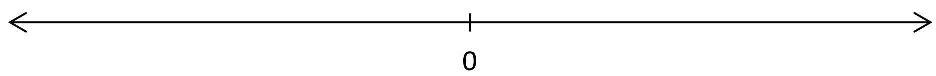
Real Numbers and Inequalities

Real numbers are $0,\ 1,\ 2006,\ 2.2124,\ \pi, \sqrt{2},$ and so on. They are points on the number line.



Q. Mark the numbers 5 and -2.1 on the number line.

Inequalities express the order of real numbers. For example

$a < b$	a is smaller than b.
$a < b < c$	a is smaller than b, and b is smaller than c.
$a \geq b$	a is greater than or equal to b.
$-10 < -3$	
	2 is between -1 and 5.

Sets and intervals are commonly seen in this course.

$\{\, x \mid x > 3 \,\}$	The set of all x such that x is greater than 3.
$\{\, x \mid -2 < x < 5 \,\}$	The set of all x such that x is between -2 and 5.
$\{\, x \mid -1 \leq x \leq 2 \,\}$	The set of all x such that x is between -1 and 2, including endpoints.

Those sets on the left hand side are called intervals.

$[a, b]$	$\{x \mid a \leq x \leq b\}$	The closed interval from a to b.
$(3,5)$		The open interval from 3 to 5.
$(-1,10]$		The left-open interval from -1 to 10.
$[-2,5)$		The right-open interval from -2 to 5.

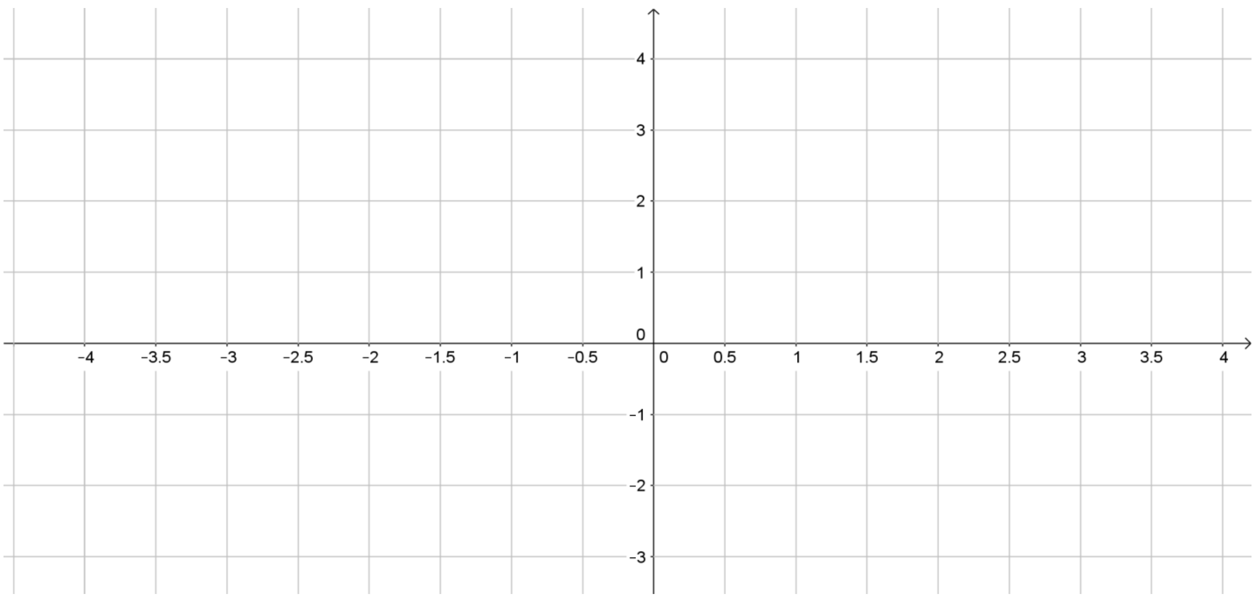
Describe intervals: http://www.mathquickeasy.com/types_of_intervals.html

In addition to finite intervals, we have infinite intervals.

$[a, \infty)$	$\{x \mid x \geq a\}$	The closed interval from a to plus infinity.
$(3\ \infty)$		
$(-\infty, 1]$		
$(-\infty, 10)$		

**You may hear that $[a, \infty)$ is called the closed ray from a to plus infinity.

On the Cartesian plane, a point is specified uniquely by an ordered pair (x, y). x is the x-coordinate, and y is the y-coordinate of that point.



Q. Mark and label the point (1,2), (-2,3), (-3,-3) and (3,-1).

Two points (x_1, y_1) and (x_2, y_2) determine a line on the Cartesian plane. The slope of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The terms Δx and Δy are called the change in x and the change in y. If we put $\Delta x = 1$ to the slope formula, we get $m = \Delta y$. In other words, slope is the amount that the line rises when x is increased by1.

Q. Find the slope of the line through pairs of points below. Graph the lines.

a) (2,1) and (3,4)
b) (-1,10) and (5,7)

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Very often we have to find the equation of a line. There are two forms of a line equation.

a. Slope-intercept form of a line.
   If m is the slope of a line and b is the y-intercept of the line, then the line equation is given by $y = mx + b$

b. Point-slope form of a line.
   If m is the slope of a line and (x_1, y_1) is a point on the line, then the line equation is given by $y - y_1 = m(x - x_1)$.

A general linear equation of a line is in the form of $ax + by = c$ for some constants a, b, c. Whenever you see this equation, immediately you know it represents a line on the Cartesian plane.

We also have the following facts about line equations.

c. A horizontal line is given by the equation $y = a$.
A vertical line is given by the equation $x = b$.
The slope of a horizontal line is 0, and that of a vertical line is undefined.

d. Two lines l_1 and l_2 are
  (a) parallel to each other if they have the same slope, i.e. $m_1 = m_2$.
  (b) perpendicular to each other if $m_1 = -\frac{1}{m_2}$.

Q. Write down the linear equation of the line passing through (5,3) and (7,-1) in the slope-intercept form. >

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Q. Given the linear equation $2x - 3y = 12$, find the slope m and the y-intercept b. Draw the graph.

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Exponents

For any positive integer n, $x^n = x \cdot x \cdot x \cdots x$ for n many x on the right hand side. In the term x^n, x is called the base, and n is called the exponent or power.

Formula	Example
$x^m \cdot x^n = x^{m+n}$	
$\frac{x^m}{x^n} = x^{m-n}$	
$(x^m)^n = x^{m \cdot n}$	
$(xy)^n = x^n \cdot y^n$	
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	

When $x \neq 0$, we have the below formulas for zero and negative exponents.

Formula	Example
$x^0 = 1$	
$x^{-1} = \frac{1}{x}$	
$x^{-n} = \frac{1}{x^n}$	

Formulas for roots and fractional exponents are stated below.

Formula	Example
$x^{1/2} = \sqrt{x}$	
$x^{1/3} = \sqrt[3]{x}$	
$x^{1/n} = \sqrt[n]{x}$	
$x^{m/n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$	

Q. Evaluate each expression.

a. $\left(\frac{3}{4}\right)^{-1}$
b. $(-27)^{2/3}$
c. $16^{3/4}$

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Q. Write each expression in power form ax^b for numbers a and b.

a. $\frac{18}{\left(3\sqrt[3]{x}\right)^2}$

b. $\frac{10\sqrt{x}}{2\sqrt[3]{x}}$

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Q. Simplify the expression. $\frac{(5xy^4)^2}{25x^3y^3}$.

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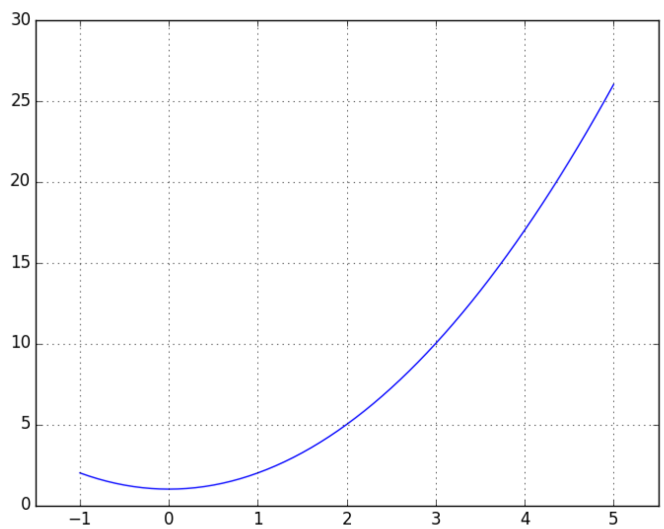
Linear and Quadratic Functions

A function f is assigning each number x to another number $f(x)$. For example, the function $f(x) = x^2$ assigns 1 to $f(1) = 1^2 = 1$,
2 to $f(2) = 2^2 = 4$ and so on.

Domain of f = the set of all values x at which $f(x)$ is well defined.

Range of f = the set of all values $f(x)$.

Q. Let $f(x) = x^2 + 1, -1 \leq x \leq 5$. Find the domain and range of $f(x)$.



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Q. Find the domain and range of the following functions.

a) $f(x) = x^2 + 4$.

b) $f(x) = \sqrt{x-2}$

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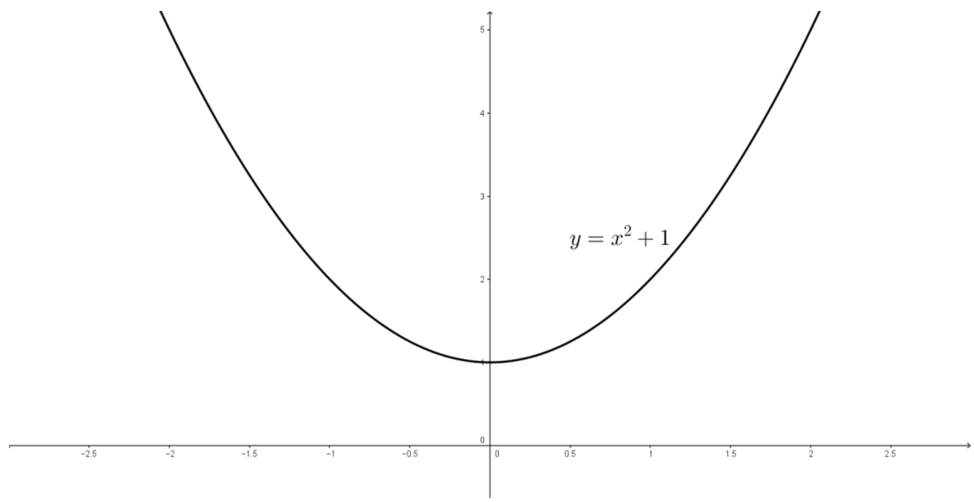
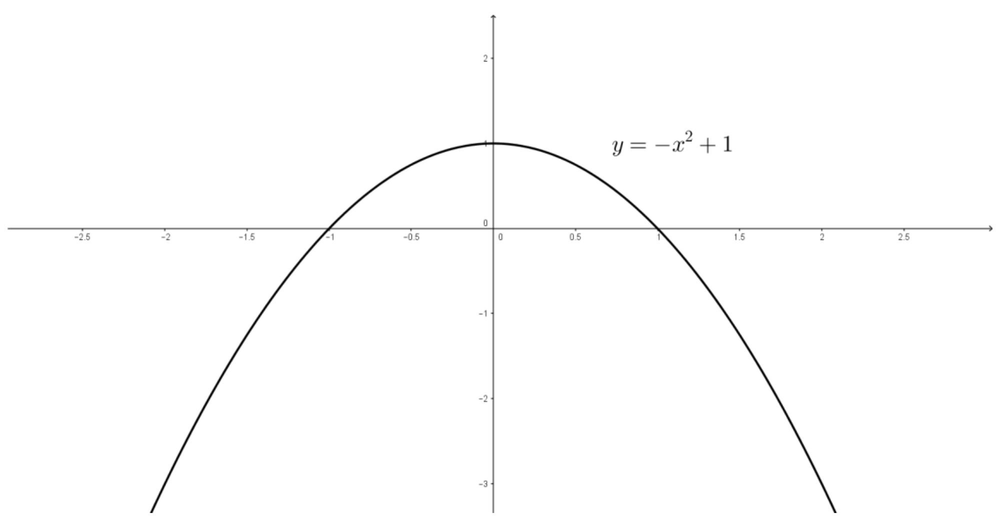
A linear function is a function in the form $f(x) = mx + c$ for some constants m and c. The graph of $f(x)$ is a line, of slope m and y-intercept c.

A quadratic function is a function in the form $f(x) = ax^2 + bx + c$ for some constants a, b, c, where $a \neq 0$. Its graph is called a parabola.

Example. The graph of $f(x) = x^2 + 1$ and $f(x) = -x^2 + 1$.





Q. Identify the constants a, b, c in the function $f(x) = 3x^2 + 2x + 5$.

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Suppose we have a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

If	*then*	Example
$a > 0$	$f(x)$ opens upward.	$f(x) = x^2 + 1$
$a < 0$	$f(x)$ opens downward.	$f(x) = -x^2 + 1$

The vertex of the parabola $f(x) = ax^2 + bx + c$, has x-coordinate $x = -\frac{b}{2a}$. Therefore, its vertex is the point $x = -\frac{b}{2a}$, $y = f\left(-\frac{b}{2a}\right)$.

Q. Find the x-coordinate of the vertex of $f(x) = 3x^2 + 2x + 5$.

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Solving a quadratic equation $ax^2 + bx + c = 0$ is exactly finding the x-intercept(s) of the quadratic function $f(x) = ax^2 + bx + c$.

There are two ways of solving a quadratic equation. By (1) factorization, or by (2) quadratic formula, which is stated as follows.

The solution to $ax^2 + bx + c = 0$ is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ whenever $b^2 - 4ac \geq 0$

Q. Solve the following quadratic equations.

a. $x^2 - 3x + 2 = 0$

b. $x^2 - 2x - 8 = 0$

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Q. Find the domain of the function $f(x) = \frac{1}{x^2-4x+3}$.

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Example. (P.40) A company that installs automobile CD players finds that, if the company installs x CD players per day, then its costs will be $C(x) = 130x + 5000$ and its revenue will be $R(x) = -2x^2 + 500x$ (both in dollars). Find the profit function $P(x)$ of the company.

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Functions: Polynomial, Rational and Exponential

Polynomials are functions in the forms

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

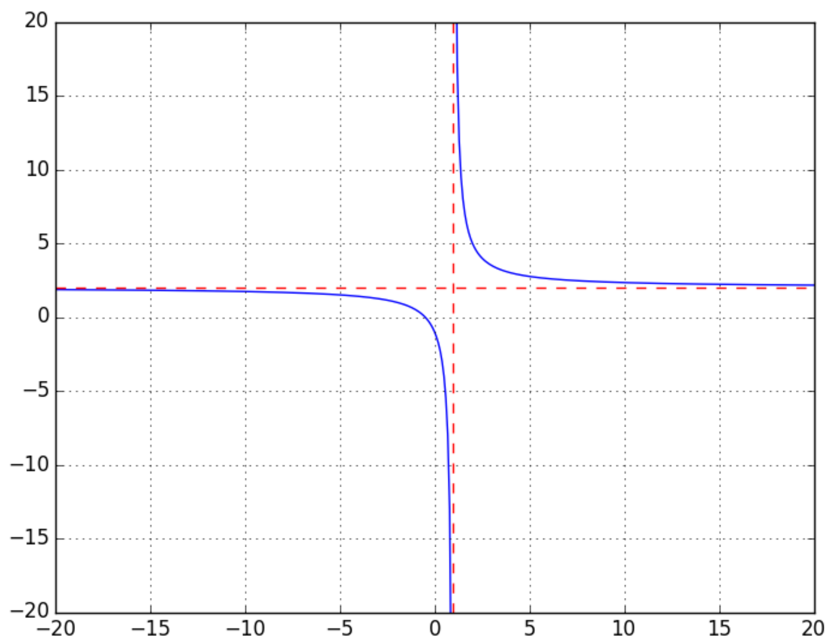
For example, $f(x) = 3x^3 + 4x^2 + 5x + 7$ is a polynomial of degree 3.

A rational function is a quotient of two polynomials. For example,

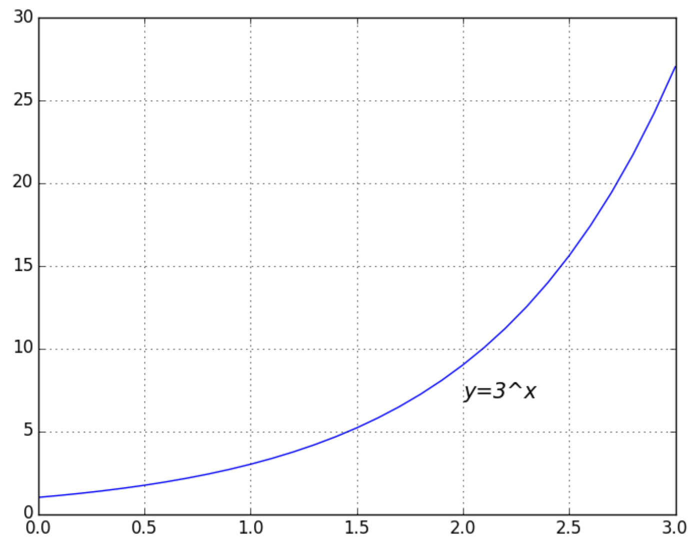
$$f(x) = \frac{2x+1}{x-1}.$$

The domain of a rational function is the set of numbers x at which the bottom polynomial is non-zero. Using the above rational function,

$Domain\ of\ f(x) = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty)$.



An exponential function is a function in the form $f(x) = a^x$ for some non-zero number a. For example, $f(x) = 3^x$ or $f(x) = 0.5^x$.



The absolute value function is

$$f(x) = |x| = \begin{cases} x & when\ x \geq 0 \\ -x & when\ x < 0 \end{cases}$$

Q. Draw the graph of the absolute value function.

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A piecewise linear function is a function defined by different formulas on different intervals. The function $f(x) = |x|$ above is a good example of piecewise linear functions.

Q. Express $f(x) = |x + 1|$ in the form of a piecewise linear function.

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A composite function is a function defined by the composition of two functions. Suppose we have two functions $f(x) = x + 1$ and $g(x) = 2x$.

The composition of f with g, (or the function $f \circ g$) is defined by

$$(f \circ g)(x) = f(g(x)).$$

For example, $f(g(1)) = f(2) = 2 + 1 = 3$. We are finding (1) g(1) first, and then (2) use x=g(1)=2 as an input and put it to the formula f(x)=x+1. In general,

$$(f \circ g)(x) = f(g(x)) = f(2x) = 2x + 1\,.$$

Note $(g \circ f)(x) = g(f(x)) = g(x + 1) = 2(x + 1) = 2x + 2 \neq (f \circ g)(x)$. $f \circ g$ and $g \circ f$ are basically two different functions.

Q, Let $f(x) = x^2 + 1$. Find $f(x + 1)$.

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The difference quotient of a function $f(x)$ is the quantity

$$\frac{f(x+h) - f(x)}{h}.$$

The top is the change in f when x increases to $x + h$. The bottom captures the change in x, which is h by default.

Q. Let $f(x) = x^2$. Find its difference quotient and simplify your answer.

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