Functions of Several Variables (2)

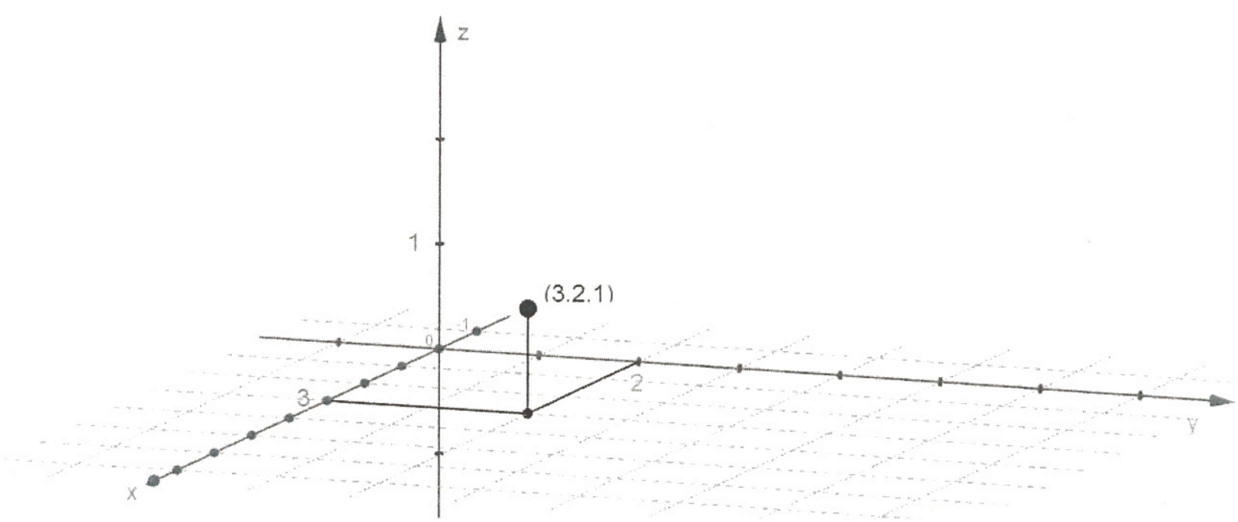
We start with mentioning more on the graph of a function $f(x,y)$. Then we go to see some application problems, and discuss more on partial differentiation.

The three dimensional coordinate system is the space that consists of three coordinate axes. They are x-axis, y-axis and z-axis. In this space, denoted by $\mathbb{R}^3$, we can talk about both left-and-right, front-and-back, and up-and-down, just like this space that we are living in.

A point inside the three dimensional coordinates is represented by its x-coordinate, y-coordinate and z-coordinate. For example, a point is

$$(3,2,1).$$

Here 3 is the x-coordinate, 2 is the y-coordinate and 1 is the z-coordinate of this point (3,2,1).



The graph of a function $f(x, y)$ consists of points (x, y, z) in the three dimensional coordinate system of which

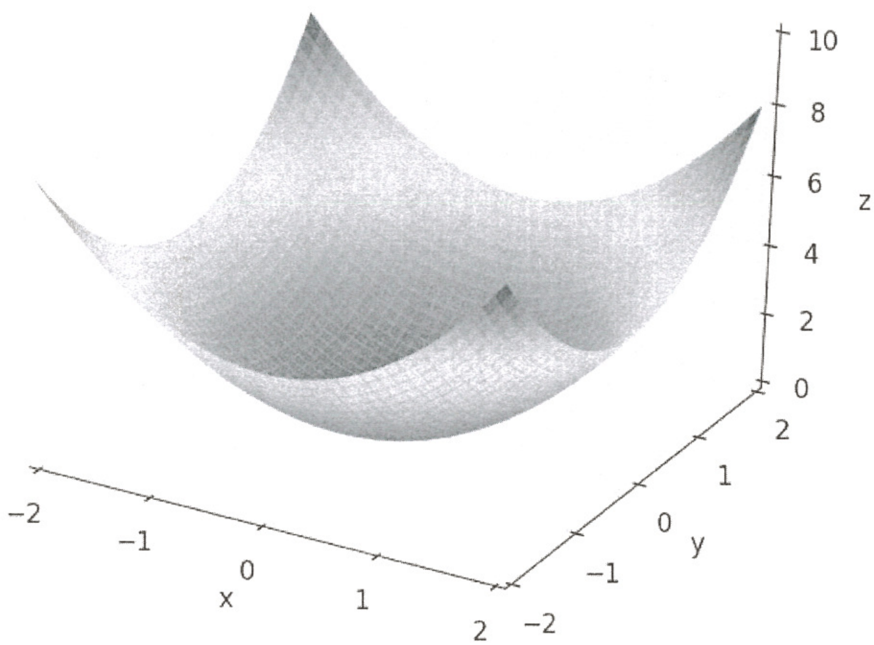
1. the ordered pair (x, y) lies on the domain of $f(x, y)$,
2. and $z = f(x, y)$

In set language, the graph of $f(x, y)$ is

$$\{(x, y, z) \mid (x, y) \text{ is in the domain of } f(x, y), z = f(x, y)\}.$$

Example. Let $f(x, y) = x^2 + y^2$.

$z = x^2 + y^2$

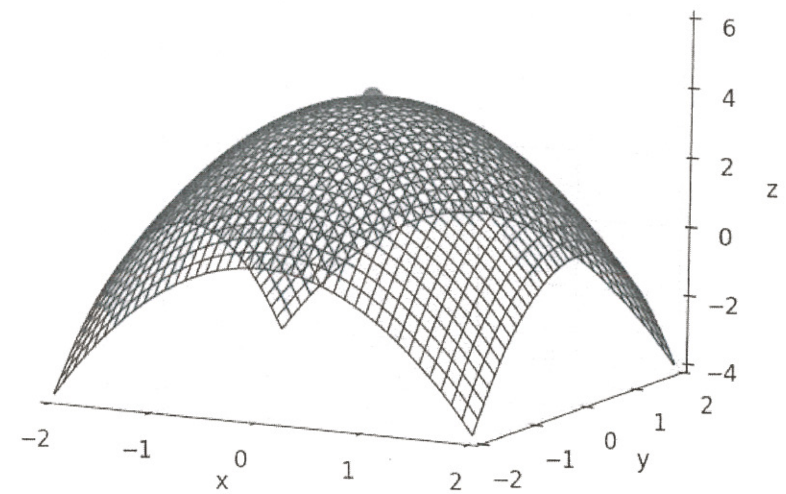


There are some points on the graph of $f(x, y)$ that draw extra attention. They are (1) relative maximum point(s), (2) relative minimum point(s), and (3) saddle point(s).

Relative maximum point. [textbook, p.469]
A point (a, b, c) on the surface $z = f(x, y)$ is a relative maximum point if

$$f(a, b) \geq f(x, y)$$

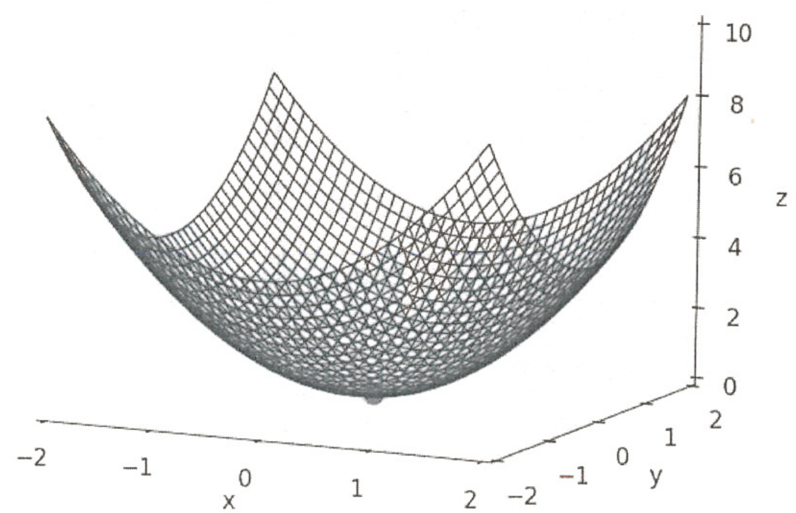
for all (x,y) close to (a,b).



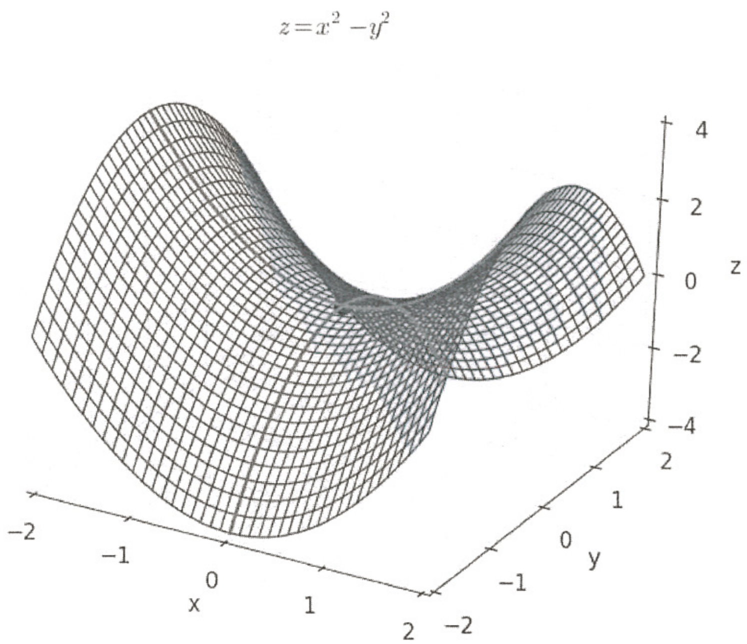
Relative minimum point. [textbook, p.469]
A point (a, b, c) on the surface $z = f(x, y)$ is a relative minimum point if

$$f(a, b) \leq f(x, y)$$

for all (x,y) close to (a,b).



Saddle point. [textbook, p.469]
A saddle point (a, b, c) on the surface $z = f(x, y)$ is the highest point along one curve on the surface, but is the lowest point along another curve on the surface,



A relative extreme point is either a relative maximum point or a relative minimum point. A saddle point is **not** a relative extreme point.

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We continue our discussion on partial derivatives. We can interpret partial derivatives as rates of change. [textbook, p.478]

Given a function of two variables $f(x, y)$, $f_x(x, y)$ is the instantaneous rate of change of f with respect to x when y is held constant. Similarly, $f_y(x, y)$ is is the instantaneous rate of change of f with respect to y when x is held constant.

For example, when $C(x, y)$ is the cost function for x units of product A and y units of product B. Then, $C_x(x, y)$ is the marginal cost function for product A, keeping production level of product B the same. On the other hand, $C_y(x, y)$ is the marginal cost function for product B, keeping production level of product A the same.

Q. A company sells donuts and bagels. It costs $0.5 to make a donut and $1 to make a bagel. The fixed cost is $50 per day. Find the cost function (per day), and use it to find the cost of producing 100 donuts and 30 bagels.

> Let x be the no. of donuts sold per day
> Let y be the no of bagels sold per day.
The cost function is
> $C(x,y) = 50 + 0.5x + y$ (in dollars)
>
$C(100,30) = 50 + 0.5(100) + 30$
> $= \$130$ #
>

Q. An electronics company's profit $P(x, y)$ from making x DVD players and y CD players per day is given by

$$P(x, y) = 2x^2 - 3xy + 3y^2 + 150x + 75y + 200\,.$$

1. Find the marginal profit function for DVD players.

2. Evaluate your answer to part (a) at x=200 and y=300.
Interpret your answer.

> $P_x = \frac{\partial}{\partial x}(2x^2 - 3xy + 3y^2 + 150x + 75y + 200)$
> $= 4x - 3y + 150.$
>
> $P_x(200, 300) = 4(200) - 3(300) + 150$
> $= 50.$
>
> Interpretation
> The profit is increasing at a rate of $50 per
> unit of DVD players when producing
200 DVD players and 300 CD players per day. #
>

For a function $f(x, y)$, $f_x(x, y)$ and $f_y(x, y)$ are so-called the first order partial derivatives. At the same time, we have higher-order partial derivatives. Very often we look at the first and second order partials of a function $f(x, y)$ only.

The second order partials of a function $f(x, y)$ are

$$f_{xx} = \frac{\partial^2}{\partial x^2} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2}{\partial y^2} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2}{\partial y\,\partial x} f = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx} = \frac{\partial^2}{\partial x\,\partial y} f = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Q. Let $f(x, y) = 4x^2 - 3x^3y^2 + 5y^5$.
Find the second-order partials f_{xx}, f_{xy}, f_{yx} and f_{yy}.

> $f_x = 8x - 9x^2y^2$
> $f_y = -6x^3y + 25y^4$
>
> $f_{xx} = (f_x)_x = 8 - 18xy^2$
> $f_{yy} = (f_y)_y = -6x^3 + 100y^3$
> $f_{xy} = (f_x)_y = -18x^2y$
> $f_{yx} = (f_y)_x = -18x^2y$ #
>
>
>
>
>

We have a useful formula.

$$\boxed{f_{xy} = f_{yx}}$$

Q. Let $f(x, y) = (xy + 1)^3$.
Find the second-order partials f_{xx}, f_{xy}, f_{yx} and f_{yy}.

> $f_x = \frac{\partial}{\partial x}(xy+1)^3 = 3(xy+1)^2 \cdot \frac{\partial}{\partial x}(xy+1) = 3y(xy+1)^2$
>
> $f_y = \frac{\partial}{\partial y}(xy+1)^3 = 3(xy+1)^2 \cdot \frac{\partial}{\partial y}(xy+1) = 3x(xy+1)^2$.
>
> $f_{xx} = 3y \frac{\partial}{\partial x}(xy+1)^2 = 3y \cdot 2(xy+1) \cdot \frac{\partial}{\partial x}(xy+1)$
>
> $= 6y(xy+1) \cdot y = 6y^2(xy+1)$
>
> $f_{yy} = 3x \frac{\partial}{\partial y}(xy+1)^2 = 3x \cdot 2(xy+1) \cdot \frac{\partial}{\partial y}(xy+1)$
>
> $= 6x(xy+1) \cdot x = 6x^2(xy+1)$
>
> $f_{xy} = 3 \frac{\partial}{\partial y}(y \cdot (xy+1)^2) = 3\left(\frac{\partial}{\partial y}(y)\right) \cdot (xy+1)^2 + 3y \frac{\partial}{\partial y}(xy+1)^2$
>
> $= 3(xy+1)^2 + 3y \cdot 2(xy+1) \cdot \frac{\partial}{\partial y}(xy+1)$
>
> $= 3(xy+1)^2 + 6xy(xy+1)$ $\#$ $(= f_{yx})$

Finally we mention that partial differentiation is applicable on functions of two or more variables. For example now,

$$f(x, y, z) = xyz\,.$$

It has three (first order) partial derivatives,

$$\frac{\partial f}{\partial x}\,, \quad \frac{\partial f}{\partial y} \quad and \quad \frac{\partial f}{\partial z}.$$

The second-order partials of $f(x, y, z)$ are f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yx}, f_{yz}, f_{zy}, f_{xz}, f_{zx}. There are nine of them with $f_{xy} = f_{yx}$, $f_{yz} = f_{zy}$ and $f_{xz} = f_{zx}$.

We use the same skill to find these partials: differentiate with respect to one variable every time, while treating other variables as constant.

Q. $f(x,y) = \frac{xy}{x+y}$. Find f_x and f_y.

> $f_x = \frac{(x+y)(y)-(xy)(1)}{(x+y)^2} = \frac{xy+y^2-xy}{(x+y)^2}$
>
> $= \frac{y^2}{(x+y)^2}$ #
>
> $f_y = \frac{(x+y)(x)-(xy)(1)}{(x+y)^2} = \frac{x^2+xy-xy}{(x+y)^2}$
>
> $= \frac{x^2}{(x+y)^2}$ #
>
>
>
>
>

Q. $f(x,y) = (x^2 + xy + 1)^4$. Find f_x and f_y.

> $f_x = 4(x^2+xy+1)^3 \frac{\partial}{\partial x}(x^2+xy+1)$
>
> $= 4(x^2+xy+1)^3 \cdot (2x+y)$
>
> $f_y = 4(x^2+xy+1)^3 \frac{\partial}{\partial y}(x^2+xy+1)$
>
> $= 4x(x^2+xy+1)^3$ #
>
>
>
>
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