*Business Calculus – Week 2*

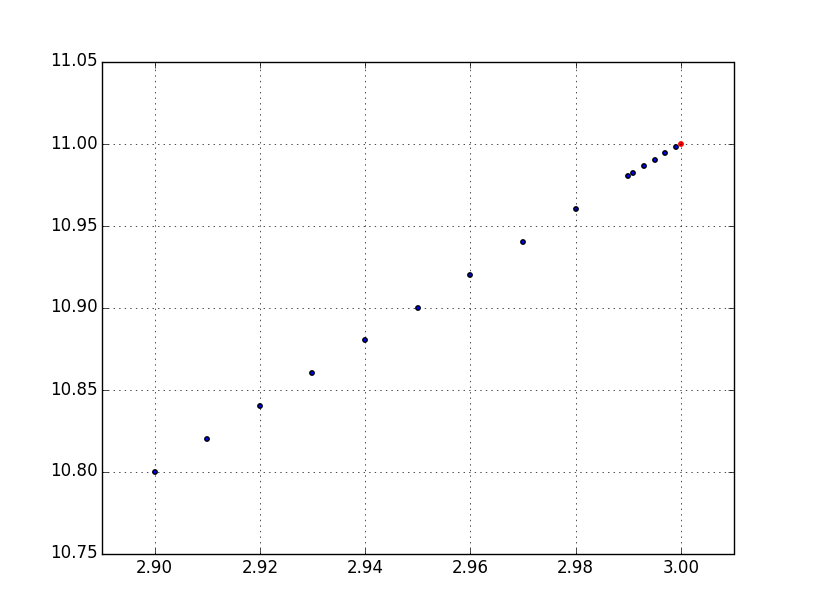
Limits and Continuity

The notation means approaches 3. It describes the process that the variable is getting closer and closer to . For example,

and so on. Let be a function of . In this section, we are interested in finding this limit as approaches 3 of ,

It means the value that the function goes to when approaches 3. Take as an example. A very simple method to get this limit is to draw a table.

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| 2.9 | 10.8 |
| 2.99 | 10.98 |
| 2.999 | 10.998 |
| … | … |



The blue dots represent and other values of at the corresponding . We can see that the blue dots are approaching the red dot. The red dot is the point . So we say

The function approaches 11 when approaches . In a more rigorous context, the concept of *limit* is defined as follows.

The statement

means that the value can be arbitrarily close to a number when is sufficiently close to .

Yet, basically, we are concerning where the goes to, when approaches some number , and then call that resulting number, the limit as approaches of , written as .

**1. One-sided limit**

Back to the notation , there are two different ways for to get close to the number . can approach 3 from the right or from the left.

3

3.1

2.9

The notation means that approaches 3 from the right (red arrow). For example, , , , and so on. These ’s are (i) getting close to and (2) all **larger** than . So we say these ’s are approaching from the right.

On the other hand, means that approaches 3 from the left (blue arrow). For example, , , , and so on. These ’s are (i) getting close to and (2) all **smaller** than . So we say these ’s are approaching from the left.

The limit as approaches from the right of ,

is the number where goes to, when approaches from the right.

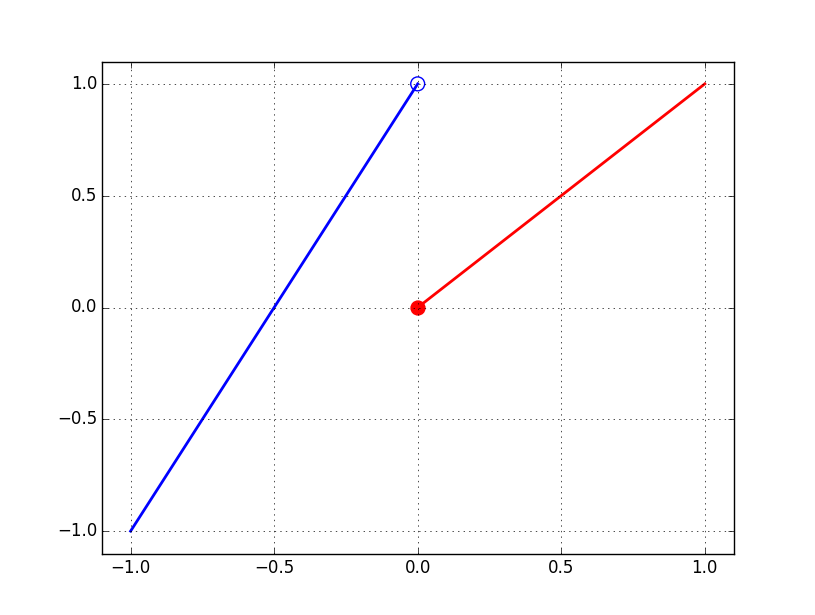
The limit as approaches from the left of ,

is the number where goes to, when approaches from the left.

is called the right-hand limit, and is called the left-hand limit. They are one-sided limits. We call the two-sided limits.

Q. Let be a piecewise linear function on the interval .

Find and from its graph.



. We say that does not exist (DNE) whenever these two one-sided limits don’t agree each other. We have the following facts.

If both and exist, and

For some number , then

If (i) any of the one-sided limit or doesn’t exist,   
or (ii) both of them exist but not equal each other, then

**2. Finding limits by substitution**

In the previous example, . This number 11, is exactly

So we are putting into to get this limit . In many situations, finding a limit as approaches is as easy as putting to be the number .

(A) When is a polynomial, and is a real number, we have

For example, . Then,

Note that for any constant , the constant function suits the above criterion. So we have , whatever the number is.

(B) When is a rational function, and is a real number such that the bottom polynomial is non-zero at . That is, . Then,

For example, . Then, , and we have

(C) When is the square root function, and is a non-negative number (), then,

This method also works for functions like , but then we require instead. Equivalently, . For example, ,

**3. Rules of Limits**

We have seen these two rules,

There are four rules of limit concerning addition, subtraction, multiplication and division between functions.

When both and exist,

When both and exist, and ,

Q. Find the following limits.

(a)

(b)

(c)

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**4. Finding limits by factorization.**

Some limits cannot be found by a direct substitution. However, we can still find these limits by simplifying the expression. For example, let

We are finding . By direct substitution,

which is not any number. So, direct substitution doesn’t work. Note

by the identity . Back to the limit,

This limit turns out to be 4.

Q. Find the .

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Q. Find the following limit.

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**5. Limits involving infinity**

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Infinity is the concept that it is larger than any number on the number line. But the infinity itself is not a number. Similarly, is smaller than any number on the number line. In some context, is represented by to emphasize the positive sign.

The notation means approaches infinity. That is, is getting larger than any real numbers (red arrow). For example,

and so on. In a similar fashion, the notation means that is getting smaller than any real number (blue arrow). For example,

and so on.

The limit as approaches of ,

is the number where goes to, when is arbitrarily large.

The limit as approaches of ,

is the number where goes to, when is arbitrarily small.

Q. Find the limit, .

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Q. Find the limit, .

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Q. Find the limit, .

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Infinity also appears when we are finding the value of a limit.

means the value of is getting larger than any real number when approaches from the right.

means the value of is getting larger than any real number when approaches from the left.

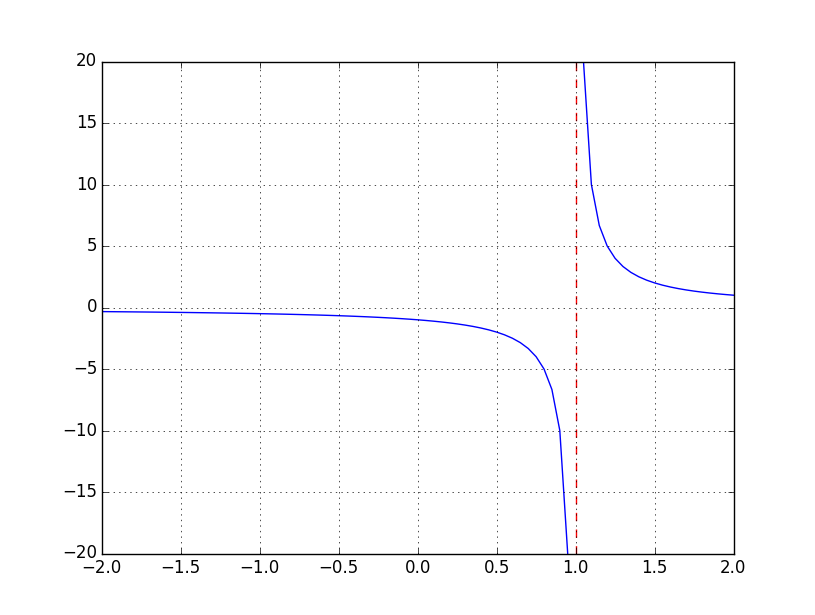
when both and .

For example, let . We are finding

Tracing from the right hand side of the graph of below,

Since or is not a number, we say that this limit

If you know that your limit to be found is an or a in any quiz or exam, please specify your answer. Don’t just say that the limit doesn’t exist. Instead, write down or in your answer.



Q. Find the limit .

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6. Continuity

A function is continuous at a number if   
(1) is in the domain of ,   
(2) exists,   
and (3) .

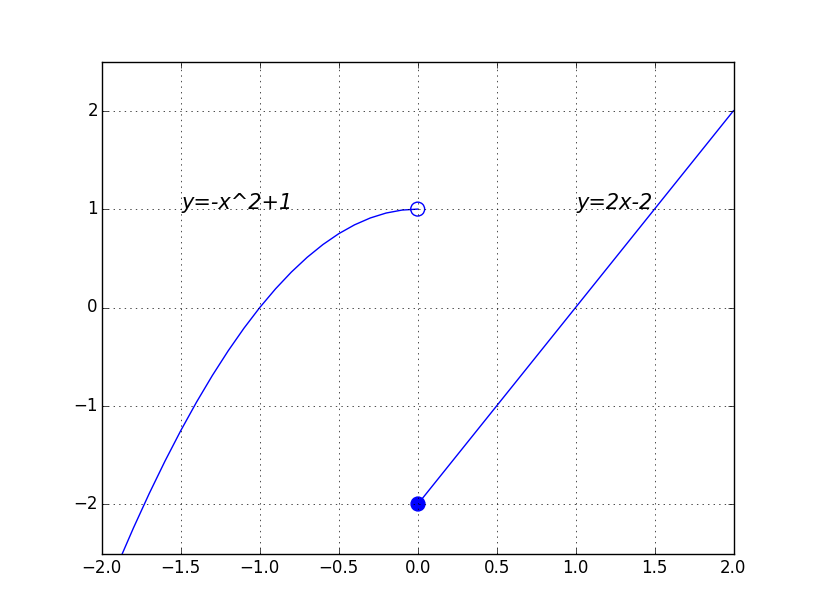
is discontinuous at if is not continuous at .

We say a function is continuous, if is continuous at every number on the real line. All polynomials are continuous at every number . Rational functions are continuous at a number if .

For example, is continuous on the real line. On the other hand, the function is continuous at any number .

Q. Given the graph of the piecewise function

find the following limits or state that they does not exist.



(a) , (b) , (c)

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Q. Let be a piecewise function defined as follows.

State where is discontinuous. Sketch the graph of .

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Rates of Change, Slopes, and Derivatives

Suppose is a function depending on a variable . The **rate of change** of the function describes how changes with respect to the change in .

The average rate of change of between numbers and is

The instantaneous rate of change of at the number is

If measures certain distance travelled by an object, or position of an object, then average rate of change usually means average speed or average velocity over a period of time. Instantaneous rate of change measures the speed or velocity of this object at a particular moment.

Q. A ball is thrown straight up from a height of 192 feet with an initial velocity of 64 feet/second. Its height at time (in seconds), , is given by

in feet. Find the average rate of change (=average velocity) of between and .

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The above quantities can be traced back to the graph of .

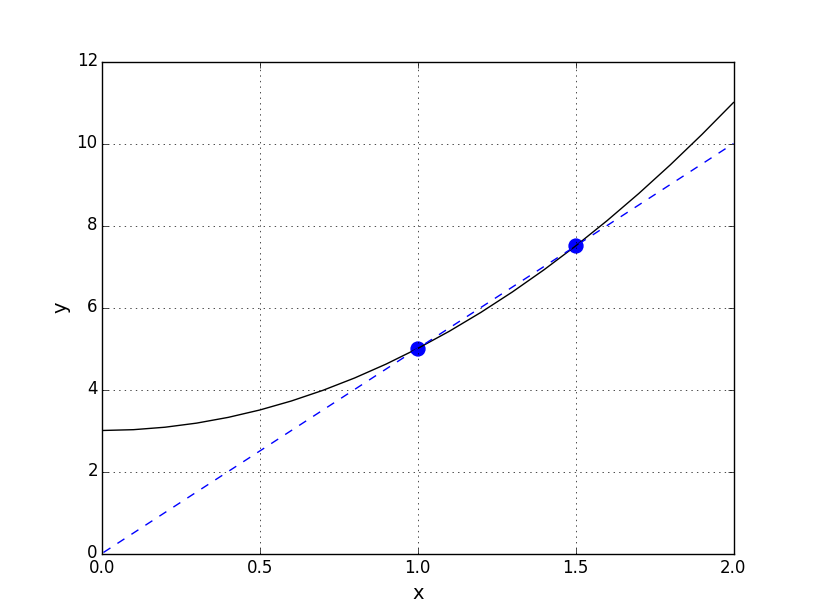
is the slope of the line passing through and .

is the slope of the the **tangent line** to at .

For example, we let . Let . .

When . we have . The average rate of change is

The slope of the line passing through and is .



We call this blue line the secant line to through the points and . It cuts the curve at two different points.

Q. Find when .

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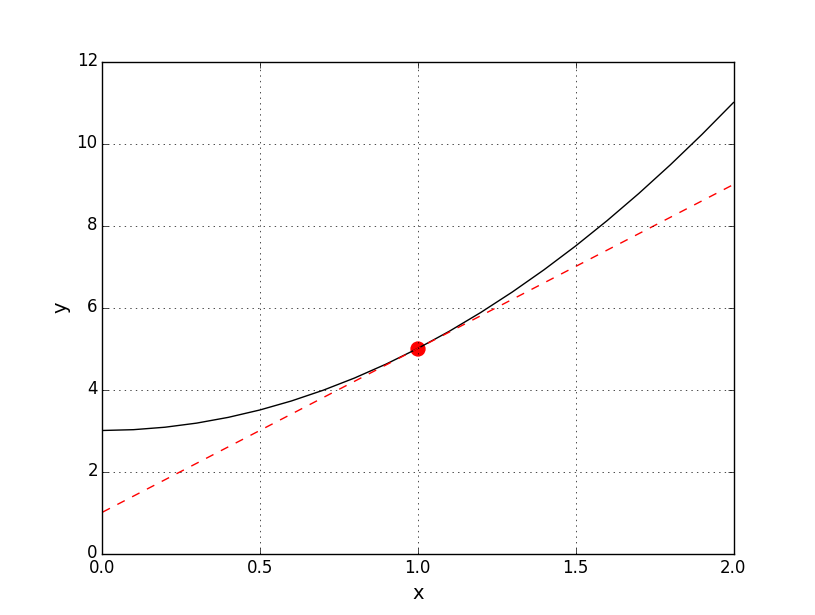
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The slope of the tangent line to at is then .



We call this red line the **tangent** line to at , since it cuts the curve at exactly one point. This point is .

Given a function , the **derivative of**  is a new function which returns the instantaneous rate of change of , or the slope of the tangent line to at any number . Here comes the definition of the derivative.

The derivative of at a number , written as is defined by

It gives the instantaneous rate of change of at , and also the slope of the tangent line to at .

Let be a particular number. What does the derivative of at , , tell us that if we increase -value by units, correspondingly the value of will change by units.

The derivative of , , is very often seen as . If we are putting , and hence finding , then we write as . If is a function of , then or denotes the derivative (function) of , so as .

Q. Using the previous example, .

(a) Find and the line equation of the tangent line to at .

(b) Find by the definition of the derivative.

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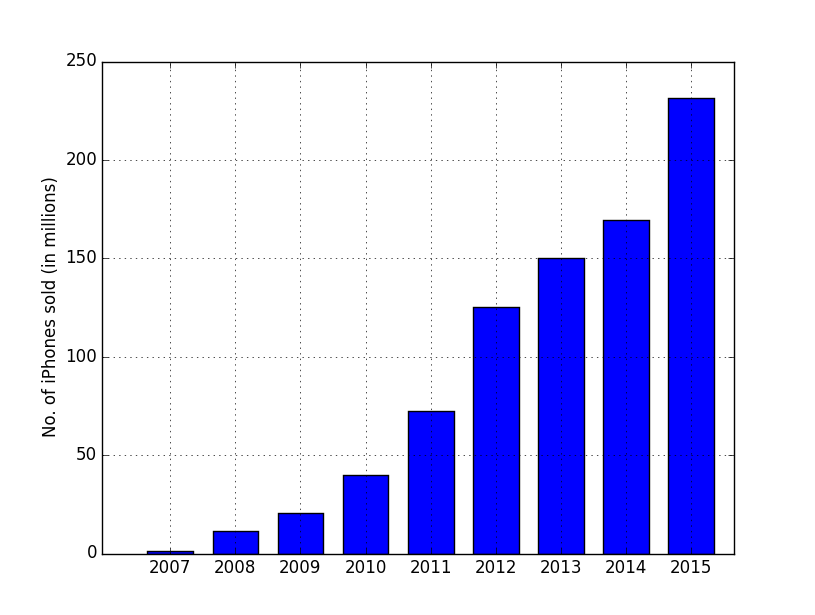
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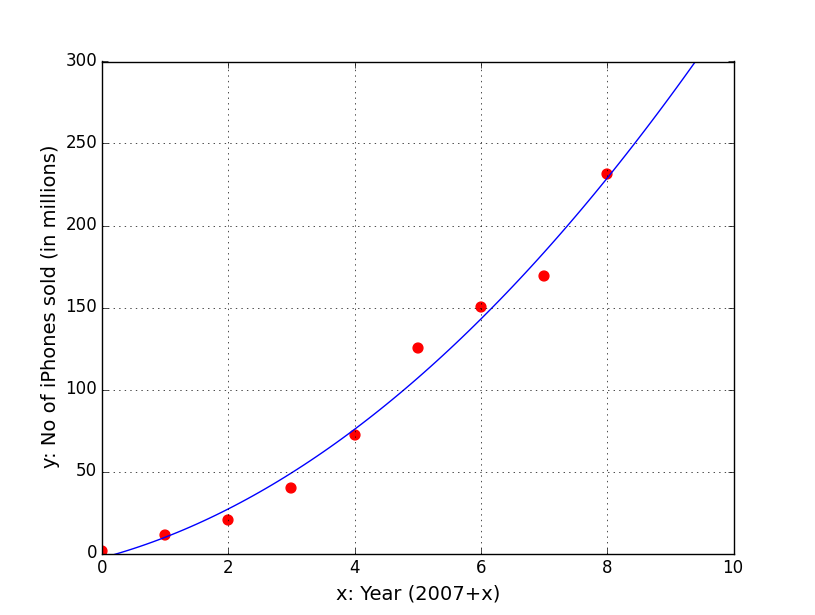
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In your text-book, on P.93, there is an example about the annual sales of Macintosh. We are rephrasing that example by a different context, the annual global sales of iPhones between 2007 and 2015.



Let be the number of years after Year 2007. means the year 2007, means the year 2008, and so on. means the year 2015. The annual sales are approximated by the following function.



is a realistic approximation of the annual sales of iPhones between year 2007 and year 2015.

Q. Find the derivative of . Find and interpret your answer.

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Therefore,

*Interpretation.* represents the year 2015.

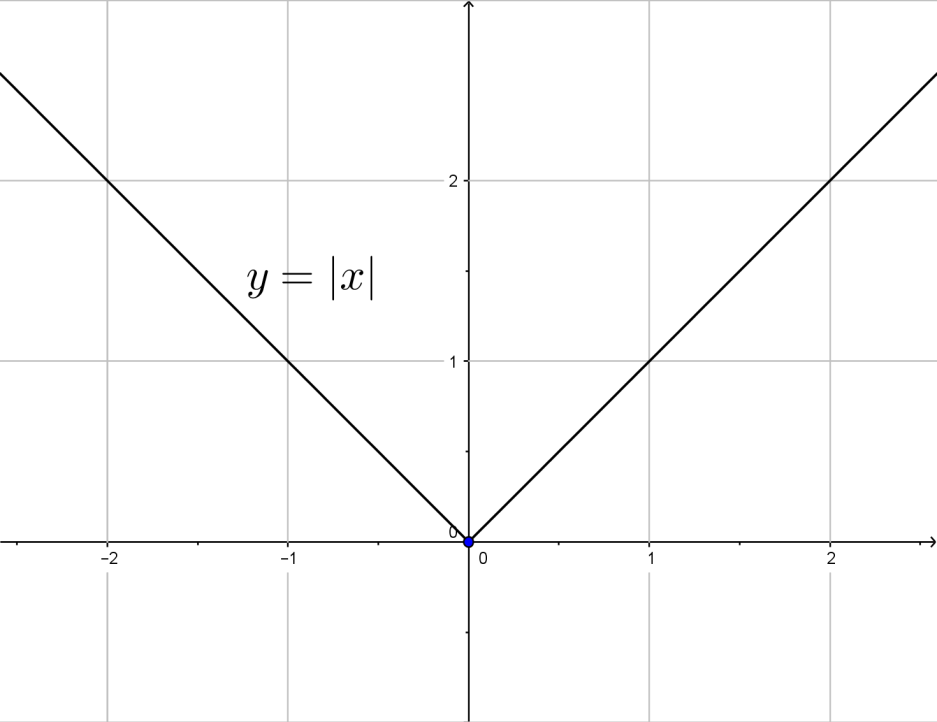
In 2015, the global sales of iPhones is increasing at the rate of 47.629 million units per year.

The above interpretation consists of several parts.

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| --- | --- | --- |
| In 2015 | means | at |
| the global sales of iPhones | means |  |
| increasing | means |  |
| at the rate of 47.629 | means |  |
| million units per year | means | the unit of |

\*If , we use “decreasing” or “falling” instead of “increasing”.  
 “ ” means falling at the rate of 10.0 (million units per year).

Some functions are not differentiable at a certain -value, . Graphically, it means that the graph of the function has a corner point at . Here comes an example.



The absolute value function has a corner point at . It is not differentiable at . The underlying reason is that the two-sided limit

doesn't exist. Since we have to define , in this case, doesn’t exist. We say that is not differentiable at .

Differentiation Formulas

Very often we use formulas to differentiate a function, instead of using the definition of derivatives, if the function itself is nice.

1. Constant rule. If is a constant,

For example, .

2. Power rule. If the function is for any constant exponent ,

In particular , and .

3. Constant-multiple rule. If is a constant, and is a function,

For example. .

4.Sum-Difference rule. For two functions and .

Sume-Difference rule can be applide even though we have three or more functions in a sum (or in a difference). For example,

Q. Differentiate the following functions.

(a)

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(b)

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(c)

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(d)

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Q Find . Find the tangent line to at .

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We mention an application here, marginal analysis. Suppose a company has certain revenue function, cost function and profit function.

We have .

The marginal cost function, is the derivative of the cost function.

The marginal revenue function, is the derivative of the revenue function.

The marginal profit function, is the derivative of the profit function.

Q. Suppose a company is producing a mini optical mouse. Let be the number of units of mouses produced. Let the cost function (in dollars) be

and the revenue function (in dollars) be

(a) Find the profit function , and the marginal profit function .

(b) Find . Interpret your answer.

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(b) .

*Interpretation.*

When 5 units of mouses have been produced, the profit is increasing at a rate of $15 per unit of mouses produced.

You may also say:

When 5 units of mouses have been produced, the profit increases by $15 per unit of mouses produced.

When 5 units of mouses have been produced, the company gained $15 for an additional mouse being produced and sold.

The above interpretation captures everything we mentioned before about an interpretation.

|  |  |  |
| --- | --- | --- |
| When 5 units of mouses have been produced, | means | at |
| the profit | means |  |
| increasing | means |  |
| at the rate of 15 | means |  |
| dollars by unit of mouses | means | the unit of |