substitute u for x

Integration by Substitution

In this section, we learn a new integration technique: integration by substitution. Suppose we have a function $g(x)$. A change in $g(x)$ is denoted by Δg. When this change is very small, we denote it by

$$dg\,.$$

dg is called the differential of $g(x)$. Since g is a function of x, we have

$$\boxed{dg = g'(x) \cdot dx\,.}$$

It means that when x is a number a and the change in x (Δx or dx) is 0.001, the change in $g(x)$ (Δg or dg) is given by

$$dg = g'(a) \cdot dx = g'(a) \cdot (0.001)\,.$$

Eg $g(x) = x^2$, $a = 20$
$dg = g'(20) \cdot dx = 40\,dx$
$= 40 \cdot 0.001 = 0.04$

The right hand side is just a number. Basically, we only use the differentials dg and dx within integration. Elsewhere we represent the changes of x and $g(x)$ by Δx and Δg.

> Integration by substitution.
>
> Suppose $f(u)$ and $u(x)$ are two functions. Putting $u = u(x)$, we have
>
> $$\int f\big(u(x)\big) \cdot u'(x)\, dx = \int f(u)\, du\,.$$

Note that $u = u(x)$ gives $du = u'(x) \cdot dx$ as in above discussion. So we are replacing (1) $u(x)$ by u, and (2) $u'(x)\,dx$ by du.

If we are looking at definite integrals, we use the below formula.

$$\boxed{\int_a^b f\big(u(x)\big) \cdot u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du}$$

The substitution remains $u = u(x)$.

In the following, we study 3 particular cases.

(I) Suppose $u(x)$ is a polynomial, and n is a number

$$\int [u(x)]^n \cdot u'(x)\,dx = \int u^n\,du = \frac{1}{n+1}u(x)^{n+1} + C$$

Q. Find $\int (x^2+1)^3 2x\,dx$. "substitute u for x^2+1" "substitute du for $2x\,dx$"

> substitute $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x \cdot dx$

> $\int (x^2+1)^3 2x\,dx = \int u^3\,du = \frac{u^4}{4} + C$

> $= \frac{(x^2+1)^4}{4} + C$ #

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Q. Find $\int (x^2+10)^4 x\ dx$.

> substitute $u = x^2+10 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x \cdot dx$

> $\int (x^2+10)^4 x\,dx = \frac{1}{2}\int (x^2+10)^4 \cdot 2x\,dx$

> $= \frac{1}{2}\int u^4\,du = \frac{1}{2}\left(\frac{u^5}{5}\right) + C$

> $= \frac{1}{10}(x^2+10)^5 + C$ #

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Q. Find $\int \sqrt{x^2+5}\,(2x)\,dx$.

substitute $u = x^2+5 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x \cdot dx$

$$\int \sqrt{x^2+5} \cdot 2x\,dx = \int (x^2+5)^{\frac{1}{2}} \cdot 2x\,dx$$

$$= \int u^{\frac{1}{2}}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^2+5)^{3/2} + C \quad \#$$

Q. Find $\int \frac{5x}{(x^2+1)^3}\,dx.$

substitute $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$$\int \frac{5x}{(x^2+1)^3}\,dx = \int \frac{\frac{5}{2} \cdot 2x}{(x^2+1)^3}\,dx = \frac{5}{2}\int \frac{1}{(x^2+1)^3}\,2x\,dx$$

$$= \frac{5}{2}\int u^{-3}\,du = \frac{5}{2}\left(-\frac{1}{2}u^{-2}\right) + C$$

$$= -\frac{5}{4}u^{-2} + C$$

$$= -\frac{5}{4}(x^2+1)^{-2} + C \quad \#$$

(II) Suppose $u(x)$ is a polynomial.

$$\int e^{u(x)} \cdot u'(x)\, dx = \int e^u\, du \;= e^{u(x)} + C$$

Q. Find $\int e^{x^3} \cdot x^2\, dx$

> substitute $u = x^3 \Rightarrow \frac{du}{dx} = 3x^2 \Rightarrow du = 3x^2 \cdot dx$

> $\int e^{x^3} \cdot x^2 dx = \frac{1}{3} \int e^{x^3} \cdot (3x^2) dx$

> $= \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C$

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> $= \frac{1}{3} e^{x^3} + C$ #

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Q. Find $\int_0^2 te^{t^2}\, dt$.

> substitute $u = t^2 \Rightarrow \frac{du}{dt} = 2t \Rightarrow du = 2t\, dt$
$t = 0 \Rightarrow u = 0^2 = 0$

> $t = 2 \Rightarrow u = 2^2 = 4$

> $\int_0^2 te^{t^2} dt = \frac{1}{2} \int_0^2 e^{t^2} \cdot 2t\, dt$

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> $= \frac{1}{2} \int_0^4 e^u du = \frac{1}{2} \left(e^u \Big|_{u=0}^{u=4} \right)$

> $= \frac{1}{2} (e^4 - e^0) = \frac{1}{2} (e^4 - 1)$ #

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Q. Find $\int \frac{1}{\sqrt{x}} e^{\sqrt{x}}\, dx.$

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Q. Find $\int x^3 e^{-x^4}\, dx.$

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(III) Suppose $u(x)$ is a polynomial.

$$\int \frac{1}{u(x)} \cdot u'(x)\, dx = \int \frac{1}{u}\, du = \ln|u(x)| + C$$

Q. Find $\int \frac{x^3}{x^4+2}\, dx$.

> Substitute $u = x^4 + 2 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow du = 4x^3\, dx$
>
> $\int \frac{x^3}{x^4+2} dx = \frac{1}{4} \int \frac{4x^3}{x^4+2} dx = \frac{1}{4} \int \frac{1}{x^4+2} \cdot 4x^3 dx$
>
> $= \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln|u| + C$
>
> $= \frac{1}{4} \ln|x^4+2| + C$
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> $= \frac{1}{4} \ln(x^4+2) + C$ #
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Q. Find $\int_4^5 \frac{dx}{x-3}$.

> Substitute $u = x-3 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$
>
> $x = 4 \Rightarrow u = 4-3 = 1$
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> $x = 5 \Rightarrow u = 5-3 = 2$
>
> $\int_4^5 \frac{1}{x-3} dx = \int_1^2 \frac{1}{u} du$
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> $= (\ln|u|)\Big|_{u=1}^{u=2} = \ln 2 - \cancel{\ln 1}^{\,0}$
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> $= \ln 2$ #
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Q*. Find

$$\int_{1}^{2} \frac{e^{x}}{1+e^{x}}\,dx$$

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Q*. Find

$$\int \frac{\sqrt{\ln(x^{3})}}{x}\,dx$$

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