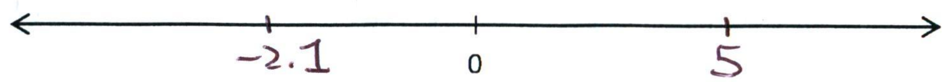
Real Numbers and Inequalities

Real numbers are $0,\ 1,\ 2006,\ 2.2124,\ \pi, \sqrt{2},$ and so on. They are points on the number line.



Q. Mark the numbers 5 and -2.1 on the number line.

Inequalities express the order of real numbers. For example

$a < b$	a is smaller than b.
$a < b < c$	a is smaller than b, and b is smaller than c.
$a \geq b$	a is greater than or equal to b.
$-10 < -3$	-10 is smaller than -3.
$-1 < 2 < 5$	2 is between -1 and 5.

Sets and intervals are commonly seen in this course.

$\{x \mid x > 3\}$	The set of all x such that x is greater than 3.
$\{x \mid -2 < x < 5\}$	The set of all x such that x is between -2 and 5.
$\{x \mid -1 \leq x \leq 2\}$	The set of all x such that x is between -1 and 2, including endpoints.

Those sets on the left hand side are called intervals.

$[a, b]$	$\{x \mid a \leq x \leq b\}$	The closed interval from a to b.
$(3,5)$	$\{x \mid 3 < x < 5\}$	The open interval from 3 to 5.
$(-1,10]$	$\{x \mid -1 < x \leq 10\}$	The left-open interval from -1 to 10.
$[-2,5)$	$\{x \mid -2 \leq x < 5\}$	The right-open interval from -2 to 5.

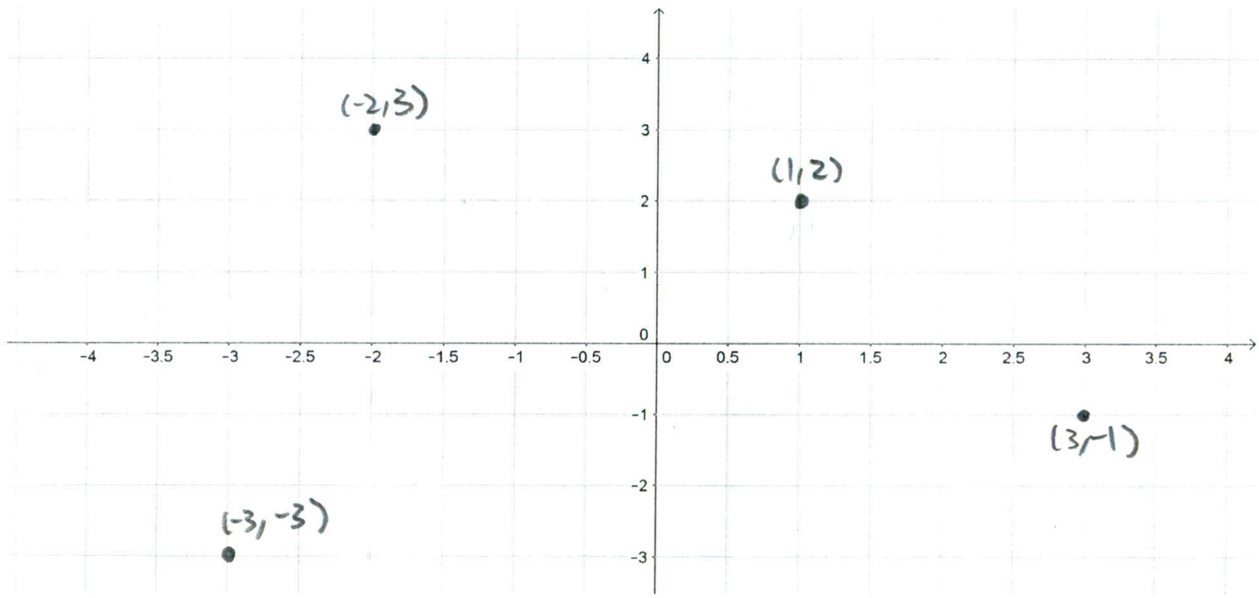
Describe intervals: http://www.mathquickeasy.com/types_of_intervals.html

In addition to finite intervals, we have infinite intervals.

$[a, \infty)$	$\{x \mid x \geq a\}$	The closed interval from a to plus infinity.
$(3\ \infty)$	$\{x \mid x > 3\}$	The open interval from 3 to plus infinity.
$(-\infty, 1]$	$\{x \mid x \leq 1\}$	The closed interval from minus infinity to 1.
$(-\infty, 10)$	$\{x \mid x < 10\}$	The open interval from $-\infty$ to 10.

**You may hear that $[a, \infty)$ is called the closed ray from a to plus infinity.

On the Cartesian plane, a point is specified uniquely by an ordered pair (x, y). x is the x-coordinate, and y is the y-coordinate of that point.



Q. Mark and label the point (1,2), (-2,3), (-3,-3) and (3,-1).

Two points (x_1, y_1) and (x_2, y_2) determine a line on the Cartesian plane. The slope of the line through (x_1, y_1) and (x_2, y_2) is

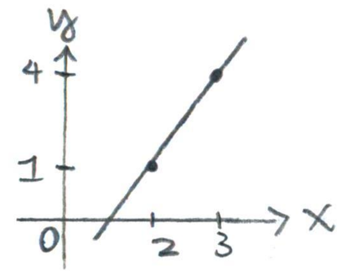
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

The terms Δx and Δy are called the change in x and the change in y. If we put $\Delta x = 1$ to the slope formula, we get $m = \Delta y$. In other words, slope is the amount that the line rises when x is increased by1.

Q. Find the slope of the line through pairs of points below. Graph the lines.

a) (2,1) and (3,4)
b) (-1,10) and (5,7)

> a) $m = \frac{4-1}{3-2} = \frac{3}{1} = 3$



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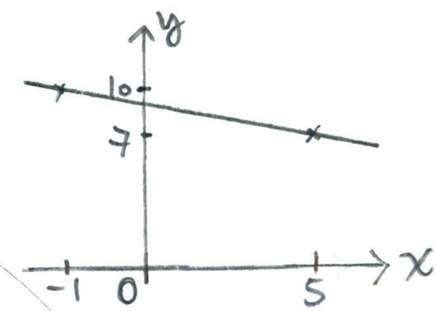
> b) $m = \frac{7-10}{5-(-1)} = \frac{-3}{6} = -\frac{1}{2}$



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Very often we have to find the equation of a line. There are two forms of a line equation.

a. Slope-intercept form of a line.
   If m is the slope of a line and b is the y-intercept of the line, then the line equation is given by $y = mx + b$

b. Point-slope form of a line.
   If m is the slope of a line and (x_1, y_1) is a point on the line, then the line equation is given by $y - y_1 = m(x - x_1)$.

A general linear equation of a line is in the form of $ax + by = c$ for some constants a, b, c. Whenever you see this equation, immediately you know it represents a line on the Cartesian plane.

We also have the following facts about line equations.

c. A horizontal line is given by the equation $y = a$.
A vertical line is given by the equation $x = b$.
The slope of a horizontal line is 0, and that of a vertical line is undefined.

d. Two lines l_1 and l_2 are
 (a) parallel to each other if they have the same slope, i.e. $m_1 = m_2$.
 (b) perpendicular to each other if $m_1 = -\frac{1}{m_2}$.

Q. Write down the linear equation of the line passing through (5,3) and (7,-1) in the slope-intercept form. >

> Find slope : $m = \frac{-1-3}{7-5} = -2$

> Let $y = -2x + b$ be the line equation

> for some constant b to be determined.

> Put $x = 5$, $y = 3$,

> $3 = -10 + b$

> $b = 10 + 3 = 13$

$\therefore$ The line equation is
$y = -2x + 13$ #

Q. Given the linear equation $2x - 3y = 12$, find the slope m and the y-intercept b. Draw the graph.

> $2x - 3y = 12$

> $-3y = 12 - 2x$

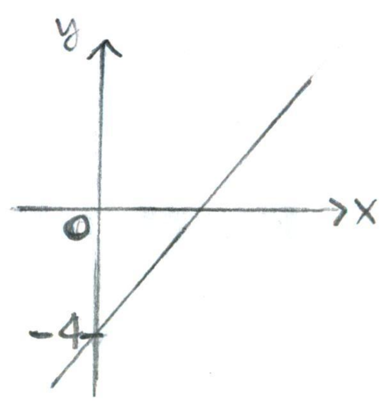
> $y = -\frac{1}{3}(12 - 2x)$

> $y = -\frac{12}{3} + \frac{2}{3}x$

> $y = -4 + \frac{2}{3}x$

> $\therefore$ $m = 2/3$

y-intercept $= -4$



Exponents

For any positive integer n, $x^n = x \cdot x \cdot x \cdots x$ for n many x on the right hand side. In the term x^n, x is called the base, and n is called the exponent or power. Eg/ $2^3 = 2 \cdot 2 \cdot 2 = 8$

Formula	Example
$x^m \cdot x^n = x^{m+n}$	$2^2 \cdot 2^3 = 4 \cdot 8 = 32 = 2^5$
$\frac{x^m}{x^n} = x^{m-n}$	$\frac{3^3}{3^1} = \frac{27}{3} = 9 = 3^2$
$(x^m)^n = x^{m \cdot n}$	$(5^2)^3 = 25^3 = 15625 = 5^6$
$(xy)^n = x^n \cdot y^n$	$(2 \cdot 3)^2 = 6^2 = 36 = 4 \cdot 9 = 2^2 \cdot 3^2$
$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{1}{2}\right)^4 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16} = \frac{1^4}{2^4}$

When $x \neq 0$, we have the below formulas for zero and negative exponents.

Formula	Example
$x^0 = 1$	$7^0 = 1$, $(\sqrt{3})^0 = 1$
$x^{-1} = \frac{1}{x}$	$5^{-1} = \frac{1}{5}$, $\left(\frac{1}{2}\right)^{-1} = \frac{1}{\left(\frac{1}{2}\right)} = 2$
$x^{-n} = \frac{1}{x^n}$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

Formulas for roots and fractional exponents are stated below.

Formula	Example
$x^{1/2} = \sqrt{x}$	$10^{1/2} = \sqrt{10}$, $49^{1/2} = \sqrt{49} = 7$
$x^{1/3} = \sqrt[3]{x}$	$8^{1/3} = \sqrt[3]{8} = 2$ $(2^3 = 8)$
$x^{1/n} = \sqrt[n]{x}$	$16^{1/4} = \sqrt[4]{16} = 2$ $(2^4 = 16)$
$x^{m/n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$	$4^{\frac{3}{2}} = (4^{\frac{1}{2}})^3 = 2^3 = 8$

$4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$

Q. Evaluate each expression.

a. $\left(\frac{3}{4}\right)^{-1}$

b. $(-27)^{2/3}$

c. $16^{3/4}$

> a. $\left(\frac{3}{4}\right)^{-1} = \frac{1}{(3/4)} = \frac{1}{(3/4)} \times \frac{4}{4} = \frac{4}{3}$ #

> b. $(-27)^{2/3} = \left((-27)^{\frac{1}{3}}\right)^{2} = (-3)^{2}$ $[(-3)^3 = -27]$

> $= (-3)\cdot(-3) = 9$ #

> c. $16^{3/4} = (16^{1/4})^{3} = 2^{3} = 8$ # $[2^4 = 16]$

> (It's not easy to find 16^3 by hand)

>

Q. Write each expression in power form ax^b for numbers a and b.

a. $\dfrac{18}{(3\sqrt[3]{x})^2}$

b. $\dfrac{10\sqrt{x}}{2\sqrt[3]{x}}$

> a. $\frac{18}{(3\sqrt[3]{x})^2} = \frac{18}{9x^{\frac{2}{3}}} = 2x^{-\frac{2}{3}}$ #

>

> b. $\frac{10\sqrt{x}}{2\sqrt[3]{x}} = \frac{10}{2}\cdot x^{\frac{1}{2}}\cdot x^{-\frac{1}{3}} = 5x^{\frac{1}{2}-\frac{1}{3}} = 5x^{\frac{1}{6}}$ #

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(not mentioned in class)

~~Q~~. Simplify the expression. $\frac{(5xy^4)^2}{25x^3y^3}$.

> $\frac{(5xy^4)^2}{25x^3y^3} = \frac{5^2x^2y^8}{25x^3y^3} = \frac{25}{25}\cdot x^{2-3}\cdot y^{8-3}$

>

> $= x^{-1}y^5$ #

>

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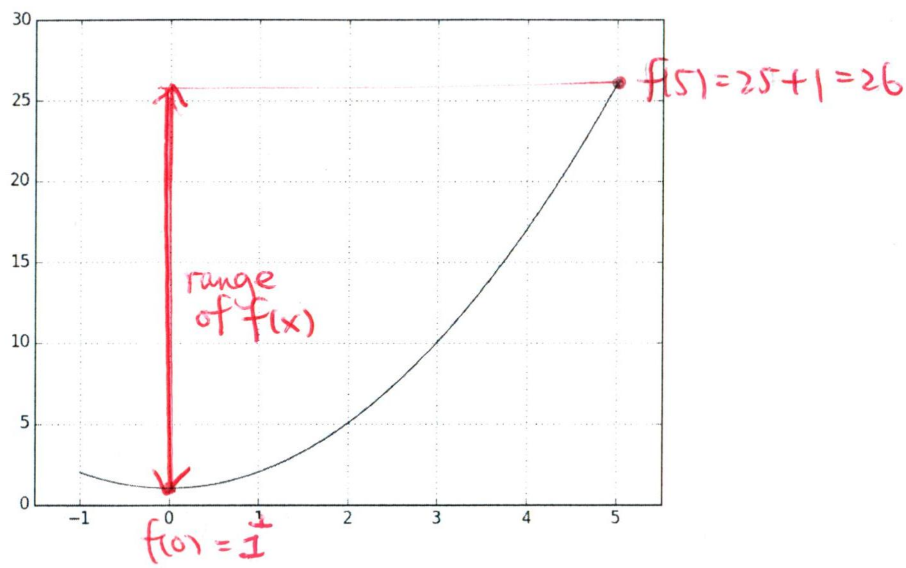
Linear and Quadratic Functions

A function f is assigning each number x to another number $f(x)$. For example, the function $f(x) = x^2$ assigns 1 to $f(1) = 1^2 = 1$, 2 to $f(2) = 2^2 = 4$ and so on.

Domain of f = the set of all values x at which $f(x)$ is well defined.

Range of f = the set of all values $f(x)$.

Q. Let $f(x) = x^2 + 1$, $-1 \leq x \leq 5$. Find the domain and range of $f(x)$.



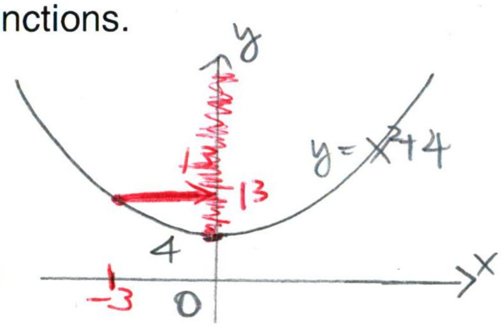
> Domain = $[-1, 5]$

> Range = [min-value, max-value] = $[1, 26]$

Q. Find the domain and range of the following functions.

a) $f(x) = x^2 + 4$.

b) $f(x) = \sqrt{x - 2}$

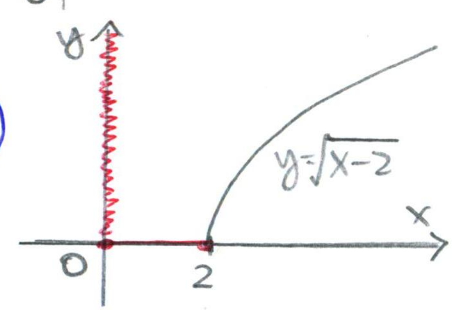


> a) Domain = $\mathbb{R} = (-\infty, \infty)$

> Range = $[4, \infty)$ ←

> b) Domain = $[2, \infty)$ $(x \geq 2)$

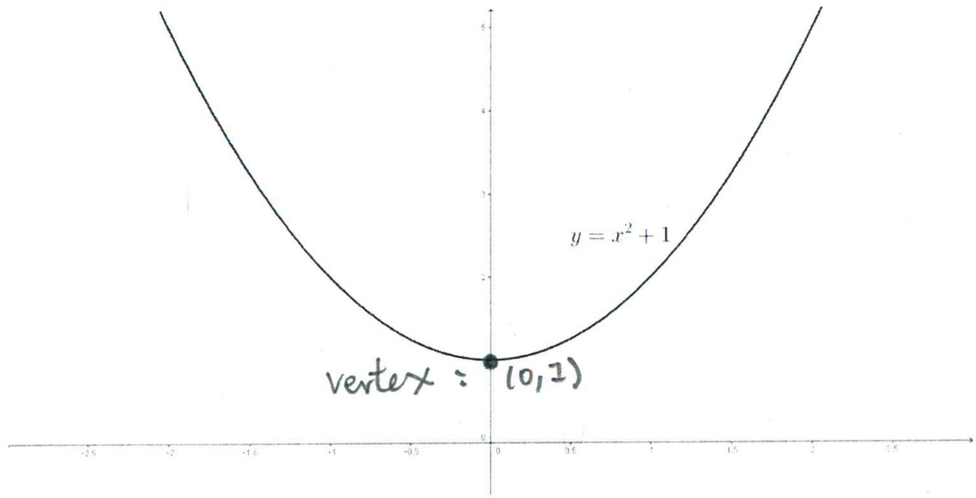
Range = $[0, \infty)$ ←

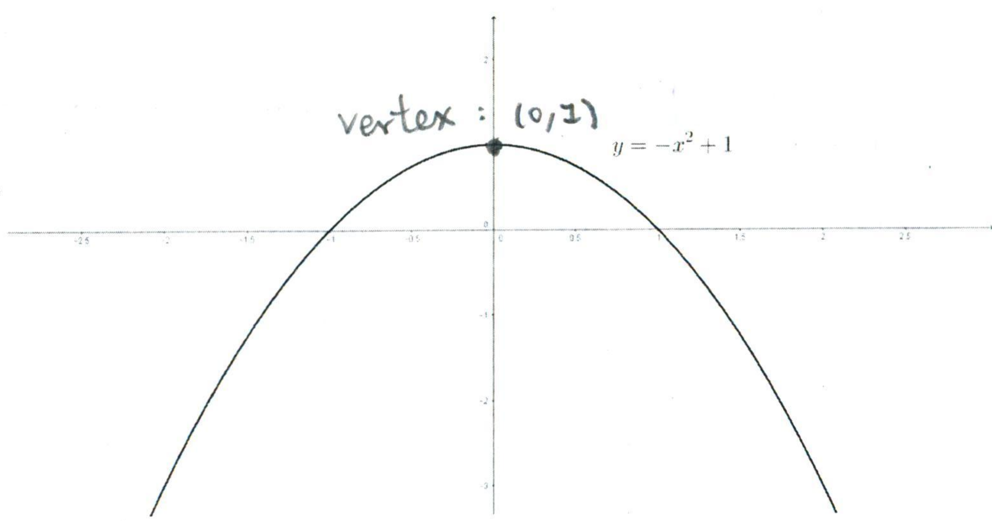


A linear function is a function in the form $f(x) = mx + c$ for some constants m and c. The graph of $f(x)$ is a line, of slope m and y-intercept c.

A quadratic function is a function in the form $f(x) = ax^2 + bx + c$ for some constants a, b, c, where $a \neq 0$. Its graph is called a parabola.

Example. The graph of $f(x) = x^2 + 1$ and $f(x) = -x^2 + 1$.





A Q. Identify the constants a, b, c in the function $f(x) = 3x^2 + 2x + 5$.

> $f(x) = 3x^2 + 2x + 5$

> $a = 3, \quad b = 2, \quad c = 5$

>

Suppose we have a quadratic function $f(x) = ax^2 + bx + c$, $a \neq 0$.

turning up along x-axis

If	then	Example
$a > 0$	$f(x)$ opens upward.	$f(x) = x^2 + 1$
$a < 0$	$f(x)$ opens downward.	$f(x) = -x^2 + 1$

turning down along x-axis

The vertex of the parabola $f(x) = ax^2 + bx + c$, has x-coordinate $x = -\frac{b}{2a}$.

Therefore, its vertex is the point $x = -\frac{b}{2a}, y = f\left(-\frac{b}{2a}\right)$.

I Q. Find the x-coordinate of the vertex of $f(x) = 3x^2 + 2x + 5$.

> $x = -\frac{b}{2a} = \frac{-2}{2 \cdot 3} = -\frac{1}{3}$

> Its vertex is the point

> $x = -\frac{1}{3}, \quad y = f(-\frac{1}{3}) = 3(-\frac{1}{3})^2 + 2(-\frac{1}{3}) + 5$

> $= (3 \cdot \frac{1}{9}) - \frac{2}{3} + 5 = \frac{14}{3}$ #

Solving a quadratic equation $ax^2 + bx + c = 0$ is exactly finding the x-intercept(s) of the quadratic function $f(x) = ax^2 + bx + c$.

There are two ways of solving a quadratic equation. By (1) factorization, or by (2) quadratic formula, which is stated as follows.

The solution to $ax^2 + bx + c = 0$ is $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ whenever $b^2 - 4ac \geq 0$

They

Q. Solve the following quadratic equations. (Use factorization or Quadratic formula)

a. $x^2 - 3x + 2 = 0$

b. $x^2 - 2x - 8 = 0$

> a. $x^2 - 3x + 2 = 0$

> $(x-1)(x-2) = 0$

> $x = 1$ or 2 #

Quadratic formula

$a = 1,\ b = -3,\ c = 2$

$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$

$= \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2}$

$= \dfrac{4}{2}$ or $\dfrac{2}{2} = 2$ or 1 #

> b. $x^2 - 2x - 8 = 0$

> $(x-4)(x+2) = 0$

> $x = 4$ or -2 #

skip

Q. Find the domain of the function $f(x) = \frac{1}{x^2 - 4x + 3}$.

> $x^2 - 4x + 3 = 0$

> $(x-3)(x-1) = 0$

> $x = 1$ or 3

> $\therefore$ Domain of $f = \{x \mid x \neq 1,\ x \neq 3\}$

> $= (-\infty, 1) \cup (1, 3) \cup (3, \infty)$

>

union sign

Example. (P.40) A company that installs automobile CD players finds that, if the company installs x CD players per day, then its costs will be $C(x) = 130x + 5000$ and its revenue will be $R(x) = -2x^2 + 500x$ (both in dollars). Find the profit function $P(x)$ of the company.

> $P(x) = R(x) - C(x)$

> $= (-2x^2 + 500x) - (130x + 5000)$

> $= -2x^2 + (500 - 130)x - 5000$

$= -2x^2 + 370 - 5000$ #

Functions: Polynomial, Rational and Exponential

Polynomials are functions in the forms

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

For example, $f(x) = 3x^3 + 4x^2 + 5x + 7$ is a polynomial of degree 3.

A rational function is a quotient of two polynomials. For example,

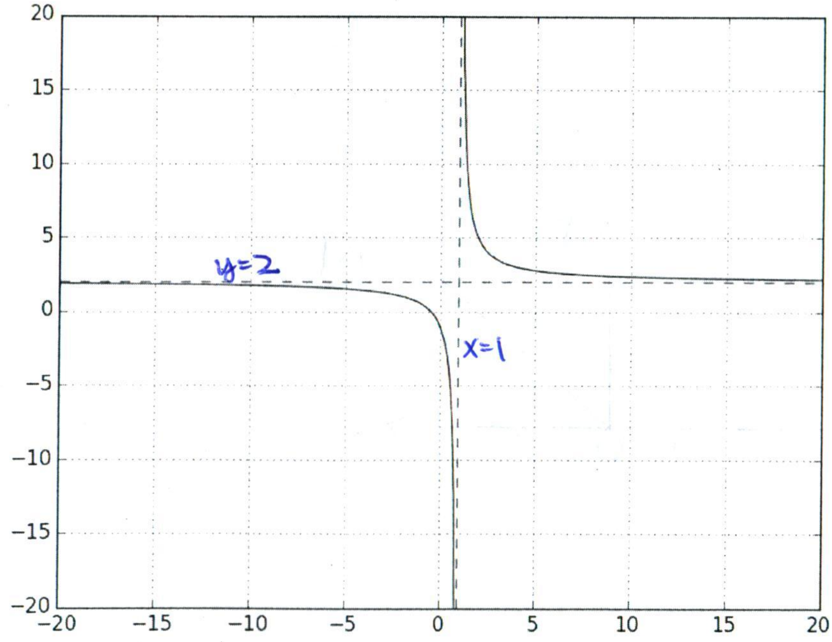
$$f(x) = \frac{2x+1}{x-1}.$$

bottom: $x-1$

The domain of a rational function is the set of numbers x at which the bottom polynomial is non-zero. Using the above rational function,

$x-1=0$
$\Leftrightarrow x=1$

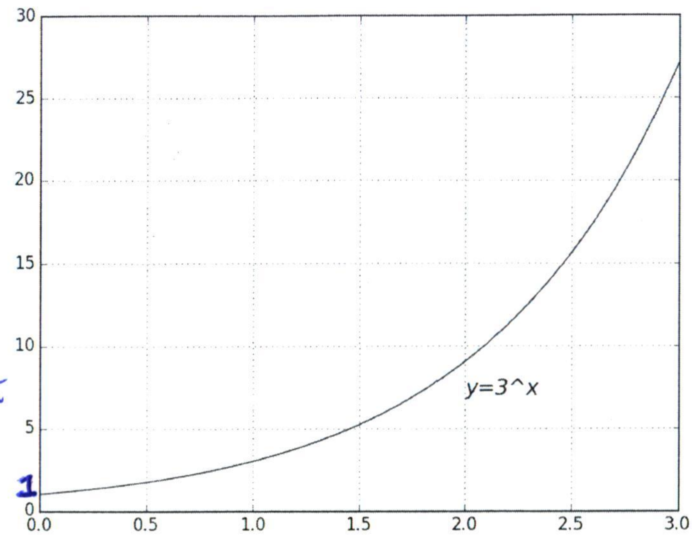
$$Domain\ of\ f(x) = \{x \mid x \neq 1\} = (-\infty, 1) \cup (1, \infty).$$



$y = \frac{2x+1}{x-1}$

An exponential function is a function in the form $f(x) = a^x$ for some non-zero number a. For example, $f(x) = 3^x$ or $f(x) = 0.5^x$.

What is the y-intercept of $f(x) = 3^x$?



goes up forever !!

The absolute value function is → It returns the positive part of a number. Eg $|-5| = 5$, $|12| = 12$

$$f(x) = |x| = \begin{cases} x & when\ x \geq 0 \\ -x & when\ x < 0 \end{cases}$$

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Q. Draw the graph of the absolute value function.

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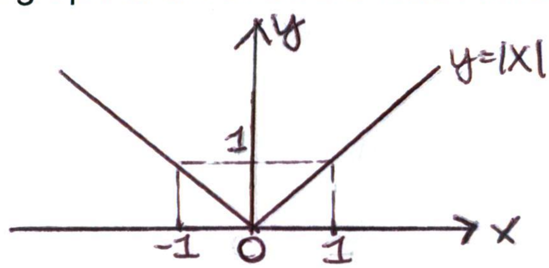

A piecewise linear function is a function defined by different ~~formulas~~ linear functions on different intervals. The function $f(x) = |x|$ above is a good example of piecewise linear functions.

I Q. Express $f(x) = |x + 1|$ in the form of a piecewise linear function.

> $|y| = y$ when $y \geq 0$

> $|y| = -y$ when $y < 0$

Put $y = x+1$. $|x+1| = x+1$ when $x+1 \geq 0 \iff x \geq -1$

$|x+1| = -(x+1)$ when $x+1 < 0 \iff x < -1$

$$\therefore \boxed{f(x) = |x+1| = \begin{cases} x+1 & when\ x \geq -1 \\ -x-1 & when\ x < -1 \end{cases}} \ \#$$

A composite function is a function defined by the composition of two functions. Suppose we have two functions $f(x) = x + 1$ and $g(x) = 2x$.

(f circle g , f of g)

The composition of f with g, (or the function $\boxed{f \circ g}$) is defined by

$$(f \circ g)(x) = f(g(x)).$$

For example, $f(g(1)) = f(2) = 2 + 1 = 3$. We are finding (1) g(1) first, and then (2) use x=g(1)=2 as an input and put it to the formula f(x)=x+1. In general,

$$(f \circ g)(x) = f(g(x)) = f(2x) = 2x + 1\,.$$

Note $(g \circ f)(x) = g(f(x)) = g(x + 1) = 2(x + 1) = 2x + 2 \neq (f \circ g)(x)\,.$
$f \circ g$ and $g \circ f$ are basically two different functions.

I Q. Let $f(x) = x^2 + 1$. Find $f(x + 1)$.

> $f(\boxed{x+1}) = (\boxed{x+1})^2 + 1$.
> Replace every x in $[f(x) = x^2+1]$ by $\boxed{x+1}$.
> $\therefore f(x+1) = (x+1)^2 + 1 = (x^2+2x+1)+1$
> $f(x+1) = x^2 + 2x + 2$ #

$(x+1)^2 = (x+1)(x+1)$
$(x+1)^2 = x^2 + 2x + 1$

The difference quotient of a function $f(x)$ is the quantity

$$\frac{f(x + h) - f(x)}{h}.$$

The top is the change in f when x increases to $x + h$. The bottom captures the change in x, which is h by default. $h = (x+h) - x$.

I Q. Let $f(x) = x^2$. Find its difference quotient and simplify your answer.

> $f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$
> $f(x) = x^2$
> $\therefore \frac{f(x+h) - f(x)}{h} = \frac{(x^2+2hx+h^2) - x^2}{h}$
> $= \frac{2hx + h^2}{h} = 2x + h$ #

Use the identity
$(a+b)^2 = a^2 + 2ab + b^2$.
Put $x = a$, $b = h$.