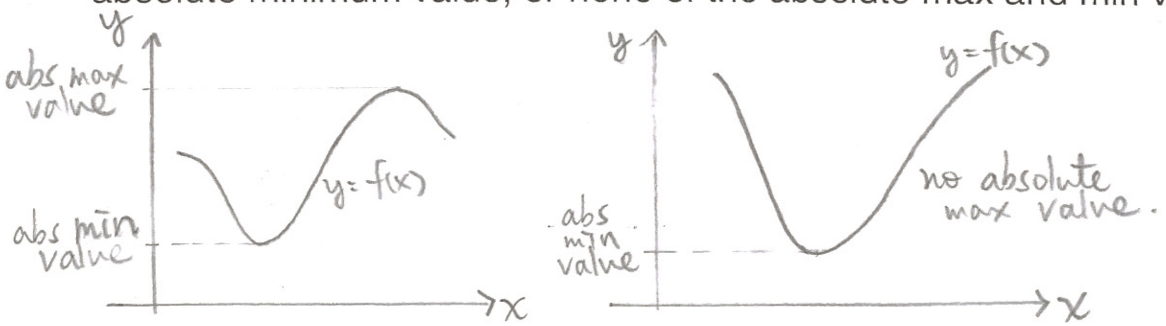
Optimization

Optimizing a function $f(x)$ means finding its maximum or minimum value over the domain of $f(x)$.

> The absolute maximum value of $f(x)$ is the largest value of the function on its domain. The absolute minimum value of $f(x)$ is the smallest value of the function on its domain.
>
> An absolute extreme value of $f(x)$ refers to the absolute maximum value or the absolute minimum value of $f(x)$.

It can happen that a function $f(x)$ has no absolute maximum value, or no absolute minimum value, or none of the absolute max and min values.



Let $f(x)$ be a continuous function, and let $[a, b]$ be a closed interval inside the domain of $f(x)$. Then, $f(x)$ must have the absolute maximum value, and the absolute minimum value on $[a, b]$.

Example. Let $f(x) = x^3 - 6x^2 + 9x + 8$. Find the absolute maximum and absolute minimum values of $f(x)$ on the closed interval $[-1,2]$.

> Find absolute max/min values of $f(x)$ on $[a, b]$.
>
> Step 1: Find all critical numbers of $f(x)$ on $[a, b]$.
>
> Step 2: Evaluate $f(x)$ at all CNs, and at the endpoints $x = a$ and $x = b$.

Step 3: Compare all the values of f(x) you get in step 2.

>
> 1. $f'(x) = 3x^2 - 12x + 9 = 3(x^2-4x+3)$
$= 3(x-1)(x-3)$
> Solve $f'(x) = 0$
$3(x-1)(x-3) = 0$
> $x = 1$ or 3
>
> Check $-1 \leq 1 \leq 2$ OK
$-1 \leq 3 \leq 2$ Wrong
> $\therefore$ The only CN on $[-1,2]$ is $x=1$.
>
> 2. Evaluate $f(x)$ at CN and endpoints
> $f(-1) = -8$, $f(1) = 12$, $f(2) = 10$.
> 3.
The maximum value of $f(x)$ is 12, occuring at $x=1$
The minimum value of $f(x)$ is -8, occuring at $x=-1$ #

Not in class

Q. Let $f(x) = 6x^2 - x^3$. Find the absolute maximum and absolute minimum values of $f(x)$ on the interval [0,5].

> 1. $f'(x) = 12x - 3x^2 = 3x(4-x)$
Solve $f'(x) = 0$.
> $3x(4-x) = 0$
> $x=0$, $x=4$
> Check $0 \leq 0 \leq 5$
$0 \leq 4 \leq 5$
> $\therefore$ CNs on $[0,5]$ are 0 and 4.
>
> 2. Evaluate $f(x)$ at CNs and endpoints
> $f(0) = 0$
> $f(4) = (6 \cdot 4^2) - 4^3 = 32$
> $f(5) = (6 \cdot 5^2) - 5^3 = 25$
>
> 3. The maximum value of $f(x)$ is 32, occuring at $x=4$.
The minimum value of $f(x)$ is 0, occuring at $x=0$.
#

The second derivative test states as follows. If $f(x)$ has a critical number c in its domain, then (1) $f''(c) > 0$ means c is a relative minimum point, and (2) $f''(c) < 0$ means c is a relative maximum point.

OK: $[1,2], [3,4), (4,\infty), [5,\infty)$
$(5,6], (12,30), (-\infty,10], (-\infty,7)$

Second-Derivative Test (Absolute max/min)

Suppose the domain of $f(x)$ is an interval, a ray or $(-\infty, \infty)$.
Let c be the ONLY critical number of $f(x)$ in its domain.

(3) If $f''(c) > 0$, then $f(x)$ attains the absolute minimum value at $x = c$ and the absolute minimum value is $f(c)$.

(4) If $f''(c) < 0$, then $f(x)$ attains the absolute maximum value at $x = c$ and the absolute maximum value is $f(c)$.

Example. Uber (Pittsburgh) finds that its price function is

$$p(x) = 125 - 3x,$$

where $p(x)$ is the price in dollars at which x Uber cars will be in service per day. (counted in frequency) It costs Uber 5 dollars to call on a Uber driver each ride, and the fixed cost is 400 dollars per day.

(1) Find the number of Uber cars that Uber should deploy per day, and the price it should charge to maximize profit.

(2) Find the maximum profit.

These two statements come from the text book P.193 & P.195.

The price function $p(x)$ gives the price p at which consumers will buy exactly x units of the products.

A maximum profit, marginal revenue $MR(x)$ equals marginal cost $MC(x)$.

>

> Revenue $R(x) = p(x) \cdot x = (125-3x)x$

>

Cost $C(x) = 5x + 400$

>

> $\therefore$ Profit $P(x) = (125-3x)x - (5x+400)$

> $P(x) = 125x - 3x^2 - 5x - 400$

$= -3x^2 + 120x - 400.$

>

> Maximize $P(x)$.

> $P'(x) = -6x + 120$

$P''(x) = -6$

>

> Solve $P'(x) = 0$

$-6x + 120 = 0$

> $6x = 120$

> $x = 20$

> 2nd-derivative test : $P''(20) = -6 < 0$

$\therefore$ $P(x)$ is maximized at $x = 20$.

>

> When $x = 20$, $p(20) = 125 - 60 = 65$

$P(20) = -3(400) + (2400) - 400 = 800.$

>

> Therefore, the UBer company should deploy

> 20 cars per day and charge them

for $65 each car..

>

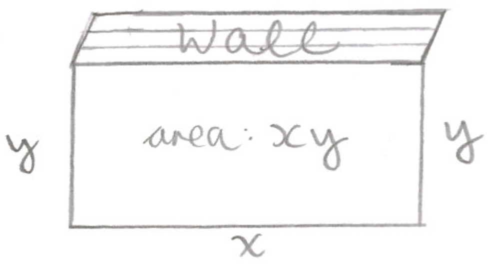
> The maximum profit is $800 per day. #

>

>

Example. [Fencing, textbook P.196]

A farmer has 2000 feet of fence and wants to build a rectangular enclosure along a straight wall. The side along this wall needs no fence. Find the dimension that make the enclosure as large as possible. Also find the maximum area.



Let x in feet be the length of rectangle. (parallel to wall)
Let y in feet be the width of rectangle
Area: $A = xy$

Maximize $A = xy$ subject to $x + 2y = 2000$.

$$x + 2y = 2000$$
$$x = 2000 - 2y$$
$$\therefore A = (2000 - 2y)y$$
$$A(y) = 2000y - 2y^2.$$

$$A'(y) = 2000 - 4y$$
$$A''(y) = -4$$

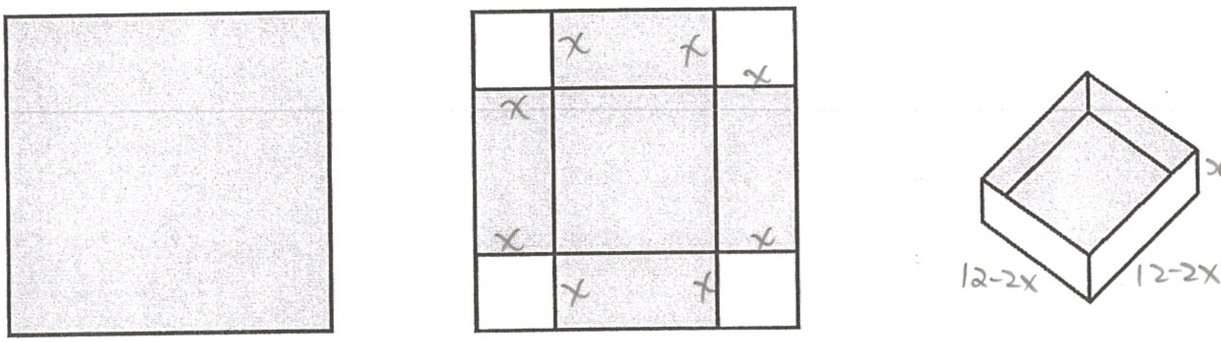
$A'(y) = 0 \Leftrightarrow 2000 - 4y = 0 \Leftrightarrow y = 500$
$A''(500) = -4 < 0$
$\therefore A(y)$ is maximized at $y = 500$.

When $y = 500$, $x = 2000 - 2(500) = 1000$
$A(500) = 2000(500) - 2(500)^2 = 500{,}000$

Therefore, the required dimensions are length $= 1{,}000$ ft and width $= 500$ ft.
The maximum area is $500{,}000$ square feet. #

Example. [Box, textbook P.197]

An open-top box is to be made from a square sheet of paper 12 inches on each side, by cutting a square from each corner and folding up the sides. Find the volume of the largest box that can be made in this way.



12 *inches*

Let x in inches be the length of the side of the square cut from each corner.

The length and width of the base of the box = $12-2x$ inches
The height of the box = x inches
$\therefore$ The volume of the box is

$$V(x) = (12-2x)(12-2x)\,x$$
$$= (144 - 48x + 4x^2)\,x$$
$$V(x) = 4x^3 - 48x^2 + 144x$$

Maximize $V(x)$.

$$V'(x) = 12x^2 - 96x + 144$$
$$V''(x) = 24x - 96$$

$$V'(x) = 0 \iff 12x^2 - 96x + 144 = 0$$
$$12(x^2 - 8x + 12) = 0$$
$$12(x-2)(x-6) = 0$$
$$x = 2 \text{ or } \cancel{6} \text{ (rejected)}$$

$0 < x < 6$.

[If we cut more than 6 inches from each corner, there will be nothing left on the paper sheet.]

$\therefore$ The only CN is $x = 2$
2nd derivative test.
$$V''(2) = 48 - 96 = -48 < 0$$
$\therefore$ $V(x)$ is maximized at $x = 2$.

When $x = 2$, $V(2) = (12-4)(12-4) \cdot 2$
$$= 64 \cdot 2 = 128$$

Therefore, the volume of the largest box is 128 inches^3.

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