*Business Calculus - Week 9*

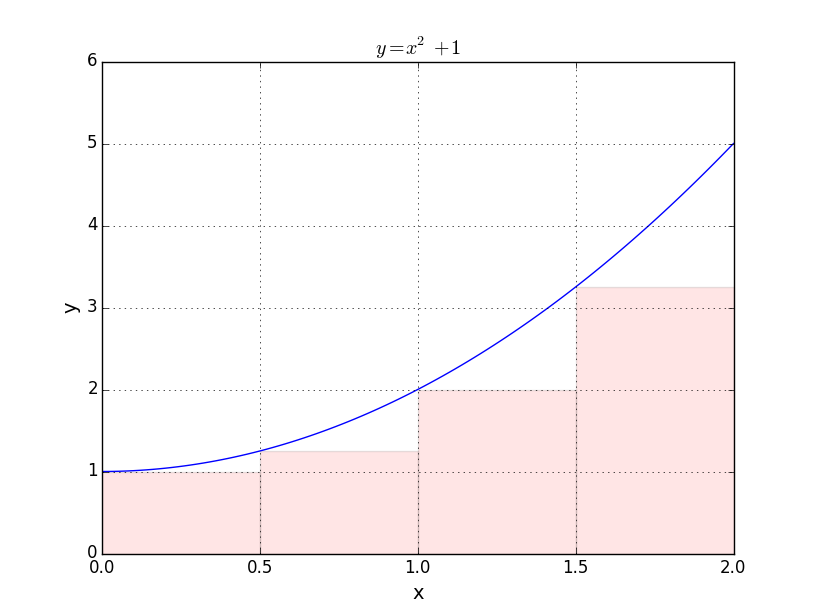
Definite Integrals and Areas

Suppose we have a non-negative function . Let be an interval within the domain of . We may consider the area under the curve and above the x-axis, bounded between two vertical lines and .

Definite integral. The area under the curve from to is called the definite integral of the function from to . It is written as

For example, let and the interval be . We are finding

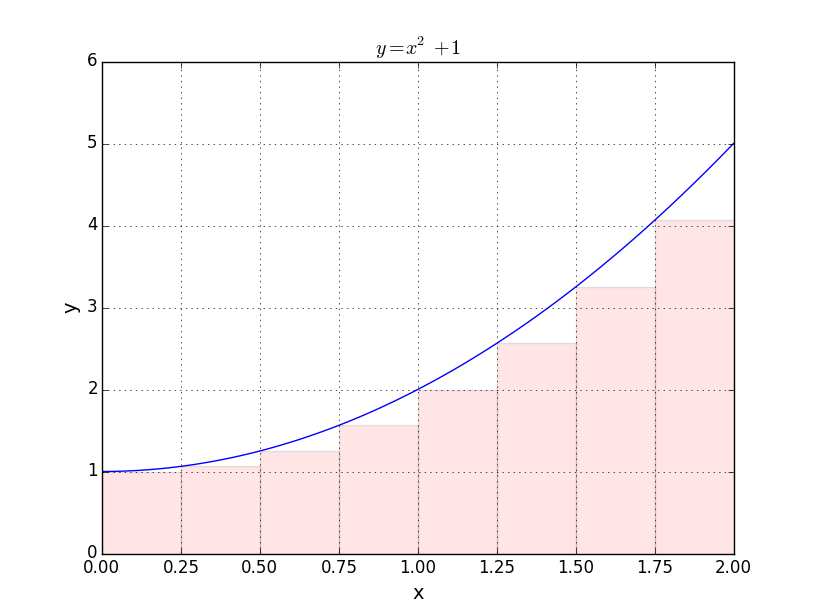
The area under from to can be approximated by rectangles below the curve . First we use four rectangles.



The pink rectangles above are called *left rectangles*, because each has a height equal to the height of the curve at the left-hand edge of the rectangle.

The approximate area by these four rectangles is

in square units. This approximate area doesn’t count the area under the curve outside the red-shaded region. It can be improved by using more left-rectangles. Let say this time we use 8 left-rectangles to estimate the area under .



The new approximate area is

This figure is closer to the definite integral than , since the unshaded region under the curve is smaller in the upper graph.

So, what is the value of ?

Fundamental Theorem of Integral Calculus.

Suppose is a continuous function on an interval . If is one antiderivative of , I.e. . Then,

Before going on, we introduce a notation here. Suppose is the above function. We set

Example. Find . We know that

So we ignore the arbitrary constant and let .

\*\* When you are finding a definite integral, always write down your antiderivative “” in your answer. Don’t just jump to a numerical figure!

Q. Find .

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Properties of definite integrals follow from properties of indefinite integral.

3. Constant-multiple rule for definite integrals. For any constant ,

4. Sum-Difference rule for definite integrals.

Q. Find the definite integral .

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Q. Find the definite integral .

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Q. Find the definite integral

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Q. Find the definite integral

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Example. [Area under curve]

Find the area under from to .

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Q. Find the area under from to .

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Example. [Cost of succession of units, textbook P.336]

For a marginal cost function , the total cost of units to is

A company’s marginal cost function is

where is the number of units. Find the total cost of producing the first hundred units.

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In general, we find the total accumulation at a given rate by definite integral.

The total accumulation at rate from to is

Example. An average child of age years grows at the rate of   
 inches per year (for . Find the total height gain from age 4 to age 9.

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Average Value and Area between Curves

The first application of definite integrals is to find the average value of a function on an interval .

Example. [Population of the United States, textbook P.346]

The population of the United States is predicted to be   
in million people, where is the number of years since 2010. Find the average population between the year 2020 and year 2030.

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Example. [Average temperature]

The temperature at time hours is for . Find the average temperature between time 0 and time 10.

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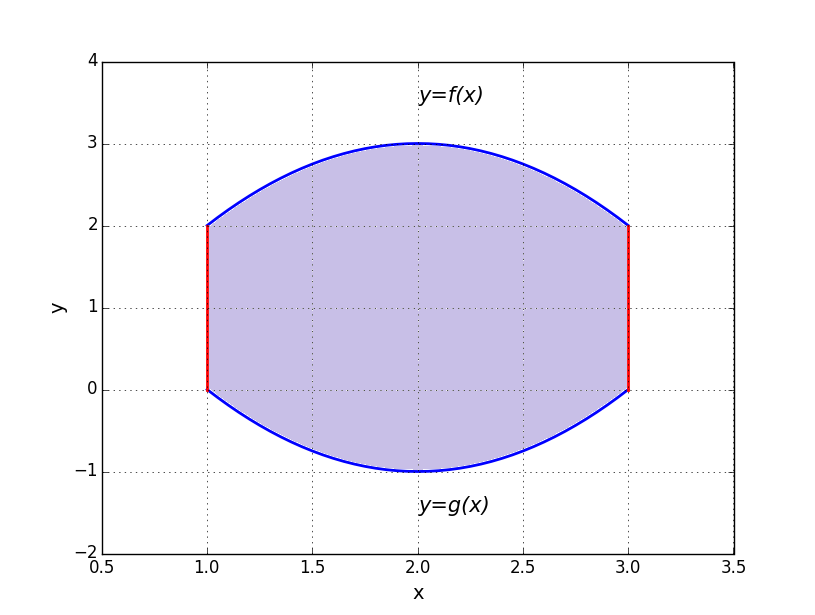
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The second application is to find the area between curves. Suppose and are two functions on the interval . Assume that at any point between and on the x-axis, we have

In this case, we call the upper curve, and the lower curve.

Area between curves. The area between two curves and on , when for all in , is given by the below formula.



The purple-shaded region is the area between and on interval . Note that is the upper function, while is the lower function on the graph above. The area is given by

Example. [Area between curves]

Find the area between and from to .

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Q. Find the area between and from to .

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Q. Find the area bounded by the curves and .

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Consumers’ Surplus, Producers’ Surplus & Gini Index

The third application is on the classical economics, Demand & Supply. Weeks before, we mentioned that a demand equation is the relation between the price of a product, and the quantity demanded at that price. We also have introduced the demand function , which gives us the quantity demanded by consumers at the price .

Now, we switch the roles of and in the demand function. We introduce the demand function (or demand curve) as a function of .

It gives us the price of a product at which units of products will be sold.

The market price is the price at which all transactions actually occur. Consumers buy good at this market price, and suppliers sell good at this market price.

\*\*Here we assume that all prices are in dollars.

Consumers’ surplus.   
Suppose is a demand function and the market price is . Let be the quantity demanded (=demand level) at price , i.e. .   
The consumers’ surplus (CS) is the area between the demand curve and the market price.

Consumers’ surplus measures the total benefit consumers get under the market price .

The supply function for a product, gives us the price at which units of the product are supplied.

Similar to consumers’ surplus, we have the producers’ surplus (PS). It measures the total benefit producers get at the market price.

Producers’ surplus.  
Suppose is a demand function and the market price is . Let be the quantity supplied (= supply level = demand level) at price , i.e. . The producers’ surplus (PS) is the area between the market price and the supply curve.

Given a demand function and a supply function , we can find out the market price and market demand. Market demand is the quantity of a product exactly sold and bought in the marketplace.  
 is the -value that solves

Then, the market price . The pair is called the market equilibrium.

Q. For the demand function and the demand level , find the consumers’ surplus.

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Q. For the supply function and the demand level , find the producers’ surplus.

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Q. Given the demand and supply functions

(a) find the market demand and market price,

(b) find the consumers’ surplus at the market demand,

(c) and find the producers’ surplus at the market demand.

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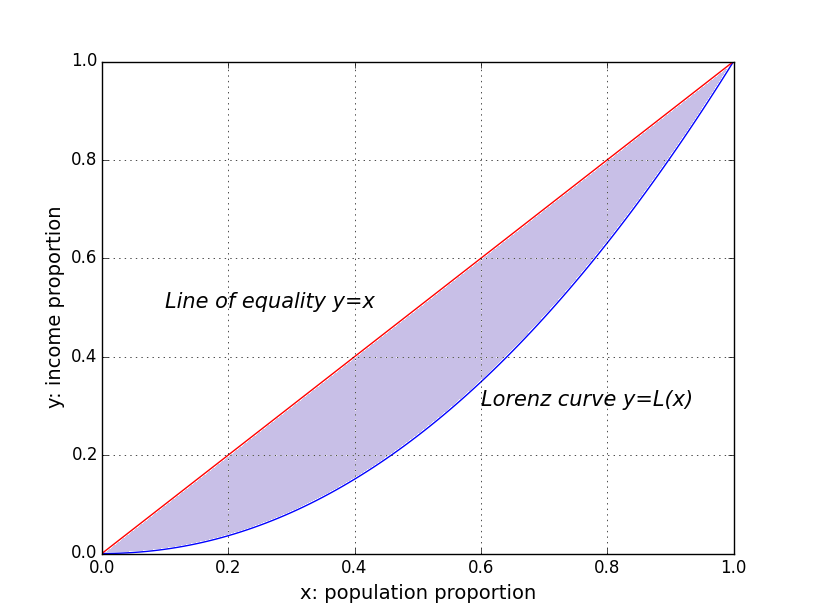
The fourth application is on income distribution. The Lorenz curve and Gini index help us measure income inequality in a city or over a country.

The Lorenz curve gives us the proportion of total income earned by the lowest of population (or of households) in income.

Suppose in 2016, the city A carries out a census and obtains the following data about household incomes.

|  |  |
| --- | --- |
| Proportion of population  (from lowest to highest) | Proportion of income |
| 0.2 | 0.032 |
| 0.4 | 0.1 |
| 0.6 | 0.226 |
| 0.8 | 0.43 |
| 1.0 | 1 |

For example, it means the bottom 20% of population earns 3.2% of the total income. We plot the income population against population proportion to obtain the Lorenz curve.



For convenience, we are not using the above data, but use them to approximate instead. We get

The line above is called the line of equality. Income equality is achieved if the Lorenz curve is exactly the line of equality .

For a Lorenz curve , Gini index is two times the area between the line of equality and the Lorenz curve.

The Gini index is always between 0 and 1.

The area between the line of equality and the Lorenz curve is shaded in purple in the above graph.

Q. Find the Gini index for the given Lorenz curve.

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Integration by Substitution

In this section, we learn a new integration technique: integration by substitution. Suppose we have a function . A change in is denoted by . When this change is very small, we denote it by

is called the differential of . Since is a function of , we have

It means that when is a number and the change in ( is , the change in is given by

The right hand side is just a number. Basically, we only use the differentials and within integration. Elsewhere we represent the changes of and by and .

Integration by substitution.

Suppose and are two functions. Putting , we have

Note that gives as in above discussion. So we are replacing (1) by , and (2) by .

If we are looking at definite integrals, we use the below formula.

The substitution remains .

In the following, we study 3 particular cases.

(I) Suppose is a polynomial, and is a number

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Q. Find .

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Q. Find

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Q. Find .

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(II) Suppose is a polynomial.

Q. Find

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Q. Find .

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Q. Find .

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Q. Find .

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(III) Suppose is a polynomial.

Q. Find .

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Q. Find .

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Q\*. Find

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Q\*. Find

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