Limits and Continuity

The notation $x \to 3$ means x approaches 3. It describes the process that the variable x is getting closer and closer to 3. For example,

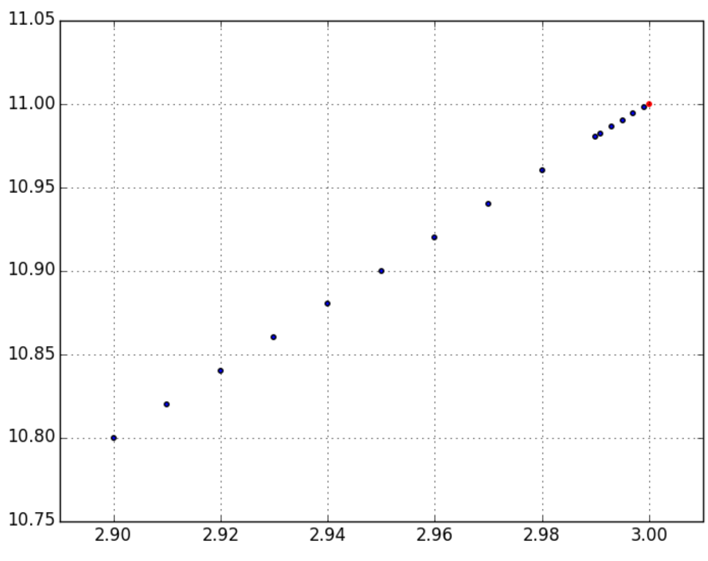
$$x = 2.9, \qquad x = 2.99, \qquad x = 2.999, \qquad x = 2.9999, \qquad x = 2.9999,$$

and so on. Let $f(x)$ be a function of x. In this section, we are interested in finding this limit as x approaches 3 of $f(x)$,

$$\lim_{x \to 3} f(x) .$$

It means the value that the function $f(x)$ goes to when x approaches 3. Take $f(x) = 2x + 5$ as an example. A very simple method to get this limit is to draw a table.

x	$f(x) = 2x + 5$
2.9	10.8
2.99	10.98
2.999	10.998
…	…



The blue dots represent $f(2.9) = 10.80$, $f(2.99) = 10.98$, $f(2.999)$ and other values of $f(x)$ at the corresponding x. We can see that the blue dots are approaching the red dot. The red dot is the point $(3,11)$. So we say

$$\lim_{x \to 3}(2x + 5) = 11\,.$$

The function $f(x) = 2x + 5$ approaches 11 when x approaches 3. In a more rigorous context, the concept of *limit* is defined as follows.

The statement

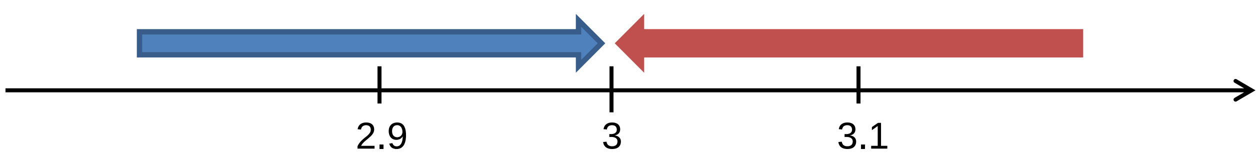
$$\lim_{x \to c} f(x) = L$$

means that the value $f(x)$ can be arbitrarily close to a number L when x is sufficiently close to c.

Yet, basically, we are concerning where the $f(x)$ goes to, when x approaches some number c, and then call that resulting number, the limit as x approaches c of $f(x)$, written as $\lim_{x \to c} f(x)$.

1. One-sided limit

Back to the notation $x \to 3$, there are two different ways for x to get close to the number 3. x can approach 3 from the right or from the left.



The notation $x \to 3^+$ means that x approaches 3 from the right (red arrow). For example, $x = 3.1$, $x = 3.01$, $x = 3.001$, $x = 3.0001$ and so on. These x's are (i) getting close to 3 and (2) all **larger** than 3. So we say these x's are approaching 3 from the right.

On the other hand, $x \to 3^-$ means that x approaches 3 from the left (blue arrow). For example, $x = 2.9$, $x = 2.99$, $x = 2.999$, $x = 2.9999$ and so on.

These x's are (i) getting close to 3 and (2) all **smaller** than 3. So we say these x's are approaching 3 from the left.

The limit as x approaches c from the right of $f(x)$, $(x > c)$

$$\lim_{x \to c^+} f(x)$$

is the number where $f(x)$ goes to, when x approaches c from the right.

The limit as x approaches c from the left of $f(x)$, $(x < c)$

$$\lim_{x \to c^-} f(x)$$

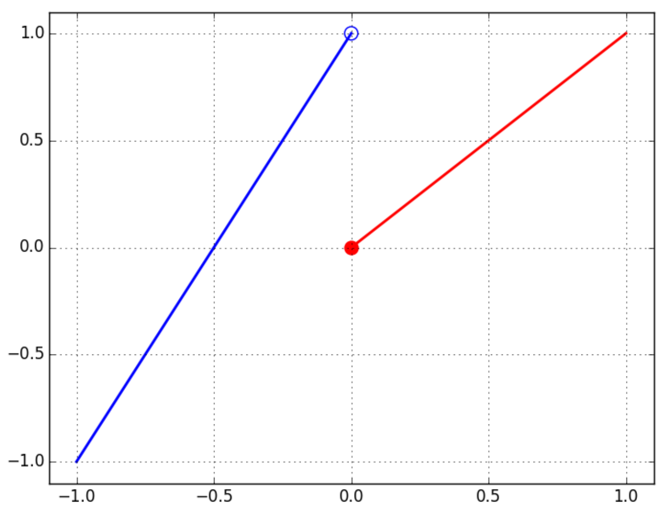
is the number where $f(x)$ goes to, when x approaches c from the left.

$\lim_{x \to c^+} f(x)$ is called the right-hand limit, and $\lim_{x \to c^-} f(x)$ is called the left-hand limit. They are one-sided limits. We call $\lim_{x \to c} f(x)$ the two-sided limits.

Q. Let $f(x)$ be a piecewise linear function on the interval $[-1,1]$.

$$f(x) = \begin{cases} x & when \quad 0 \leq x \leq 1 \\ 2x+1 & when \quad -1 \leq x < 0 \end{cases}$$

Find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$ from its graph.



$$\lim_{x\to 0^+} f(x) = \underline{\hspace{3cm}} \quad \text{and} \quad \lim_{x\to 0^-} f(x) = \underline{\hspace{3cm}}$$

$\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$. We say that $\lim_{x\to 0} f(x)$ does not exist (DNE) whenever these two one-sided limits don't agree each other. We have the following facts.

If both $\lim_{x\to c^+} f(x)$ and $\lim_{x\to c^-} f(x)$ exist, and

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L$$

For some number L, then

$$\lim_{x\to c} f(x) \text{ exists and } \lim_{x\to c} f(x) = L\,.$$

If (i) any of the one-sided limit $\lim_{x\to c^+} f(x)$ or $\lim_{x\to c^-} f(x)$ doesn't exist, or (ii) both of them exist but not equal each other, then

$$\lim_{x\to c} f(x) \text{ does not exist}\,.$$

2. Finding limits by substitution

In the previous example, $\lim_{x\to 3}(2x+5) = 11$. This number 11, is exactly

$$f(3) = 2(3) + 5 = 11.$$

So we are putting $x = 3$ into $f(x) = 2x + 5$ to get this limit $\lim_{x\to 3}(2x+5)$. In many situations, finding a limit as x approaches c is as easy as putting x to be the number c.

(A) When $f(x)$ is a polynomial, and c is a real number, we have

$$lim_{x\to c} f(x) = f(c)\,.$$

For example, $f(x) = 3x^2 + 5x + 6$. Then,

$$lim_{x\to 1}(3x^2 + 5x + 6) = 3(1)^2 + 5(1) + 6 = 3 + 5 + 6 = 14\,.$$

Note that for any constant a, the constant function $f(x) = a$ suits the above criterion. So we have $\lim_{x\to c} a = a$, whatever the number c is.

(B) When $f(x) = \frac{p(x)}{q(x)}$ is a rational function, and c is a real number such that the bottom polynomial $q(x)$ is non-zero at $x = c$. That is, $q(c) \neq 0$. Then, $lim_{x\to c} f(x) = f(c)$.

For example, $f(x) = \frac{3x+6}{5x+8}$. Then, $5(0) + 8 = 8 \neq 0$, and we have

$$\lim_{x\to 0} \frac{3x+6}{5x+8} = \frac{3(0)+6}{5(0)+8} = \frac{6}{8} = \frac{3}{4}.$$

(C) When $f(x) = \sqrt{x}$ is the square root function, and c is a non-negative number ($c \geq 0$), then, $lim_{x\to c} \sqrt{x} = \sqrt{c}$.

This method also works for functions like $f(x) = \sqrt{x+a}$, but then we require $c + a \geq 0$ instead. Equivalently, $c \geq -a$. For example, $a = 1$,

$$\lim_{x\to -0.5} \sqrt{x+1} = \sqrt{(-0.5)+1} = \sqrt{0.5} = 0.7071068\cdots.$$

3. Rules of Limits

We have seen these two rules,

$$\boxed{\lim_{x\to c} a = a \qquad \text{and} \qquad \lim_{x\to c} x^n = c^n \ .}$$

There are four rules of limit concerning addition, subtraction, multiplication and division between functions.

When both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist,

$$\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) \ ,$$

$$\lim_{x\to c}[f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) \ ,$$

$$\lim_{x\to c}[f(x)\cdot g(x)] = [\lim_{x\to c} f(x)]\cdot[\lim_{x\to c} g(x)] \ .$$

When both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist, and $\lim_{x\to c} g(x) \neq 0$,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} \ .$$

Q. Find the following limits.

(a) $\lim_{\mathrm{x}\to 3}\frac{8x+4}{7x-2}$

(b) $\lim_{x\to 0}[\ \sqrt{x+1}\ + (5x^2+4x+8)\ ]$

(c) $\lim_{x\to 1}[\ (\sqrt{x})\cdot(x^3+x^2+x+1)\ ]$

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4. Finding limits by factorization.

Some limits cannot be found by a direct substitution. However, we can still find these limits by simplifying the expression. For example, let

$$f(x) = \frac{x^2 - 4}{x - 2}.$$

We are finding $\lim_{x \to 2} f(x)$. By direct substitution,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \frac{(2)^2 - 4}{2 - 2} = \frac{0}{0},$$

which is not any number. So, direct substitution doesn’t work. Note

$$x^2 - 4 = x^2 - 2^2 = (x - 2)(x + 2)$$

by the identity $(a + b)(a - b) = a^2 - b^2$. Back to the limit,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4.$$

This limit turns out to be 4.

Q. Find the $\lim_{x \to 4} \frac{x^2 - 4x}{x - 4}$.

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Q. Find the following limit.

$$\lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5}$$

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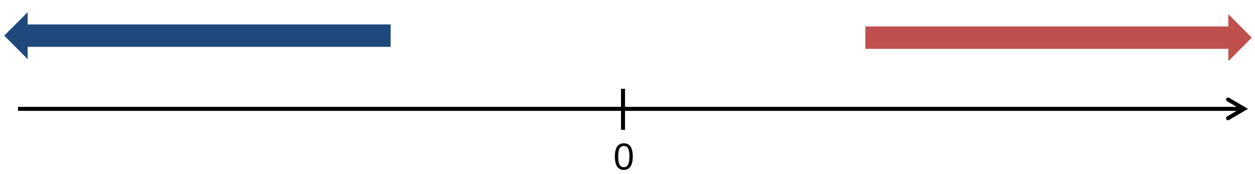
5. Limits involving infinity



Infinity ∞ is the concept that it is larger than any number on the number line. But the infinity itself is not a number. Similarly, $-\infty$ is smaller than any number on the number line. In some context, ∞ is represented by $+\infty$ to emphasize the positive sign.

The notation $x \to \infty$ means x approaches infinity. That is, x is getting larger than any real numbers (red arrow). For example,

$$x = 10, \qquad x = 100, \qquad x = 1{,}000, \qquad x = 10{,}000, \qquad x = 100{,}000, \quad \cdots$$

and so on. In a similar fashion, the notation $x \to -\infty$ means that x is getting smaller than any real number (blue arrow). For example,

$$x = -10, \qquad x = -100, \qquad x = -1{,}000, \qquad x = -10{,}000, \quad \cdots$$

and so on.

The limit as x approaches ∞ of $f(x)$,

$$\lim_{x \to \infty} f(x)$$

is the number where $f(x)$ goes to, when x is arbitrarily large.

The limit as x approaches $-\infty$ of $f(x)$,

$$\lim_{x \to -\infty} f(x)$$

is the number where $f(x)$ goes to, when x is arbitrarily small.

Q. Find the limit, $\lim_{x \to \infty} \frac{1}{x^2}$.

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Q. Find the limit, $\lim_{x \to \infty} \frac{3x+1}{5x+8}$.

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Q. Find the limit, $\lim_{x\to\infty} \frac{x+1}{x^2+4}$.

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Infinity also appears when we are finding the value of a limit.

$$\lim_{x\to c^+} f(x) = \infty$$

means the value of $f(x)$ is getting larger than any real number when x approaches c from the right.

$$\lim_{x\to c^-} f(x) = \infty$$

means the value of $f(x)$ is getting larger than any real number when x approaches c from the left.

$$\lim_{x\to c} f(x) = \infty$$

when both $\lim_{x\to c^+} f(x) = \infty$ and $\lim_{x\to c^-} f(x) = \infty$.

For example, let $(x) = \frac{1}{x-1}$. We are finding $\lim_{x\to 1^+} \frac{1}{x-1}$.

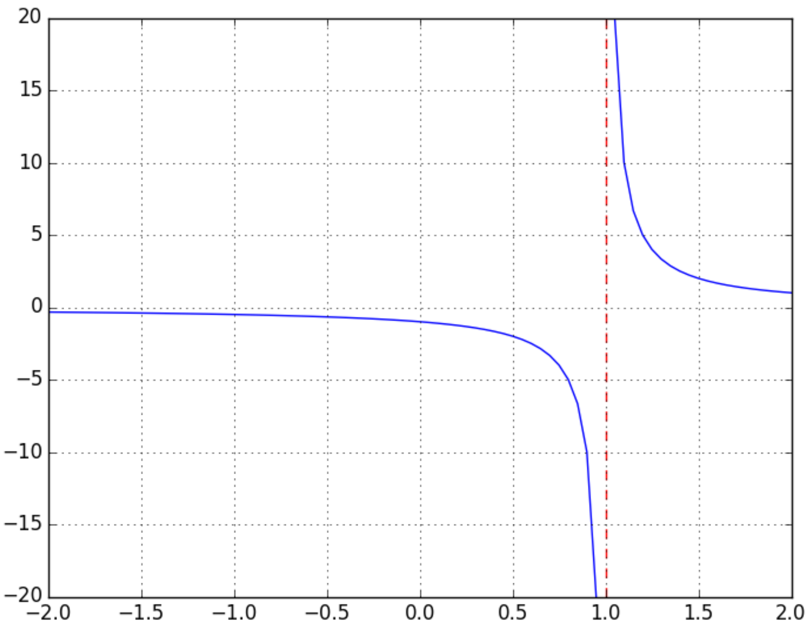
Tracing from the right hand side of the graph of $f(x) = \frac{1}{x-1}$ below,

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty .$$

Since ∞ or $-\infty$ is not a number, we say that this limit

$$\lim_{x\to 1^+} \frac{1}{x-1} \text{ does not exist.}$$

If you know that your limit to be found is an ∞ or a −∞ in any quiz or exam, please specify your answer. Don’t just say that the limit doesn’t exist. Instead, write down ∞ or −∞ in your answer.



Q. Find the limit $\lim_{x \to 1^-} \frac{x}{x-1}$.

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6. Continuity

A function $f(x)$ is continuous at a number c if
(1) c is in the domain of $f(x)$,
(2) $\lim_{x\to c} f(x)$ exists,
and (3) $\lim_{x\to c} f(x) = f(c)$.

$f(x)$ is discontinuous at c if $f(x)$ is not continuous at c.

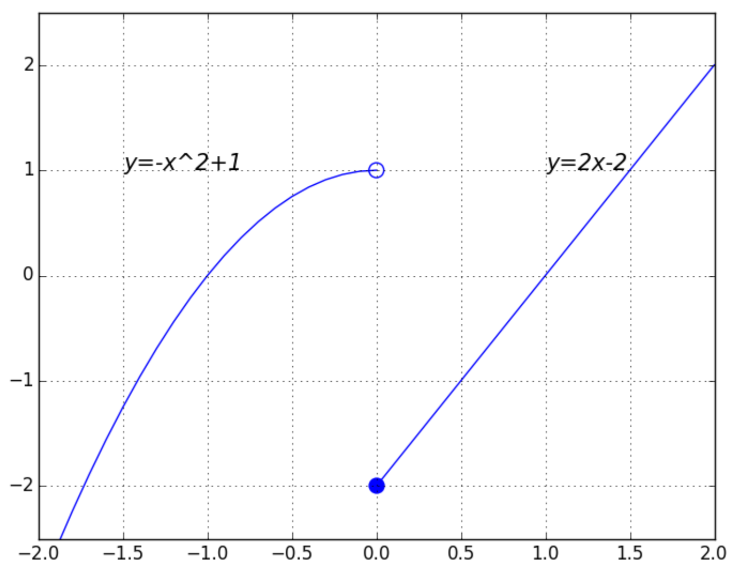
We say a function $f(x)$ is continuous, if $f(x)$ is continuous at every number c on the real line. All polynomials are continuous at every number c. Rational functions $\frac{p(x)}{q(x)}$ are continuous at a number c if $q(c) \neq 0$.

For example, $f(x) = 3x^2 + 5x + 8$ is continuous on the real line. On the other hand, the function $f(x) = \frac{1}{x-1}$ is continuous at any number $c \neq 1$.

Q. Given the graph of the piecewise function

$$f(x) = \begin{cases} x^2 + 1 & when \quad x < 0 \\ 2x - 2 & when \quad x \geq 0 \end{cases},$$

find the following limits or state that they does not exist.



(a) $\lim_{x \to 0^+} f(x)$, (b) $\lim_{x \to 0^-} f(x)$, (c) $\lim_{x \to 0} f(x)$

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Q. Let $f(x)$ be a piecewise function defined as follows.

$$f(x) = \begin{cases} x^2 & when \quad x \leq 0 \\ 2 - x & when \quad 0 < x \leq 1 \\ x & when \quad x > 1 \end{cases}$$

State where $f(x)$ is discontinuous. Sketch the graph of $f(x)$.

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Rates of Change, Slopes, and Derivatives

Suppose $f(x)$ is a function depending on a variable x. The **rate of change** of the function $f(x)$ describes how $f(x)$ changes with respect to the change in x.

The average rate of change of $f(x)$ between numbers a and $a + h$ is

$$\frac{f(a+h) - f(a)}{h}.$$

The instantaneous rate of change of $f(x)$ at the number a is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

If $f(x)$ measures certain distance travelled by an object, or position of an object, then average rate of change usually means average speed or average velocity over a period of time. Instantaneous rate of change measures the speed or velocity of this object at a particular moment.

Q. A ball is thrown straight up from a height of 192 feet with an initial velocity of 64 feet/second. Its height at time t (in seconds), $0 \leq t \leq 6$, is given by

$$h(t) = -16t^2 + 64t + 192$$

in feet. Find the average rate of change (=average velocity) of $h(t)$ between $t = 0$ and $t = 1$.

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The above quantities can be traced back to the graph of $f(x)$.

$$\frac{f(a+h)-f(a)}{h}$$

is the slope of the line passing through $\big(a, f(a)\big)$ and $\big(a+h, f(a+h)\big)$.

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\ .$$

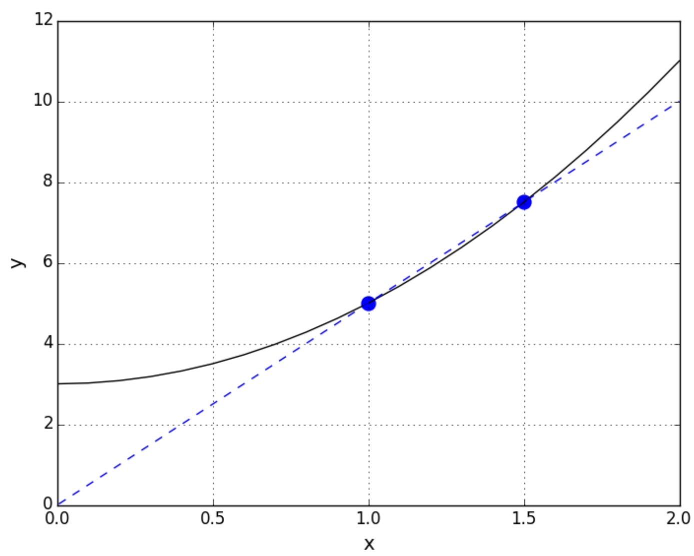
is the slope of the the **tangent line** to $f(x)$ at $x = a$.

For example, we let $f(x) = 2x^2 + 3$. Let $a = 1$. $f(1) = 5$.

When $h = 0.5$. we have $a + h = 1.5$. The average rate of change is

$$\frac{f(a+h)-f(a)}{h} = \frac{f(1.5)-f(1)}{0.5} = \frac{7.5-5}{0.5} = 5\ .$$

The slope of the line passing through $(1,5)$ and $(1.5,7.5)$ is $m = 5$.



We call this blue line the secant line to $f(x)$ through the points $(1,5)$ and $(1.5,7.5)$. It cuts the curve $y = 2x^2 + 3$ at two different points.

Q. Find $\lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$ when $f(x) = 2x^2 + 3$.

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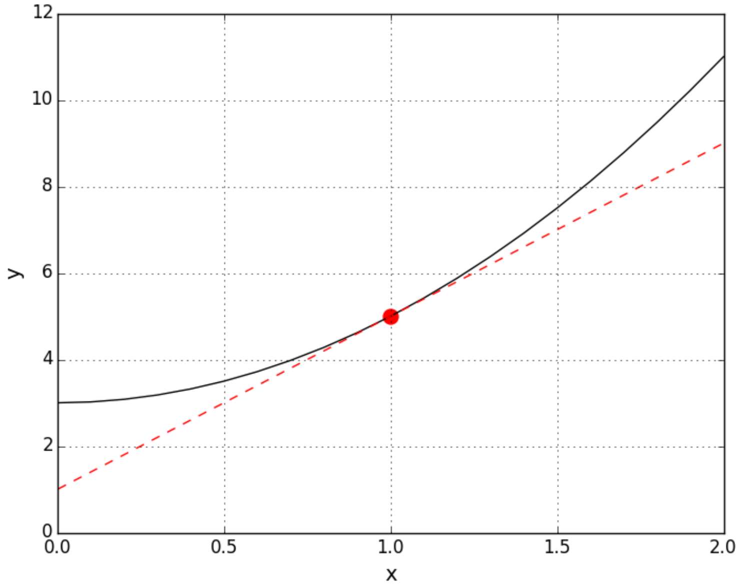
The slope of the tangent line to $f(x)$ at $x = 1$ is then $m =$ ______ .



We call this red line the **tangent** line to $f(x)$ at $x = 1$, since it cuts the curve $y = 2x^2 + 3$ at exactly one point. This point is $(1, f(1))$.

Given a function $f(x)$, the **derivative of $\boldsymbol{f(x)}$** is a new function which returns the instantaneous rate of change of $f(x)$, or the slope of the tangent line to $f(x)$ at any number x. Here comes the definition of the derivative.

The derivative of $f(x)$ at a number x, written as $f'(x)$ is defined by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

It gives the instantaneous rate of change of $f(x)$ at x, and also the slope of the tangent line to $f(x)$ at x.

Let a be a particular number. What does the derivative of f at $x = a$, $f'(a)$, tell us that if we increase x-value by h units, correspondingly the value of $f(x)$ will change by $h \cdot f'(a)$ units.

The derivative of $f(x)$, $f'(x)$, is very often seen as $\frac{df}{dx}$. If we are putting $x = a$, and hence finding $f'(a)$, then we write $f'(a)$ as $\frac{df}{dx}(a)$. If y is a function of x, then y' or $y'(x)$ denotes the derivative (function) of y, so as $\frac{dy}{dx}$.

Q. Using the previous example, $f(x) = 2x^2 + 3$.

(a) Find $f'(1)$ and the line equation of the tangent line to $f(x)$ at $x = 1$.

(b) Find $f'(x)$ by the definition of the derivative.

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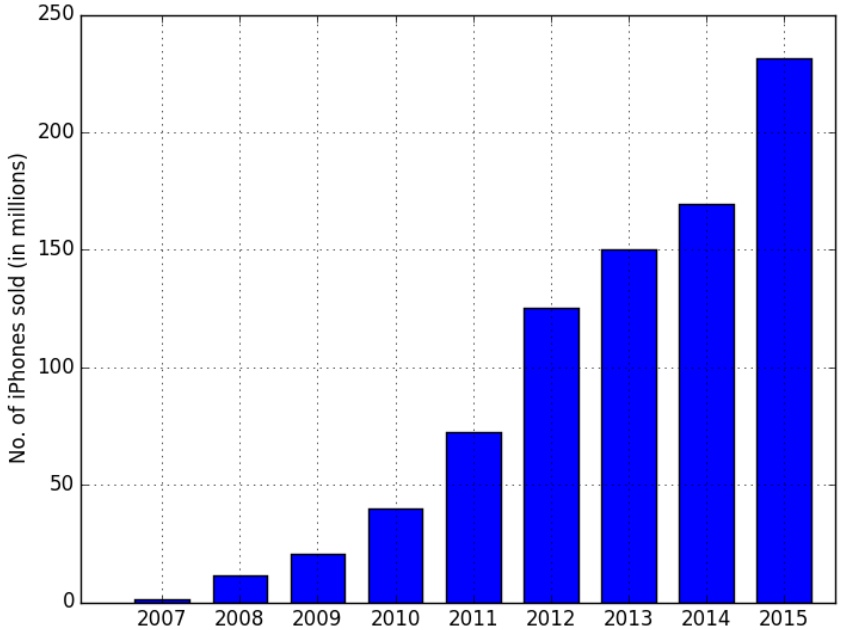
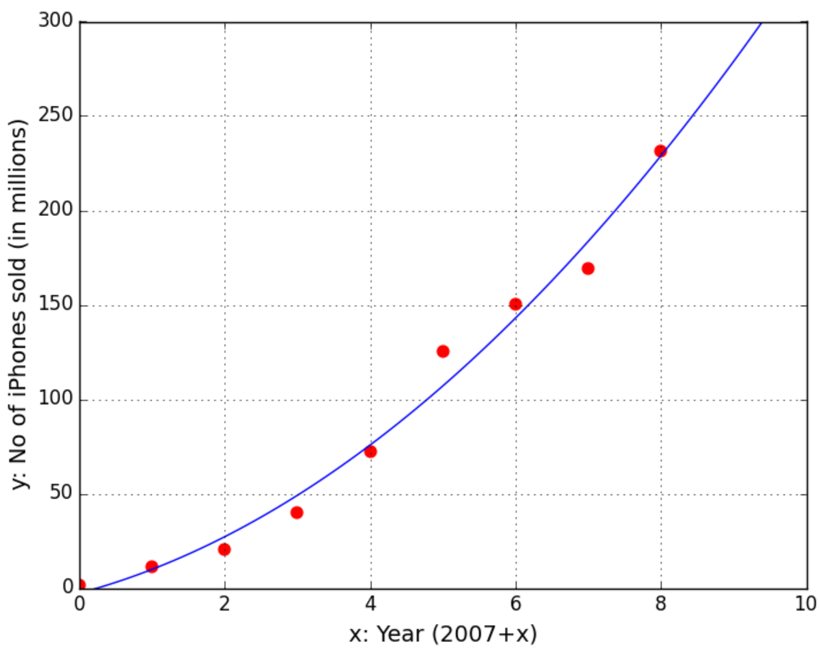
In your text-book, on P.93, there is an example about the annual sales of Macintosh. We are rephrasing that example by a different context, the annual global sales of iPhones between 2007 and 2015.



Let x be the number of years after Year 2007. $x = 0$ means the year 2007, $x = 1$ means the year 2008, and so on. $x = 8$ means the year 2015. The annual sales are approximated by the following function.

$$f(x) = 2.335x^2 + 10.269x - 2.727\,.$$



$f(x)$ is a realistic approximation of the annual sales of iPhones between year 2007 and year 2015.

Q. Find the derivative of $f(x)$. Find $f'(8)$ and interpret your answer.

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Therefore, $f'(x) = 4.67x + 10.269$.

$f'(8) = 4.67(8) + 10.269 = 47.629$

Interpretation. $x = 8$ represents the year 2015.

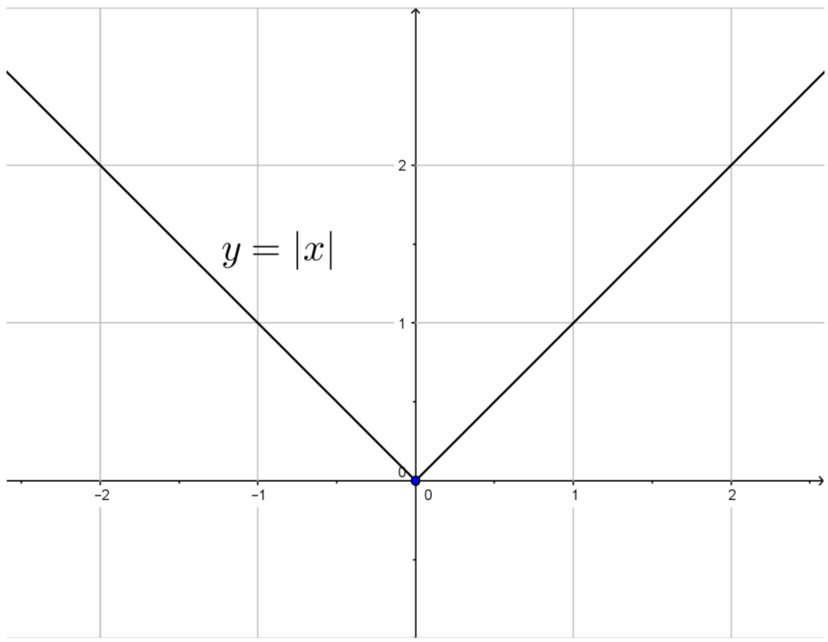
In 2015, the global sales of iPhones is increasing at the rate of 47.629 million units per year.

The above interpretation consists of several parts.

In 2015	means	at $x = 8$
the global sales of iPhones	means	$f(x)$
increasing	means	$f'(8) > 0$
at the rate of 47.629	means	$f'(8) = 47.629$
million units per year	means	the unit of $f'(8)$

*If $f'(8) < 0$, we use “decreasing” or “falling” instead of “increasing”.
“ $f'(8) = -10.0$ ” means falling at the rate of 10.0 (million units per year).

Some functions are not differentiable at a certain x-value, $x = a$. Graphically, it means that the graph of the function $f(x)$ has a corner point at $x = a$. Here comes an example.



The absolute value function $f(x) = |x|$ has a corner point at $x = 0$. It is not differentiable at $x = 0$. The underlying reason is that the two-sided limit

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

doesn't exist. Since we have to define $f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h}$, in this case, $f'(0)$ doesn't exist. We say that $|x|$ is not differentiable at $x = 0$.

Differentiation Formulas

Very often we use formulas to differentiate a function, instead of using the definition of derivatives, if the function itself is nice.

1. Constant rule. If c is a constant,

$$\frac{d}{dx}c = 0 .$$

For example, $\frac{d}{dx}(2016) = 0$.

2. Power rule. If the function is x^n for any constant exponent n,

$$\frac{d}{dx}x^n = nx^{n-1}$$

In particular $\frac{d}{dx}x = 1$, and $\frac{d}{dx}x^2 = 2x$.

3. Constant-multiple rule. If c is a constant, and $f(x)$ is a function,

$$\frac{d}{dx}\big(c \cdot f(x)\big) = c \cdot f'(x).$$

For example. $\frac{d}{dx}(2016x) = 2016 \cdot \frac{d}{dx}(x) = 2016$.

4.Sum-Difference rule. For two functions $f(x)$ and $g(x)$.

$$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x) ,$$

$$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x) .$$

Sume-Difference rule can be applide even though we have three or more functions in a sum (or in a difference). For example,

$$\frac{d}{dx}(10 + x + x^2) = \frac{d}{dx}(10) + \frac{d}{dx}(x) + \frac{d}{dx}(x^2) = 0 + 1 + 2x = 1 + 2x .$$

Q. Differentiate the following functions.

(a) $f(x) = x^{3/2}$

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(b) $f(x) = x\sqrt{x}$

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(c) $f(x) = x^{5/2} - 2x + 5$

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(d) $f(x) = \frac{7x^2+5x+3}{\sqrt{x}}$

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Q Find $f'(x)$. Find the tangent line to $f(x)$ at $x = 1$.

$$f(x) = \frac{1}{x^{3/2}}$$

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We mention an application here, marginal analysis. Suppose a company has certain revenue function, cost function and profit function.

$$R(x) = \text{Total revenue gained by selling } x \text{ units}$$

$$C(x) = \text{Total cost of producing } x \text{ units}$$

$$P(x) = \text{Total profit gained (or loss incurred) by selling } x \text{ units.}$$

We have $P(x) = R(x) - C(x)$.

The marginal cost function, is the derivative of the cost function.

$$MC(x) = C'(x)$$

The marginal revenue function, is the derivative of the revenue function.

$$MR(x) = R'(x)$$

The marginal profit function, is the derivative of the profit function.

$$MP(x) = P'(x)$$

Q. Suppose a company is producing a mini optical mouse. Let x be the number of units of mouses produced. Let the cost function (in dollars) be

$$C(x) = 8x + 60\,,$$

and the revenue function (in dollars) be

$$R(x) = 25x - 0.2x^2\,.$$

(a) Find the profit function $P(x)$, and the marginal profit function $MP(x)$.

(b) Find $MP(5)$. Interpret your answer.

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(b) $MP(5) = P'(5) = -0.4(5) + 17 = 15$.

Interpretation.

When 5 units of mouses have been produced, the profit is increasing at a rate of \$15 per unit of mouses produced.

You may also say:

When 5 units of mouses have been produced, the profit increases by \$15 per unit of mouses produced.

When 5 units of mouses have been produced, the company gained \$15 for an additional mouse being produced and sold.

The above interpretation captures everything we mentioned before about an interpretation.

When 5 units of mouses have been produced,	means	at $x = 5$
the profit	means	$P(x)$
increasing	means	$MP(5) = P'(5) > 0$
at the rate of 15	means	$MP(5) = P'(5) = 15$
dollars by unit of mouses	means	the unit of $MP(5)$