Functions of Several Variables

The function $f(x,y) = x^2 + y^2$ is so called *a function of two variables.* These two variables are x and y. For each ordered pair (x,y), the function f returns a real number $f(x,y)$.

Let say $(x,y) = (2,3)$. It means $x = 2$ and $y = 3$. We then have

$$f(2,3) = (2)^2 + (3)^2 = 4 + 9 = 13.$$

The domain of a function $f(x,y)$ is the set of all ordered pairs (x,y) at which $f(x,y)$ is well defined.

Also, the range of $f(x,y)$ is the set of all values $f(x,y)$ given by ordered pairs (x,y) inside the domain of $f(x,y)$.

Example. The domain of $f(x,y) = x^2 + y^2$ is

$$\{(x,y) \mid all\ possible\ ordered\ pairs\ (x,y)\}$$

Q. Let $f(x,y) = \frac{\sqrt{x}}{y}$.

(1) Find the domain of $f(x,y)$.
(2) Find $f(4,3)$.

> domain of $f(x,y) = \{(x,y) \mid x \geq 0$ and $y \neq 0\}$
>
> $f(4,3) = \frac{\sqrt{4}}{3} = \frac{2}{3}$ #
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Q. Find the domain of the function $f(x,y) = \frac{1}{x-y}$.

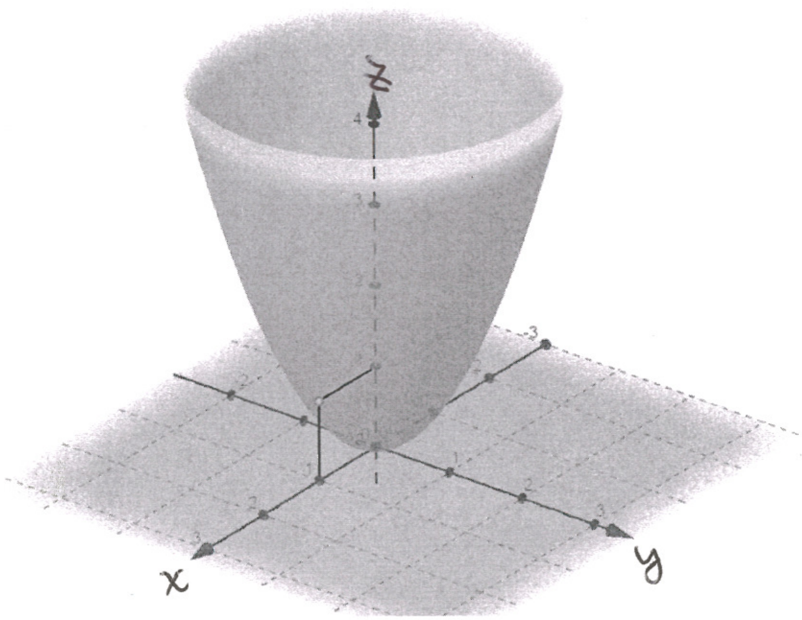
> domain of $f(x,y) = \{(x,y) \mid x-y \neq 0\}$
>
> $= \{(x,y) \mid x \neq y\}$ #

The graph of a two-variable function is displayed on the three dimensional coordinate system. The graph of $f(x,y)$ is the surface $z = f(x,y)$ on (x,y,z) coordinate space. For example,

= coordinate system

$$f(x,y) = x^2 + y^2 .$$

In the below figure, the red line (pointing left) is the x-axis, the green line (pointing right) is the y-axis, and the blue line (pointing up) is the z-axis. The plane on the floor is the xy-plane in this (x,y,z) coordinate space.



The purple-shaded surface is the graph of $f(x,y)$, denoted by $z = x^2 + y^2$. For example, I pick the ordered pair $(1,0)$ on the xy-plane, and then draw a vertical line above $(1,0)$ until it hits the surface at a point.

We know that $f(1,0) = 1$. So the z-coordinate of that point is 1. That point is $(1,0,1)$. ← $x=1$ $y=0$ $z=1$

Differentiation on a two variable function is by taking partial derivatives. Given a function $f(x, y)$, it has two partial derivatives, one coming from x and other coming from y, which are

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

Partial derivatives. At a point (x, y) in the domain of $f(x, y)$.

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

We are finding partial derivatives by differentiation formulas.

1. When finding $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate everything with respect to x. All differentiation formulas apply.
2. When finding $\frac{\partial f}{\partial y}$, we treat x as a constant and differentiate everything with respect to y. All differentiation formulas apply.

There are two ways to write down partial derivatives.

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y)$$

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y)$$

Q. Let $f(x, y) = x^4 y^3$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

> $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^4 y^3) = y^3 \frac{\partial}{\partial x}(x^4)$
$= y^3 (4x^3) = 4x^3 y^3.$

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> $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^4 y^3) = x^4 \frac{\partial}{\partial y}(y^3)$
$= x^4 (3y^2) = 3x^4 y^2.$ #

Q. Let $f(x, y) = 4x^3 - 3x^2y^2 - 2y^2$.

(1) Find f_x and f_y.

(2) Find $f_x(-1,1)$ and $f_y(-1,1)$.

> (1) $f_x = \frac{\partial}{\partial x}(4x^3) - \frac{\partial}{\partial x}(3x^2y^2) - \cancel{\frac{\partial}{\partial x}(2y^2)}^{\,0}$
> $= \frac{\partial}{\partial x}(4x^3) - y^2 \frac{\partial}{\partial x}(3x^2)$
> $= 12x^2 - y^2(6x)$
> $= 12x^2 - 6xy^2$
>
> $f_y = \cancel{\frac{\partial}{\partial y}(4x^3)}^{\,0} - \frac{\partial}{\partial y}(3x^2y^2) - \frac{\partial}{\partial y}(2y^2)$
> $= -x^2 \frac{\partial}{\partial y}(3y^2) - \frac{\partial}{\partial y}(2y^2)$
> $= -x^2(6y) - (4y)$
> $= -6x^2y - 4y$
>
> (2) $f_x(-1,1) = 12(-1)^2 - 6(-1)(1) = 18$
> $f_y(-1,1) = -6(-1)^2(1) - 4(1) = -10$ #

Q. Let $f(x, y) = y(\ln x) + xe^y$. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

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