Lagrange Multipliers and Constrained Optimization

In the previous lecture, we are optimizing a function $f(x,y)$. For example, we find the relative extreme values of a function

$$f(x,y) = x^3 - y^2 - 3x + 6y\,.$$

Some optimization problems are with certain constraints. For example, we are maximizing

$$f(x,y) = 2x + 2xy + y \quad \text{subject to} \quad 2x + y = 100.$$

The function $f(x,y)$ can get as large as possible in value, so it doesn't make sense to maximize $f(x,y)$ per se. However, when we further require that $2x + y = 100$, $f(x,y)$ attains a maximum value at some point (x,y) at which $2x + y$ is 100.

In general, a constrained optimization problem is of the form:

maximize(or minimize) $f(x,y)$, subject to $g(x,y) = 0$.

The condition "$g(x,y) = 0$" is the constraint on the optimization problem. The method of Lagrange multipliers is a method to solve constrained optimization problems.

> Lagrange multiplier. [textbook, p.513]
> Maximize (or minimize) $f(x,y)$ subject to $g(x,y) = 0$.
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> 1. Write $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.
> 2. Set the partial derivatives of $F(x,y)$ to be zero.
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> $$F_x = 0,\ \ F_y = 0 \quad \text{and} \quad F_z = 0.$$
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> Solve for critical points.
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> 3. The solution to the oirginal problem (if exists) will occur at one of these critical points.

The question will tell you whether max value/min value/both values exist in the constrained optimization problem.

Q. Use Lagrange multipliers to find the maximum and minimum values of

$$f(x, y) = 2xy$$

subject to the constraint $x^2 + y^2 = 18$.

> $F(x, y, \lambda) = 2xy + \lambda(x^2 + y^2 - 18)$

Solve

> $$\begin{cases} F_x = 0 : & 2y + 2x\lambda = 0 \quad \text{— ①} \\ F_y = 0 : & 2x + 2y\lambda = 0 \quad \text{— ②} \\ F_\lambda = 0 : & x^2 + y^2 - 18 = 0 \quad \text{— ③} \end{cases}$$

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> ① : $2y + 2x\lambda = 0 \Rightarrow \lambda = \frac{-2y}{2x} = \frac{y}{x}$ Solve eqtn① for λ

> ② : $2x + 2y\lambda = 0 \Rightarrow \lambda = \frac{-2x}{2y} = \frac{x}{y}$ Solve eqtn② for λ

> $(\lambda =) \frac{y}{x} = \frac{x}{y}$ equating $\lambda = \frac{y}{x}$ and $\lambda = \frac{x}{y}$

> $y^2 = x^2$

> ③ : $x^2 + y^2 - 18 = 0$ Put $y^2 = x^2$ into ③

> $2x^2 = 18$

> $x^2 = 9$

$x = 3$ or -3

$x \backslash y$	3	-3
3	(3,3)	(3,-3)
-3	(-3,3)	(-3,-3)

> $x = 3 \Rightarrow y^2 = (3^2) = 9 \Rightarrow y = 3$ or -3

$x = -3 \Rightarrow y^2 = (-3)^2 = 9 \Rightarrow y = 3$ or -3

> $\therefore$ CPs are $(3, 3)$, $(3, -3)$, $(-3, 3)$ and $(-3, 3)$

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> $f(3,3) = 18$, $f(3, -3) = -18$

$f(-3, 3) = -18$, $f(-3, -3) = 18$

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> Therefore, the maximum value of $f = 18$, occuring at $(3, 3)$ and $(-3, 3)$.

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> The minimum value of $f = -18$, occuring at $(3, -3)$ and $(-3, 3)$.

Q. Use Lagrange multipliers to find the maximum value of

$$f(x,y) = 2x + 2xy + y$$

subject to the constraint $2x + y = 100$. (The maximum value exists.)

> $F(x,y,\lambda) = (2x+2xy+y) + \lambda(2x+y-100)$
> Solve

$$\begin{cases} F_x = 0 : & 2 + 2y + 2\lambda = 0 \quad \text{—— ①} \\ F_y = 0 : & 2x + 1 + \lambda = 0 \quad \text{—— ②} \\ F_\lambda = 0 : & 2x + y - 100 = 0 \quad \text{—— ③} \end{cases}$$

> ① : $2\lambda = -2-2y \Rightarrow \lambda = -1-y$
> ② : $\lambda = -1-2x$

$$-1-y = -1-2x$$
$$y = 2x$$

> ③ :

$$2x + y - 100 = 0$$
$$4x - 100 = 0$$
$$x = 25$$

> $x = 25 \Rightarrow y = 2x = 50$
> ∴ The only CP is $(25, 50)$.
> It is a maximum point.

> Therefore, the maximum value of $f(x,y)$
> $= f(25,50)$
> $= 2(25) + 2(25)(50) + 50$
> $= 2{,}600$ #

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Q. Use Lagrange multipliers to find the minimum value of

$$f(x, y) = 5x^2 + 6y^2 - xy$$

subject to the constraint $x + 2y = 24$. (The minimum value exists.)

> $F(x, y, z) = (5x^2 + 6y^2 - xy) + \lambda(x + 2y - 24)$

Solve

> $$\begin{cases} F_x = 0: & 10x - y + \lambda = 0 & — ① \\ F_y = 0: & 12y - x + 2\lambda = 0 & — ② \\ F_\lambda = 0: & x + 2y - 24 = 0 & — ③ \end{cases}$$

> ①: $\lambda = -10x + y$

> ②: $2\lambda = x - 12y \Rightarrow \lambda = \frac{1}{2}x - 6y$

> $$-10x + y = \frac{1}{2}x - 6y$$
> $$-\frac{21}{2}x = -7y$$
> $$x = +\frac{14}{21}y$$
> $$x = +\frac{2}{3}y$$

> ③: $$x + 2y - 24 = 0$$
> $$\frac{2}{3}y + 2y = 24$$
> $$\frac{8}{3}y = 24$$
> $$y = 9$$

> $y = 9 \Rightarrow x = \frac{2}{3}y = 6$

> $\therefore$ The only CP is $(6, 9)$.

It is a minimum point.

> Therefore, the minimum value of $f(x, y)$

> $= f(6, 9)$

$= 5(36) + 6(81) - 6 \cdot 9$

> $= 612$ #

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Q. Use Lagrange multipliers to find the maximum value of

$$f(x, y) = e^{(x+2)(y-3)}$$

subject to the constraint $x + 3y = 1$. (The maximum value exists.)

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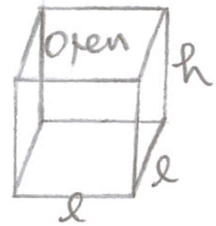
Q. A cardboard box with a square base and without a lid, is to a have a volume of 32,000 cm^3. Find the dimensions that minimize the amount of cardboard used.



Let l be the length of the base in cm.
Let h be the height of the box in cm.

Amount of cardboard used
= Surface area $= l^2 + 4hl$

$\therefore$ minimize $S(h, l) = l^2 + 4hl$
Subject to $hl^2 = 32000$ $\quad$ (Vol = 32000 cm^3)

$F(h, l, \lambda) = (l^2 + 4hl) + \lambda(hl^2 - 32000)$

Solve
$$\begin{cases} F_h = 0 : & 4l + \lambda l^2 = 0 \quad \text{—— ①} \\ F_l = 0 : & 2l + 4h + 2\lambda hl = 0 \quad \text{—— ②} \\ F_\lambda = 0 : & hl^2 - 32000 = 0. \quad \text{—— ③} \end{cases}$$

① : $\lambda l^2 = -4l \Rightarrow \lambda = \frac{-4}{l}$
② : $2\lambda hl = -(2l + 4h) \Rightarrow \lambda = \frac{-(l+2h)}{hl}$

$$-\frac{4}{l} = -\frac{l+2h}{hl}$$
$$4h = l + 2h$$
$$l = 2h$$

③ : $hl^2 - 32000 = 0$
$$4h^3 - 32000 = 0$$
$$h = (8000)^{\frac{1}{3}} = 20$$

$h = 20 \Rightarrow l = 2h = 40$.
$\therefore$ The only CP is $h = 20$, $l = 40$.
It is a minimum point.

Therefore, the required dimensions are :
length = 40 cm and height = 20 cm #
(* 40 cm × 40 cm × 20 cm)