Antiderivatives and Indefinite Integrals

Anti-differentiation, or finding anti-derivatives, is to recover a function from its derivative. For example, the derivative of x^2 is $2x$, so we say:

An antiderivative of $2x$ is x^2.

Note $x^2 + 1$ and $x^2 + 17$ are also antiderivatives of $2x$. The most general antiderivative of $2x$ is $x^2 + C$, where C is a constant.

The indefinite integral of a function $f(x)$ is the most general antiderivative of $f(x)$, written as

$$\int f(x)\,dx\,.$$

We have the following.

$$\boxed{\int f(x)\,dx = g(x) + C \quad \text{if and only if} \quad g'(x) = f(x)\,.}$$

Example. $\int 2x\,dx = x^2 + C$.

We say that the indefinite integral of $2x$ is $x^2 + C$.

$\int$ Is called the integral sign. $2x\,dx$ is the integrand in this example. The constant C is called an arbitrary constant, since it can be any value.

Note that we exactly have $\frac{d}{dx}(x^2 + C) = 2x + 0 = 2x$.

We use integration rules to do integration.

1. Constant rule for integration.

$$\int k\,dx = kx + C, \quad \text{when } k \text{ is a constant.}$$

In particular, we have $\int 1dx = x + C$.

2. Power rule for integration.

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad \text{when } n \neq -1.$$

Example.

$$\int x^2\,dx = \frac{1}{2+1}x^{2+1} + C = \frac{1}{3}x^3 + C$$

Q. Find the indefinite integral

$$\int \sqrt{x}\,dx.$$

>

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{1}{(1+\frac{1}{2})}X^{1+\frac{1}{2}} + C$$

$$= \frac{1}{(\frac{3}{2})}X^{\frac{3}{2}} + C = \frac{2}{3}X^{3/2} + C \quad \#$$

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Q. Find the indefinite integral

$$\int \frac{dx}{x^3}.$$

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$$\int \frac{dx}{X^3} = \int \frac{1}{X^3}\,dx = \int X^{-3}\,dx$$

$$= \frac{1}{-3+1}X^{-3+1} + C$$

$$= -\frac{1}{2}X^{-2} + C \quad \#$$

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3. Constant-multiple rule for integration. For any constant k,

$$\int k \cdot f(x)\, dx = k \int f(x)\, dx + C\,.$$

For example, $\int 2016x\, dx = 2016 \int x\, dx = 2016 \cdot \frac{1}{2} \cdot x^2 + C = 1008x^2 + C$.

4. Sum-Difference rule for integration. For two functions $f(x)$ and $g(x)$,

$$\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx\,.$$

$$\int [f(x) - g(x)]\, dx = \int f(x)\, dx - \int g(x)\, dx\,.$$

Example. Find the indefinite integral,

$$\int x^2 + x^3\, dx\,.$$

> $\int x^2 + x^3 dx = \int x^2 dx + \int x^3 dx$ [SD-I]
>
> $= \frac{1}{2+1} x^{2+1} + \frac{1}{3+1} x^{3+1} + C$
>
> $= \frac{1}{3} x^3 + \frac{1}{4} x^4 + C$ #
>
>

Q. Find $\int 5t + 7\, dt$.

> $\int 5t + 7\, dt = \int 5t\, dt + \int 7\, dt$ [SD-I]
>
> $= 5 \int t\, dt + 7t + C$ $\quad \int 5t\, dt = 5 \int t\, dt$ [CM-I]
> $\int 7\, dt = 7t + C$ [constant-I]
>
> $= 5 \cdot \frac{1}{2} t^2 + 7t + C$
>
> $= \frac{5}{2} t^2 + 7t + C$ # [We only need one "+C" term on the right-hand side.]

Q. Find $\int 6x^2 - 3x^{-2} + 4\, dx$.

> $= \int 6x^2 dx - \int 3x^{-2} dx + \int 4\, dx$
>
> $= 6\int x^2 dx - 3\int x^{-2} dx + \int 4\, dx$
>
> $= 6 \cdot \frac{1}{3} x^3 - 3 \cdot \frac{1}{(-1)} x^{-1} + 4x + C$
>
> $= 2x^3 + 3x^{-1} + 4x + C$ #
>

Q. Find $\int x^2 (x+6)^2\, dx$.

> $= \int x^2 (x^2 + 12x + 36)\, dx$
>
> $= \int x^4 + 12x^3 + 36x^2\, dx$
>
> $= \int x^4 dx + 12\int x^3 dx + 36\int x^2 dx$
>
> $= \frac{1}{5} x^5 + 12 \cdot \frac{1}{4} x^4 + 36 \cdot \frac{1}{3} x^3 + C$
>
> $= \frac{1}{5} x^5 + 3x^4 + 12x^3 + C$ #

Q. Find

$$\int \frac{6t^2 - t}{t}\, dt\ .$$

>
> $\int \frac{6t^2 - t}{t} dt = \int \frac{6t^2}{t} - \frac{t}{t} dt = \int 6t - 1\, dt$
>
> $= 6\int t\, dt - \int 1\, dt = 6 \cdot \frac{1}{2} \cdot t^2 - t + C$
>
> $= 3t^2 - t + C$ #
>
>

Q. Find

$$\int \left(6\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right) dx\,.$$

> $= \int 6x^{\frac{1}{2}} + x^{-\frac{1}{3}}\, dx$
>
> $= 6\int x^{\frac{1}{2}} dx + \int x^{-\frac{1}{3}} dx$
>
> $= 6 \cdot \frac{1}{(3/2)} x^{\frac{3}{2}} + \frac{1}{(2/3)} x^{\frac{2}{3}} + C$
>
> $= 4x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{2}{3}} + C$ #
>

There are some application problems in this section.

Example. [Recovering cost function from marginal cost function]

A company's marginal cost function is

$$MC(x) = 20x^{3/2} - 15x^{2/3} + 1\,,$$

where x is the number of units, and fixed costs are \$4000. Find the cost function.

> $C(x) = \int MC(x)\, dx = \int 20x^{\frac{3}{2}} - 15x^{\frac{2}{3}} + 1\, dx$
>
> $= 20\int x^{\frac{3}{2}} dx - 15\int x^{\frac{2}{3}} dx + \int 1\, dx$
>
> $= 20 \cdot \frac{2}{5} x^{\frac{5}{2}} - 15 \cdot \frac{3}{5} x^{\frac{5}{3}} + x + K$ [use K as the arbitrary constant here.]
>
> $= 8x^{\frac{5}{2}} - 9x^{\frac{5}{3}} + x + K$
>
> Fixed costs are \$4000.
>
> $C(0) = 4000$
>
> $0 - 0 + 0 + K = 4000$
>
> $\therefore\ K = 4000.$

Therefore, the cost function is

$C(x) = 8x^{5/2} - 9x^{5/3} + x + 4000.$ #

Long time ago we learnt that, if there is a car riding on a straight line,

$$s(t) = \text{distance travelled by the car at time } t,$$

$$v(t) = \text{velocity of the car at time } t,$$

$$\text{and} \quad a(t) = \text{acceleration of the car at time } t,$$

then we have $v(t) = s'(t)$ and $a(t) = v'(t) = s''(t)$.

In this manner, we can recover the distance travelled function $s(t)$ by velocity function $v(t)$.

Q. A Porsche 997 Turbo Cabriolet can accelerate from a standing start to a speed of $v(t) = -0.24t^2 + 18t$ feet per second, after t seconds $(0 \le t < 40)$.

1. Find a formula for the distance that it will travel from the starting point in the first t seconds. [Hints: integrate velocity to find distance, and then use the fact that distance is 0 at time $t = 0$.]

2. Use the formula in (1) to find the distance that the car will travel in the first 10 seconds.

$$1. \quad s(t) = \int v(t)\,dt = \int -0.24t^2 + 18t \, dt$$

$$= -0.24 \int t^2 \, dt + 18 \int t \, dt$$

$$= -0.24 \cdot \frac{1}{3} t^3 + 18 \cdot \frac{1}{2} t^2 + C$$

$$= -0.08 t^3 + 9t^2 + C.$$

Put $\quad s(0) = 0$

$$0 + 0 + C = 0$$

$$\therefore C = 0$$

Therefore, $s(t) = -0.08t^3 + 9t^2$ #

$$2. \quad s(10) = -0.08 \cdot 10^3 + 9 \cdot 10^2$$

$$= 820 \text{ feet} \quad \#$$

Integration using Logarithmic and Exponential Functions

5. Integral of e^x.

$$\int e^x \, dx = e^x + C$$

When a is a constant, we have

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C\,.$$

Remark: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ when $n \neq -1$.

6. Integral of $\frac{1}{x}$.

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

The absolute sign in $|x|$ appears so that the above formula works for positive x-values and negative x-values.

Q. Find $\int e^{3x} \, dx$.

> $\int e^{3x} dx = \frac{1}{3} e^{3x} + C$ # [a=3]

>

Q. Find $\int e^{x/4} \, dx$.

> $\int e^{\frac{x}{4}} dx = \frac{1}{(\frac{1}{4})} e^{\frac{x}{4}} + C = 4e^{\frac{x}{4}} + C$ #

>

>

Q. Find $\int \frac{3\, dx}{x}$.

> $\int \frac{3dx}{x} = 3\int \frac{dx}{x} = 3\int \frac{1}{x} dx = 3\ln|x| + C$ #

>

Q. Find the indefinite integral

$$\int e^{2x} - \frac{2}{x}\, dx\,.$$

> $= \int e^{2x}\, dx - 2\int \frac{1}{x}\, dx$

> $= \frac{1}{2}e^{2x} - 2\ln|x| + C$ #

$$\left[\begin{array}{l} \int e^{2x}dx = \frac{1}{2}e^{2x} + C \\ \int \frac{1}{x}dx = \ln|x| + C \end{array}\right]$$

>

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>

We have three relevant application problems in this section.

Example. [Epidemic, textbook P.321]

An influenza epidemic hits a large city and spreads at the rate of $12e^{0.2t}$ new cases per day, where t is the number of days since the epidemic began. The epidemic began with 4 cases.

1. Find a formula for the total number of flu cases in the first t days of the epidemic.

2. Use your formula to find the number of cases during the first 30 days.

> 1. Let $f(t)$ be the total no. of flu cases in the first t days.

> $f(t) = \int 12e^{0.2t}dt = 12\int e^{0.2t}dt = \frac{12}{0.2}e^{0.2t} + C$

> $= 60e^{0.2t} + C$

> Put $f(0) = 4$

> $60(1) + C = 4$

> $C = -56$

$\therefore f(t) = 60e^{0.2t} - 56$ #

>

2. $f(30) = 60 \cdot e^{0.2 \cdot 30} - 56$

$= 60e^{6} - 56 = 24149.73$

≈ 24150 #

Example. [Total sales]

A college bookstore runs a sale on its least popular mathematics books. The sales rate (books sold per day) on day t of the sale is predicted to be

$$\frac{60}{t} \quad (for\ t \geq 1).$$

where $t = 1$ corresponds to the beginning of the sale, at which time none of the inventory of 350 books had been sold.

1. Find a formula for the number of books sold up to day t.

2. Will the store have sold its inventory of 350 books by day $t = 30$?

> 1. Let $S(t)$ be the total sales in first t days

> $S(t) = \int \frac{60}{t}\,dt = 60 \int \frac{1}{t}\,dt = 60 \ln|t| + C$

>

> The initial number of sales is 0.

> $S(1) = 0$

> $60 \ln(1) + C = 0$

> $0 + C = 0$

> $C = 0$

> $\therefore S(t) = 60 \ln|t| = 60 \ln t \quad (t > 0)$ #

>

> 2.

> $S(30) = 60 \cdot \ln 30 = 204.07$

> ≈ 204 books.

>

> Therefore, the store won't have sold its inventory

> of 350 books by day 30. #

>

>

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Example. [Consumption of natural resources, textbook P.324]

The annual world consumption of silver is predicted to be $22.3e^{0.01t}$ thousand metric tons per year, where t is the number of years since 2014.

Find a formula for the total silver consumption within t years of 2014 and estimate when the known world reserves of 540 thousand metric tons will be exhausted.

> Let $C(t)$ be the total silver consumption within t years of 2014.

> $C(t) = \int 22.3 e^{0.01t} dt = 22.3 \int e^{0.01t} dt$

> $= 22.3 \cdot \frac{1}{0.01} e^{0.01t} + C$

> $= 2230 e^{0.01t} + C$

> The total consumed in the first zero year is 0.

> $C(0) = 0$

> $2230 + C = 0$

> $C = -2230$

> $\therefore C(t) = 2230 e^{0.01t} - 2230$ in metric tons.

> Set $C(t) = 540$

> $2230 e^{0.01t} - 2230 = 540$

> $2230 e^{0.01t} = 2770$

> $e^{0.01t} = \frac{2770}{2230} = 1.242$

> Take ln to the both sides,

> $\ln(e^{0.01t}) = \ln(1.242)$

> $0.01t = \ln(1.242)$

> $t = \frac{1}{0.01} \ln(1.242) = 21.7$

> Therefore, the known world reserves of 540 thousand metric tons will be exhausted in about 22 years after 2014, which means in about the year 2036 #