* No application problems are on substitution or IBP

Integration by Parts

Integration by substitution is a useful technique but it doesn't always work. For example now we have an integral

$$\int xe^x \, dx\,.$$

By what we knew last week, probably we let $u(x) = x$ as it is the exponent of e above. However, $u'(x) = 1$ and we have an extra x inside the integrand. It means substitution doesn't work.

In this case, we may shift to another important integration technique: integration by parts.

Integration by parts. Suppose $u(x)$ and $v(x)$ are two functions.

$$\int u(x) \cdot v'(x) \, dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x) \, dx$$

Remind that we have differentials $dv = v'(x)\,dx$ and $du = u'(x)\,dx$. Sometimes the above formula is written as

$$\int u \, dv = u \cdot v - \int v \, du\,.$$

If we are looking at definite integrals, we use the following formula.

$$\int_a^b u(x) \cdot v'(x) \, dx = \big(u(x) \cdot v(x)\big)\Big|_a^b - \int v(x) \cdot u'(x) \, dx$$

The first term on the right hand side is

$$\big(u(x) \cdot v(x)\big)\Big|_a^b = u(b) \cdot v(b) - u(a) \cdot v(a).$$

However, we are not really using the definite-integral formula here. Instead, we can first find the indefinite integral by IBP, and then do the evaluation.

There are two typical kind of integrals able to be found by IBP.

(IV) Suppose a is a constant. Let $u = x$ and $dv = e^{ax}dx$. We have

$$\int x \cdot e^{ax}\, dx = \left(x \cdot \frac{1}{a}e^{ax}\right) - \int e^{ax} dx\,.$$

Q. Find the integral $\int 20xe^x\, dx$.

> $\int 20 x e^x dx = 20 \int x e^x dx$
>
> $u(x) = x \Rightarrow u'(x) = 1 \Rightarrow du = 1\,dx = dx$
> $dv = e^x dx \Rightarrow v'(x) = e^x \Rightarrow v(x) = \int e^x dx = e^x \cancel{+C}$
>
> $= 20\left(xe^x - \int e^x dx\right)$ $[u \cdot v - \int v\, du]$
>
> $= 20(xe^x - e^x) + C$
>
> $= 20xe^x - 20e^x + C$ #
>
>

Q. Find the integral $\int xe^{-2x}\, dx$.

> $u(x) = x \Rightarrow u'(x) = 1 \Rightarrow du = dx$
> $dv = e^{-2x}dx \Rightarrow v'(x) = e^{-2x} \Rightarrow$
> $\Rightarrow v(x) = \int e^{-2x} dx = -\frac{1}{2}e^{-2x} \cancel{+C}$
>
> $\int xe^{-2x}dx = x \cdot (-\frac{1}{2}e^{-2x}) - \int (-\frac{1}{2}e^{-2x})dx$
>
> $= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$
>
> $= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$ #
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(V) Suppose p is a number and $p \neq -1$. Let $u(x) = \ln x$ and $v(x) = x^p dx$.

$$\int (\ln x) \cdot x^p dx = \left((\ln x) \cdot \frac{x^{p+1}}{p+1} \right) - \int \frac{x^p}{p+1}\, dx$$

Q. Find the integral $\int x \ln x \ dx$.

> $u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$
> $dv = x\,dx \Rightarrow v'(x) = x \Rightarrow v(x) = \frac{1}{2}x^2 \cancel{+C}$
>
> $\int x \ln x\, dx = (\ln x)(\frac{1}{2}x^2) - \int (\frac{1}{2}x^2) \cdot \frac{1}{x} dx$
>
> $= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$
>
> $= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$ #
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Q. Find the integral $\int_1^5 \ln x \ dx$.

> Find $\int \ln x\, dx$ first
> $u(x) = \ln x \Rightarrow u'(x) = \frac{1}{x} \Rightarrow du = \frac{1}{x} dx$
> $dv = dx = 1 \cdot dx \Rightarrow v'(x) = 1 \Rightarrow v(x) = x \cancel{+C}$
>
> $\int \ln x\, dx = (\ln x) \cdot x - \int x \cdot \frac{1}{x} dx$
>
> $= x \ln x - \int 1\, dx = x \ln x - x + C$
>
> $\therefore \int_1^5 \ln x\, dx = (x \ln x - x)\Big|_1^5$
>
> $= (5 \ln 5 - 5) - (\cancel{\ln 1}^{0} - 1)$

$= (5 \ln 5) - 4$ #

Q. Find the integral $\int x^{-2} \cdot \ln x \; dx$.

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Q. Find the integral $\int \sqrt{x} \cdot \ln x \; dx$.

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Here we introduce an application of IBP. In week 7, we have a concept about the present value (PV) of a future amount, when interest rate is compounded continuously.

Suppose a future payment is of P dollars at an annual interest rate r compounded continuously, and to be paid in t years. Then,

$$\boxed{PV = P \cdot e^{-rt}.}$$

When a company is running a business, it may produce a steady income of $C(t)$ dollars per year, where t is the number of years from now. Suppose this income stream is to be over the next T years.

This income stream is called *continuous*, if this future income is paid at every moment of time in the future. Or you may think of it as if the number of payments over the coming T years is huge.

PV of a continuous income stream.
The present value of a continuous income stream $C(t)$ dollars per year, where t is the number of years from now, for the next T years at a continuous annual interest rate r, is given by

$$PV = \int_0^T C(t) \cdot e^{-rt}\, dt\,.$$

Q. Suppose a company generates a continuous income stream of \$10,000 per year, where t is the number of years from now. Find the present value of this continuous income stream for the next 10 years at the continuous annual interest rate of 8%.

> $C(t) = 10{,}000, \quad T = 10, \quad r = 0.08$

> $PV = \int_0^{10} C(t)\, e^{-0.08t} dt = \int_0^{10} 10{,}000 \cdot e^{-0.08t} dt$

> $= 10{,}000 \int_0^{10} e^{-0.08t} dt = 10{,}000 \left(\frac{-1}{0.08} e^{-0.08t} \right) \Big|_0^{10}$

> $= 10{,}000 \left[\left(\frac{-1}{0.08} e^{-0.8} \right) - \left(\frac{-1}{0.08} \right) \right]$

$= 10{,}000 \cdot \frac{1}{0.08} \cdot (1 - e^{-0.8}) = \$ 68834.0$ #

Here come further questions on IBP.

Q. Find the integral

$$\int_0^4 z(z-4)^6\,dz\,.$$

for a constant a

$* \int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + C$ when $n \neq -1$ or 0

[You can use substitution or IBP.]

> $u(z) = z \Rightarrow u'(z) = 1 \Rightarrow du = dz$
> $dv = (z-4)^6 dz \Rightarrow v'(z) = (z-4)^6$
> $\Rightarrow v(z) = \int (z-4)^6\, dz = \frac{(z-4)^7}{7} + \cancel{C}$
>
> $\int z(z-4)^6\, dz = z \cdot \frac{(z-4)^7}{7} - \int \frac{(z-4)^7}{7}\, dz$
> $= \frac{z(z-4)^7}{7} - \frac{1}{7}\,\frac{(z-4)^8}{8} + C$
> $= \frac{z(z-4)^7}{7} - \frac{(z-4)^8}{56} + C$
>
> $\therefore \int_0^4 z(z-4)^6\, dz = \left(\frac{z(z-4)^7}{7} - \frac{(z-4)^8}{56}\right)\Big|_0^4$
> $= \underset{\text{at } z=4}{(0-0)} - \underset{\text{at } z=0}{\left(0 - \frac{4^8}{56}\right)} = \frac{4^8}{56} = \frac{8192}{7}$ #
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> use substitution
> Substitute $u = z-4 \Rightarrow u'(z) = 1 \Rightarrow du = dz$
> When $u = z-4$, $z = u+4$
> $z = 0 \Rightarrow u = -4$
> $z = 4 \Rightarrow u = 0$
>
> $\therefore \int_0^4 z(z-4)^6\, dz = \int_{-4}^0 (u+4)u^6\, du = \int_{-4}^0 u^7 + 4u^6\, du$
> $= \left(\frac{u^8}{8} + \frac{4u^7}{7}\right)\Big|_{-4}^0 = (0+0) - \left(\frac{(-4)^8}{8} + \frac{4(-4)^7}{7}\right)$
>
$= \frac{8192}{7}$ #

Q. Find the integral

$$\int \frac{\ln x}{x}\,dx.$$

> $= \int \ln x \cdot \frac{1}{x}\,dx$
>
> Substitute $u = \ln x \Rightarrow u'(x) = \frac{1}{x} \Rightarrow du = \frac{1}{x}dx$
>
> $= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$ #
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Q. Find the integral $\int x^2 e^x\,dx$.

> use IBP. $u(x) = x^2 \Rightarrow u'(x) = 2x \Rightarrow du = 2x\,dx$
> $dv = e^x dx \Rightarrow v'(x) = e^x \Rightarrow v(x) = e^x \cancel{+K}$.
>
> $= x^2e^x - \int 2xe^x\,dx$
>
> Use IBP again to find $\int xe^x dx$
>
> $u(x) = x \Rightarrow du = dx$
> $dv = e^x dx \Rightarrow v(x) = e^x$
>
> $= x^2e^x - 2\int xe^x dx$
>
> $= x^2e^x - 2\left(xe^x - \int e^x dx\right)$
>
> $= x^2e^x - 2xe^x + 2\int e^x dx$
>
> $= x^2e^x - 2xe^x + 2e^x + C$ #