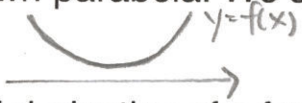
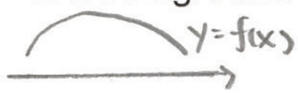
Graphing using the First and Second Derivatives

A curve on the xy-plane is concave up if it curls upward. It is like an opening up parabola. A curve is concave down if it curls downward, like an opening down parabola. We say that a straight line is flat, having no concavity.

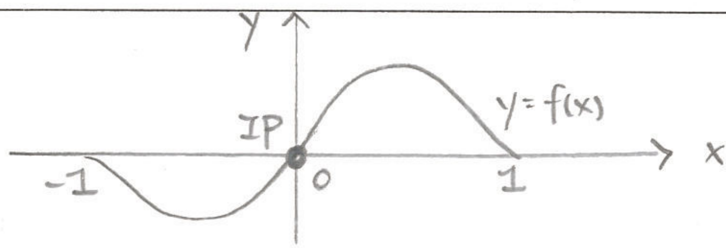




The second derivative of a function $f(x)$ tells us about the concavity of the graph of $f(x)$. (the state of f(x) being concave up or down)

> Suppose $f(x)$ is a function, and an open interval (a, b) is contained in the domain of $f(x)$.
>
> (1) If $f''(x) > 0$ on (a, b), then $f(x)$ is concave up on (a, b).
>
> (2) If $f''(x) < 0$ on (a, b). Then $f(x)$ is concave down on (a, b).

> An inflection point (IP) is a point on the graph of $f(x)$ where concavity changes. At an inflection point $(x\,, y)$, $f''(x) = 0$ or undefined.
>
> A possible inflection point is a point on the graph of $f(x)$ of which the x-value solves $f''(x) = 0$.



Example. Let $f(x) = x^4 + 8x^3 + 18x^2 + 8$. Find the followings.

1. Find all critical numbers of $f(x)$.
2. Find all intervals of increase and intervals of decrease.
3. Find x-values of all relative extrema of $f(x)$.
4. Find the x-values of all possible inflection points of $f(x)$.
5. Find all concave-up intervals and concave-down intervals of $f(x)$.
6. Find the x-values of all inflection points of $f(x)$.

Step 1 to Step 6

1. Find $f'(x)$ and $f''(x)$.
2. Solve $f'(x) = 0$ to find all CNs.
   Solve $f''(x) = 0$ to find all possible IPs (x-values).
3. Make a sign diagram for the first derivative $f'(x)$.
4. Find intervals of increase and decrease.
   Locate relative maximum and relative minimum (x-values).
5. Make a sign diagram for the second derivative $f''(x)$.
   Find concave-up and concave-down intervals.
6. Locate IPs (x-values).

> 1. $f'(x) = 4x^3 + 24x^2 + 36x$
> $= 4x(x^2 + 6x + 9)$
> $= 4x(x+3)^2$
>
> $f''(x) = \frac{d}{dx}(4x^3 + 24x^2 + 36x)$
> $= 12x^2 + 48x + 36$
> $= 12(x^2 + 4x + 3)$
> $= 12(x+1)(x+3).$
>
> 2. $f'(x) = 0 \iff 4x(x+3)^2 = 0$
> $x = 0$ or $x = -3$.
> $\therefore$ CNs are $x = 0$ and $x = -3$.
>
> $f''(x) = 0 \iff 12(x+1)(x+3) = 0$
> $x = -1$ or -3
> $\therefore$ Possible IPs are $x = -1$ and $x = -3$.
>
>
>

3.

x	$(-\infty,-3)$	-3	$(-3,0)$	0	$(0,\infty)$
$f'(x)$	$-$ $x=-5$	0	$-$ $x=-2$	0	$+$ $x=1$
$f(x)$	decrease	X	decrease	rel min point	increase

Put test points to $f'(x)$.

$$f'(-5) = 4(-5)(-2)^2 = -80 < 0$$
$$f'(-2) = 4(-2)(1)^2 = -8 < 0$$
$$f'(1) = 4(1)(16) = 64 > 0$$

5.

x	$(-\infty,-3)$	-3	$(-3,-1)$	-1	$(-1,\infty)$
$f''(x)$	$+$ $x=-5$	0	$-$ $x=-2$	0	$+$ $x=1$
$f(x)$	CU	IP	CD	IP	CU

Put test points to $f''(x)$

$$f''(-5) = 12(-4)(-2) = 96 > 0$$
$$f''(-2) = 12(-1)(1) = -12 < 0$$
$$f''(1) = 12(2)(4) = 96 > 0.$$

4 & 6.

CNs : $x=0$ and $x=-3$

Intervals of increase : $(0,\infty)$

Intervals of decrease : $(-\infty,-3)$, $(-3,0)$

relative maximum : None

relative minimum : $x=0$.

IPs : $x=-3$ and $x=-1$

Concave-up intervals : $(-\infty,-3)$, $(-1,\infty)$

Concave-down intervals : $(-3,-1)$

#.

Now we can (7.) sketch the graph of $f(x) = x^4 + 8x^3 + 18x^2 + 8$.

On the graph, we have to label all relative extreme points, inflection points and intercepts. Here we have to find the y-values of those relative maximum/minimum points and that of IPs.

[DON'T work on asymptote if it is not mentioned in the question.]

CNs : $x = 0$, $f(0) = 8$

$x = -3$, $f(-3) = (-3)^4 + 8(-3)^3 + 18(9) + 8$

$= 35$.

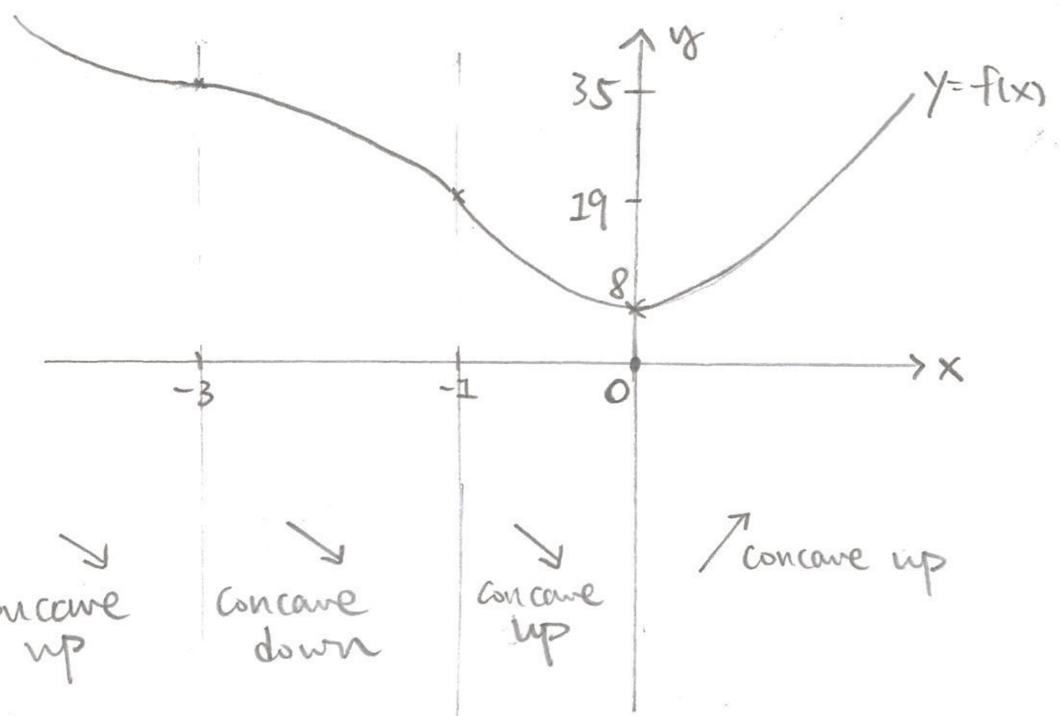
IPs : $x = -1$, $f(-1) = (-1)^4 + 8(-1)^3 + 18(-1)^2 + 8$

$= 1 - 8 + 18 + 8 = 19$.

$x = -3$, $f(-3) = 35$.

x-intercepts : (skip)

y-intercept : $y = f(0) = 8$.

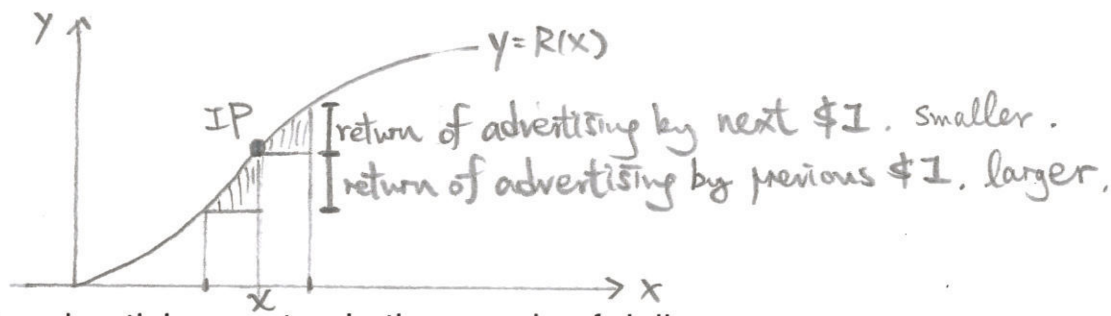


* * f(x) has horizontal tangent at 0 and -3

Here we mention an application of inflection point and concavity,

Suppose a company has its revenue function $R(x)$, where x represents the amount in dollars invested in advertising. (Or we can use its profit function $P(x)$.) An inflection point where $R(x)$ changes from concave up to concave down, is called a point of diminishing returns.

Basically it means after this point, further investment will bring decreasing returns but not increasing returns. The additional dollar spent on advertising starts to be not as worth as the previous dollar spent on advertising.



Q. By increasing its advertising cost x, in thousands of dollars, a company discovers that it can increase the sales y, in thousands of dollars, of a product according to the model.

$$y = 10 - 3x + 24x^2 - x^3, \quad for\ 0 \leq x \leq 15$$

Find the point of diminishing returns for this product, and explain what this means.

> $y' = \frac{d}{dx}(10 - 3x + 24x^2 - x^3) = -3 + 48x - 3x^2$

> $y'' = \frac{d}{dx}(-3 + 48x - 3x^2) = 48 - 6x.$

> Find possible IPs. $y'' = 0 \Leftrightarrow 48 - 6x = 0 \Leftrightarrow x = 8.$

x	$(0,8)$	8	$(8,15)$
y''	$+$ $x=1$	0	$-$ $x=10$

> $\therefore x = 8$ is an IP.

> $\Rightarrow$ The point of diminishing return is $x = 8$ in thousands of dollars.

This means spending over $8,000 in advertising cost will not result in good use of money toward advertising.

The last section is about the second-derivative test. The second derivative of a function also gives us information about its critical numbers.

Second-Derivative Test
Let c be a critical number of a function $f(x)$. Suppose $f''(c)$ is well defined.
(1) If $f''(c) > 0$, then $f(x)$ has a relative minimum at $x = c$.
(2) If $f''(c) < 0$, then $f(x)$ has a relative maximum at $x = c$.

Q. Find all critical numbers for the function

$$f(x) = -x^3 + 27x + 2\,.$$

Use the second-derivative test to classify each critical number as a relative maximum point, a relative minimum point, or neither of them.

> $f'(x) = -3x^2 + 27$
> $f''(x) = -6x$.
>
> Solve $f'(x) = 0$
> $-3x^2 + 27 = 0$
> $3x^2 = 27$
> $x^2 = 9$
> $x = 3$ or -3
> $\therefore$ CNs are $x=3$ and $x=-3$.
>
> 2nd-derivative test.
>
> $f''(3) = -6 \times 3 = -18 < 0$
> $f''(-3) = -6 \times (-3) = 18 > 0$
>
> $\therefore$ $f(x)$ has a relative max at $x=3$
> $f(x)$ has a relative min at $x=-3$ #

Example. It Is a more involving example that can be skipped.

$$f(x) = \frac{x}{x^2 - 1}.$$

Given the function $f(x)$, we have to

(1) make a sign diagram for the first derivative,
(2) make a sign diagram for the second derivative,
(3) sketch the graph of $f(x)$, showing relative max/min points, IPs, intercepts, and all asymptotes.

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