Definite Integrals and Areas

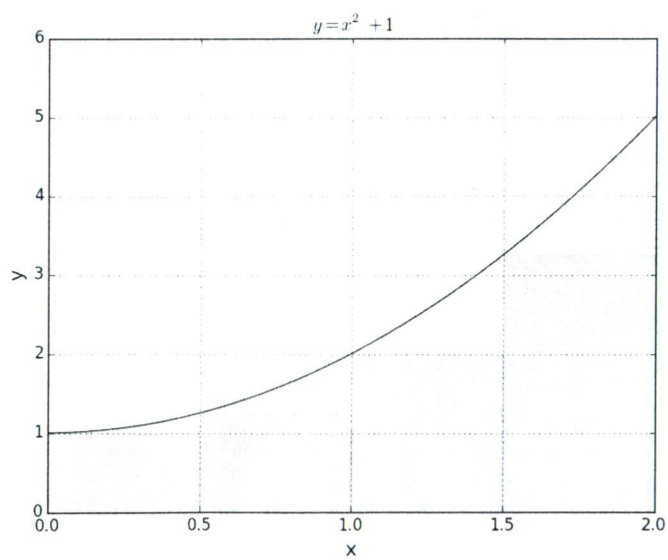
Suppose we have a non-negative function $f(x)$. Let $[a, b]$ be an interval within the domain of $f(x)$. We may consider the area under the curve $y = f(x)$ and above the x-axis, bounded between two vertical lines $x = a$ and $x = b$.

Definite integral. The area under the curve $y = f(x)$ from a to b is called the definite integral of the function $f(x)$ from a to b. It is written as $$\int_a^b f(x)\,dx\,.$$

For example, let $f(x) = x^2 + 1$ and the interval be $[0,2]$. We are finding

$$\int_0^2 x^2 + 1\,dx\,.$$

The area under $f(x) = x^2 + 1$ from 0 to 2 can be approximated by rectangles below the curve $y = x^2 + 1$. First we use four rectangles.

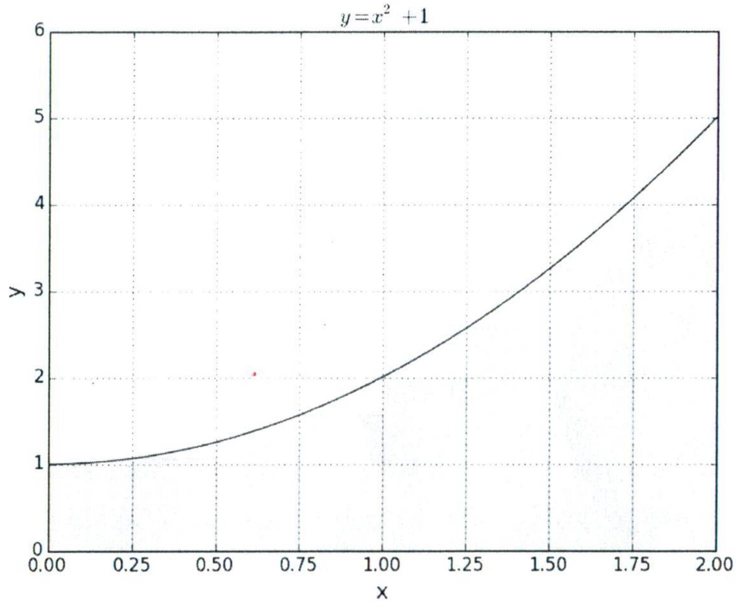


The pink rectangles above are called *left rectangles*, because each has a height equal to the height of the curve at the left-hand edge of the rectangle.

The approximate area by these four rectangles is

$$S_4 = 0.5\big(f(0) + f(0.5) + f(1) + f(1.5)\big) = 0.5(1 + 1.25 + 2 + 3.25) = 3.75$$

in square units. This approximate area doesn't count the area under the curve $y = x^2 + 1$ outside the red-shaded region. It can be improved by using more left-rectangles. Let say this time we use 8 left-rectangles to estimate the area under $y = x^2 + 1$.



The new approximate area is

$$S_8 = 0.25(1 + 1.0625 + 1.25 + \cdots + 3.25 + 4.0625) = 4.1875\,.$$

This figure is closer to the definite integral $\int_0^2 x^2 + 1\, dx$ than $S_4 = 3.75$, since the unshaded region under the curve $y = x^2 + 1$ is smaller in the upper graph.

So, what is the value of $\int_0^2 x^2 + 1\, dx$?

Fundamental Theorem of Integral Calculus.

Suppose $f(x)$ is a continuous function on an interval $[a,b]$. If $F(x)$ is one antiderivative of $f(x)$, I.e. $F'(x) = f(x)$. Then,

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Before going on, we introduce a notation here. Suppose $F(x)$ is the above function. We set

$$F(x)|_a^b = F(b) - F(a)\,.$$

Example. Find $\int_0^2 x^2\,dx$. We know that

$$\int x^2\,dx = \frac{1}{3}x^3 + C\,.$$

So we ignore the arbitrary constant C and let $F(x) = \frac{1}{3}x^3$.

$$\int_0^2 x^2\,dx = \left(\frac{1}{3}x^3\right)\Big|_0^2 = \left(\frac{2^3}{3}\right) - \left(\frac{0}{3}\right) = \frac{8}{3}.$$

** When you are finding a definite integral, always write down your antiderivative "$F(x)$" in your answer. Don't just jump to a numerical figure!

Q. Find $\int_0^2 x^2 + 1\,dx$.

> $\int x^2+1\,dx = \int x^2 dx + \int 1\,dx = \frac{1}{3}x^3 + x + C$

>

> $\therefore \int_0^2 x^2+1\,dx = \left(\frac{1}{3}x^3 + x\right)\Big|_0^2$

> $= \left(\frac{8}{3}+2\right) - (0) = \frac{14}{3} \approx 4.67$ #

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Properties of definite integrals follow from properties of indefinite integral.

3. Constant-multiple rule for definite integrals. For any constant k,

$$\int_a^b c \cdot f(x)\, dx = c \cdot \int_a^b f(x)\, dx\,.$$

4. Sum-Difference rule for definite integrals.

$$\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

Q. Find the definite integral $\int_2^4 (1 + x^{-2})\, dx$.

> $= \left(x + \frac{1}{(-1)} x^{-1}\right)\Big|_2^4 = \left(x - \frac{1}{x}\right)\Big|_2^4$
>
> $= \left(4 - \frac{1}{4}\right) - \left(2 - \frac{1}{2}\right) = \frac{9}{4} = 2.25$ #

Q. Find the definite integral $\int_0^1 12e^{3x}\, dx$.

> $= 12 \int_0^1 e^{3x} dx = 12 \cdot \left(\frac{1}{3} e^{3x}\right)\Big|_0^1$
>
> $= 12\left(\frac{1}{3}e^3 - \frac{1}{3}\underbrace{e^0}_{=1}\right) = 4e^3 - 4$ #
>

Q. Find the definite integral

$$\int_1^2 \frac{(x+1)^2}{x} dx\,.$$

> $= \int_1^2 \frac{x^2 + 2x + 1}{x} dx = \int_1^2 x + 2 + \frac{1}{x}\, dx$
>
> $= \left(\frac{x^2}{2} + 2x + \ln|x|\right)\Big|_1^2$
>
> $= (2 + 4 + \ln 2) - \left(\frac{1}{2} + 2 + \cancel{\ln 1}^{\,0}\right) = \frac{7}{2} + \ln 2$ #
>
>

Q. Find the definite integral

$$\int_0^7 \frac{8x^2+9}{\sqrt{x}}\,dx\,.$$

> $= \int_3^7 8x^{\frac{3}{2}} + 9x^{-\frac{1}{2}}\,dx$

> $= \left(8 \cdot \frac{2}{5} x^{5/2} + 9 \cdot 2 \cdot x^{1/2}\right)\Big|_0^7$

>

> $= \left(\frac{16}{5} x^{5/2} + 18x^{1/2}\right)\Big|_0^7$

> $= \left(\frac{16}{5} 7^{5/2} + 18 \cdot 7^{1/2}\right) - (0)$

> $= \frac{16}{5} 7^{5/2} + 18 \cdot 7^{1/2} \approx 462.477$ #

Example. [Area under curve]

Find the area under $f(x) = \frac{1}{x}$ from $x = 1$ to $x = 13$.

* Note $\frac{1}{x}$ is non-negative and continuous on $[1, 13]$.

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Soln

> Area $= \int_1^{13} \frac{1}{x}\,dx = (\ln|x|)\Big|_1^{13} = \ln 13 - \ln 1$

> $= \ln 13$ square units #

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Q. Find the area under $f(x) = \frac{\sqrt{x}+1}{x}$ from $x = 1$ to $x = 2$.

> Area $= \int_1^2 \frac{\sqrt{x}+1}{x}\,dx = \int_1^2 \frac{1}{\sqrt{x}} + \frac{1}{x}\,dx$

> $= \int_1^2 x^{-1/2} + x^{-1}\,dx = \left(2x^{1/2} + \ln|x|\right)\Big|_1^2$

>

> $= \left(2(2^{1/2}) + \ln 2\right) - \left(2 + \cancel{\ln 1}^{\,0}\right)$

$= (2\sqrt{2} - 2) + \ln 2$ square units #

Example. [Cost of succession of units, textbook P.336]

For a marginal cost function $MC(x)$, the total cost of units a to b is

$$\int_a^b MC(x)\,dx\,.$$

A company's marginal cost function is

$$MC(x) = 8e^{-0.01x} + 4,$$

where x is the number of units. Find the total cost of producing the first hundred units.

> Total cost of units 0 to 100

> $= \int_0^{100} MC(x)\,dx = \int_0^{100} 8e^{-0.01x} + 4\,dx$

> $= \left(\frac{8}{-0.01} e^{-0.01x} + 4x\right)\Big|_0^{100}$

> $= \left(-800 e^{-0.01x} + 4x\right)\Big|_0^{100}$

> $= (-800e^{-1} + 400) - (-800 + 0)$

> $= 1200 - 800e^{-1} \approx 905.70$

> Therefore, the cost to produce the first 100 units is \$905.70. #

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In general, we find the total accumulation at a given rate by definite integral.

The total accumulation at rate $f(x)$ from a to b is $$\int_a^b f(x)\,dx\,.$$

Example. An average child of age x years grows at the rate of $6x^{-1/2}$ inches per year (for $2 \le x \le 16$). Find the total height gain from age 4 to age 9.

> $f(x) = 6x^{-1/2}$ (in inches per year)
>
> Total height gain from age 4 to age 9
>
> $= \int_4^9 f(x)\,dx = \int_4^9 6x^{-1/2}\,dx$
>
> $= \left(6 \cdot 2x^{1/2}\right)\Big|_4^9$
>
> $= \left(12x^{1/2}\right)\Big|_4^9 = (12 \cdot 3) - (12 \cdot 2)$
>
> $= 36 - 24 = 12$ inches.
>
> Therefore, the total height gain from age 4 to age 9 is 12 inches. #
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