Implicit Differentiation and Related Rates

Suppose x and y are two variables. Many times y is a function of x, and then we say $y = f(x)$. Variables x and y are related explicitly in this case. However, sometimes x and y are related implicitly. For example, we know that x and y are related by

$$x^2 + y^2 = 16\,.$$

y fails to be a function of x in this equation. We call this equation an implicit equation, and y is called an implicit function of x. In this section, we find

$$\frac{dy}{dx}$$

from an implicit equation. The process is called implicit differentiation.

Generalized power rule. $$\frac{d}{dx}(y^n) = n \cdot y^{n-1} \cdot \frac{dy}{dx}$$

Example. Use implicit differentiation to find $\frac{dy}{dx}$ when $7x^2 + y^2 = 16$.

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$$7x^2 + y^2 = 16$$

> Differentiate with respect to x,

> $$\frac{d}{dx}(7x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

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> $$14x + 2y\frac{dy}{dx} = 0$$

> $$2y\frac{dy}{dx} = -14x$$

> $$\frac{dy}{dx} = \frac{-14x}{2y} = \frac{-7x}{y} \quad \#$$

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Q. Find the equation of the tangent line to $7x^2 + y^2 = 16$ at $(1,3)$.

> Slope $= \left.\frac{dy}{dx}\right|_{(1,3)} = \frac{-7}{3}$

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> $\therefore$ The tangent line to $7x^2+y^2=16$ at $(1,3)$ is

$$y-3 = \left(-\frac{7}{3}\right)(x-1)$$

> $$y = -\frac{7}{3}x + \frac{16}{3} \quad \#$$

Steps of Implicit Differentiation

1. Differentiate every term with respect to x.
2. Collect on terms with $\frac{dy}{dx}$ on the LHS. Put other terms to the RHS.
3. Factor out $\frac{dy}{dx}$ on the left. Solve by division.

Q. Find the slope of the tangent to the curve

$$2x^3 + 3xy + y^2 = 12$$

at $(1,2)$.

> differentiate with respect to x.

> $$\frac{d}{dx}(2x^3) + \frac{d}{dx}(3xy) + \frac{d}{dx}(y^2) = \frac{d}{dx}(12)$$

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> $$6x^2 + \left[\frac{d}{dx}(3x)\right]\cdot y + (3x)\cdot\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

> $$6x^2 + 3y + 3x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

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> $$(3x+2y)\frac{dy}{dx} = -6x^2 - 3y$$

> $$\frac{dy}{dx} = \frac{-6x^2-3y}{3x+2y}$$

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> $\therefore$ Slope $= \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{-6-3(2)}{3+2(2)} = \frac{-12}{7} \quad \#$

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In economics, a demand equation is the relation between the price p of a product, and the quantity x that consumers will demand at that price.

Q. A company's demand equation is

$$x = \sqrt{2000 - p^2},$$

where p is the price in dollars. Find $\frac{dp}{dx}$ when $p = 40$. Interpret your answer.

> $$x^2 = 2000 - p^2$$

> differentiate with respect to x,

> $$\frac{d}{dx}(x^2) = \frac{d}{dx}(2000) - \frac{d}{dx}(p^2)$$

> $$2x = 0 - 2p\frac{dp}{dx}$$

> $$2p\frac{dp}{dx} = -2x$$

> $$\frac{dp}{dx} = -\frac{x}{p}.$$

> When $p=40$, $x = \sqrt{2000-(40)^2}$

> $= \sqrt{2000 - 1600}$

> $= \sqrt{400} = 20$

> $$\therefore \left.\frac{dp}{dx}\right|_{x=20,\, p=40} = -\frac{20}{40} = -0.5$$

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> Interpretation

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> When the price is \$40, the price is decreasing at the rate of \$0.5 per unit of product.

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> In other words, the price has to drop by about \$0.5 for an additional unit to be demaned, when the price is \$40. #

Example. [Sales]

The number of printer cartridges that a store will sell per week, x in number, and their price, p in dollars, are related by the equation

$$x^2 = 4500 - 5p^2 \, .$$

If the price is falling at the rate of $1 per week, find how the sales will change when the current price is $20.

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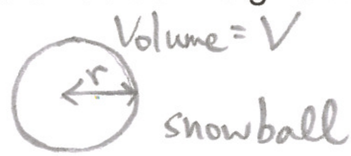
Implicit differentiation is useful when we are finding related rates.

Example. [Snowball]

A large snowball is melting so that its radius is decreasing at the rate of 2 inches per hour. How fast is the volume decreasing at the moment when the radius is 3 inches?

[Hint: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.]



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> At time t in hour, let $r(t)$ be the radius in inches,
> and $V(t)$ be the volume in cm^3 of the snowball.

> $$V(t) = \frac{4}{3}\pi\, r(t)^3$$

> differentiate wrt t,

> $$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r(t)^2 \frac{dr}{dt}\right)$$

> Put $\frac{dr}{dt} = -2$ and $r = 3$.

> $$\frac{dV}{dt} = \frac{4}{3}\pi\left(3 \cdot 9 \cdot (-2)\right) = -72\pi$$

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> Therefore, when the radius is 3 inches,
> the volume of the snowball is decreasing
> at a rate of 72π inch^3 per hour.

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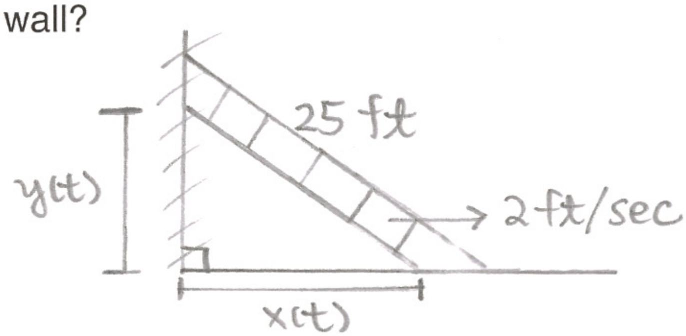
Example. [Ladder]

A ladder of 25 feet long is leaning against the wall of a house. The base of the ladder is being pulled away from the wall at a rate of 2 feet per second.

How fast is the top of the ladder moving down the wall when its base is 7 feet from the wall?



> At time t in second,
> let x(t) be the distance between the bottom of the ladder and the wall in feet,
> and let y(t) be the height of the ladder in feet.

$$x(t)^2 + y(t)^2 = 25^2 = 625$$

differentiate with respect to t,

$$2x(t)\frac{dx}{dt} + 2y(t)\underbrace{\frac{dy}{dt}}_{\text{find it.}} = 0$$

When $x(t) = 7$,

$$49 + y(t)^2 = 625$$
$$y(t) = \sqrt{576} = 24$$

Put $x(t) = 7$, $y(t) = 24$, $\frac{dx}{dt} = 2$

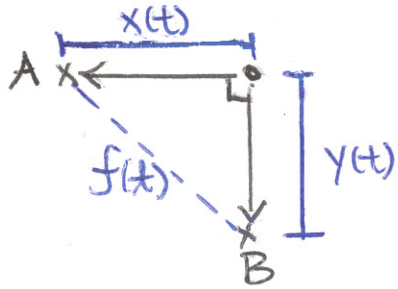
$$\therefore \quad 2(7)(2) + 2(24)\frac{dy}{dt} = 0$$
$$48\frac{dy}{dt} = -28$$
$$\frac{dy}{dt} = -\frac{7}{12}$$

Therefore, when the base is 7 feet from the wall, the top is moving down the wall at a rate of $\frac{7}{25}$ feet per second. #

Example. [Distance]

Two people on bikes start riding from the same point at the same time. Person A starts riding west at a rate of 3 meters per second, and Person B starts riding south at 4 meters per second.

At what rate is the distance separating these two people changing when it is 2 seconds after they started riding?



At time t in second,
let $x(t)$ in m be the horizontal distance between A and starting point,
and let $y(t)$ in m be the vertical distance between B and starting point.
Let $f(t)$ be the distance separating A and B at time t.

$$f(t)^2 = x(t)^2 + y(t)^2$$

differentiate with respect to t,

$$2f(t)\frac{df}{dt} = 2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}.$$

When $t=2$, $x(2) = 3\cdot 2 = 6$

$y(2) = 4\cdot 2 = 8$

$f(2)^2 = x(2)^2 + y(2)^2 = 36 + 64 = 100$

$\therefore\ f(2) = 10.$

Put $x(2)=6$, $y(2)=8$, $f(2)=10$,

$\frac{dx}{dt}(2) = 3$, $\frac{dy}{dt}(2) = 4$.

$$(2\cdot 10)\frac{df}{dt}(2) = (2\cdot 6)\cdot 3 + (2\cdot 8)\cdot 4$$

$$\frac{df}{dt}(2) = \frac{100}{20} = 5.$$

Therefore, the distance separating A and B is increasing at a rate of 5 m per second when it's 2 seconds after start. #