Non-differentiable Functions

We start with revising the meaning of *continuity*.

A function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x)$ exists and

$$\lim_{x\to c} f(x) = f(c)\,.$$

We say $f(x)$ is discontinuous at $x = c$ if it is NOT continuous at c. There are several conditions that guarantee $f(x)$ discontinuous at a number c.

A function $f(x)$ is discontinuous at a number c when one of the following conditions applies.

(1) $f(c)$ does not exist. (c is not in the domain of f(x))

(2) The two-sided limits $\lim_{x\to c} f(x)$ doesn't exist. $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$

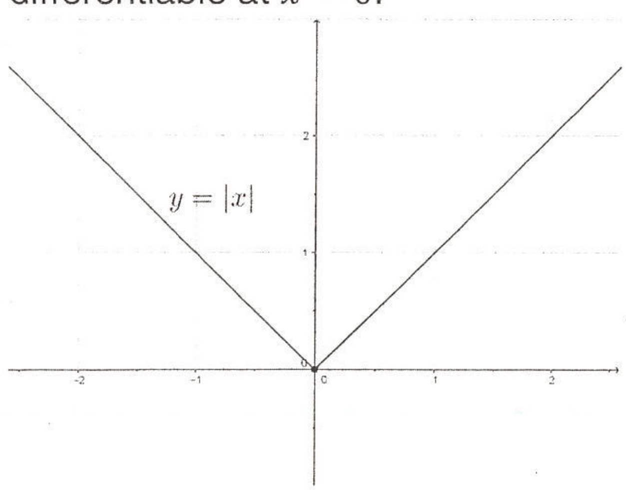
(3) $f(c)$ doesn't equal the limit $\lim_{x\to c} f(x)$. $f(c) \neq \lim_{x\to c} f(x)$

The graph of a continuous function is special. You can draw and complete it without lifting your hand from the paper.

Let us go back to differentiation.

A function $f(x)$ is differentiable at a point $x = c$ if $f'(c)$ exists.

Previously we mentioned that the absolute function $f(x) = |x|$ is not differentiable at $x = 0$.

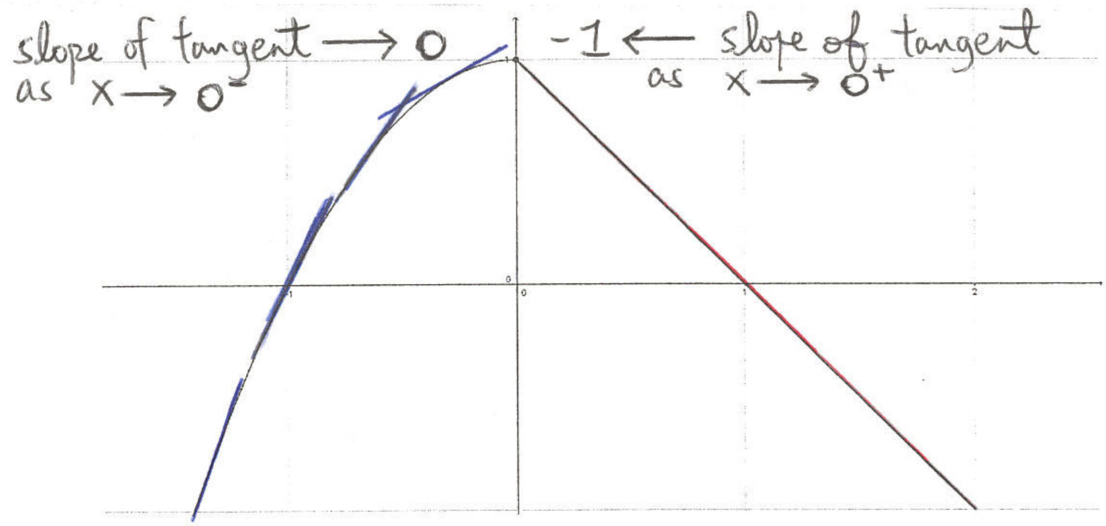


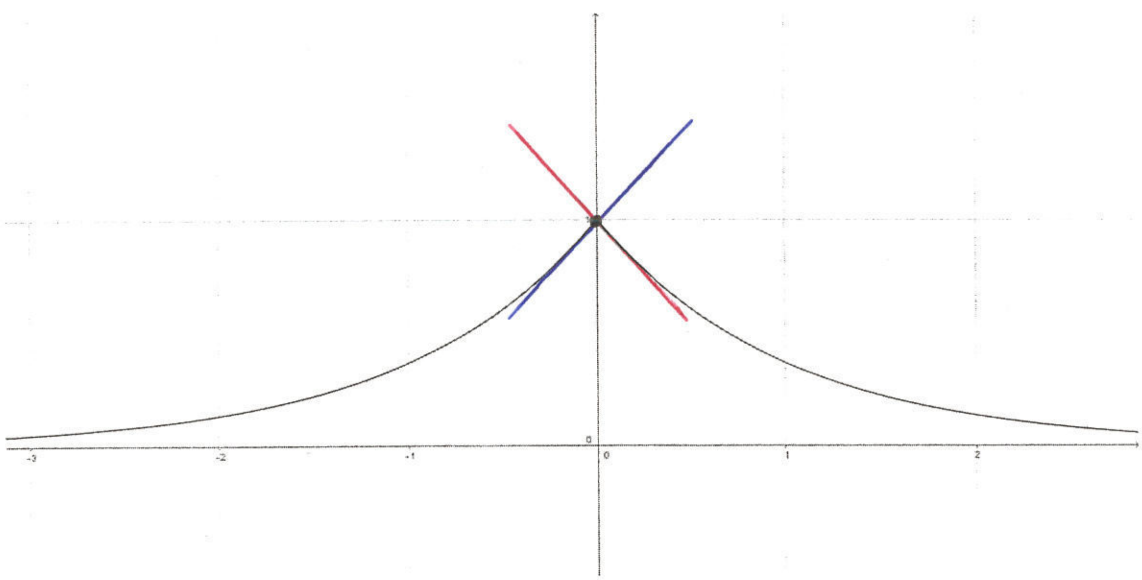
A function $f(x)$ is not differentiable at a number c if one of the following conditions applies.

(1) $f(x)$ has a corner point at c. Basically, a corner point is formed when

$$\lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h} \neq \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}.$$

We can look at different kinds of corner points.



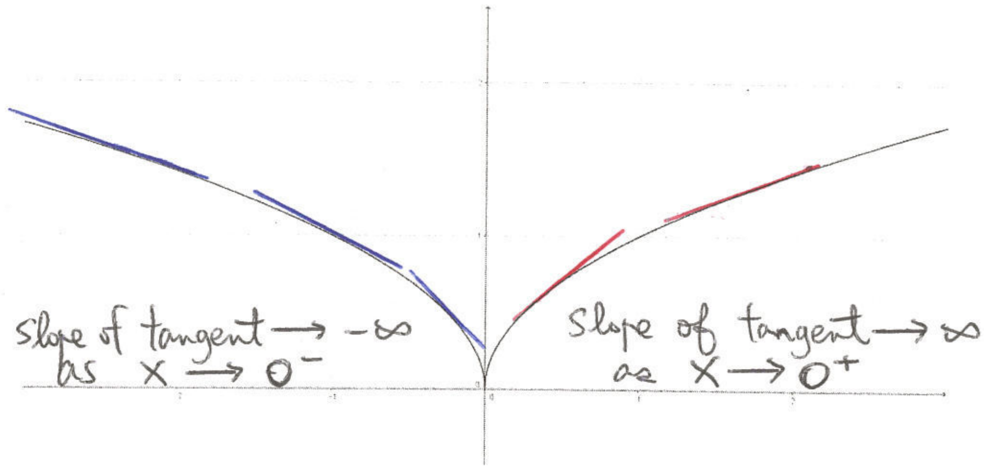


These two functions are having a corner point at $x = 0$. Note that they are continuous at $x = 0$.

(2) $f(x)$ has a vertical tangent line at $x = c$. It means $f'(c) = \pm\infty$.

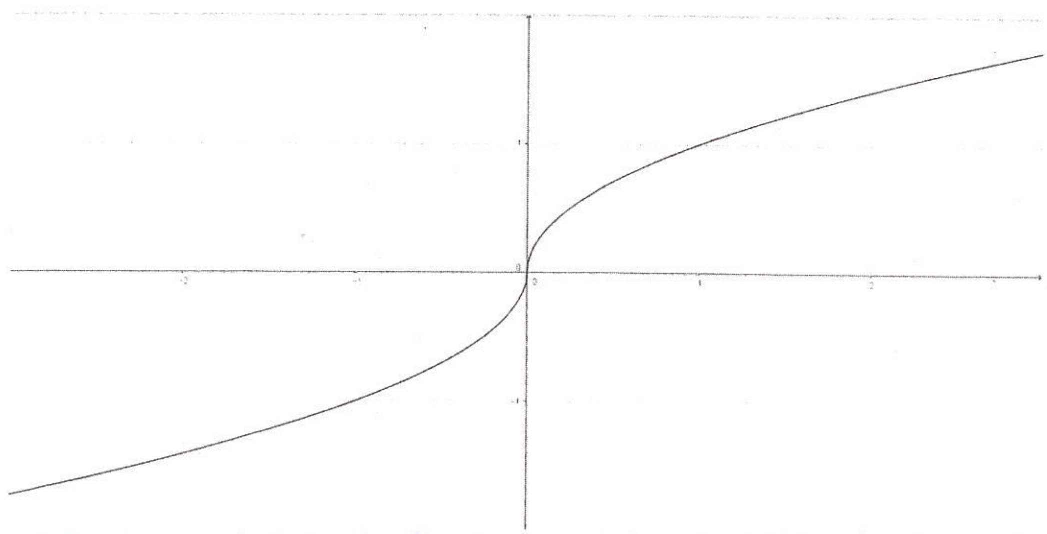
We can see what's exactly happening from the graph of $f(x)$. For example.

$$f(x) = \begin{cases} \sqrt{x} & when \quad x > 0 \\ 0 & when \quad x = 0 \\ \sqrt{-x} & when \quad x < 0 \end{cases} .$$



The tangent line to $f(x)$ at 0 is the vertical line $x = 0$. We say that $f(x)$ is not differentiable at 0 because $f'(0)$ doesn't exist.

We also say that $f(x)$ has a vertical tangent line at $x = 0$ in this case. The slope of the tangent line is approaching $\pm\infty$ as x approaches 0 from the left or from the right.



Discontinuous at number c if
(1) f(c) doesn't exist
OR (2) $\lim_{x\to c^+} f(x) \neq \lim_{x\to c^-} f(x)$
OR (3) $f(c) \neq \lim_{x\to c} f(x)$

(3) $f(x)$ is discontinuous at $x = c$.

We have the following fact

If $f(x)$ is differentiable at $x = c$, then $f(x)$ is continuous at $x = c$.

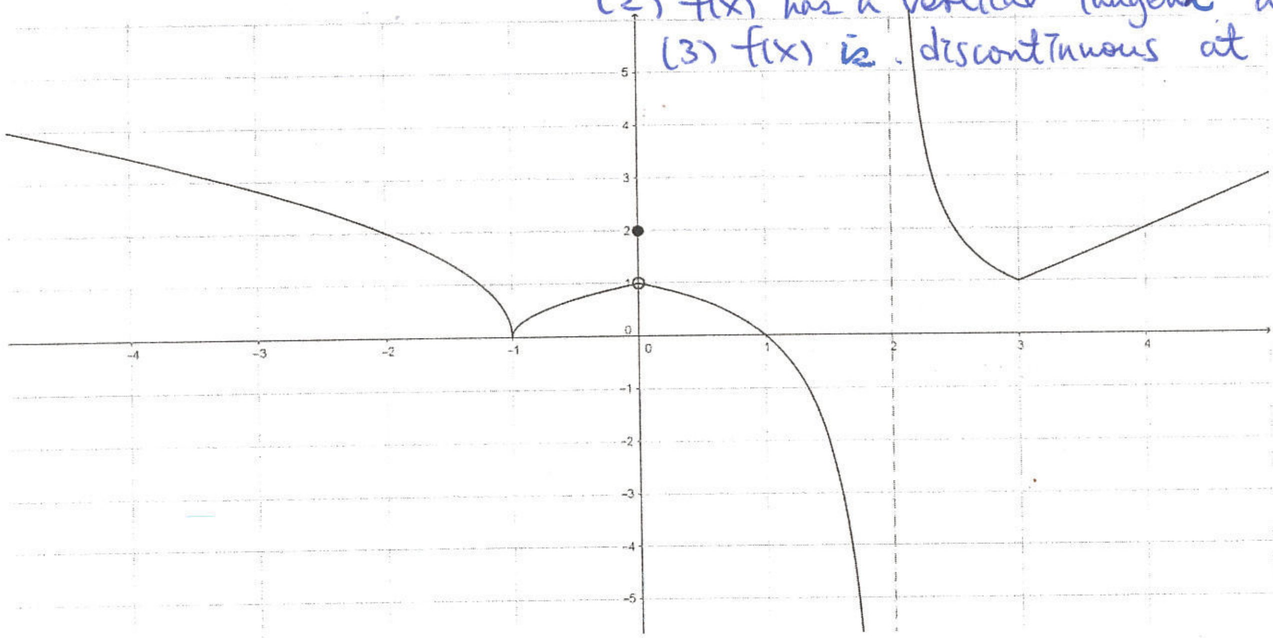
Q. Below is the graph of a function $f(x)$.

(a) At what x-value(s) does $f(x)$ appear to be not continuous?

(b) At what x-value(s) does $f(x)$ appear to be not differentiable?

f(x) is not differentiable at c if
(1) f(x) has a corner point at c
(2) f(x) has a vertical tangent at c
(3) f(x) is discontinuous at c.

Explain your answer.



> (a) f(x) is discontinuous at $\underline{x=0}$ and $\underline{x=2}$.
> $x=0$: $f(0)=2$, $\lim_{x\to 0} f(x) = 1 \neq f(0)$.
> $x=2$: $f(2)$ doesn't exist.
> [Or we can say $\lim_{x\to 2} f(x)$ DNE.]
>
> (b) f(x) is not differentiable at

$\underline{x=0,\ x=2}$, $\underline{x=-1}$ and $\underline{x=3}$.

f(x) is discontinuous at 0 & 2

vertical tangent

corner point.