*Business Calculus – Week 3*

Product and Quotient Rules

Last week we learnt four differentiation formulas. This week, we start with two more. The product rule and the quotient rule.

The product rule is used when we have to differentiate a product of two functions, and the quotient rule is used when we have to differentiate a quotient of two functions.

5. Product rule. For two functions and ,

For example, let and .

Be cautious that .

6. Quotient rule. . For two functions and ,

Q. Differentiate the following functions

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(b)

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(c)

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(d)

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(e)

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(f)

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(g)

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(h)

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Derivative and Approximation

We open up this section with a remark. The derivatives of a function represents how fast changes with respect to . That is, is the instantaneous rate of change of at number . In principle,

When increases from to , the value of changes by about units. Similarly, when decreases from to , the value of changes by about units.

Let be a function differentiable at . Then,

In other words, is about .

If we let , then the above equation is

For example, . We have known that and . So we have

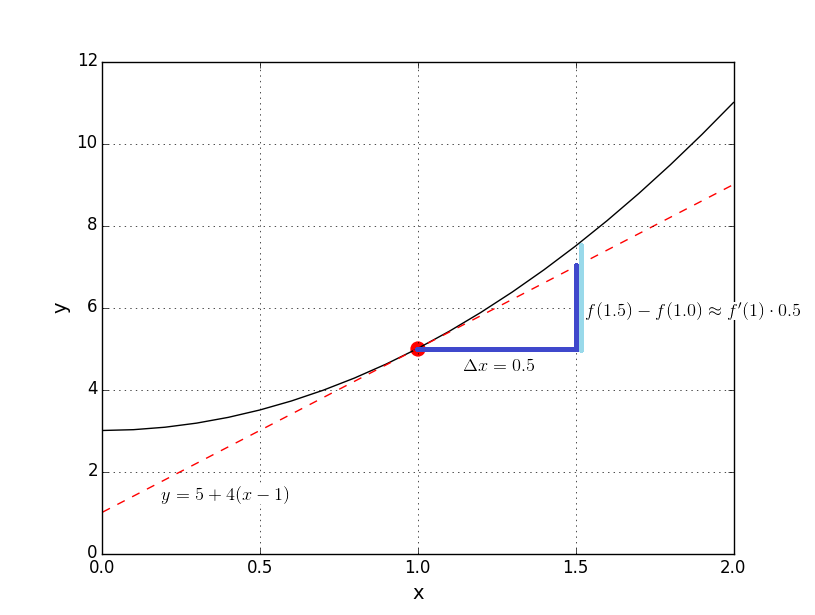
The following table examine how accurate this approximation of is.

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This approximation is about the exact value when is small.

The linear approximation of at is the function

Note that is the tangent line to at . Using the previous example, . The tangent line to at is close to the graph of when is around .



The light blue line represents the quantity and the dark blue line represents the quanitity . This difference between them is getting smaller, as goes from to .

From the graph above, the tangent line at is sloping up, and the function is increasing at . Here, “increasing” means that goes up as goes up.

When , the function is increasing at . Moreover, increases by units, for every additional unit of .

When , the function is decreasing at . Moreover, decreases by units, for every additional unit of .

Last week we talked about the cost function of producing units of a product, as well as the revenue function and the profit function. Here we have new concepts about average cost, average revenue and average profit.

The average cost function gives the average cost per unit.

The marginal average cost function is the derivative of .

The marginal average revenue function is the derivative of .

The marginal average profit function is the derivative of .

Q. [Week 2, P.23] Suppose a company is producing a mini optical mouse. Let be the number of units of mouses produced. Let the cost function (in dollars) be

and the revenue function (in dollars) be

(c) Find the average cost function, ,

(d) and the marginal average cost function, .

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Q. A company can produce computer flash memory devices at a cost of $7 each, while fixed costs are $100 per day. So the cost function is

(a) Find the average cost function, .

(b) Find the marginal average cost function, .

(c) Evaluate at . Interpret your answer.

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Higher-order Derivatives

We can also differentiate the derivative function of a function .

The second derivative of the function is the derivative of .

The third derivative of the function is the derivative of .

In this manner, we define higher-order derivatives of ,

The n-th derivative of is the derivative of the (n-1)-th derivative of .

Starting from the fourth derivative, we denote the n-th derivative function by instead of adding prime until we get to .

is often written as , and is often written as . Similarly, is written as .

Q. Find and for the following function.

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Q. Find , and for the following function.

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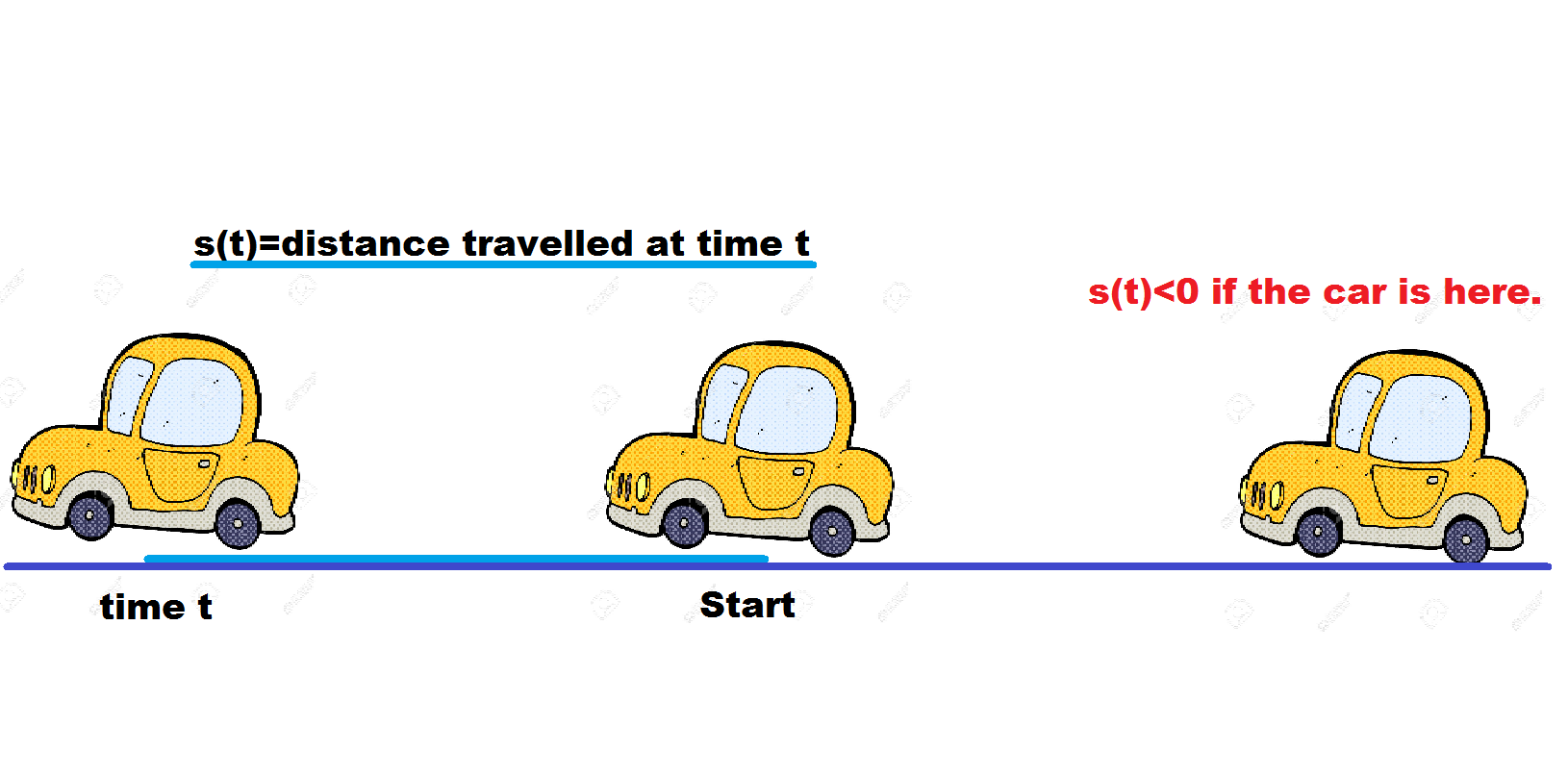
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As an application of the second derivative, we talk about velocity and acceleration. Suppose a car is moving from its starting point along a straight line. At time (in hour), let be the distance travelled by the car in miles.



If the car is behind its starting point, we say that this distance travelled is negative at certain time .

The distance function measures the distance travelled (in miles) by the car at time t (in hours).

The velocity function measures the velocity (i.e. signed speed) of the car at time t (in miles per hour). It is the derivative of .

The acceleration function measures the instantenous rate of change of the velocity of the car at time t (in ) . It is the derivative of .

When the acceleration is positive, the car is speeding up. That is, it is going faster and faster. For example, at time it is going at . Then at time it may go at . When is negative, the car is slowing down. It is going slower and slower.

Q. A truck is driving along a straight road. After hours its distance (in miles) from the starting time at time (in hours) is given by

(a) Find the velocity of the truck after 2 hours.

(b) Find the acceleration of the truck after 4 hours.

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Recall that

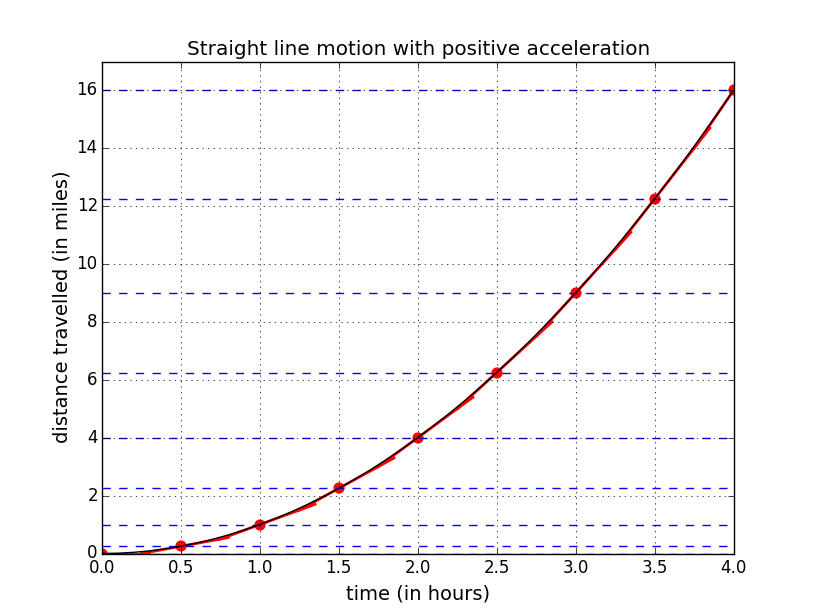
1. When , the function is increasing at .
2. When , the function is decreasing at .

Similarly, we get information about the behaviour of near , by looking at the signs of and .

Suppose you are driving a car along a straight road with acceleration. . First the car is at rest , and .

In the first hour, we move by 1 miles. miles. In the second hour, our car is getting faster, so we move by 4-1=3 miles. Again, our car is much faster in the third hour, and we move by 9-4 = 5 miles in this period of time.

This straight road motion can be described by the following graph, in distance travelled by the car against time.



What do we learn from this acceleration example?

1. When , the velocity keeps increasing. Note the is the slope of the tangent line (red line) at time . The tangent line gets steeper as time goes. So the slope is increasing.
2. When and (we are not moving backward), the graph of resembles this shape of an openning up parabola. Your distance travelled within an hour increases as time goes.

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|  |  | Rate of change of | Slope of the tangent line to function |
|  |  | The quantity is increasing at an increasing rate. | The tangent line is sloping up. The slope increases as increases. |
|  |  | The quantity is increasing at an decreasing rate. | The tangent line is sloping up. The slope decreases as increases. |
|  |  | The quantity is decreasing at an decreasing rate. That is it is getting closer to level off. | The tangent line is sloping down. The slope increases as increases. That is the tangent line is getting flatter. |
|  |  | The quantity is decreasing at an increasing rate. | The tangent line is sloping down. The slope decreases as increases. |

The Chain Rule and the Generalized Power Rule

Here we have two more differentiation rules.

7. Chain rule. For two functions and ,

Suppose and . Then,

We know by power rule. Also . Therefore,

When for some number , . In this case, we can use the generalized power rule to find .

8. Generalized power rule.

In the above example, .

We are treating ) as a whole in and apply the power rule to the later, pretending that is the variable in power rule

So we after . Don’t forget to multiply it by at the end.

Q. Suppose and . Find and its derivative by the chain rule.

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Q. Use the generalized power rule to find the derivative of

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Q. Find .

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1. Constant rule.

2. Power rule.

3. Constant-multiple rule.

4.Sum-Difference rule.

5. Product rule.

6. Quotient rule.

7. Chain rule.

8. Generalized power rule.

Practice on differentiation rules.

Q. Find .

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Q. Find .

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Q. [Webwork HW4, Q13] find .

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Q. [Webwork HW4, Q7]

Suppose , , and .

1. Find if .
2. Find if .

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