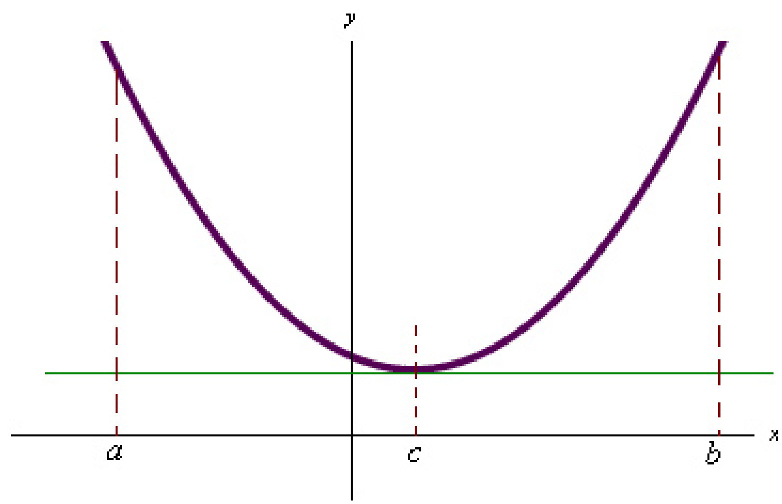
Any number in the interval will give the same results. Hence, from Theorem 6, f is decreasing on $(-\infty, 0]$ and increasing on $[0, +\infty)$.

Theorem 7. ***The First-Derivative Test for Relative Extrema***
Let the function f be continuous at all points of the open interval (a, b) containing the number c, and suppose that f' exists at all points of (a, b) except possibly at c:

(i) if $f'(x) > 0$ for all values of x in some open interval having c as its right endpoint, and if $f'(x) < 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative maximum value at c;



(ii) if $f'(x) < 0$ for all values of x in some open interval having c as its right endpoint, and if $f'(x) > 0$ for all values of x in some open interval having c as its left endpoint, then f has a relative minimum value at c.



To obtain the relative extrema of a function $f(x)$, we follow these steps:

1. Compute $f'(x)$.

2. Determine the critical numbers of f.

3. Apply the first-derivative test.

Example 16. Plot the graph of $f(x) = x^3 - 9x^2 + 15x - 5$ and determine its relative exrema.

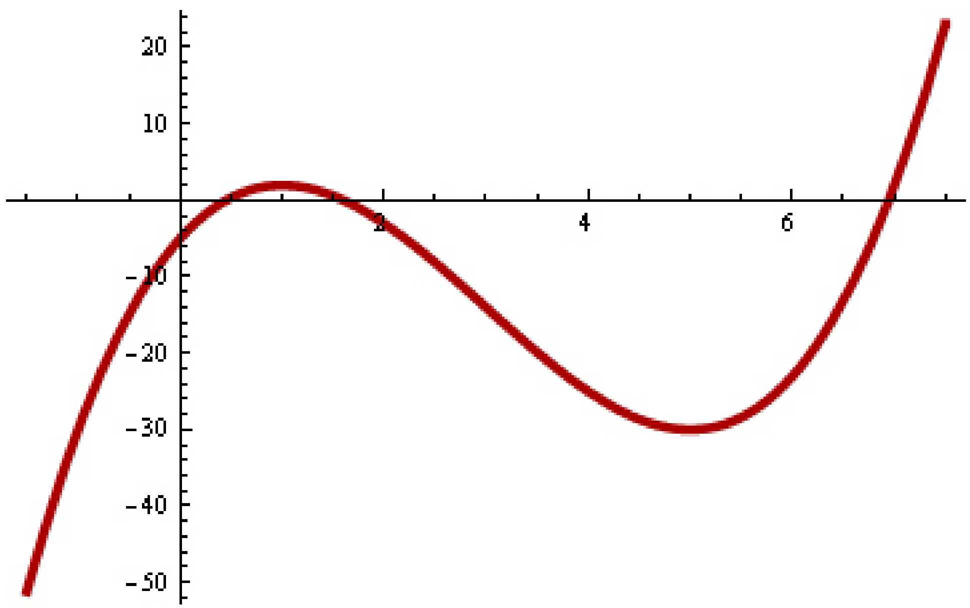
Solution. We first compute $f'(x)$ and then find all of its critical numbers.

$$\begin{aligned} f'(x) &= 3x^2 - 18x + 15 \\ &= 3(x-5)(x-1). \end{aligned}$$

Thus, the critical numbers are 1 and 5. To determine whether f has a relative extremum at either of these numbers, we apply the first-derivative test. The results are summarized in the following table.

We see from the table that 2 is a relative maximum value of f occurring at 1, and -30 is a relative minimum value of f occurring at 5. A sketch of the graph is shown.

	$f(x)$	$f'(x)$	Conclusion
$x < 1$		$+$	f is increasing
$x = 1$	2	0	f has a relative maximum value
$1 < x < 5$		$-$	f is decreasing
$x = 5$	-30	0	f has a relative minimum value
$x > 5$		$+$	f is increasing



Graph of $f(x) = x^3 - 9x^2 + 15x - 5$

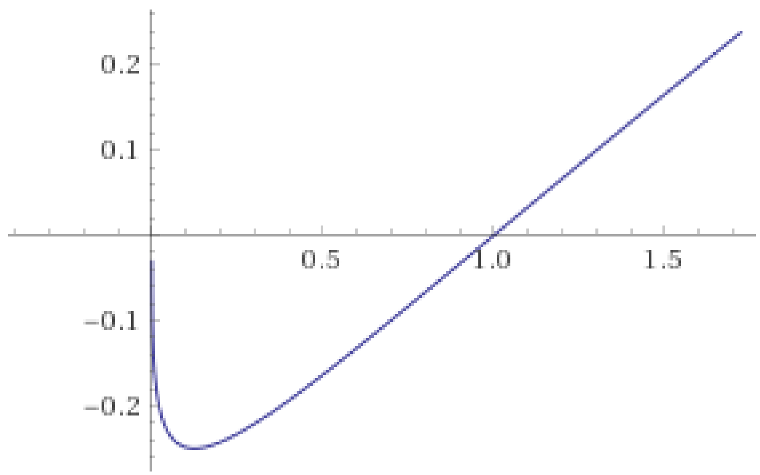
Example 17. Plot the graph of $f(x) = x^{2/3} - x^{1/3}$ and determine its relative exrema.

Solution. We first compute $f'(x)$ and then find all of its critical numbers.

$$\begin{aligned} f'(x) &= \frac{2}{3}x^{-1/3} - \frac{1}{3}x^{-2/3} \\ &= \frac{2x^{1/3} - 1}{3x^{2/3}} \end{aligned}$$

Thus, the critical numbers are 0 (since f' does not exist at this point) and $\frac{1}{8}$ (since $f' = 0$ at this point).

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$-$	f is decreasing
$x = 0$	0	DNE	f has no relative extremum value
$0 < x < \frac{1}{8}$		$-$	f is decreasing
$x = \frac{1}{8}$	$-\frac{1}{4}$	0	f has a relative minimum value
$x > \frac{1}{8}$		$+$	f is increasing

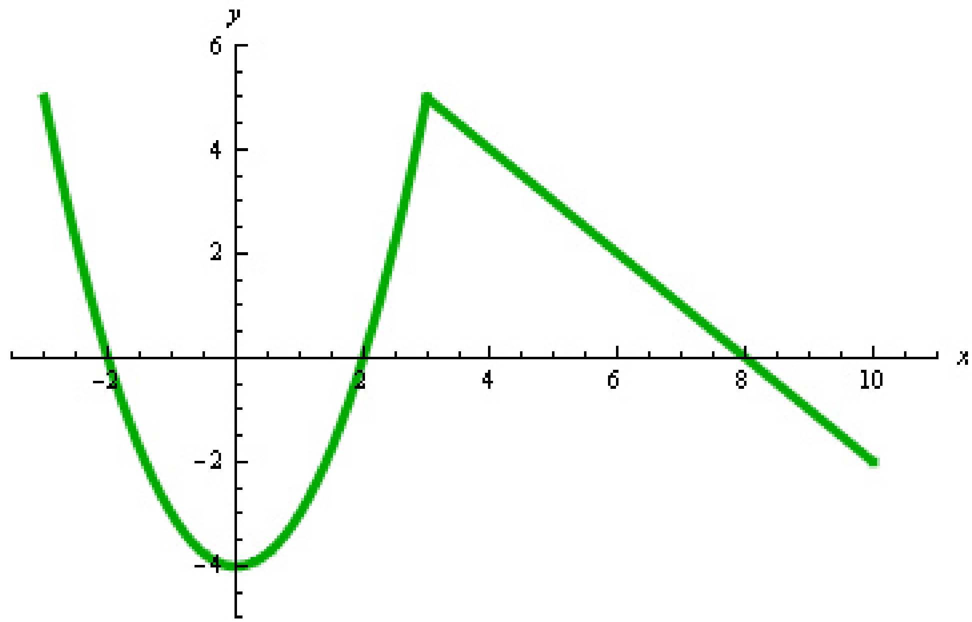


Graph of $f(x) = x^{2/3} - x^{1/3}$

Example 18. Plot the graph of $f(x) = \begin{cases} x^2 - 4, & \text{if } x \leq 3 \\ 8 - x, & \text{otherwise.} \end{cases}$ and determine its relative exrema.

Solution. We first compute $f'(x)$ and then find all of its critical numbers. $f'(x) = \begin{cases} 2x, & \text{if } x < 3 \\ -1, & \text{if } x > 3 \end{cases}$ .Thus, the critical numbers are 0 and 3 (f' does not exist here).

	$f(x)$	$f'(x)$	Conclusion
$x < 0$		$-$	f is decreasing
$x = 0$	-4	0	f has a relative minimum value
$0 < x < 3$		$+$	f is increasing
$x = 3$	5	DNE	f has a relative maximum value
$x > 3$		$-$	f is decreasing



Graph of $f(x) = \begin{cases} x^2 - 4, & \text{if } x \leq 3 \\ 8 - x, & \text{otherwise.} \end{cases}$

Exercises.

In Exercises 1 through 10, (a) plot the graph, and determine from the graph: (b) the relative extrema of f, (c) the value of x at which the relative extrema occur, (d) the intervals on which f is increasing, and (e) the intervals on which f is decreasing. Confirm analytically the information you obtained graphically.

1. $f(x) = x^2 - 4x - 1$
2. $f(x) = x^3 - x^2 - x$
3. $f(x) = \frac{1}{4}x^4 - x^3 + x^2$
4. $f(x) = 4\sin\frac{1}{2}x; x \in [-2\pi, 2\pi]$
5. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$
6. $f(x) = (1-x)^2(1+x)^3$
7. $f(x) = x - 3x^{1/3}$
8. $f(x) = x^{2/3} - x^{1/3}$
9. $f(x) = x^{5/4} + 10x^{1/4}$
10. $f(x) = x^{5/3} - 10x^{2/3}$

In Exercises 11 through 20, do the following analytically: (a) find the relative extrema of f; (b)determine the values of x at which the relative extrema occur; (c) determine the intervals on which f is increasing, (d) determine the intervals on which f is decreasing. Support your answers graphically.

11. $f(x) = 2x^3 - 9x^2 + 2$

12. $f(x) = x^3 - 3x^2 - 9x$

13. $f(x) = \frac{1}{5}x^5 - \frac{5}{3}x^3 + 4x + 1$

14. $f(x) = x^5 - 5x^3 - 20x - 2$

15. $f(x) = x + \dfrac{1}{x^2}$

16. $f(x) = 2x + \dfrac{1}{2x}$

17. $f(x) = 2x\sqrt{3 - x}$

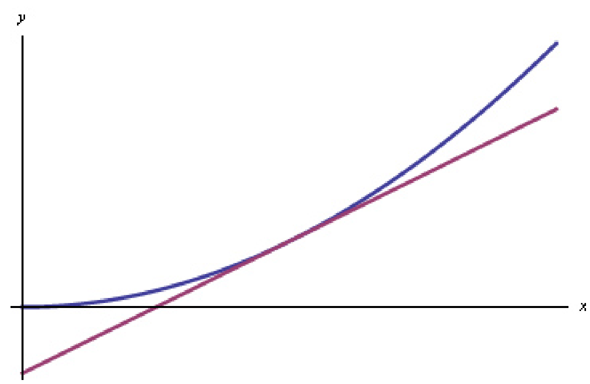
18. $f(x) = 2 - 3(x - 4)^{2/3}$

19. $f(x) = \frac{1}{2}\sec 4x; x \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$

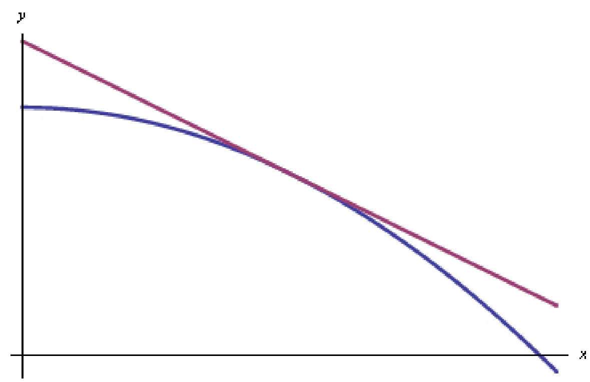
20. $f(x) = x^{1/3}(x + 4)^{-2/3}$

Concavity, Points of Inflection and the Second Derivative Test

Definition 8. The graph of a function f is said to be concave upward at the point $(c, f(c))$ if $f'(c)$ exists and if there is an open interval I containing c such that for all value of $x \neq c$ in I the point $(x, f(x))$ on the graph is above the tangent line to the graph at $(c, f(c))$.



Definition 9. The graph of a function f is said to be concave downward at the point $(c, f(c))$ if $f'(c)$ exists and if there is an open interval I containing c such that for all value of $x \neq c$ in I the point $(x, f(x))$ on the graph is below the tangent line to the graph at $(c, f(c))$.



Theorem 8. *Let f be a function that is differentiable on some open interval containing c. Then*

(i) if $f''(c) > 0$, the graph of f is concave upward at $(c, f(c)$;

(ii) if $f''(c) < 0$, the graph of f is concave downward at $(c, f(c)$.

Definition 10. The point $(c, f(c)$ is a point of inflection of the graph of f if the graph has a tangent line there, and if there exists an open interval I containing c such that if x is in I, then either

(i) $f''(x) < 0$ if $x < c$, and $f''(x) > 0$ if $x > c$; or

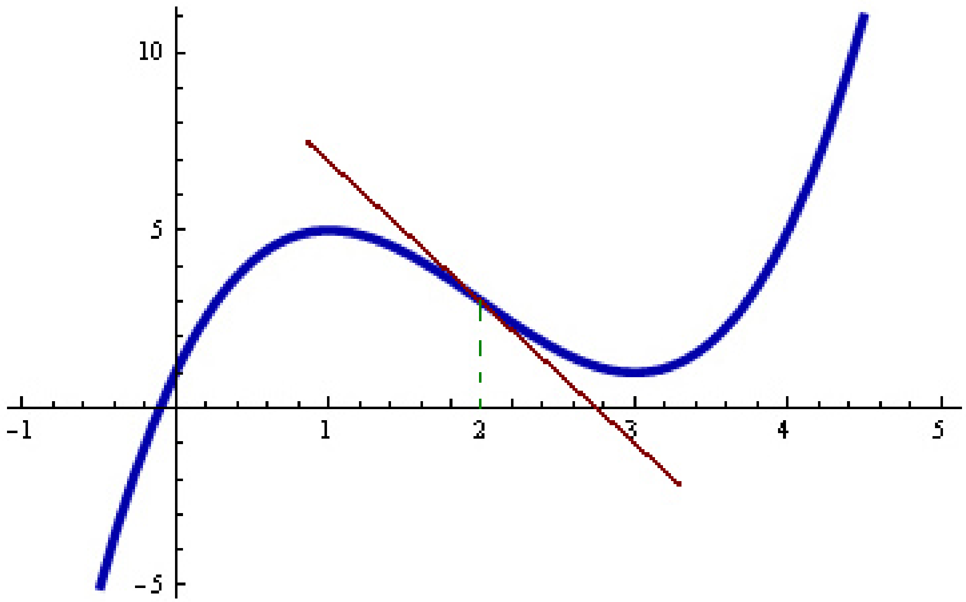
(ii) $f''(x) > 0$ if $x < c$, and $f''(x) < 0$ if $x > c$.

Theorem 9. *Suppose the function f is differentiable on some open interval containing c, and $(c, f(c))$ is a point of inflection of the graph of f. Then if $f''(c)$ exists, $f''(c) = 0$.*

Example 19. Verify the above theorem for the function $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution. We have $f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$. Thus, the only possible point of inflection is $x = 2$.

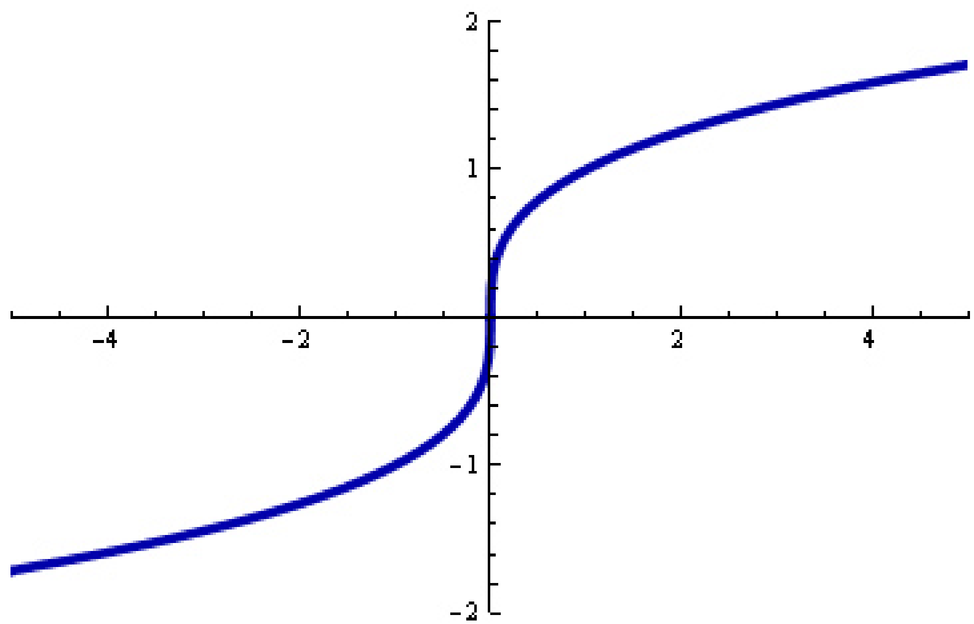
	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < 2$			$-$	graph is concave downward
$x = 2$	3	$-$	0	graph has a point of inflection
$x > 2$			$+$	graph is concave upward



Example 20. Verify the above theorem for the function $f(x) = x^{1/3}$.

Solution. We have $f'(x) = \frac{1}{3}x^{-2/3}$ and $f''(x) = -\frac{2}{9}x^{-5/3}$. Neither $f'(0)$ and $f''(0)$ exist. Note that the tangent at $x = 0$ is the y-axis.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < 0$		$+$	$+$	f is increasing; graph is concave upward
$x = 0$	0	DNE	DNE	graph has a point of inflection
$x > 0$		$+$	$-$	f is increasing; graph is concave downward



Theorem 10. ***Second Derivative Test for Relative Extrema.*** *Let c be a critical value of a function f at which $f'(c) = 0$, and let $f''(x)$ exist for all values of x in some open interval containing c.*

(1) If $f''(c) < 0$, then f has a relative maximum value at c.

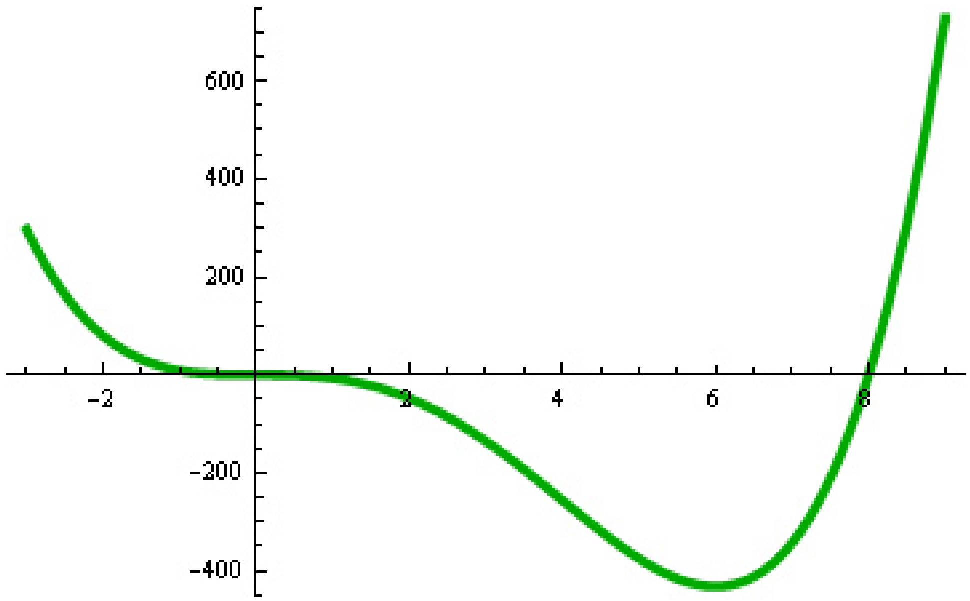
(2) If $f''(c) > 0$, then f has a relative minimum value at c.

Example 21. Use the Second Derivative Test to find the relative extrema of the function $g(x) = x^4 - 8x^3$.

Solution. We compute g' and $g'' \Rightarrow g'(x) = 4x^3 - 24x^2 = 4x^2(x-6)$. Now,
$g''(x) = 12x^2 - 48x = 12x(x-4)$. The critical numbers are 0 and 6.

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$x = 0$	0	0	0	No conclusion can be made.
$x = 6$	-432	0	$+$	g has a relative minimum value

Using the First Derivative Test, we have $g'(x) < 0$ when $x < 0$ and $g'(x) < 0$ when $0 < x < 6$ and so, g has no relative extremum value at $x = 0$.



Summary of Sketching Graphs of Functions

We now apply what we have discussed in the preceding sections in identifying the properties of graphs of functions by using their derivatives. Here, we obtain the relative extremum points, points of inflection, the intervals on which the graph is increasing or decreasing, and values on which the graph is concave upward or downward.

Summary of Sketching Graphs of Functions

1. Determine the domain of f.

2. Find any x and y intercepts. When finding the x intercepts you may need to approximate the roots of the equation $f(x) = 0$ on your calculator.

3. Test for symmetry with respect to the y axis and origin.

4. Check for any possible horizontal, vertical, or oblique asymptotes.

5. Compute $f'(x)$ and $f''(x)$.

6. Determine the critical numbers of f. These are the values in the domain of the function for which either $f'(x)$ does not exist or is equal to 0.

7. Apply the first-derivative test or the second-derivative test to determine whether at a critical number there is a relative maximum value, a relative minimum value, or neither.

8. Determine the intervals on which f is increasing by finding the values of x for which $f'(x) > 0$; determine the intervals on which f is decreasing by finding the values of x for which $f'(x) < 0$.

9. Find the critical numbers of f' to obtain possible points of inflection. At each of these check to see if $f''(x)$ changes sign and if the graph has a tangent line there to determine if there actually is a point of inflection.

10. Check for concavity of the graph. Find the values of x for which $f''(x)$ is positive to obtain points at which the graph is concave upward; to obtain points at which the graph is concave downward find the values of x for which $f''(x)$ is negative.

11. Find the slope of each inflectional tangent if that is helpful.

Example 22. Graph each of the following.

1. $f(x) = 2x^3 + 3x^2 - 12x + 1$

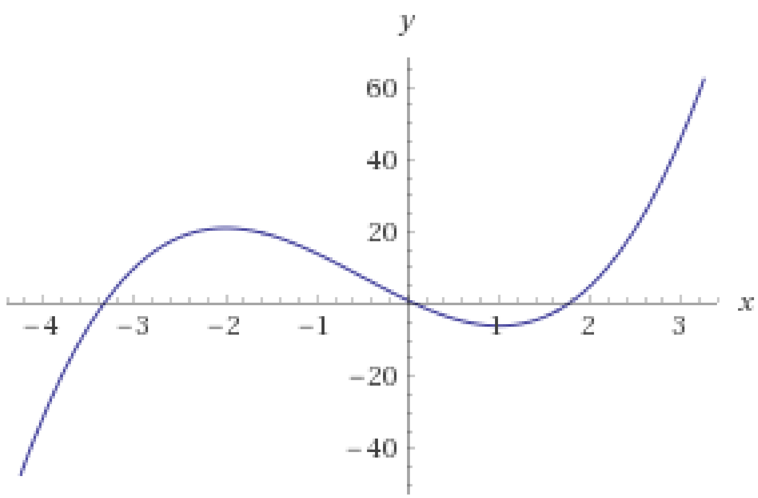
   Solution. We will use the first derivative test and the second derivative test to aid us in sketching the graph.

   $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1)$

   The critical numbers are -2 and 1.

   $f''(x) = 12x + 6 = 6(2x + 1)$ and a possible point of inflection at $x = -\frac{1}{2}$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < -2$		$+$	$-$	f is increasing, concave downward
$x = -2$	21	0	$-$	f has a relative maximum value
$-2 < x < -\frac{1}{2}$		$-$	$-$	f is decreasing, concave downward
$x = -\frac{1}{2}$	7.5	$-$	0	inflection point
$-\frac{1}{2} < x < 1$		$-$	$+$	f is decreasing, concave upward
$x = 1$	-6	0	$+$	f has a relative minimum value
$x > 1$		$+$	$+$	f is increasing, concave upward



`Graph of` $f(x) = 2x^3 + 3x^2 - 12x + 1$

2. $g(x) = \dfrac{x}{x^2+4}$

Solution. From the lesson on horizontal and vertical asymptotes, $f(x)$ has no vertical asymptote but it has a horizontal asymptote $y = 0$ since

$\lim_{x\to+\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

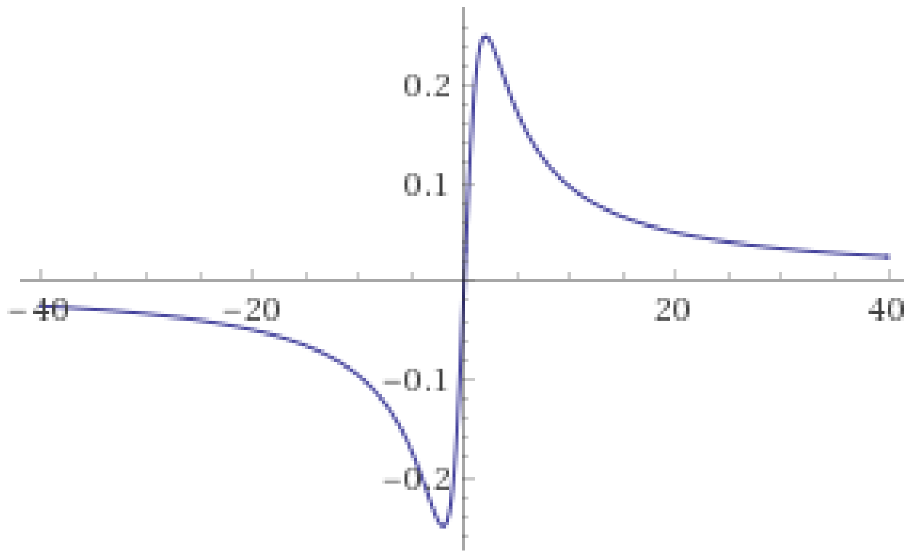
Next, we will use the first derivative test and the second derivative test.

$$f'(x) = \frac{4-x^2}{(x^2+4)^2}$$

The critical numbers are -2 and 2.

$f''(x) = \dfrac{2x(x^2-12)}{(x^2+4)^3}$ with roots at $-2\sqrt{3}, 0, 2\sqrt{3}$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$x < -2\sqrt{3}$		$-$	$-$	f is decreasing, concave downward
$x = -2\sqrt{3}$	≈ -0.217	$-$	0	inflection point
$-2\sqrt{3} < x < -2$		$-$	$+$	f is decreasing, concave upward
$x = -2$	-0.25	0	$+$	f has a relative minimum value
$-2 < x < 0$		$+$	$+$	f is increasing, concave upward
$x = 0$	0	$+$	0	inflection point
$0 < x < 2$		$+$	$-$	f is increasing, concave downward
$x = 2$	0.25	0	-	f has a relative maximum value
$2 < x < 2\sqrt{3}$		$-$	-	f is decreasing, concave downward
$x = 2\sqrt{3}$	≈ 0.217	$+$	0	inflection point
$x > 2\sqrt{3}$		$+$	$+$	f is increasing, concave upward



Graph of $f(x) = \dfrac{x}{x^2+4}$

3. $g(x) = 2\sin 3x$; $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

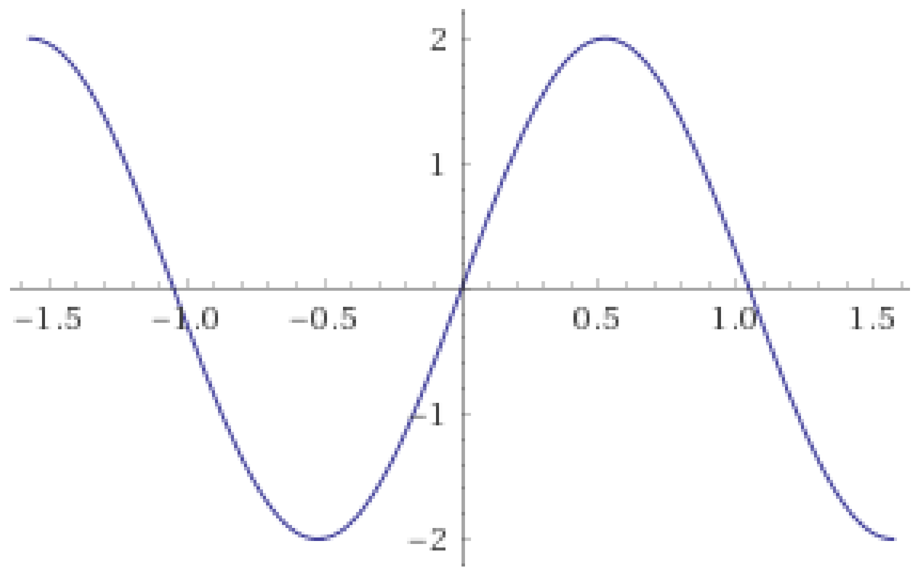
Solution. We use the first derivative test and the second derivative test.

$f'(x) = 6\cos 3x$

The critical numbers are $-\dfrac{\pi}{6}$ and $\dfrac{\pi}{6}$.

$f''(x) = -18\sin 3x$ with roots at $-\dfrac{\pi}{3}, 0$ and $\dfrac{\pi}{3}$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\frac{\pi}{2} < x < -\frac{\pi}{3}$		$-$	$-$	f is decreasing, concave downward
$x = -\frac{\pi}{3}$	0	$-$	0	inflection point
$-\frac{\pi}{3} < x < -\frac{\pi}{6}$		$-$	$+$	f is decreasing, concave upward
$x = -\frac{\pi}{6}$	-2	0	$+$	f has a relative minimum value
$-\frac{\pi}{6} < x < 0$		$+$	$+$	f is increasing, concave upward
$x = 0$	0	$+$	0	inflection point
$0 < x < \frac{\pi}{6}$		$+$	$-$	f is increasing, concave downward
$x = \frac{\pi}{6}$	2	0	$-$	f has a relative maximum value
$\frac{\pi}{6} < x < \frac{\pi}{3}$		$-$	$-$	f is decreasing, concave downward
$x = \frac{\pi}{3}$	0	$-$	0	inflection point
$\frac{\pi}{3} < x < \frac{\pi}{2}$		$-$	$-$	f is decreasing, concave upward



Graph of $g(x) = 2\sin 3x$; $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Exercises. Graph each of the following.

1. $f(x) = x^2 - 4x - 1$
2. $f(x) = x^3 - 9x^2 + 15x - 5$
3. $f(x) = 2x^3 - x^2 + 3x - 1$
4. $f(x) = x^4 + 4x$
5. $f(x) = x^5 - 5x^3 - 20x - 2$
6. $f(x) = 2x + \dfrac{1}{2x}$
7. $f(x) = \sqrt{x} - \dfrac{1}{\sqrt{x}}$
8. $f(x) = \dfrac{x-2}{x+2}$
9. $f(x) = (x-3)^4$
10. $f(x) = \dfrac{9}{x} + \dfrac{x^2}{9}$

Additional Applications of Absolute Extrema

Theorem 11. *Suppose the function f is continuous on the interval I containing the number c. If $f(c)$ is a relative extremum of f on I and c is the only number in I for which f has a relative extremum, then $f(c)$ is an absolute extremum of f on I. Furthermore,*

(i) if $f(c)$ is a relative maximum value of f on I, then $f(c)$ is an absolute maximum value of f on I;

(ii) if $f(c)$ is a relative minimum value of f on I, then $f(c)$ is an absolute minimum value of f on I.

Example 23. A closed box with a square base is to have a volume of 2000 cubic inches. The material for the top and bottom of the box is to cost $\$3$ per square inch, and the material for the sides is to cost $\$1.50$ per square inch. If the cost of the material is to be the least, find the dimensions of the box.

Solution. Let

x =the number of inches in the length of a side of the square base;

y= the number of inches in the depth of the box;

C= the number of dollars in the cost of the material.

The total number of square inches in the combined area of the top and bottom is $2x^2$, and for the sides it is $4xy$; so we have

$$C = 3(2x^2) + \frac{3}{2}(4xy)$$

Because the volume of the box is the product of the area of the base and the depth, we have

$$x^2y = 2000$$

Solving for y in terms of x and substituting into $C = 3(2x^2) + \frac{3}{2}(4xy)$, we get

$$C = 6x^2 + \frac{12,000}{x}$$

x is in the interval $(0, +\infty)$, and C is a function of x, which is continuous on $(0, +\infty)$.From the above equation we obtain

$$C'(x) = 12x - \frac{12,000}{x^2}$$

$C'(x)$ does not exist when $x = 0$, but 0 is not in $(0, +\infty)$. Hence, the only critical numbers will be those obtained by setting $C'(x) = 0$. The only real solution is 10; thus, 10 is the only critical number. To determine if $x = 10$ makes C a relative minimum we apply the second-derivative test.

$$C''(x) = 12 + \frac{24,000}{x^3}$$

The results of the second-derivative test are summarized below.

	$C'(x)$	$C''(x)$	Conclusion
$x = 10$	0	$+$	C has a relative minimum value

C is a continuous function of x on $(0, +\infty)$. Because the one and only relative extremum of C on $(0, +\infty)$; is at $x = 10$, it follows from Theorem 11(ii) that this relative minimum value of C is the absolute minimum value of C. Hence, we conclude that the total cost of the material will be the least when the side of the square base is 10 in. and the depth is 20 in.

Example 24. If a closed tin can of specific volume is to be in the form of a right-circular cylinder, find the ratio of the height to the base radius if the least amount of material is to be used in its manufacture.

Solution. We wish to find a relationship between the height and the base radius of the right-circular cylinder in order for the total surface area to be an absolute minimum for a fixed volume. Therefore, we consider the volume of the cylinder a constant. Let

$V =$ the number of cubic units in the volume of a cylinder (a constant).

$r=$ the number of units in the base radius of the cylinder; $0 < r < +\infty$;

$h=$ the number of units in the height of the cylinder; $0 < h < +\infty$;

$S=$ the number of square units in the total surface area of the cylinder.

We have the following equations:

$$S = 2\pi r^2 + 2\pi r h$$

$$V = \pi r^2 h$$

Because V is a constant, we could solve $V = \pi r^2 h$ for eithr r or h in terms of the other and substitute into $S = 2\pi r^2 + 2\pi r h$,which will give us S as a function of one variable. The alternative method is to consider S as a function of two variables r and h; however, r and h are not independent of each other. That is, if we choose r as the independent variable, then S depends on r; also, h depends on r.

Differentiating S and V with respect to r and bearing in mind that h is a function of r, we have

$$D_r S = 4\pi r + 2\pi h + 2\pi r D_r h$$

and

$$D_r V = 2\pi r h + \pi r^2 D_r h$$

Because V is a constant, $D_r V = 0$; therefor

$$0 = 2\pi r h + \pi r^2 D_r h$$

with $r \neq 0$, and we can divide by r and solve for $D_r h$, thus obtaining

$$D_r h = -\frac{2h}{r}$$

By substitution,

$$D_r S = 2\pi \left[2r + h + r \left(-\frac{2h}{r} \right) \right]$$

$$D_r S = 2\pi(2r - h)$$

To find when S has a relative minimum value, we set $D_r S = 0$ and obtain $2r - h = 0$, which gives us

$$r = \tfrac{1}{2}h$$

To determine if this relationship between r and h makes S a relative minimum, we apply the second-derivative test.

$$D_{r^2} S = 2\pi(2 - D_r h)$$

By substitution,

$$D_{r^2} S = 2\pi\left[2 - \left(-\frac{2h}{r}\right)\right] = 2\pi\left(2 + \frac{2h}{r}\right)$$

The results of the second-derivative test are summarized below.

	D_rS	$D_{r^2}S$	Conclusion
$r = \frac{1}{2}h$	0	$+$	S has a relative minimum value

S is a continuous function of r on $(0, +\infty)$. Because the one and only relative extremum of S on $(0, +\infty)$; is at $r = \frac{1}{2}h$, it follows from Theorem 11(ii) that S has an absolute minimum value at $r = \frac{1}{2}h$. Therefore, the total surface area of the tin can will be least for a specific volume when the ratio of the height to the base radius is 2.

Exercises. Solve each of the following:

1. If a closed tin can of volume 60 in^3 is to be in the form of a right-circular cylinder, find analytically the base radius of the can if the least amount of tin is to be used in the manufacture.

2. A closed box with square base is to have a volume of 2000 in^3. The material for the top and bottom of the box is to cost 3 cents per square inch and the material for the sides is to cost 1.5 cents per square inch. Find the dimensions of the box so that the total cost of material is least.

3. If a closed tin can of fixed volume is to be in the form of a right-circular cylinder, find the ratio of the height to the base radius if the least amount of material is to be used in its manufacture.

4. A right-circular cylinder it to be inscribed in a sphere of given radius. Find the ratio of the height to the base radius of the cylinder having the largest surface area.

5. A rectangular field, having an area of 2700 yd^2, is to be enclosed by a fence, and an additional fence is to be used to divide the field down the middle. If the cost of the fence down the middle is $\$2$ per running yard, and the fence along the sides costs $\$3$ per running yard, find the dimensions of the field so that the cost of the fencing will be the least.

6. A rectangular open tank is to have a square base, and its volume is to be 125yd^3. The cost per square yard for the bottom is $\$8$ and for the sides is $\$4$. Find the dimensions of the tank in order for the cost of the material to be the least.

7. A box manufacturer is to produce a closed box of specific volume whose base is a rectangle having a length that is three times its width. Find the most economical dimensions.