Average Value and Area between Curves

The first application of definite integrals is to find the average value of a function $f(x)$ on an interval $[a, b]$.

$$\text{Average value of } f(x) \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Example. [Population of the United States, textbook P.346]

The population of the United States is predicted to be $P(t) = 310e^{0.0073t}$ in million people, where t is the number of years since 2010. Find the average population between the year 2020 and year 2030.

> Integrate from $t=10$ (Year 2020) to $t=20$ (Year 2030)

> Average population $= \dfrac{1}{20-10}\displaystyle\int_{10}^{20} 310\, e^{0.0073t}\,dt$

> $= \dfrac{310}{10}\displaystyle\int_{10}^{20} e^{0.0073t}\,dt$

>

> $= 31\left(\dfrac{1}{0.0073} e^{0.0073t}\right)\Big|_{10}^{20}$

> $= 31\left(\dfrac{1}{0.0073} e^{0.0073(20)} - \dfrac{1}{0.0073} e^{0.0073(10)}\right)$

> $= 346.0$ million people. #

Example. [Average temperature]

The temperature at time t hours is $T(t) = -0.3t^2 + 4t + 60$, for $0 \le t \le 12$. Find the average temperature between time 0 and time 10.

> Average temperature $= \displaystyle\int_0^{10} (-0.3t^2 + 4t + 60)\,dt$

> $= \dfrac{1}{10}\left(-0.1t^3 + 2t^2 + 60t\right)\Big|_{t=0}^{t=10}$

>

> $= \dfrac{1}{10}\left((-100 + 200 + 600) - (0)\right) = 70$ #

>

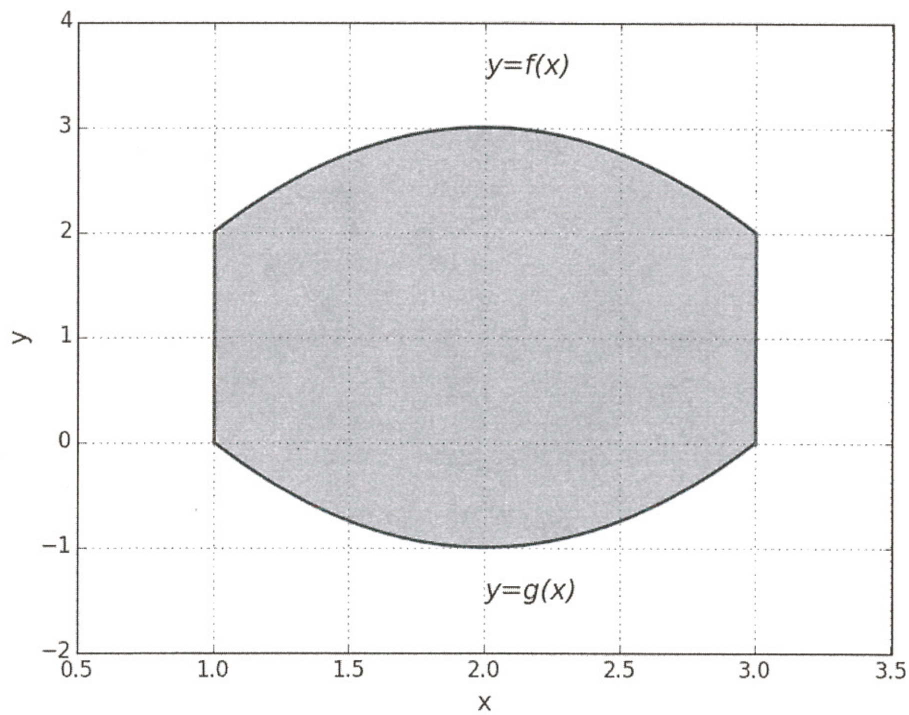
The second application is to find the area between curves. Suppose $f(x)$ and $g(x)$ are two functions on the interval $[a.b]$. Assume that at any point x between a and b on the x-axis, we have

$$f(x) \geq g(x).$$

In this case, we call $y = f(x)$ the upper curve, and $y = g(x)$ the lower curve.

Area between curves. The area between two curves $y = f(x)$ and $y = g(x)$ on $[a, b]$, when $f(x) \geq g(x)$ for all x in $[a, b]$, is given by the below formula.

$$Area\ between\ f(x)\ and\ g(x)\ on\ [a, b] = \int_a^b [f(x) - g(x)]\, dx$$

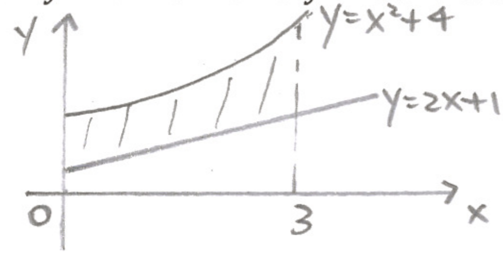


The purple-shaded region is the area between $f(x)$ and $g(x)$ on interval $[1,3]$. Note that $f(x)$ is the upper function, while $g(x)$ is the lower function on the graph above. The area is given by

$$\int_1^3 [f(x) - g(x)]\, dx.$$

Example. [Area between curves]

Find the area between $y = x^2 + 4$ and $y = 2x + 1$ from $x = 0$ to $x = 3$.



upper: $y=x^2+4$

lower: $y=2x+1$

\>

\>

\>

\> $\text{Area} = \int_0^3 (x^2+4) - (2x+1)\,dx = \int_0^3 x^2-2x+3\,dx$

\> $= \left(\frac{1}{3}x^3 - x^2 + 3x\right)\Big|_0^3 = (9-9+9)-(0)$

\> $= 9$ square units #

\>

Q. Find the area between $y = 3x^2 - 3$ and $y = 2x + 5$ from $x = 0$ to $x = 3$.

1. Find intersection point

\> Solve $3x^2-3 = 2x+5 \iff 3x^2-2x-8=0$ $\quad 0 \le x \le 3$

\> $\iff x = \frac{2 \pm \sqrt{4+4(3)(8)}}{6} = \frac{2 \pm 10}{6} = 2$ or $\cancel{-\frac{4}{3}}$

\>

2. Pick test points to determine upper/lower functions.

\> On $[0,2]$, pick $x=1$, $\begin{matrix} 3x^2-3 = 3-3 = 0 \\ 2x+5 = 2+5 = 7 \end{matrix}$

\> $\therefore y=2x+5$ is upper, $y=3x^2-3$ is lower on $[0,2]$.

\> On $[2,3]$, pick $x=3$, $\begin{matrix} 3x^2-3 = 27-3 = 24 \\ 2x+5 = 6+5 = 11 \end{matrix}$

\> $\therefore y=3x^2-3$ is upper, $y=2x+5$ is lower on $[2,3]$.

\> 3. $\text{Area} = \int_0^2 (2x+5)-(3x^2-3)\,dx + \int_2^3 (3x^2-3)-(2x+5)\,dx$

\> $= \int_0^2 2x-3x^2+8\,dx + \int_2^3 3x^2-2x-8\,dx$

\> $= \left(x^2-x^3+8x\right)\Big|_0^2 + \left(x^3-x^2-8x\right)\Big|_2^3$

\> $= (12-0) + (-6+12) = 18$ square units #

[Try "plot $y=3x^2-3$ and $y=2x+5$ from $x=0$ to $x=3$" on Wolfram-alpha]

Q. Find the area bounded by the curves $y = 7x$ and $y = 9x^2$.

1. Find intersection points

Solve $7x = 9x^2$

$$x(7-9x) = 0$$

$$x = 0 \quad \text{or} \quad \frac{7}{9}$$

$\therefore$ integrate on $[0, \frac{7}{9}]$.

2. Pick test points to determine upper/lower functions.

On $[0, 7/9]$, pick $x = \frac{1}{2}$.

$$7x = 7 \cdot \frac{1}{2} = 3.5$$

$$9x^2 = 9 \cdot \frac{1}{4} = 9/4 = 2.25$$

$\therefore$ $y = 9x^2$ is upper, $y = 7x$ is lower on $[0, 7/9]$.

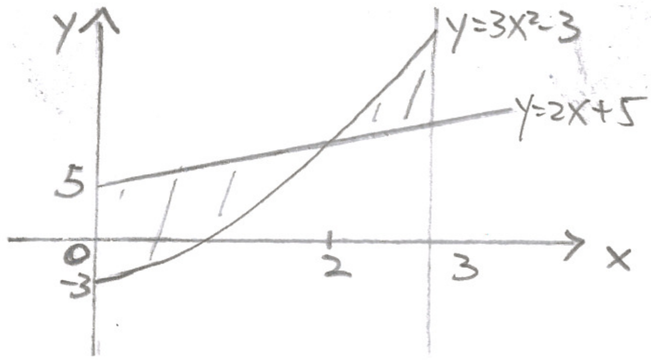
3.

$$\text{Area} = \int_0^{7/9} 7x - 9x^2 \, dx = \left(\frac{7}{2}x^2 - 3x^3\right)\Big|_0^{7/9}$$

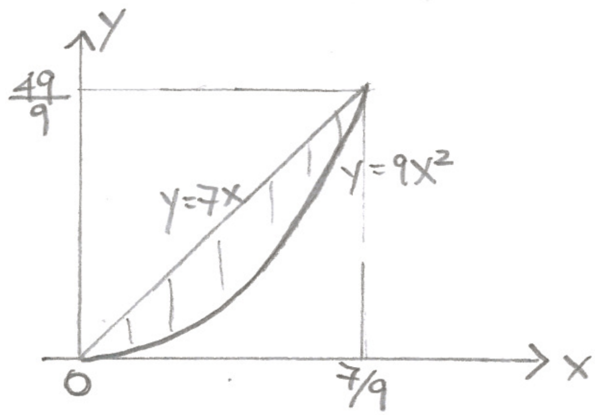
$$= \left(\frac{343}{162} - \frac{1029}{729}\right) - (0)$$

$$= \frac{343}{486} = 0.70576 \text{ square units } \#$$

P.3



P.4



Consumers' Surplus, Producers' Surplus & Gini Index

The third application is on the classical economics, Demand & Supply. Weeks before, we mentioned that a demand equation is the relation between the price p of a product, and the quantity demanded x at that price. We also have introduced the demand function $x = D(p)$, which gives us the quantity demanded x by consumers at the price p.

Now, we switch the roles of x and p in the demand function. We introduce the demand function (or demand curve) as a function of x.

$$p = d(x)$$

It gives us the price of a product at which x units of products will be sold.

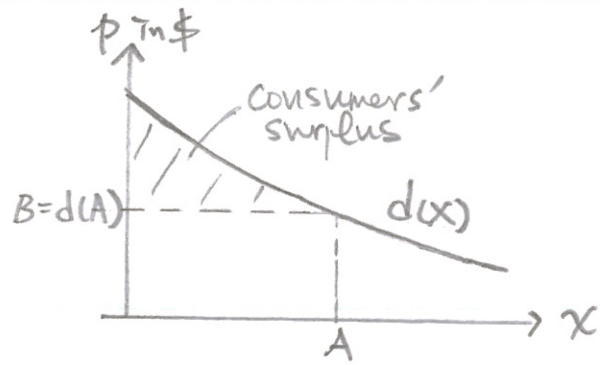
The market price is the price at which all transactions actually occur. Consumers buy good at this market price, and suppliers sell good at this market price.

**Here we assume that all prices are in dollars.

Consumers' surplus.
Suppose $d(x)$ is a demand function and the market price is B. Let A be the quantity demanded (=demand level) at price B, i.e. $B = d(A)$.
The consumers' surplus (CS) is the area between the demand curve and the market price.

$$CS = \int_0^A [d(x) - B]\, dx$$

Consumers' surplus measures the total benefit consumers get under the market price B.



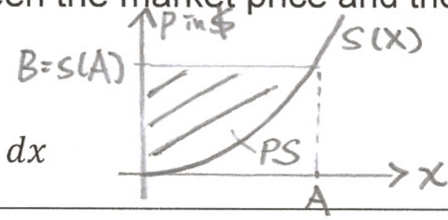
The supply function $s(x)$ for a product, gives us the price at which x units of the product are supplied.

$$p = s(x)$$

Similar to consumers' surplus, we have the producers' surplus (PS). It measures the total benefit producers get at the market price.

Producers' surplus.
Suppose $d(x)$ is a demand function and the market price is B. Let A be the quantity supplied (= supply level = demand level) at price B, i.e. $B = s(A)$. The producers' surplus (PS) is the area between the market price and the supply curve.

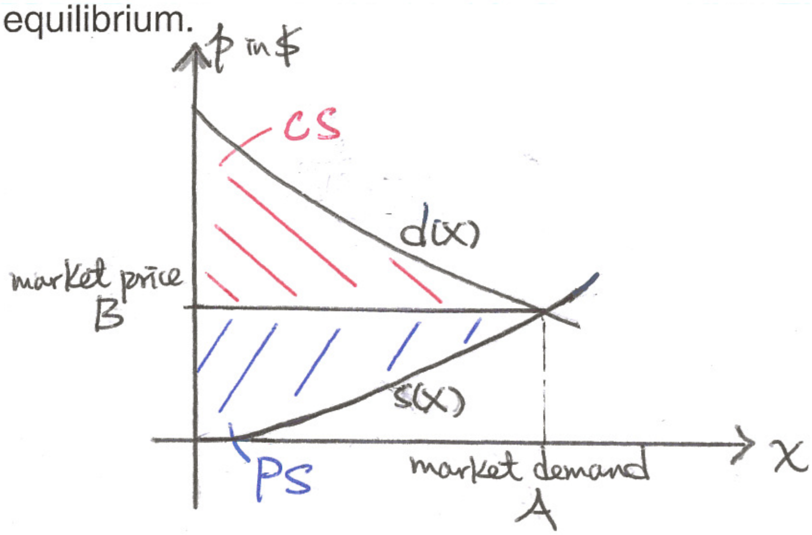
$$PS = \int_0^A [B - s(x)]\, dx$$



Give a demand function $p = d(x)$ and a supply function $p = s(x)$. We can find out the market price and market demand. Market demand A is the quantity of a product exactly sold and bought in the marketplace.
A is the x-value that solves

$$d(x) = s(x).$$

Then, the market price $B = d(A) = s(A)$. The pair $(x = A, p = B)$ is called the market equilibrium.



Q. For the demand function $d(x) = 4000 - 12x$ and the demand level $x = 80$, find the consumers' surplus.

\> demand level $x = 80$

\> $\therefore$ market price $= d(80) = 4000 - 960 = 3040$

\> $CS = \int_0^{80} d(x) - 3040 \, dx$

\>

\> $= \int_0^{80} 960 - 12x \, dx$

\> $= \left(960x - 6x^2\right)\Big|_0^{80}$

\>

\> $= (76800 - 38400) - (0)$

\> $= 38,400$ in dollars #

\>

Q. For the supply function $s(x) = 0.03x^2$ and the demand level $x = 200$, find the producers' surplus.

\> demand level $x = 200$

\> market price $= s(200) = 0.03 \cdot (200)^2 = 1200$

\> $PS = \int_0^{200} 1200 - s(x) \, dx$

\>

\> $= \int_0^{200} 1200 - 0.03x^2 \, dx$

\> $= \left(1200x - 0.01x^3\right)\Big|_0^{200}$

\>

\> $= (240,000 - 80,000) - (0)$

\>

\> $= \$160,000$ #

Q. Given the demand and supply functions

$$d(x) = 300 - 0.4x \quad and \quad s(x) = 0.2x,$$

(a) find the market demand and market price,

(b) find the consumers' surplus at the market demand,

(c) and find the producers' surplus at the market demand.

>
(a) Solve $d(x) = s(x)$

$$300 - 0.4X = 0.2X$$

>
$$300 = 0.6X$$

>
$$X = 500$$

>
$\therefore$ market demand $= 500$

market price $= d(500) = 300 - 0.4(500) = 100$ (in dollars) #

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>
(b)

$$CS = \int_0^{500} d(x) - 100\, dx = \int_0^{500} (300 - 0.4x) - 100\, dx$$

>
$$= \int_0^{500} 200 - 0.4x\, dx = \left(200X - 0.2X^2\right)\Big|_0^{500}$$

>
$$= \$50{,}000 \text{ \#}$$

>

>

>
(c)

$$PS = \int_0^{500} 100 - s(x)\, dx = \int_0^{500} 100 - 0.2x\, dx$$

>
$$= \left(100X - 0.1X^2\right)\Big|_0^{500}$$

>

>
$$= \$25{,}000 \text{ \#}$$

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