Optimizing Functions of Several Variables

Given a function $f(x, y)$, we first define a critical point of $f(x, y)$.

A point (a, b) in the domain of $f(x, y)$ is a critical point of $f(x, y)$ if

$$\frac{\partial f}{\partial x}(a, b) = 0 \quad \text{and} \quad \frac{\partial f}{\partial y}(a, b) = 0.$$

We still call the xy-coordinate (a, b) of a relative maximum/minimum point (a, b, c) by a relative maximum/minimum point. In particular, all relative maximum, points, relative minimum points and saddle points of $f(x, y)$ are CPs of $f(x, y)$.

The second derivative test is used for classifying a CP. When $f(x, y)$ is a function of two variables, this second derivative test is called the D-test.

D-test. Suppose (a, b) is a critical point of a function $f(x, y)$. Let

$$D = f_{xx}(a, b) \cdot f_{yy}(a, b) - \left[f_{xy}(a, b)\right]^2.$$

Then, (a, b) is a

1. relative maximum point if $D > 0$ and $f_{xx}(a, b) < 0$.

2. relative minimum point if $D > 0$ and $f_{xx}(a, b) > 0$.

3. saddle point if $D < 0$.

Finding a relative maximum/minimum point, or a saddle point of $f(x, y)$ means that we

1. find CPs of $f(x, y)$ by setting $f_x = 0$ and $f_y = 0$,
2. find the second-order partials of $f(x, y)$ and the D-value at CPs.
3. and look at f_{xx} at CPs if necessary.

Q. Find the relative extreme values of the function $f(x,y)$.

$$f(x,y) = 2x^2 + 3y^2 + 2xy + 4x - 8y$$

$f_x = 4x + 2y + 4$

$f_y = 6y + 2x - 8$

Solve

$$\begin{cases} (f_x = 0) \quad 4x + 2y + 4 = 0 \quad \text{—— ①} \\ (f_y = 0) \quad 6y + 2x - 8 = 0 \quad \text{—— ②} \end{cases}$$

Find a pair (x,y) st eqtns ① and ② are satisfied.

① − 2×② :

$$(4x + 2y + 4) - 2(6y + 2x - 8) = 0$$

$$(2y - 12y) + (4 + 16) = 0$$

$$-10y + 20 = 0$$

$$y = 2.$$

①: $4x + 2y + 4 = 0$

①: $4x + 2(2) + 4 = 0$

$$4x + 8 = 0$$

$$x = -2$$

∴ a CP is $(-2, 2)$.

$f_{xx} = 4, \quad f_{xy} = f_{yx} = 2, \quad f_{yy} = 6$

$D = f_{xx} f_{yy} - f_{xy}^2 = 24 - 4 = 20$

∴ $D(-2,2) = 20 > 0$ and $f_{xx}(-2,2) = 4 > 0$

$(-2,2)$ is a relative minimum point of $f(x,y)$.

The relative minimum value is

$$f(-2,2) = 2(4) + 3(4) + 2(-2)(2) + 4(-2) - 8(2)$$

$$= 8 + 12 - 8 - 8 - 16$$

$$= -12 \quad \#$$

Q. Find the relative extreme values of the function $f(x,y)$.

$$f(x,y)=x^3-y^2-3x+6y$$

$f_x = 3x^2-3$

$f_y = -2y+6$

Solve $\begin{cases} 3x^2-3=0 & \text{—①} \\ -2y+6=0 & \text{—②} \end{cases}$

①: $3x^2-3=0 \Rightarrow x^2=1 \Rightarrow x=1$ or -1

②: $-2y+6=0 \Rightarrow y=6/2=3$.

$\therefore$ CPs are $(1,3)$ and $(-1,3)$.

$f_{xx}=6x$, $f_{xy}=0$, $f_{yy}=-2$

$D = f_{xx}f_{yy}-f_{xy}^2 = (6x)(-2)-0 = -12x$

$D(1,3) = -12 < 0$

$(1,3)$ is a saddle point of $f(x,y)$.

$D(-1,3) = 12 > 0$, $f_{xx}(-1,3) = -6 < 0$

$(-1,3)$ is a relative maximum point of $f(x,y)$.

The relative maximum value is

$f(-1,3) = (-1)-(9)-(-3)+(18)$

$= 11$ # at $(-1,3)$.

Q. Find the relative extreme values of the function $f(x, y)$.

$$f(x, y) = -x^2 - y^3 - 6x + 3y + 4$$

> $f_x = -2x - 6$

> $f_y = -3y^2 + 3$

> Solve $\begin{cases} -2x-6=0 & \text{——①} \\ -3y^2+3=0 & \text{——②} \end{cases}$

>

> ①: $-2x-6=0 \Rightarrow -2x=6 \Rightarrow x=-3$

> ②: $-3y^2+3=0 \Rightarrow y^2=1 \Rightarrow y=1$ or -1

> $\therefore$ CPs are $(-3, 1)$ and $(-3, -1)$.

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> $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -6y$

> $D = f_{xx} \cdot f_{yy} - f_{xy}^2 = 12y$.

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> $D(-3, 1) = 12 > 0$, $f_{xx}(-3, 1) = -2 < 0$

> $(-3, 1)$ is a relative maximum point of $f(x, y)$.

> The relative maximum value at $(-3, 1)$ is

> $f(-3, 1) = -(9) - (1) - (-18) + 3 + 4$

$= 15$

>

> $D(-3, -1) = -12 < 0$

> $(-3, -1)$ is a saddle point of $f(x, y)$. #

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Q. [Company's Profit, textbook P.496]

A company manufactures two products. The price function for product A is

$$p = 12 - \frac{1}{2}x, \quad for\ 0 \le x \le 24.$$

The price function for product B is

$$q = 20 - y, \quad for\ 0 \le y \le 20.$$

Both p and q are in thousands of dollars. x are y are the amounts of products A and B produced, respectively. The cost function is

$$C(x, y) = 9x + 16y - xy + 7$$

in thousands of dollars. Find the quantities and the prices of the two products that maximize profit. Also find the maximum profit.

> Profit function

> $P(x) = [(12-\frac{1}{2}x)x + (20-y)y] - (9x + 16y - xy + 7)$

> $P(x) = -\frac{1}{2}x^2 - y^2 + xy + 3x + 4y - 7$

> Maximize $P(x)$.

> $P_x = -x + y + 3$

> $P_y = -2y + x + 4$

> Solve $\begin{cases} (P_x = 0) & -x + y + 3 = 0 \text{ —— } ① \\ (P_y = 0) & -2y + x + 4 = 0 \text{ —— } ② \end{cases}$

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> $2 \times ① + ②$:

> $2(-x+y+3) + (-2y + x + 4) = 0$

> $-2x + \cancel{2y} + 6 - \cancel{2y} + x + 4 = 0$

> $-x + 10 = 0$

$x = 10$

>

> ① : $-x + y + 3 = 0$

$-10 + y + 3 = 0$

$y - 7 = 0$

$y = 7$

$\therefore$ The only CP is $(10, 7)$.

$P_{xx} = -1$, $P_{xy} = 1$, $P_{yy} = -2$

$D = P_{xx} P_{yy} - P_{xy}^2 = 2 - 1 = 1 > 0$

$D(10,7) = 1 > 0$ $\quad P_{xx}(10,7) = -1 < 0$

$\therefore$ $P(x)$ is maximized at $(10, 7)$.

When $x = 10$, $y = 7$,

$p = 12 - \frac{1}{2}(10) = 7$ in thousand dollars

$q = 20 - (7) = 13$ in thousand dollars.

$P(10,7) = -\frac{1}{2}(10^2) - (7^2) + (70) + 3(10) + 4(7) - 7$

$= 22$. in thousand dollars.

Therefore, the profit is maximized when 10 units of product A are produced, sold for \$7,000 each, and 7 units of product B are produced, sold for \$13,000 each.

The maximized profit is \$22,000. #

Q. [Drug Dosage, textbook P.497]

In a laboratory test, the combined antibiotic effect of x milligrams of medicine A and y milligrams of medicine B is given by the function

$$f(x, y) = xy - x^2 - y^2 + 11x - 4y + 120\,.$$

Here $0 \le x \le 55$ and $0 \le y \le 60$. Find the amounts of the two medicines that maximize the antibiotic effect.

\> maximize $f(x,y)$

\> $f_x = y - 2x + 11$

\> $f_y = x - 2y - 4$

\> Solve $\begin{cases} (f_x = 0) & y - 2x + 11 = 0 \text{ —— ①} \\ (f_y = 0) & x - 2y - 4 = 0 \text{ —— ②} \end{cases}$

\>

\> ① + 2×② :

$(y - 2x + 11) + 2(x - 2y - 4) = 0$

\> $(y - 4y) + (11 - 8) = 0$

\> $-3y + 3 = 0$

$y = 1$

\>

\> ① : $y - 2x + 11 = 0$

$1 - 2x + 11 = 0$

\> $-2x = -12$

\> $x = 6$

$\therefore$ The only CP is $(6, 1)$.

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\> $f_{xx} = -2, \quad f_{xy} = 1, \quad f_{yy} = -2$

\> $D = f_{xx} f_{yy} - f_{xy}^2 = 4 - 1 = 3$

$D(6,1) = 3 > 0$ and $f_{xx}(6,1) = -2 < 0$

\> $f(x,y)$ is maximized at $(6, 1)$.

\>

Therefore, the antibiotic effect is maximized when 6 milligrams of medicine A and 1 milligram of medicine B are used. #