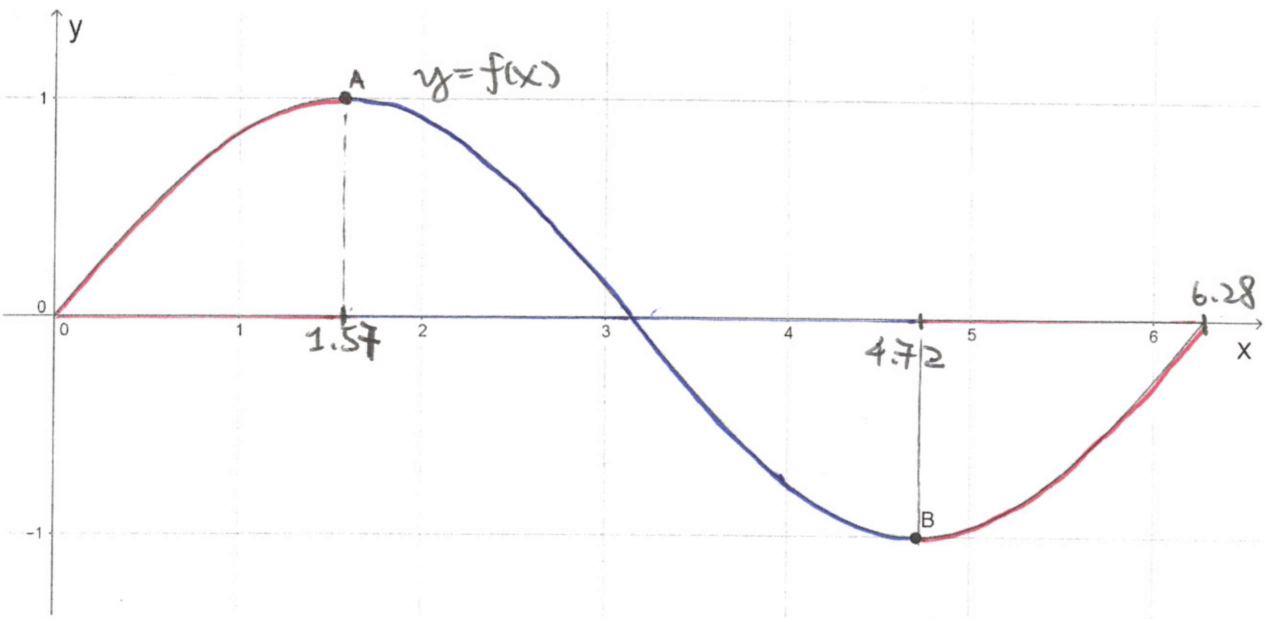
Graphing using the First Derivative

The first derivatives tell us whether a function is increasing or decreasing in a region of x-values.

Suppose (a, b) is an interval contained in the domain of a function $f(x)$. We say $f(x)$ is increasing on (a, b) if the value $f(x)$ keeps rising as x goes from a to b. We say $f(x)$ is decreasing on (a, b) if the value $f(x)$ keeps falling as x goes from a to b.

> Given a function $f(x)$ and an open interval (a, b) inside the domain of $f(x)$.
>
> 1. $f(x)$ is increasing on (a, b) if $f'(x) > 0$ for all x in (a, b).
>
> 2. $f(x)$ is decreasing on (a, b) if $f'(x) < 0$ for all x in (a, b).

In this example, $f(x)$ is increasing on the interval $(0\,, 1.57)$ and on the interval $(4.71\,, 6.28)$. It is decreasing on the interval $(1.57\,, 4.72)$.



The point $A(1.57\,, 1)$ on the graph of $f(x)$ is the highest point. Before $x = 1.57$, $f(x)$ is increasing, and after $x = 1.57$, $f(x)$ is decreasing. We call this point A a relative maximum point of $f(x)$.

We have the following more formal definition.

Suppose $f(x)$ is a function, and a number $x = c$ is in its domain.

1. $f(x)$ has a relative maximum point at c if $f(c) \geq f(x)$ for any x close to c.

2. $f(x)$ has a relative minimum point at c if $f(c) \leq f(x)$ for any x close to c.

In the textbook (P.164), there are two exact statements about relative maximum and relative minimum. But there we say $f(x)$ has a relative maximum/minimum **value**.

Let's use the above example to illustrate everything.

1. The point $A(1.57\,,1)$ is called a relative maximum point of $f(x)$.

2. The number c is 1.57 here. $f(x)$ has a relative maximum point at 1.57.

3. We can also say that $f(x)$ has a relative maximum value at 1.57.

4. Sometimes, relative maximum point refers to the x-value 1.57. [Wiki]

5. The relative maximum value of $f(x)$ at 1.57 refers to the y-value of the relative maximum point. $A(1.57\,,1)$. It is the value 1 here.

By this logic, $f(x)$ has a relative minimum point at $x = 4.71$, The relative minimum value at 4.71 is $y = -1$.

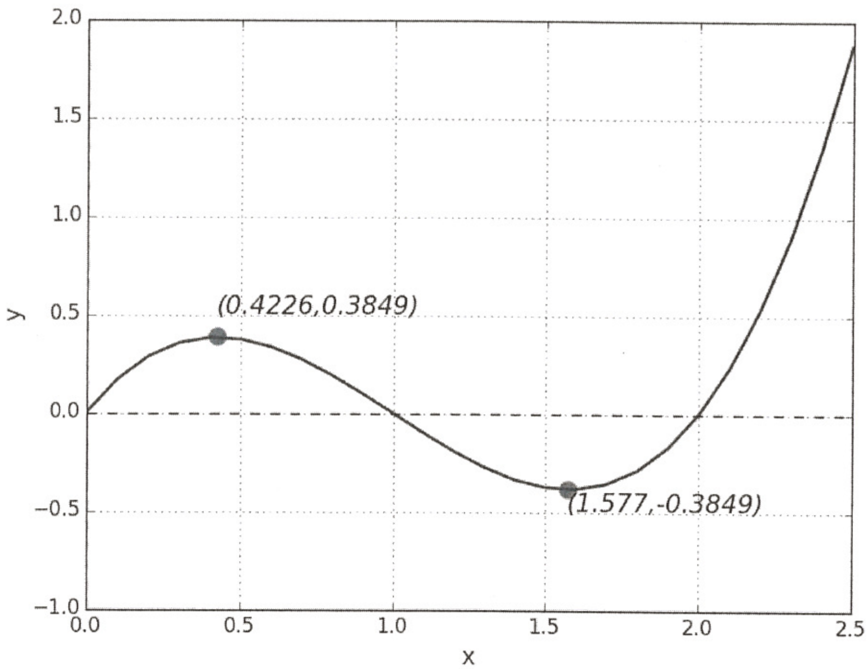
A relative maximum point is like the peak of a mountain. This mountain can be just a small hill within a mountain range, and its peak is still a relative maximum point.

The condition "$f(c) \geq f(x)$ for any x close to c" means that, this point is at least higher than any surrounding points, but not necessarily one of the highest points along the graph of $f(x)$.

Similarly, a relative minimum point is the bottom of a valley. It needs not be the lowest point on the graph of $f(x)$.

6. A relative maximum/minimum point is also named as *a relative extreme point*, or a *relative extremum* (plural: *extrema*), when not specifying whether it is a relative maximum or a relative minimum.

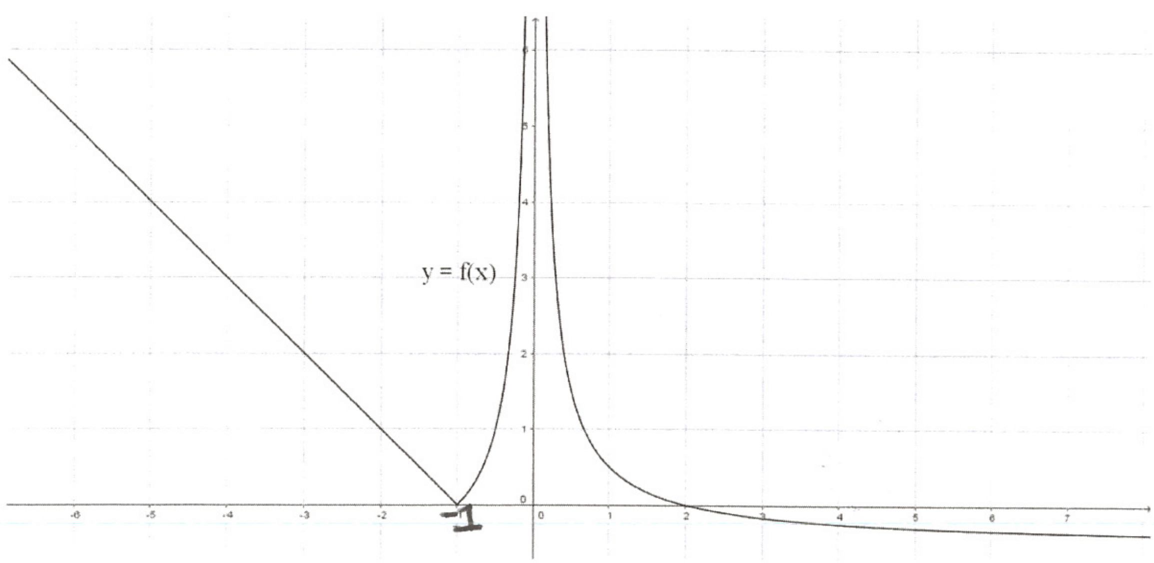
Below is the graph of $f(x) = x^3 - 3x^2 + 2x,\ 0 \le x \le 2.5$.



Q. Write down the x-value(s) of the relative maximum and relative minimum points of $f(x)$.

Relative maximum: X= 0.4226 Relative minimum: : X= 1.577

Q. Find the x-value(s) of the relative extrema of $f(x)$ below.



Relative minimum : X = -1

Relative maximum : None

** $f(x)$ ~~does~~/doesn't have to be differentiable at a relative maximum point or a relative minimum point.

A critical number of $f(x)$ is a number x in the domain of $f(x)$ at which

1. $f'(x) = 0$ or 2. $f'(x)$ is undefined. [$f(x)$ is well-defined.]

In particular, all relative maximum points and relative minimum points are critical numbers. The converse is not true. A critical number doesn't have to be a relative maximum or relative minimum.

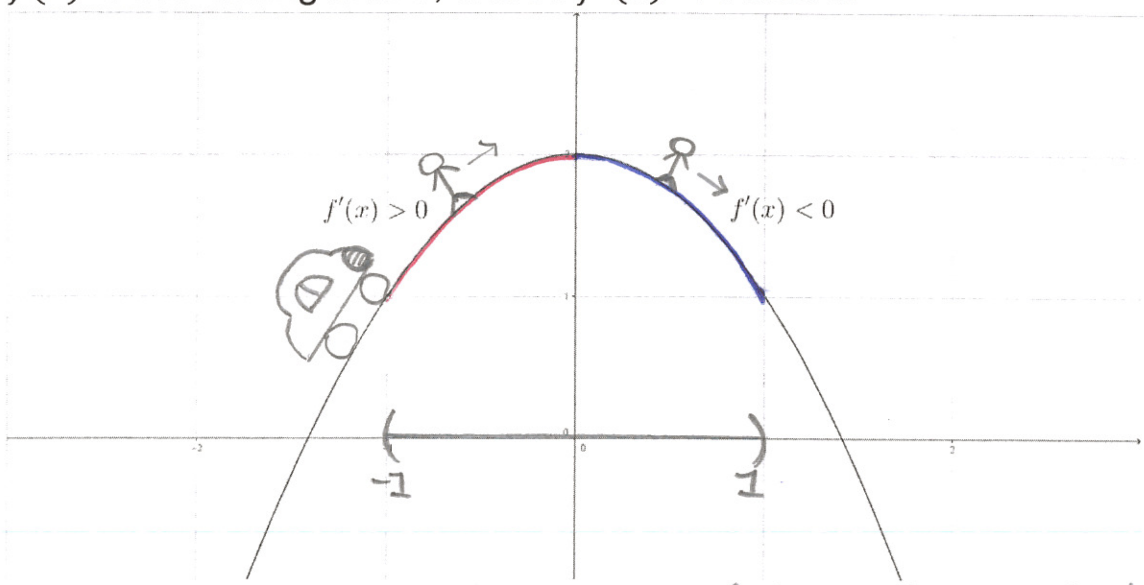
Suppose c is a critical number of $f(x)$. How can we know that c is a relative maximum, a relative minimum, or none of them?

First-Derivative Test. Suppose $f(x)$ has a critical number (CN) c.

1. If $f'(x) > 0$ just before c and $f'(x) < 0$ just after c, then c is a relative maximum point of $f(x)$.

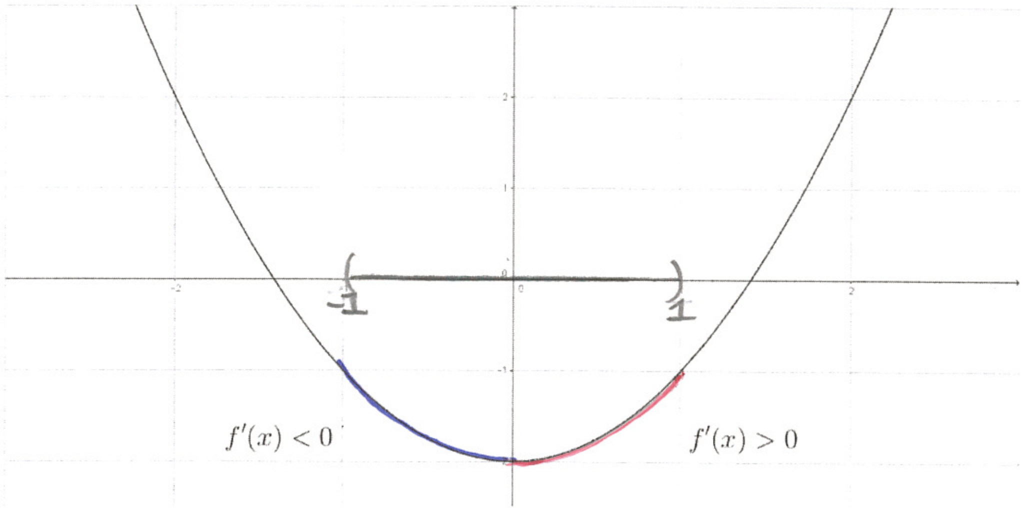
2. If $f'(x) < 0$ just before c and $f'(x) > 0$ just after c, then c is a relative minimum point of $f(x)$.

A relative maximum is the peak of a mountain. We have to climb up the mountain to reach the top. $f(x)$ is increasing before the number c, meaning that $f'(x) > 0$ before c. After we have reached the top, we have to go down. $f(x)$ is decreasing after c, and so $f'(x) < 0$ after c.



looking at a small interval surrounding c

As long as $f'(x)$ changes its sign from positive to negative across a critical number c over an open interval containing c, (being the interval (-1,1) in the above graph and $c = 0$), we are sure that c is a relative maximum.



Now, giving a function $f(x)$, we try to (1) find all its critical numbers, and (2) identify relative maximum or relative minimum points from those CNs. This process is done by the method of drawing a **sign diagram**.

Q. Find the critical numbers of the function $f(x)$.

$$f(x) = x^3 + 6x^2 - 36x - 60$$

Use sign diagrams for the derivatives to find all open intervals of increase and decrease.

The solution is as follows.

1. Find the critical numbers of $f(x)$.

$$f'(x) = 3x^2 + 12x - 36 = 3(x + 6)(x - 2)$$

Solve $f'(x) = 0$.

$3x^2+12x-36=0$

$3(x+6)(x-2)=0$

$x=-6$ or 2.

Sign diagram

x	$(-\infty, 6)$	-6	$(-6, 2)$	2	$(2, \infty)$
$f'(x)$		0		0	
$f(x)$					

$\therefore$ CNs are $x = 2$ and $x = -6$.

2. Make a sign diagram for the derivative.

x	$(-\infty\,, -6)$	-6	$(-6\,, 2)$	2	$(2\,, \infty)$
$f'(x)$	+ $x=-7$	0	− $x=0$	0	+ $x=3$
$f(x)$	increasing		decreasing		increasing

[Blue] what we know from the below "test-point-method".

We have to determine the signs of $f'(x)$ on intervals $(-\infty, -6)$, $(-6\,, 2)$ and $(2\,, \infty)$ respectively. (Sign means positive + or negative −) Here we use **test points** to find the correct signs.

> Suppose (a, b) is in the domain of a function $f(x)$. If there is no critical number in the open interval (a, b), then $f'(x)$ remains positive or remains negative all over the interval (a, b).

On $(-\infty\,, -6)$, we can pick $x = -7$.

$$f'(-7) = 3(-7+6)(-7-2) = 3 \cdot (-1) \cdot (-9) = 27$$

$f'(-7) > 0$ means that $f(x)$ is increasing on $(-\infty\,, -6)$.

>

> On $(-6, 2)$, pick $x = 0$.

> $x=0$, $f'(0) = 3(0+6)(0-2) = 3 \cdot 6 \cdot (-2) = -36 < 0$

> $\therefore f(x)$ is decreasing on $(-6, 2)$.

>

> On $(2, \infty)$, pick $x = 3$

> $x=3$, $f'(3) = 3(3+6)(3-2) = 3 \cdot 9 \cdot 1 = 27 > 0$

$\therefore f(x)$ is increasing on $(2, \infty)$.

> At the CN $x = -6$,
> $f' > 0$ before -6 and $f' < 0$ after -6
> $\therefore x = -6$ is a relative maximum.
>
> At $x = 2$, $f' < 0$ before 2 and $f' > 0$ after 2
> $\therefore x = 2$ is a relative minimum.
>

Therefore, we have the following

Interval(s) of increase: $(-\infty, -6)$, $(2, \infty)$

Interval(s) of decrease: $(-6, 2)$

Critical numbers: $x = -6$, $x = 2$

Relative maximum points (x-values): $x = -6$

Relative minimum points (x-values): $x = 2$

Q. Find the critical numbers of the function $f(x)$.

$$f(x) = 3x^4 - 8x^3 + 6x^2$$

Use sign diagrams for the derivatives to find all open intervals of increase and decrease.

Find CNs

> $f'(x) = \frac{d}{dx}(3x^4 - 8x^3 + 6x^2) = 12x^3 - 24x^2 + 12x$
> $f'(x) = 0 \Leftrightarrow 12x^3 - 24x^2 + 12x = 0$
> $12x(x^2 - 2x + 1) = 0$
> $12x(x-1)^2 = 0$
> $x = 0$ or 1
> $\therefore$ CNs are 0 and 1.

Make sign diagram

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, \infty)$
$f'(x)$	$-$ $\quad x=-1$	0	$+$ $\quad x=\frac{1}{2}$	0	$+$ $\quad x=2$
$f(x)$	decreasing		increasing	X	increasing

Put test points to $f'(x)$

$$f'(-1) = 12(-1)^3 - 24(-1)^2 + 12(-1)$$
$$= -12 - 24 - 12 = -48 < 0$$

$$f'(\tfrac{1}{2}) = 12(\tfrac{1}{2})^3 - 24(\tfrac{1}{2})^2 + 12(\tfrac{1}{2})$$
$$= \frac{12}{8} - \frac{24}{4} + 6 = \frac{12}{8} = 1.5 > 0$$

$$f'(2) = 12(2)^3 - 24(2)^2 + 12(2)$$
$$= 24 > 0$$

$\therefore$ $f(x)$ has a relative minimum at $x = 0$

$f(x)$ doesn't have a relative maximum

exclude $x=1$ here!

Intervals of increase : $(0, 1)$ and $(1, \infty)$

Intervals of decrease : $(-\infty, 0)$

#

In this course we are graphing rational functions. We have to first find the critical numbers (relative maximum and relative minimum), and intervals of increase or decrease, which is mentioned above.

Another important part to graph a rational function, is to find out vertical asymptotes and horizontal asymptotes of a rational function.

A function $f(x)$ has a horizontal asymptote (HA) $y = c$ if

$$\lim_{x \to \infty} f(x) = c \quad or \quad \lim_{x \to -\infty} f(x) = c.$$

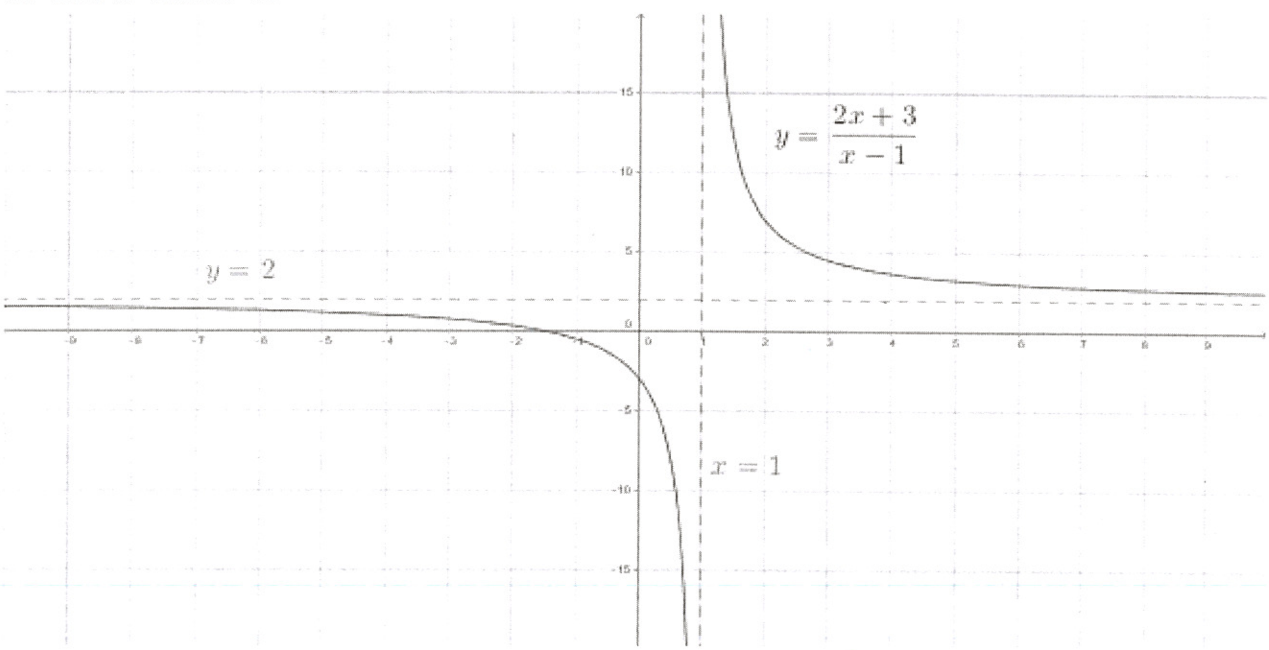
A function can have two different horizontal asymptotes at most. However, a rational function can have ZERO or ONE asymptote.

The word "asymptote" above refers to the line $y = c$ on the xy-plane. It is horizontal in nature. Sometimes this horizontal asymptote can refer to the y-value c.

A rational function $\frac{p(x)}{q(x)}$ has a vertical asymptote (VA) $x = c$ if

$$p(c) \neq 0 \quad and \quad q(c) = 0\,.$$

Again, this vertical asymptote $x = c$ can refer to the vertical line on xy-plane or the x-value c.



In general, function $f(x)$ has a vertical asymptote $x = c$ if

(1) $\lim_{x\to c^+} f(x) = \pm\infty$, and (2) $\lim_{x\to c^-} f(x) = \pm\infty$.

Since our intention is to work on rational functions, we always stick to the framed box definition of a vertical asymptote.

Q. Find the horizontal asymptote of $f(x) = \frac{4x}{x-2}$.

> $\lim_{x\to\infty} \frac{4x}{x-2} = \lim_{x\to\infty} \frac{4(\frac{x}{x})}{(\frac{x}{x})-(\frac{2}{x})} = \lim_{x\to\infty} \frac{4}{1-(\frac{2}{x})} = \frac{4}{1-0} = 4.$
>
> $\lim_{x\to-\infty} \frac{4x}{x-2} = \lim_{x\to-\infty} \frac{4}{1-(\frac{2}{x})} = \frac{4}{1-0} = 4$
>
> $\therefore$ H.A. is $y=4$ #

Q. Find the horizontal asymptote of $f(x) = \frac{11x}{x^2-25}$.

> $\lim_{x\to\infty} \frac{11x}{x^2-25} = \lim_{x\to\infty} \frac{(\frac{11x}{x^2})}{(\frac{x^2}{x^2})-\frac{25}{x^2}} = \lim_{x\to\infty} \frac{(\frac{11}{x})}{1-(\frac{25}{x^2})} = \frac{0}{1-0} = 0$
>
> $\lim_{x\to-\infty} \frac{11x}{x^2-25} = \lim_{x\to-\infty} \frac{(\frac{11}{x})}{1-\frac{25}{x^2}} = \frac{0}{1-0} = 0$
>
> $\therefore$ H.A. is $y=0$.

Q. Find all the vertical asymptotes of $f(x)$.

$$f(x) = \frac{12}{x^2 - 2x - 3}$$

> $p(x) = 12$, $q(x) = x^2-2x-3$.
>
> Solve $x^2-2x-3 = 0$
>
> $(x-3)(x+1) = 0$
>
> $x = 3$ or -1
>
> Check $p(3) = 12 \neq 0$
>
> $p(-1) = 12 \neq 0$

$\therefore$ V.A.s are $x=3$ and $x=-1$ #