Antiderivatives and Indefinite Integrals

Anti-differentiation, or finding anti-derivatives, is to recover a function from its derivative. For example, the derivative of x^2 is $2x$, so we say:

An antiderivative of $2x$ is x^2.

Note $x^2 + 1$ and $x^2 + 17$ are also antiderivatives of $2x$. The most general antiderivative of $2x$ is $x^2 + C$, where C is a constant.

The indefinite integral of a function $f(x)$ is the most general antiderivative of $f(x)$, written as

$$\int f(x)\,dx\,.$$

We have the following.

$$\boxed{\int f(x)\,dx = g(x) + C \quad \text{if and only if} \quad g'(x) = f(x)\,.}$$

Example. $\int 2x\,dx = x^2 + C$.

We say that the indefinite integral of $2x$ is $x^2 + C$.

$\int$ Is called the integral sign. $2x\,dx$ is the integrand in this example. The constant C is called an arbitrary constant, since it can be any value.

Note that we exactly have $\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x$.

We use integration rules to do integration.

1. Constant rule for integration.
$\int k\,dx = kx + C, \quad \text{when } k \text{ is a constant.}$

In particular, we have $\int 1 dx = x + C$.

2. Power rule for integration.

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad \text{when } n \neq -1.$$

Example.

$$\int x^2 \, dx = \frac{1}{2+1} x^{2+1} + C = \frac{1}{3} x^3 + C$$

Q. Find the indefinite integral

$$\int \sqrt{x} \, dx.$$

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Q. Find the indefinite integral

$$\int \frac{dx}{x^3}.$$

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3. Constant-multiple rule for integration. For any constant k,

$$\int k \cdot f(x)\, dx = k \int f(x)\, dx + C\,.$$

For example, $\int 2016x\, dx = 2016 \int x\, dx = 2016 \cdot \frac{1}{2} \cdot x^2 + C = 1008x^2 + C$.

4. Sum-Difference rule for integration. For two functions $f(x)$ and $g(x)$,

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx\,.$$

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx\,.$$

Example. Find the indefinite integral,

$$\int x^2 + x^3\, dx\,.$$

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Q. Find $\int 5t + 7\, dt$.

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Q. Find $\int 6x^2 - 3x^{-2} + 4\ dx$.

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Q. Find $\int x^2(x+6)^2\ dx$.

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Q. Find

$$\int \frac{6t^2 - t}{t}\ dt\ .$$

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Q. Find

$$\int \left(6\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)\, dx\,.$$

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There are some application problems in this section.

Example. [Recovering cost function from marginal cost function]

A company's marginal cost function is

$$MC(x) = 20x^{3/2} - 15x^{2/3} + 1\,,$$

where x is the number of units, and fixed costs are $4000. Find the cost function.

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Long time ago we learnt that, if there is a car riding on a straight line,

$$s(t) = \text{distance travelled by the car at time } t,$$

$$v(t) = \text{velocity of the car at time } t,$$

$$\text{and} \quad a(t) = \text{acceleration of the car at time } t,$$

then we have $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

In this manner, we can recover the distance travelled function $s(t)$ by velocity function $v(t)$.

Q. A Porsche 997 Turbo Cabriolet can accelerate from a standing start to a speed of $v(t) = -0.24t^2 + 18t$ feet per second, after t seconds $(0 \leq t < 40)$.

1. Find a formula for the distance that it will travel from the starting point in the first t seconds. [Hints: integrate velocity to find distance, and then use the fact that distance is 0 at time $t = 0$.]

2. Use the formula in (1) to find the distance that the car will travel in the first 10 seconds.

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Integration using Logarithmic and Exponential Functions

5. Integral of e^x.

$$\int e^x \, dx = e^x + C$$

When a is a constant, we have

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C \,.$$

6. Integral of $\frac{1}{x}$.

$$\int \frac{1}{x} \, dx = \ln |x| + C$$

The absolute sign in $|x|$ appears so that the above formula works for positive x-values and negative x-values.

Q. Find $\int e^{3x} \, dx$.

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Q. Find $\int e^{x/4} \, dx$.

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Q. Find $\int \frac{3 \, dx}{x}$.

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Q. Find the indefinite integral

$$\int e^{2x} - \frac{2}{x}\, dx\,.$$

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We have three relevant application problems in this section.

Example. [Epidemic, textbook P.321]

An influenza epidemic hits a large city and spreads at the rate of $12e^{0.2t}$ new cases per day, where t is the number of days since the epidemic began. The epidemic began with 4 cases.

1. Find a formula for the total number of flu cases in the first t days of the epidemic.

2. Use your formula to find the number of cases during the first 30 days.

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Example. [Total sales]

A college bookstore runs a sale on its least popular mathematics books. The sales rate (books sold per day) on day t of the sale is predicted to be

$$\frac{60}{t} \quad (for\ t \geq 1).$$

where $t = 1$ corresponds to the beginning of the sale, at which time none of the inventory of 350 books had been sold.

1. Find a formula for the number of books sold up to day t.

2. Will the store have sold its inventory of 350 books by day $t = 30$?

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Example. [Consumption of natural resources, textbook P.324]

The annual world consumption of silver is predicted to be $22.3e^{0.01t}$ thousand metric tons per year, where t is the number of years since 2014.

Find a formula for the total silver consumption within t years of 2014 and estimate when the known world reserves of 540 thousand metric tons will be exhausted.

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