

## Functions of Several Variables

The function  $f(x, y) = x^2 + y^2$  is so called *a function of two variables*.

These two variables are  $x$  and  $y$ . For each ordered pair  $(x, y)$ , the function  $f$  returns a real number  $f(x, y)$ .

Let say  $(x, y) = (2, 3)$ . It means  $x = 2$  and  $y = 3$ . We then have

$$f(2, 3) = (2)^2 + (3)^2 = 4 + 9 = 13.$$

The domain of a function  $f(x, y)$  is the set of all ordered pairs  $(x, y)$  at which  $f(x, y)$  is well defined.

Also, the range of  $f(x, y)$  is the set of all values  $f(x, y)$  given by ordered pairs  $(x, y)$  inside the domain of  $f(x, y)$ .

Example. The domain of  $f(x, y) = x^2 + y^2$  is

$$\{(x, y) \mid \text{all possible ordered pairs } (x, y)\}$$

Q. Let  $f(x, y) = \frac{\sqrt{x}}{y}$ .

(1) Find the domain of  $f(x, y)$ .

(2) Find  $f(4, 3)$ .

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Q. Find the domain of the function  $f(x, y) = \frac{1}{x-y}$ .

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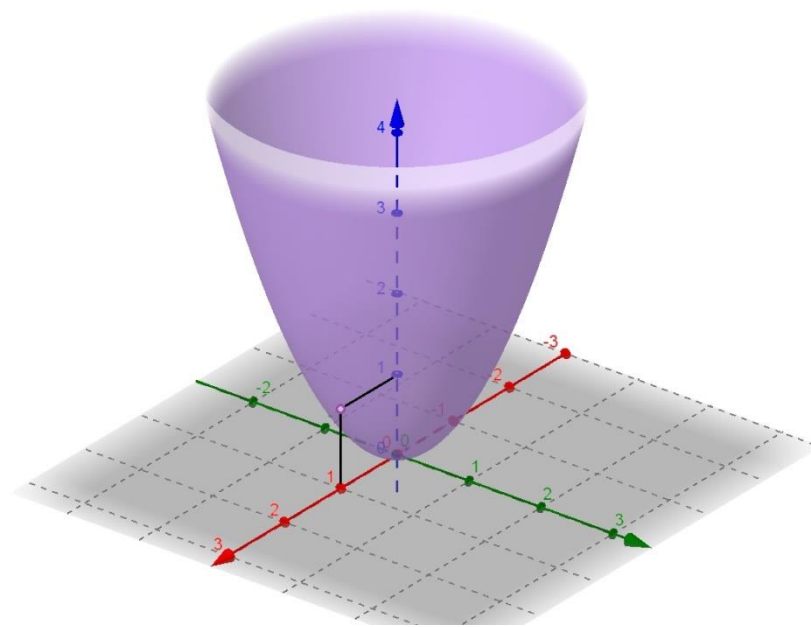
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The graph of a two-variable function is displayed on the three dimensional coordinate system. The graph of  $f(x, y)$  is the surface  $z = f(x, y)$  on  $(x, y, z)$  coordinate space. For example,

$$f(x, y) = x^2 + y^2 .$$

In the below figure, the red line (pointing left) is the x-axis, the green line (pointing right) is the y-axis, and the blue line (pointing up) is the z-axis. The plane on the floor is the xy-plane in this  $(x, y, z)$  coordinate space.



The purple-shaded surface is the graph of  $f(x, y)$ , denoted by  $z = x^2 + y^2$ . For example, I pick the ordered pair  $(1, 0)$  on the xy-plane, and then draw a vertical line above  $(1, 0)$  until it hits the surface at a point.

We know that  $f(1, 0) = 1$ . So the z-coordinate of that point is 1. That point is  $(1, 0, 1)$ .

Differentiation on a two variable function is by taking partial derivatives. Given a function  $f(x, y)$ , it has two partial derivatives, one coming from  $x$  and other coming from  $y$ , which are

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

Partial derivatives. At a point  $(x, y)$  in the domain of  $f(x, y)$ .

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y} = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

We are finding partial derivatives by differentiation formulas.

1. When finding  $\frac{\partial f}{\partial x}$ , we treat  $y$  as a constant and differentiate everything with respect to  $x$ . All differentiation formulas apply.
2. When finding  $\frac{\partial f}{\partial y}$ , we treat  $x$  as a constant and differentiate everything with respect to  $y$ . All differentiation formulas apply.

There are two ways to write down partial derivatives.

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y)$$

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y)$$

Q. Let  $f(x, y) = x^4 y^3$ . Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

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Q. Let  $f(x, y) = 4x^3 - 3x^2y^2 - 2y^2$ .

(1) Find  $f_x$  and  $f_y$ .

(2) Find  $f_x(-1, 1)$  and  $f_y(-1, 1)$ .

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Q. Let  $f(x, y) = y (\ln x) + xe^y$ . Find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .

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