Laplacian Operator (II)

4. Third covariant derivatives of function

Let (M, g) be an n-dimensional Riemannian manifold. Let $\{e_1, e_2, \cdots, e_n\}$ be an orthonormal frame on M. Suppose f is a smooth real-valued function on M. Recall that the first and second covariant derivatives of f are defined by

$$\nabla f \,=\, f_j\, e_j \quad \text{and} \quad \nabla^2 f \,=\, f_{ij}\, \omega_j \otimes e_i \,=\, \big(df_i + f_j\, \omega_{ji}\big) \otimes e_i$$

Let $S = \nabla^2 f$. Note that

$$\begin{aligned} (\nabla_X S)(Y) \,&=\, \nabla_X\big(S(Y)\big) - S(\nabla_X Y) \,=\, \nabla_X \nabla_Y (\nabla f) \,-\, \nabla_{\nabla_X Y}(\nabla f)\,, \\ (\nabla_Y S)(X) \,&=\, \nabla_Y\big(S(X)\big) - S(\nabla_Y X) \,=\, \nabla_Y \nabla_X (\nabla f) \,-\, \nabla_{\nabla_Y X}(\nabla f)\,. \end{aligned}$$

Therefore,

$$(\nabla_X S)(Y) \,-\, (\nabla_Y S)(X) \,=\, R(X, Y)\, \nabla f\,. \tag{1}$$

We are computing for ∇S as follows.

$$\begin{aligned} \nabla S \,&=\, \nabla\big(f_{ij}\, \omega_j \otimes e_i\big) \\ &=\, df_{ij}\, \omega_j \otimes e_i \,+\, f_{ij}\, d\omega_j \otimes e_i \,-\, f_{ij}\, \omega_j \wedge \nabla e_i \\ &=\, df_{ij}\, \omega_j \otimes e_i \,+\, f_{ij}\, \big(\omega_{jk} \wedge \omega_k\big) \otimes e_i \,-\, f_{ij}\, \omega_j \wedge \big(\omega_{ik} \otimes e_k\big) \\ &=\, \big(df_{ij} \,+\, f_{ik}\, \omega_{kj} \,+\, f_{kj}\, \omega_{ki}\big) \,\wedge\, \omega_j \otimes e_i\,. \end{aligned}$$

If we write $\nabla^3 f \,=\, f_{ijk}\, \omega_k \wedge \omega_j \otimes e^i$, then we have

$$f_{ijk}\, \omega_k \,=\, df_{ij} \,+\, f_{ip}\, \omega_{pj} \,+\, f_{qj}\, \omega_{qi}\,. \tag{2}$$

In the following, we justify the equalities

$$f_{ij} \,=\, g\big(\nabla_{e_j}(\nabla f)\,,\, e_i\big) \quad \text{and} \quad f_{ijk} = g\big((\nabla S)(e_k, e_j)\,,\, e_i\big)\,.$$

They clarify that the definition of covariant derivatives here match our usual understanding.

$$\begin{aligned} \nabla_{e_j}(\nabla f) \,&=\, \nabla_{e_j}\big(f_k\, e_k\big) \\ &=\, e_j\, e_k(f)\, e_k \,+\, f_k\, \nabla_{e_j} e_k \\ &=\, e_j\, e_k(f)\, e_k \,+\, f_k\, \Gamma^l_{jk}\, e_l \\ &=\, \big(e_j\, e_i(f) \,+\, f_k\, \Gamma^i_{jk}\big)\, e_i \\ &=\, f_{ij}\, e_i \end{aligned}$$

So the first identity is proven. Moreover, we have

$$
\begin{aligned}
(\nabla S)(e_k, e_j) &= \big(\nabla_{e_k}(\nabla^2 f)\big)(e_j) \\
&= \nabla_{e_k}\big(\nabla_{e_j}(\nabla f)\big) - (\nabla^2 f)(\nabla_{e_k} e_j) \\
&= \nabla_{e_k}\big(f_{ij}\, e_i\big) - \Gamma^l_{kj}\, \nabla_{e_l}(\nabla f) \\
&= e_k(f_{ij})\, e_i + f_{ij}\, \Gamma^l_{ki}\, e_l - \Gamma^l_{kj}\, f_{pl}\, e_p \\
&= \big(e_k(f_{ij}) + f_{pj}\, \Gamma^i_{kp} - f_{ip}\, \Gamma^p_{kj}\big)\, e_i \\
&= \big(df_{ij}(e_k) + f_{pj}\, \omega_{pi}(e_k) + f_{ip}\, \omega_{pj}(e_k)\big)\, e_i \\
&= f_{ijk}\, e_i\, .
\end{aligned}
$$

Therefore, the second identity also holds. In equation (1), we could put $X = e_k$ and $Y = e_j$.

$$
\begin{aligned}
(\nabla_{e_k} S)(e_j) - (\nabla_{e_j} S)(e_k) &= R(e_k, e_j)\, \nabla f \\
\Longrightarrow \quad f_{ijk}\, e_i - f_{ikj}\, e_i &= f_p\, R_{kjpq}\, e_q
\end{aligned}
$$

By comparison, we have

$$
f_{ijk} - f_{ikj} = f_p\, R_{ipjk}\, . \tag{3}
$$

This identity is called the Ricci identity.

5. Covariant derivatives of differential forms

The covariant derivatives of the covectors ω_j in the dual frame can be found by:

$$
(\nabla_X \omega_j)(e_i) = X\big(\omega_j(e_i)\big) - \omega_j(\nabla_X e_i) = -\omega_j(\nabla_X e_i)\, .
$$

So we have $\nabla_X(\omega_j) = -\omega_j(\nabla_X e_i)\, \omega_i$ and hence

$$
\nabla(\omega_j) = -\Gamma^j_{ki}\, \omega_k \otimes \omega_i = -\omega_{ij} \otimes \omega_i = \omega_{ji} \otimes \omega_i\, .
$$

Let ω be a p-form on the n-dimensional manifold M. We may let

$$
\omega = \sum_{i_1, \cdots, i_p = 1}^{n} a_{i_1 \cdots i_{p-1} i_p}\, \omega_{i_p} \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \omega_{i_1}\, ,
$$

assuming that $a_{i_1 \cdots i_{p-1} i_p} \neq 0$ only when $i_p < i_{p-1} < \cdots < i_1$.

For any vector X, we have

$$
\begin{aligned}
\nabla_X \omega &= da_{i_1 \cdots i_p}(X)\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} + a_{i_1 \cdots i_p}\, \big(\nabla_X(\omega_{i_p})\big) \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + a_{i_1 \cdots i_p}\, \omega_{i_p} \wedge \big(\nabla_X(\omega_{i_{p-1}})\big) \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \cdots + a_{i_1 \cdots i_p}\, \omega_{i_p} \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \big(\nabla_X(\omega_{i_1})\big) \\
&= da_{i_1 \cdots i_p}(X)\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} + a_{i_1 \cdots i_p}\, \omega_{i_p j_p}(X)\, \omega_{j_p} \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + a_{i_1 \cdots i_p}\, \omega_{i_{p-1} j_{p-1}}(X)\, \omega_{i_p} \wedge \omega_{j_{p-1}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \cdots + a_{i_1 \cdots i_p}\, \omega_{i_1 j_1}(X)\, \omega_{i_p} \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \omega_{j_1}\, .
\end{aligned}
$$

We would write

$$\nabla_X \omega \;=\; \big(da_{i_1\cdots i_p} + \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p}\,\omega_{j_r i_r}\big)(X)\;\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\,.$$

Therefore,

$$\nabla\omega \;=\; \Big(da_{i_1\cdots i_p} \;+\; \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p}\,\omega_{j_r i_r}\Big)\otimes\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\,. \tag{4}$$

If we let $\nabla\omega \;=\; a_{i_1\cdots i_p,j}\,\omega_j\otimes\big(\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\big)$, it results in

$$a_{i_1\cdots i_p,j}\,\omega_j \;=\; da_{i_1\cdots i_p} \;+\; \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p}\,\omega_{j_r i_r}\,.$$

The coefficient $a_{i_1\cdots i_p,j}$ is found by

$$a_{i_1\cdots i_p,j} \;=\; e_j\big(a_{i_1\cdots i_p}\big) \;+\; \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p}\,\Gamma^{i_r}_{j j_r}\,.$$

For the moment, we may define every coefficient $a_{j_1\cdots j_p}$ by

$$a_{j_1\cdots j_p} \;=\; \det\Big[\omega_{j_r}(e_{i_s})\Big]\,a_{i_1\cdots i_p}$$

where $(i_1,\cdots,i_p)$ is a rearrangement of $(j_1,\cdots,j_p)$ such that $i_p < i_{p-1} < \cdots < i_1$. As a result

$$\omega \;=\; a_{i_1\cdots i_p}\,\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\,.$$

Note that we have

$$\omega\big(e_{j_p},\cdots,e_{j_1}\big) \;=\; \det\Big[\omega_{j_r}(e_{i_s})\Big]\,\omega\big(e_{i_p},\cdots,e_{i_1}\big) \;=\; \det\Big[\omega_{j_r}(e_{i_s})\Big]\,a_{i_1\cdots i_p} \;=\; a_{j_1\cdots j_p}\,.$$

Therefore,

$$\begin{aligned}
&(\nabla_{e_j}\omega)\big(e_{i_p},e_{i_{p-1}},\cdots,e_{i_1}\big)\\
=\;& e_j\Big(\omega\big(e_{i_p},\cdots,e_{i_1}\big)\Big) \;-\; \sum_{r=1}^{p}\omega\Big(e_{i_p},\cdots,\nabla_{e_j}e_{i_r},\cdots,e_{i_1}\Big)\\
=\;& e_k\big(a_{j_1\cdots j_p}\big) \;-\; \Gamma^{j_r}_{j i_r}\,a_{i_1\cdots j_r\cdots i_p}\\
=\;& e_k\big(a_{j_1\cdots j_p}\big) \;+\; \Gamma^{i_r}_{j j_r}\,a_{i_1\cdots j_r\cdots i_p}\\
=\;& a_{i_1\cdots i_p,j}\,.
\end{aligned}$$

In the symmetric representation of a differential form, we validate that

$$a_{i_1\cdots i_p,j} \;=\; (\nabla_{e_j}\omega)\big(e_{i_p},e_{i_{p-1}},\cdots,e_{i_1}\big)\,.$$

If we define $\omega = \sum a_{i_1 \cdots i_p}\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}$ note that

$$\begin{aligned}
d\omega &= \sum_{i_1, \cdots, i_p} da_{i_1 \cdots i_p} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{i_1, \cdots, i_p} \sum_r (-1)^{p-r}\, a_{i_1 \cdots i_p}\, \omega_{i_p} \wedge \cdots \wedge d\omega_{i_r} \wedge \cdots \wedge \omega_{i_1} \\
&= \sum da_{i_1 \cdots i_p} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum (-1)^{p-r}\, a_{i_1 \cdots i_r \cdots i_p}\, \omega_{i_r j_r} \wedge \omega_{j_r} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&= \sum da_{i_1 \cdots i_p} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum a_{i_1 \cdots i_r \cdots i_p}\, \omega_{i_r j_r} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{j_r} \wedge \cdots \wedge \omega_{i_1} \\
&= \sum \Big(da_{i_1 \cdots i_p} + \sum a_{i_1 \cdots j_r \cdots i_p}\, \omega_{j_r i_r} \Big) \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&= \sum a_{i_1 \cdots i_p, k}\, \omega_k \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \,.
\end{aligned}$$

As a result, we have

$$d\omega = \sum_{i_1, \cdots, i_p, k} a_{i_1 \cdots i_p, k}\, \omega_k \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \,.$$

For any vector field Y, $\nabla_Y \omega$ is a p-form on M. The second covariant derivative of ω is found by

$$\big(\nabla_X \nabla\omega\big)(Y) \,=\, \nabla_X(\nabla_Y\omega) \,-\, \nabla_{\nabla_X Y}\,\omega\,.$$

Compute for the first term.

$$\begin{aligned}
& \Big(\nabla_X(\nabla_Y\omega)\Big)(u_1,\cdots,u_p) \\
= \;& X\Big((\nabla_Y\omega)(u_1,\cdots,u_p)\Big) \,-\, \sum_{r=1}^{p}(\nabla_Y\omega)\big(u_1,\cdots,\nabla_X u_r,\cdots,u_p\big) \\
= \;& X\Big(Y\big(\omega(u_1,\cdots,u_p)\big)\Big) \,-\, \sum_{s=1}^{p} X\Big(\omega\big(u_1,\cdots,\nabla_Y u_s,\cdots,u_p\big)\Big) \\
& -\sum_{r=1}^{p} Y\Big(\omega\big(u_1,\cdots,\nabla_X u_r,\cdots,u_p\big)\Big) \,+\, \sum_{r\neq s}\omega\big(u_1,\cdots,\nabla_X u_r,\cdots,\nabla_Y u_s,\cdots,u_p\big) \\
& +\sum_{r=1}^{p}\omega\big(u_1,\cdots,\nabla_Y\nabla_X u_r,\cdots,u_p\big)
\end{aligned}$$

Compute for the second term.

$$\begin{aligned}
& \big(\nabla_{\nabla_X Y}\,\omega\big)(u_1,\cdots,u_p) \\
= \;& (\nabla_X Y)\Big(\omega\big(u_1,\cdots,u_p\big)\Big) \,-\, \sum_{r=1}^{p}\omega\big(u_1,\cdots,\nabla_{\nabla_X Y}u_r,\cdots,u_p\big)
\end{aligned}$$

Note that by symmetry, we have

$$\begin{aligned}
& \Big(\nabla_X(\nabla_Y\omega) \,-\, \nabla_Y(\nabla_X\omega)\Big)(u_1,\cdots,u_p) \\
= \;& [X,Y]\Big(\omega(u_1,\cdots,u_p)\Big) \,+\, \sum_{r=1}^{p}\omega\Big(u_1,\cdots,(\nabla_Y\nabla_X-\nabla_X\nabla_Y)u_r,\cdots,u_p\Big)\,.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
& \Big(\nabla_{\nabla_X Y}\,\omega-\nabla_{\nabla_Y X}\,\omega\Big)(u_1,\cdots,u_p) \\
= \;& [X,Y]\Big(\omega(u_1,\cdots,u_p)\Big) \,-\, \sum_{r=1}^{p}\omega\big(u_1,\cdots,\nabla_{[X,Y]}u_r,\cdots,u_p\big)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Big((\nabla_X\nabla\omega)(Y) \,-\, (\nabla_Y\nabla\omega)(X)\Big)(u_1,\cdots,u_p) \\
= \;& \sum_{r=1}^{p}\omega\Big(u_1,\cdots,\big(\nabla_Y\nabla_X-\nabla_X\nabla_Y+\nabla_{[X,Y]}\big)\,u_r,\cdots,u_p\Big) \\
= \;& \sum_{r=1}^{p}\omega\Big(u_1,\cdots,R(Y,X)u_r,\cdots,u_p\Big)\,.
\end{aligned}$$

We summary this result as Equation (6).

$$\Big((\nabla_X \nabla\omega)(Y) \,-\, (\nabla_Y \nabla\omega)(X)\Big)(u_1, \cdots, u_p) \,=\, \sum_{r=1}^{p} \omega\Big(u_1, \cdots, R(Y,X)u_r, \cdots, u_p\Big)\,. \qquad (5)$$

We are now ready to compute for the coefficients of the tensor $\nabla^2\omega$. Let ω be the p-form

$$\omega \,=\, a_{i_1\cdots i_p}\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\,.$$

$\nabla\omega$ can be expressed as

$$\nabla\omega \,=\, a_{i_1\cdots i_p,j}\, \omega_j \otimes \big(\omega_{i_p} \wedge \omega_{i_{p-1}} \wedge \cdots \wedge \omega_{i_1}\big)\,.$$

Take the covariant derivative of $\nabla\omega$.

$$\begin{aligned}
\nabla\big(\nabla\omega\big) \;&=\; da_{i_1\cdots i_p,j} \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \,+\, a_{i_1\cdots i_p,j}\, d\omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \\
&\quad - a_{i_1\cdots i_p,j}\, \omega_j \,\wedge\, \nabla\big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \\
&=\; da_{i_1\cdots i_p,j} \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \,+\, a_{i_1\cdots i_p,j}\, \omega_{jk} \wedge \omega_k \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \\
&\quad - \sum_{r,j_r} a_{i_1\cdots i_p,j}\, \omega_j \wedge \omega_{i_r j_r} \otimes \Big(\omega_{i_p} \wedge \cdots \wedge \omega_{j_r} \wedge \cdots \wedge \omega_{i_1}\Big) \\
&=\; da_{i_1\cdots i_p,j} \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \,+\, a_{i_1\cdots i_p,s}\, \omega_{sj} \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \\
&\quad + \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p,j}\, \omega_{j_r i_r} \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big) \\
&=\; \Big(da_{i_1\cdots i_p,j} \,+\, a_{i_1\cdots i_p,s}\, \omega_{sj} \,+\, \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p,j}\, \omega_{j_r i_r}\Big) \,\wedge\, \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big)\,.
\end{aligned}$$

If we write $\nabla^2\omega \,=\, a_{i_1\cdots i_p,jk}\; \omega_k \wedge \omega_j \otimes \big(\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\big)$, then we have

$$a_{i_1\cdots i_p,jk}\, \omega_k \,=\, da_{i_1\cdots i_p,j} \,+\, a_{i_1\cdots i_p,s}\, \omega_{sj} \,+\, \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p,j}\, \omega_{j_r i_r}\,. \qquad (6)$$

If we assume that $a_{i_1\cdots i_p} \,=\, \omega\big(e_{i_p}, \cdots, e_{i_1}\big)$ as before, we may justify that

$$a_{i_1\cdots i_p,jk} \,=\, \big((\nabla_{e_k} \nabla\omega)(e_j)\big)(e_{i_p}, \cdots, e_{i_1})\,.$$

Expanding the right hand side of the equation, we have

$$\begin{aligned}
&\big((\nabla_{e_k} \nabla\omega)(e_j)\big)(e_{i_p}, \cdots, e_{i_1}) \\
=\;& e_k\Big((\nabla_{e_j}\omega)(e_{i_p}, \cdots, e_{i_1})\Big) \,-\, (\nabla_{\nabla_{e_k} e_j}\omega)(e_{i_p}, \cdots, e_{i_1}) \,-\, \sum_{r=1}^{p} (\nabla_{e_j}\omega)\big(e_{i_p}, \cdots, \nabla_{e_k} e_{i_r}, \cdots, e_{i_1}\big) \\
=\;& e_k\big(a_{i_1\cdots i_p,j}\big) \,-\, a_{i_1\cdots i_p,s}\, \Gamma^{s}_{kj} \,-\, \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p,j}\, \Gamma^{j_r}_{k i_r} \\
=\;& e_k\big(a_{i_1\cdots i_p,j}\big) \,+\, a_{i_1\cdots i_p,s}\, \Gamma^{j}_{ks} \,+\, \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p,j}\, \Gamma^{i_r}_{k j_r} \\
=\;& a_{i_1\cdots i_p,jk}\,.
\end{aligned}$$

Put the above identity to Equation (5).

$$\begin{aligned}\Big((\nabla_{e_k}\nabla\omega)(e_j) - (\nabla_{e_j}\nabla\omega)(e_k)\Big)(e_{i_p},\cdots,e_{i_1}) &= a_{i_1\cdots i_p,jk} - a_{i_1\cdots i_p,kj} \\ \sum_{r=1}^{p}\omega\Big(e_{i_p},\cdots,R(e_j,e_k)e_{i_r},\cdots,e_{i_1}\Big) &= \sum_{r,j_r} R_{jki_rj_r}\, a_{i_1\cdots j_r\cdots i_p}\end{aligned}$$

Therefore, Equation (5) becomes:

$$a_{i_1\cdots i_p,jk} - a_{i_1\cdots i_p,kj} = \sum_{r,j_r} a_{i_1\cdots j_r\cdots i_p}\, R_{jki_rj_r} \tag{7}$$

when ω is in the symmetric representation.

6. Codifferential of differential forms

When (M,g) is an n-dimensional manifold, it has a positive volume form

$$\Omega = \omega_1\wedge\omega_2\wedge\cdots\wedge\omega_n\,.$$

The Hodge star operator $*$ on M sends any p form to an $(n-p)$ form,

$$*\big(\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\big) = \mathrm{sgn}(I,I^c)\,\omega_{i_n}\wedge\cdots\wedge\omega_{i_{p+1}}\,.$$

Here we set the p-tuple $I = (i_p,\cdots,i_1)$. I^c is the complement of I, defined by $(i_n,\cdots,i_{p+1})$ such that the set $\big\{i_1,\cdots,i_p,i_{p+1},\cdots,i_n\big\}$ coincides with $\big\{1,\cdots,n\big\}$. Moreover, we let

$$\big(\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}\big)\wedge\big(\omega_{i_n}\wedge\cdots\wedge\omega_{i_{p+1}}\big) = \mathrm{sgn}(I,I^c)\,\Omega\,.$$

As a remark, for every unordered combination of p elements in $\{1,\cdots,n\}$, there is a unique ordered permutation $I = (i_p,i_{p-1},\cdots,i_1)$ such that $i_p < i_{p-1} < \cdots < i_1$. Then, we may specify a complementary tuple of I, $I^c = \big(i_n,\cdots,i_{p+1}\big)$ with $i_n < i_{n-1} < \cdots < i_{p+1}$. Under this approach, we could specify I^c without ambiguity.

The codifferential operator δ sends a p-form to a $(p-1)$-form through

$$\delta\omega = (-1)^{n(p+1)+1} * d * \omega\,.$$

Given that $\omega = a_{i_1\cdots i_p}\,\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}$, we may prove that

$$\delta\omega = \sum_{r=1}^{p}(-1)^{r+p^2+1} a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_p}\wedge\cdots\wedge\omega_{i_{r+1}}\wedge\omega_{i_{r-1}}\wedge\cdots\wedge\omega_{i_1}\,. \tag{8}$$

For simplicity, we assume that the p-tuple $I = (k_p,\cdots,k_1)$ is fixed, so ω is defined by

$$\omega = a_{k_1\cdots k_p}\,\omega_{k_p}\wedge\cdots\wedge\omega_{k_1}\,.$$

The right hand side of Equation (8) is expanded as follows.

$$
\begin{aligned}
& \sum_{r=1}^{p}(-1)^{r+p^2+1}\, a_{i_1\cdots i_r\cdots i_p,i_r}\, \omega_{i_p}\wedge\cdots\wedge\omega_{i_{r+1}}\wedge\omega_{i_{r-1}}\wedge\cdots\wedge\omega_{i_1} \\
= \; & \sum_{r,i_r}(-1)^{r+p^2+1} e_{i_r}\left(a_{i_1\cdots i_p}\right)\omega_{i_p}\wedge\cdots\wedge\omega_{i_{r+1}}\wedge\omega_{i_{r-1}}\wedge\cdots\wedge\omega_{i_1} \\
& + \sum_{r,i_r}(-1)^{r+p^2+1}\sum_{s,j_s}\left(a_{i_1\cdots j_s\cdots i_p}\,\Gamma^{i_s}_{i_r j_s}\right)\omega_{i_p}\wedge\cdots\wedge\omega_{i_{r+1}}\wedge\omega_{i_{r-1}}\wedge\cdots\wedge\omega_{i_1} \\
= \; & \sum_{r=1}^{p}(-1)^{r+p^2+1}\, e_{k_r}\left(a_{k_1\cdots k_p}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\sum_{s\neq r}\left(a_{k_1\cdots j_s\cdots k_r\cdots k_p}\,\Gamma^{i_s}_{k_r j_s}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{i_s}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\left(a_{k_1\cdots j_r\cdots k_p}\,\Gamma^{i_r}_{i_r j_r}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
= \; & \sum_{r=1}^{p}(-1)^{r+p^2+1}\, e_{k_r}\left(a_{k_1\cdots k_p}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{(r,s):\, r\neq s}(-1)^{r+p^2+1}\, a_{k_1\cdots k_s\cdots k_r\cdots k_p}\,\Gamma^{i_s}_{k_r k_s}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{i_s}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{i_r}_{i_r k_r}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
= \; & \sum_{r=1}^{p}(-1)^{r+p^2+1}\, e_{k_r}\left(a_{k_1\cdots k_p}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r\neq s}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{k_r}_{k_r k_s}\,(-1)^{r-s+1}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_s}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r\neq s}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{k_{\alpha(s)}}_{k_r k_s}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_{\alpha(s)}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{i_r}_{i_r k_r}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
= \; & \sum_{r=1}^{p}(-1)^{r+p^2+1}\, e_{k_r}\left(a_{k_1\cdots k_p}\right)\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r\neq s}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{k_{\alpha(s)}}_{k_r k_s}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_{\alpha(s)}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{s=1}^{p}(-1)^{p^2+s}\, a_{k_1\cdots k_p}\,\Gamma^{k_r}_{k_r k_s}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_s}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{i_r}_{i_r k_r}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1}
\end{aligned}
$$

Therefore, the right hand side becomes:

$$\begin{aligned}
= \quad & \sum_{r=1}^{p}(-1)^{r+p^2+1}\, e_{k_r}\big(a_{k_1\cdots k_p}\big)\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r\neq s}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{k_\alpha}_{k_r k_s}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega^{(s)}_{k_\alpha}\wedge\cdots\wedge\omega_{k_1} \\
& + \sum_{r=1}^{p}(-1)^{r+p^2+1}\, a_{k_1\cdots k_p}\,\Gamma^{k_\alpha}_{k_\alpha k_r}\,\omega_{k_p}\wedge\cdots\wedge\widetilde{\omega_{k_r}}\wedge\cdots\wedge\omega_{k_1}\,.
\end{aligned}$$

In the second term above, $\omega^{(s)}_{k_\alpha}$ means that the term ω_{k_α} lies in the s-th position of the wedge product. In order to justify Equation (8), we specify the complement of the p-tuple I as

$$I^c \,=\, \big(k_n, k_{n-1}, \cdots, k_{p+1}\big)\,.$$

Immediately, we have

$$*\,\omega \,=\, a_{k_1\cdots k_p}\,\mathrm{sgn}(I,I^c)\,\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}}\,.$$

Therefore,

$$\begin{aligned}
d*\omega \quad = \quad & \mathrm{sgn}(I,I^c)\,da_{k_1\cdots k_p}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{\alpha=p+1}^{n} a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,(-1)^{n-\alpha}\,\omega_{k_n}\wedge\cdots\wedge d\omega_{k_\alpha}\wedge\cdots\wedge\omega_{k_{p+1}} \\
= \quad & \sum_{r=1}^{p}\mathrm{sgn}(I,I^c)\,e_{k_r}\big(a_{k_1\cdots k_p}\big)\,\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{\alpha=p+1}^{n}(-1)^{n-\alpha}\,a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,\omega_{k_\alpha j}\wedge\omega_j\wedge\omega_{k_n}\wedge\cdots\wedge\widetilde{\omega_{k_\alpha}}\wedge\cdots\wedge\omega_{k_{p+1}} \\
= \quad & \sum_{r=1}^{p}\mathrm{sgn}(I,I^c)\,e_{k_r}\big(a_{k_1\cdots k_p}\big)\,\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{\alpha=p+1}^{n}(-1)^{n-\alpha}\,a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,\Gamma^{j}_{m\,k_\alpha}\,\omega_m\wedge\omega_j\wedge\omega_{k_n}\wedge\cdots\wedge\widetilde{\omega_{k_\alpha}}\wedge\cdots\wedge\omega_{k_{p+1}} \\
= \quad & \sum_{r=1}^{p}\mathrm{sgn}(I,I^c)\,e_{k_r}\big(a_{k_1\cdots k_p}\big)\,\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{\alpha}(-1)^{n-\alpha}\,a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,\Gamma^{j}_{k_\alpha\,k_\alpha}\,\omega_{k_\alpha}\wedge\omega_j\wedge\omega_{k_n}\wedge\cdots\wedge\widetilde{\omega_{k_\alpha}}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{s=1}^{p}(-1)^{n-\alpha}\,a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,\Gamma^{j}_{k_s\,k_\alpha}\,\omega_{k_s}\wedge\omega_j\wedge\omega_{k_n}\wedge\cdots\wedge\widetilde{\omega_{k_\alpha}}\wedge\cdots\wedge\omega_{k_{p+1}} \\
= \quad & \sum_{r=1}^{p}\mathrm{sgn}(I,I^c)\,e_{k_r}\big(a_{k_1\cdots k_p}\big)\,\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& - \sum_{r,\alpha} a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,\Gamma^{k_r}_{k_\alpha\,k_\alpha}\,\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\omega_{k_{p+1}} \\
& + \sum_{s\neq r} a_{k_1\cdots k_p}\mathrm{sgn}(I,I^c)\,(-1)^{n-\alpha}\,\Gamma^{k_r}_{k_s\,k_\alpha}\,\omega_{k_s}\wedge\omega_{k_r}\wedge\omega_{k_n}\wedge\cdots\wedge\widetilde{\omega_{k_\alpha}}\wedge\cdots\wedge\omega_{k_{p+1}}\,.
\end{aligned}$$

We then compute for $(-1)^{n(p+1)+1} \, * d * \omega$. First of all,

$$(-1)^{n(p+1)+1} \, * \left(\sum_{r=1}^{p} \operatorname{sgn}(I, I^c) \, e_{k_r}\big(a_{k_1 \cdots k_p}\big) \, \omega_{k_r} \wedge \omega_{k_n} \wedge \cdots \wedge \omega_{k_{p+1}} \right)$$

$$\begin{aligned}
&= \sum_r \left[\begin{array}{l} (-1)^{n(p+1)+1} \operatorname{sgn}(I, I^c) \, e_{k_r}\big(a_{k_1 \cdots k_p}\big) \operatorname{sgn}\Big(\big(k_r, k_n, \cdots, k_{p+1}\big) \, , \, \big(k_p, \cdots, \widetilde{k_r}, \cdots, k_1\big) \Big) \\ \cdot \Big(\omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \Big) \end{array} \right] \\
&= \sum_r (-1)^{np+n+1} \, (-1)^{n-r} \operatorname{sgn}(I, I^c) \operatorname{sgn}(I^c, I) \, e_{k_r}\big(a_{k_1 \cdots k_p}\big) \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \\
&= \sum_r (-1)^{np-r+1} \, (-1)^{(n-p)p} \, e_{k_r}\big(a_{k_1 \cdots k_p}\big) \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \\
&= \sum_r (-1)^{p^2+r+1} \, e_{k_r}\big(a_{k_1 \cdots k_p}\big) \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1}
\end{aligned}$$

Apply the Hodge star operator to the second term of $d * \omega$.

$$(-1)^{n(p+1)+1} \, * \left(- \sum_r a_{k_1 \cdots k_p} \operatorname{sgn}(I, I^c) \, \Gamma^{k_r}_{k_\alpha \, k_\alpha} \, \omega_{k_r} \wedge \omega_{k_n} \wedge \cdots \wedge \omega_{k_{p+1}} \right)$$

$$\begin{aligned}
&= \sum_r \left[\begin{array}{l} (-1)^{np+n} \, a_{k_1 \cdots k_p} \operatorname{sgn}(I, I^c) \, \Gamma^{k_r}_{k_\alpha \, k_\alpha} \operatorname{sgn}\Big(\big(k_r, k_n, \cdots, k_{p+1}\big) \, , \, \big(k_p, \cdots, \widetilde{k_r}, \cdots, k_1\big) \Big) \\ \cdot \Big(\omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \Big) \end{array} \right] \\
&= \sum_r (-1)^{np+n} \, (-1)^{n-r} \, a_{k_1 \cdots k_p} \, \Gamma^{k_r}_{k_\alpha \, k_\alpha} \operatorname{sgn}(I, I^c) \operatorname{sgn}(I^c, I) \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \\
&= \sum_r (-1)^{np-r} \, (-1)^{(n-p)p} \, a_{k_1 \cdots k_p} \, \Gamma^{k_r}_{k_\alpha \, k_\alpha} \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1} \\
&= \sum_r (-1)^{p^2+r+1} \, a_{k_1 \cdots k_p} \, \Gamma^{k_\alpha}_{k_\alpha \, k_r} \, \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_{k_1}
\end{aligned}$$

We compute for the third term of $d * \omega$. .

$$(-1)^{n(p+1)+1} \, * \left(\sum_{s \neq r} a_{k_1 \cdots k_p} \operatorname{sgn}(I, I^c) \, (-1)^{n-\alpha} \, \Gamma^{k_r}_{k_s \, k_\alpha} \, \omega_{k_s} \wedge \omega_{k_r} \wedge \omega_{k_n} \wedge \cdots \wedge \widetilde{\omega_{k_\alpha}} \wedge \cdots \wedge \omega_{k_{p+1}} \right)$$

$$\begin{aligned}
&= \sum_{s \neq r} \left[\begin{array}{l} (-1)^{np+n+1} \, a_{k_1 \cdots k_p} \operatorname{sgn}(I, I^c) \, (-1)^{n-\alpha} \, \Gamma^{k_r}_{k_s \, k_\alpha} \\ \cdot \operatorname{sgn}\Big(\big(k_s, k_r, k_n, \cdots, \widetilde{k_\alpha}, \cdots, k_{p+1}\big) \, , \, \big(k_\alpha, k_p, \cdots, \widetilde{k_r}, \cdots, \widetilde{k_s}, \cdots, k_1\big) \Big) \\ \cdot \Big(\omega_{k_\alpha} \wedge \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1 \Big) \end{array} \right] \\
&= \sum_{s \neq r} \left[\begin{array}{l} (-1)^{np+\alpha+1} \, a_{k_1 \cdots k_p} \operatorname{sgn}(I, I^c) \, \Gamma^{k_r}_{k_s \, k_\alpha} \\ \cdot \operatorname{sgn}\Big(\big(k_s, k_r, k_n, \cdots, \widetilde{k_\alpha}, \cdots, k_{p+1}\big) \, , \, \big(k_\alpha, k_p, \cdots, \widetilde{k_r}, \cdots, \widetilde{k_s}, \cdots, k_1\big) \Big) \\ \cdot \Big(\omega_{k_\alpha} \wedge \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1 \Big) \end{array} \right]
\end{aligned}$$

When $s > r$,

$$
\begin{aligned}
& \operatorname{sgn}\Big(\big(k_s, k_r, k_n, \cdots, \widetilde{k_\alpha}, \cdots, k_{p+1}\big)\,,\; \big(k_\alpha, k_p, \cdots, \widetilde{k_s}, \cdots, \widetilde{k_r}, \cdots, k_1\big)\Big) \\
& \cdot \Big(\omega_{k_\alpha} \wedge \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_1\Big) \\
= \;& (-1)^{\alpha-p-1}\,(-1)^{n-s+1}\,(-1)^{n-r}\operatorname{sgn}(I^c, I)\,(-1)^{p-r-1}\,\omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \omega_1 \\
= \;& (-1)^{\alpha+s+1}\operatorname{sgn}(I^c, I)\,\omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \omega_1
\end{aligned}
$$

When $s < r$,

$$
\begin{aligned}
& \operatorname{sgn}\Big(\big(k_s, k_r, k_n, \cdots, \widetilde{k_\alpha}, \cdots, k_{p+1}\big)\,,\; \big(k_\alpha, k_p, \cdots, \widetilde{k_r}, \cdots, \widetilde{k_s}, \cdots, k_1\big)\Big) \\
& \cdot \Big(\omega_{k_\alpha} \wedge \omega_{k_p} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1\Big) \\
= \;& (-1)^{\alpha-p-1}\,(-1)^{n-r}\,(-1)^{n-s}\operatorname{sgn}(I^c, I)\,(-1)^{p-r}\omega_{k_p} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1 \\
= \;& (-1)^{\alpha+s+1}\operatorname{sgn}(I^c, I)\,\omega_{k_p} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1\,.
\end{aligned}
$$

Therefore, the third term becomes

$$
\begin{aligned}
& \sum_{s\neq r} \left[\begin{array}{l} (-1)^{np+\alpha+1}\, a_{k_1\cdots k_p}\operatorname{sgn}(I, I^c)\,\Gamma^{k_r}_{k_s\,k_\alpha} \\ \cdot\,(-1)^{\alpha+s+1}\operatorname{sgn}(I^c, I)\,\Big(\omega_{k_p} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1\Big) \end{array}\right] \\
= \;& \sum_{s\neq r}(-1)^{np+\alpha+1}\,(-1)^{\alpha+s+1}\,(-1)^{(n-p)p}\,a_{k_1\cdots k_p}\,\Gamma^{k_r}_{k_s\,k_\alpha}\,\omega_{k_p} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1 \\
= \;& \sum_{s\neq r}(-1)^{p^2+s}\,a_{k_1\cdots k_p}\,\Gamma^{k_r}_{k_s\,k_\alpha}\,\omega_{k_p} \wedge \cdots \wedge \omega^{(r)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_s}} \wedge \cdots \wedge \omega_1 \\
= \;& \sum_{r\neq s}(-1)^{p^2+r+1}\,a_{k_1\cdots k_p}\,\Gamma^{k_\alpha}_{k_r\,k_s}\,\omega_{k_p} \wedge \cdots \wedge \omega^{(s)}_{k_\alpha} \wedge \cdots \wedge \widetilde{\omega_{k_r}} \wedge \cdots \wedge \omega_1\,.
\end{aligned}
$$

By comparsion of components on both sides, we have established Equation (8).

7. Laplacian of differential forms

The Laplacian of a p-form is defined by

$$\Delta\omega \;=\; -\,d\,\delta\omega \;-\; \delta\,d\,\omega\,.$$

We first consider the case that ω is a 1-form. Let $\omega = \sum_j a_j\,\omega_j$. In terms of its covariant derivative, we have $d\omega = \sum_j a_{j,k}\,\omega_k \wedge \omega_j$. By Equation (8), we find that

$$\delta\omega \;=\; \delta\big(a_j\,\omega_j\big) \;=\; -\sum_j a_{j,j} \qquad \Longrightarrow \qquad d\,\delta\,\omega \;=\; -\sum_j da_{j,j}\,.$$

The second covariant derivative of ω is described by

$$a_{i,jk}\,\omega_k \;=\; da_{i,j} \;+\; a_{i,s}\,\omega_{sj} \;+\; a_{r,j}\,\omega_{ri}\,.$$

Note that

$$\sum_{j,k} a_{j,jk}\,\omega_k \;=\; \sum_j da_{j,j} \;+\; \sum_{j,s} a_{j,s}\,\omega_{sj} \;+\; \sum_{r,j} a_{r,j}\,\omega_{rj} \;=\; \sum_j da_{j,j}\,.$$

Therefore,

$$d\,\delta\,\omega \;=\; -a_{j,jk}\,\omega_k\,.$$

On the other hand, we have $\delta\,d\,\omega = \delta\big(a_{j,k}\,\omega_k\wedge\omega_j\big)$. Let $b_{jk} = a_{j,k}$ and consider $\beta = b_{jk}\,\omega_k\wedge\omega_j$ in the following. By Equation (8),

$$\begin{aligned}
\delta\,d\,\omega \;&=\; \sum_{\alpha=1,2} (-1)^{\alpha+5}\, b_{i_1\cdots i_\alpha\cdots i_2,i_\alpha}\,\omega_{i_2}\wedge\cdots\wedge\widetilde{\omega_{i_\alpha}}\wedge\cdots\wedge\omega_{i_1} \\
&=\; (-1)^6\, b_{j_1 i_2,j_1}\,\omega_{i_2} \;+\; (-1)^7\, b_{i_1 j_2,j_2}\,\omega_{i_1} \\
&=\; b_{jk,j}\,\omega_k \;-\; b_{kj,j}\,\omega_k \\
&=\; \big(b_{jk,j} \;-\; b_{kj,j}\big)\,\omega_k
\end{aligned}$$

The first covariant derivatives of β are found by

$$b_{jk,l}\,\omega_l \;=\; db_{jk} \;+\; b_{rk}\,\omega_{rj} \;+\; b_{js}\,\omega_{sk} \;=\; da_{j,k} \;+\; a_{r,k}\,\omega_{rj} \;+\; a_{j,s}\,\omega_{sk}\,.$$

So, $b_{jk,l} = e_l(a_{j,k}) + a_{r,k}\,\Gamma^j_{lr} + a_{j,s}\,\Gamma^k_{ls}$. Note that we also have

$$a_{j,kl}\,\omega_l \;=\; da_{j,k} \;+\; a_{j,s}\,\omega_{sk} \;+\; a_{r,k}\,\omega_{rj}\,.$$

Therefore, $b_{jk,l} = a_{j,kl}$ for every j, k, l. It also means that

$$\delta\,d\,\omega \;=\; \big(a_{j,kj} \;-\; a_{k,jj}\big)\,\omega_k$$

Adding up the above terms,

$$\begin{aligned}
\Delta\,\omega &= a_{j,jk}\,\omega_k \,-\, a_{j,kj}\,\omega_k \,+\, a_{k,jj}\,\omega_k \\
&= a_{k,jj}\,\omega_k \,+\, \big(a_{j,jk} \,-\, a_{j,kj}\big)\,\omega_k \\
&= a_{k,jj}\,\omega_k \,+\, a_r\,R_{jkjr}\,\omega_k \\
&= \big(a_{k,jj} \,-\, a_r R_{rk}\big)\,\omega_k\,.
\end{aligned}$$

Here $R_{rk} \,=\, \sum_j R_{rjjk}$ is a coefficient of the Ricci tensor. We may let

$$\nabla^*\nabla\,\omega \,=\, \sum_{j,k} a_{k,jj}\,\omega_k \quad \text{and} \quad E(\omega) \,=\, \sum a_r\,R_{rk}\,\omega_k\,.$$

As a result, $\Delta\,\omega \,=\, \nabla^*\nabla\,\omega \,-\, E(\omega)$. We are going to find $\Delta\,\omega$ and prove this result to a general p-form on the manifold M. Let $\omega \,=\, a_{i_1\cdots i_p}\,\omega_{i_p}\wedge\cdots\wedge\omega_{i_1}$.

$$\delta\,\omega \,=\, \sum_{r=1}^{p}(-1)^{p^2+r+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1}.$$

It leads to

$$\begin{aligned}
d\,\delta\,\omega &= \sum_r(-1)^{p^2+r+1}\,da_{i_1\cdots i_r\cdots i_p,i_r}\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s>r}(-1)^{p^2+r+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,(-1)^{p-s}\,\omega_{i_p}\wedge\cdots\wedge d\omega_{i_s}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s<r}(-1)^{p^2+r+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,(-1)^{p-s-1}\,\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge d\omega_{i_s}\wedge\cdots\wedge\omega_{i_1} \\
&= \sum_r(-1)^{p^2+r+1}\,da_{i_1\cdots i_r\cdots i_p,i_r}\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s>r}(-1)^{r+s+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s<r}(-1)^{r+s}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\omega_{i_1} \\
&= \sum_r(-1)^{p^2+r+1}\left[\begin{array}{l}\Big(a_{i_1\cdots i_r\cdots i_p,i_rk}\,\omega_k \,-\, a_{i_1\cdots i_r\cdots i_p,s}\,\omega_{si_r} \,-\, \sum_{l,j_l} a_{i_1\cdots j_l\cdots i_p,i_r}\,\omega_{j_li_l}\Big) \\ \wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1}\end{array}\right] \\
&\quad + \sum_{s>r}(-1)^{r+s+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s<r}(-1)^{r+s}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\omega_{i_1} \\
&= \sum_r(-1)^{p^2+r+1}\,a_{i_1\cdots i_r\cdots i_p,i_rk}\,\omega_k\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad - \sum_r(-1)^{p^2+r+1}\,a_{i_1\cdots i_r\cdots i_p,s}\,\omega_{si_r}\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad - \sum_r\sum_{l,j_l}(-1)^{p^2+r+1}\,a_{i_1\cdots j_l\cdots i_p,i_r}\,\omega_{j_li_l}\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s>r}(-1)^{r+s+1}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\omega_{i_1} \\
&\quad + \sum_{s<r}(-1)^{r+s}\,a_{i_1\cdots i_r\cdots i_p,i_r}\,\omega_{i_sj}\wedge\omega_j\wedge\omega_{i_p}\wedge\cdots\wedge\widetilde{\omega_{i_r}}\wedge\cdots\wedge\widetilde{\omega_{i_s}}\wedge\cdots\wedge\omega_{i_1}
\end{aligned}$$

$$
\begin{aligned}
d\,\delta\,\omega \;&=\; \sum_r (-1)^{p^2+r+1}\, a_{i_1\cdots i_r\cdots i_p, i_r k}\, \omega_k \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_r (-1)^{p^2+r}\, a_{i_1\cdots i_r\cdots i_p, s}\, \omega_{s i_r} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{l>r} (-1)^{p^2+r}\, a_{i_1\cdots i_r\cdots j_l\cdots i_p, i_r}\, \omega_{j_l i_l} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{l<r} (-1)^{p^2+r}\, a_{i_1\cdots j_l\cdots i_r\cdots i_p, i_r}\, \omega_{j_l i_l} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_r (-1)^{p^2+r}\, a_{i_1\cdots j_r\cdots i_p, i_r}\, \omega_{j_r i_r} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{s>r} (-1)^{r+s+1}\, a_{i_1\cdots i_r\cdots j_s\cdots i_p, i_r}\, (-1)^{p-s}\, \omega_{j_s i_s} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{s<r} (-1)^{r+s}\, a_{i_1\cdots j_s\cdots i_r\cdots i_p, i_r}\, (-1)^{p-s-1}\, \omega_{j_s i_s} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&=\; \sum_r (-1)^{p^2+r+1}\, a_{i_1\cdots i_r\cdots i_p, i_r k}\, \omega_k \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1}\,.
\end{aligned}
$$

On the other hand,

$$
\delta\, d\,\omega \;=\; \delta\left(a_{i_1\cdots i_p, j}\, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \right)
$$

We may let $\beta \;=\; d\omega \;=\; \sum b_{i_1\cdots i_p i_{p+1}}\, \omega_{i_{p+1}} \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}$. So we have

$$
\begin{aligned}
\delta\, d\,\omega \;&=\; \delta\left(a_{i_1\cdots i_o, j}\, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \right) \\
&=\; \sum_{r=1}^{p} (-1)^{(p+1)^2+r+1}\, b_{i_1\cdots i_r\cdots i_p i_{p+1}, i_r}\, \omega_{i_{p+1}} \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + (-1)^{(p+1)^2+(p+1)+1}\, b_{i_1\cdots i_p i_{p+1}, i_{p+1}}\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}
\end{aligned}
$$

Note that

$$
\begin{aligned}
b_{i_1\cdots i_p i_{p+1}, j}\, \omega_j \;&=\; db_{i_1\cdots i_p i_{p+1}} \;+\; \sum_{r, j_r} b_{i_1\cdots j_r\cdots i_p i_{p+1}}\, \omega_{j_r i_r} \;+\; b_{i_1\cdots i_p s}\, \omega_{s i_{p+1}} \\
&=\; da_{i_1\cdots i_p, i_{p+1}} \;+\; \sum_{r, j_r} a_{i_1\cdots j_r\cdots i_p, i_{p+1}}\, \omega_{j_r i_r} \;+\; a_{i_1\cdots i_p, s}\, \omega_{s i_{p+1}} \\
&=\; a_{i_1\cdots i_p, i_{p+1} k}\, \omega_k\,.
\end{aligned}
$$

So we have $\;b_{i_1\cdots i_p i_{p+1}, j} \;=\; a_{i_1\cdots i_p, i_{p+1} j}\;$ for every choice of $i_1, \cdots, i_p,\, i_{p+1},\, j$.

$$
\begin{aligned}
\delta\, d\,\omega \;&=\; \sum_{r=1}^{p} (-1)^{p^2+r}\, a_{i_1\cdots i_r\cdots i_p, j i_r}\, \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad - \sum_j a_{i_1\cdots i_p, j j}\, \omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\,.
\end{aligned}
$$

As a result, the Laplacian of ω is found by

$$\begin{aligned}
\Delta\,\omega \;&=\; -\,d\,\delta\,\omega \,-\, \delta\,d\,\omega \\
&=\; \sum_r (-1)^{p^2+r}\, a_{i_1\cdots i_r\cdots i_p,i_rk}\,\omega_k \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_r (-1)^{p^2+r+1}\, a_{i_1\cdots i_r\cdots i_p,ji_r}\,\omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum a_{i_1\cdots i_p,jj}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&=\; \sum a_{i_1\cdots i_p,jj}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_r (-1)^{p^2+r} \left(a_{i_1\cdots i_r\cdots i_p,i_rj} \,-\, a_{i_1\cdots i_r\cdots i_p,ji_r} \right) \omega_j \wedge \omega_{i_p} \wedge \cdots \wedge \widetilde{\omega_{i_r}} \wedge \cdots \wedge \omega_{i_1} \\
&=\; \sum a_{i_1\cdots i_p,jj}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1} \\
&\quad + \sum_{r,s} a_{i_1\cdots i_r\cdots j_s\cdots i_p}\, R_{i_rj_ri_sj_s}\,\omega_{i_p} \wedge \cdots \wedge \omega_{j_r} \wedge \cdots \wedge \omega_{i_1}\,.
\end{aligned}$$

For the p-form ω, we let

$$\nabla^*\nabla\,\omega \;=\; \sum a_{i_1\cdots i_p,jj}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\,,$$

$$E(\omega) \;=\; \sum_{r,s} a_{i_1\cdots j_s\cdots i_p}\, R_{i_rj_rj_si_s}\,\omega_{i_p} \wedge \cdots \wedge \omega_{j_r} \wedge \cdots \wedge \omega_{i_1}\,.$$

Therfore, we have

$$\Delta\,\omega \;=\; \nabla^*\nabla\,\omega \,-\, E(\omega)\,. \tag{9}$$

8. The Bochner formula

The Bochner formula states that if ω is a p-form,

$$\omega \;=\; \sum a_{i_1\cdots i_p}\,\omega_{i_p} \wedge \cdots \wedge \omega_{i_1}\,,$$

then we have

$$\nabla|\omega|^2 \;=\; 2 < \Delta\omega\,,\,\omega > \,+ 2\,|\nabla\,\omega|^2 \,+\, 2 < E(\omega)\,,\,\omega > \;. \tag{10}$$

Before proving Equation (10), it is worth mentioning some basic results about applying the chain rule to covariant derivatives. Let f be a smooth real-valued function on the manifold M, and let $\phi : \mathbb{R} \to \mathbb{R}$. Note that we have

$$\nabla\left(\phi \circ f\right) \;=\; \phi'(f)\,\nabla f$$

and so $(\phi \circ f)_j \;=\; \phi'(f)\,f_j$. Moreover,

$$\begin{aligned} (\phi\circ f)_{ij} &= <\nabla_{e_i}\Big(\phi'(f)\,\nabla f\Big)\,,\, e_j> \\ &= e_i\big(\phi'(f)\big)\, <\nabla f\,,\, e_j> + \phi'(f)\, f_{ij} \\ &= \phi''(f)\, f_i\, f_j \,+\, \phi'(f)\, f_{ij}\,. \end{aligned}$$

If $\alpha = \sum a_j\,\omega_j$ is a 1-form instead, and every $\phi_j : \mathbb{R}\to\mathbb{R}$ is a real-valued function, then we let

$$\beta = \sum \phi_j(a_j)\,\omega_j = \sum b_j\,\omega_j\,.$$

The covariant derivative of β is obtained by

$$\begin{aligned} b_{j,k}\,\omega_k &= db_j \,+\, b_r\,\omega_{rj} \\ &= d\big(\phi_j(a_j)\big) \,+\, \phi_r(a_r)\,\omega_{rj} \\ &= \phi_j'(a_j)\,da_j \,+\, \phi_r(a_r)\,\omega_{rj} \\ &= \phi_j'(a_j)\,\Big(a_{j,l}\,\omega_l \,-\, a_s\,\omega_{sj}\Big) \,+\, \phi_r(a_r)\,\omega_{rj} \\ &= a_{j,l}\,\phi_j'(a_j)\,\omega_l + \Big(\phi_r(a_r) \,-\, a_r\,\phi_j'(a_j)\Big)\,\omega_{rj} \end{aligned}$$

Therefore, we have

$$b_{j,k} = \phi_j'(a_j)\,a_{j,k} \,+\, \Big(\phi_r(a_r) \,-\, a_r\,\phi_j'(a_j)\Big)\,\Gamma^j_{kr}\,.$$

Back to the Bochner formula, on the left hand side of Equation (10),

$$\Delta|\omega|^2 = \Delta\big(\sum a^2_{i_1\cdots i_p}\big) = \sum (a^2_{i_1\cdots i_p})_{jj}\,.$$

By our discussion above,

$$(a^2_{i_1\cdots i_p})_{jj} = 2\big((a_{i_1\cdots i_p})_j\big)^2 \,+\, 2\,a_{i_1\cdots i_p}\,(a_{i_1\cdots i_p})_{jj}\,.$$

Therefore,

$$\Delta|\omega|^2 = 2\sum\big((a_{i_1\cdots i_p})_j\big)^2 \,+\, 2\sum a_{i_1\cdots i_p}\,(a_{i_1\cdots i_p})_{jj}\,.$$

Here We add a remark that the terms $(a_{i_1\cdots i_p})_j$ and $(a_{i_1\cdots i_p})_{jj}$ are the components of the first and second covariant derivatives of the function $a_{i_1\cdots i_p}$ respectively. In other words,

$$(a_{i_1\cdots i_p})_j \neq a_{i_1\cdots i_p,j} \quad \text{and} \quad (a_{i_1\cdots i_p})_{jj} \neq a_{i_1\cdots i_p,jj}$$

in general. To make the equalities happen, in the following we fix a point x on M. Then, we choose an orthonormal frame $\big\{e_1,\cdots,e_n\big\}$ around x such that

(1) $\nabla_{e_i}e_j = 0$ at x for all i, j;

(2) $\nabla_{e_i}\nabla_{e_i}e_j = 0$ at x for all i, j.

In order to construct this orthonormal frame, we pick an orthonormal basis for T_xM named by $\{E_1, E_2 \cdots, E_n\}$. Consider the geodesic normal coordinates at x such that E_j is represented by $e_j = (0, \cdots, 1^{(j)}, \cdots, 0)$ on T_xM. Any vector E_j is parallel-transported from x to another point $y = \exp_x(\mathbf{v})$ in the neighborhood through geodesics $\gamma(t) = \exp_x(t\,\mathbf{v})$ connecting x and y.

In particular, for any pair of E_i and E_j,

$$(\nabla_{E_i} E_j)(y) = \mathbf{0}$$

whenever y lies on the geodesic $\gamma_i(t) = \exp_x(t\, e_i)$. For any E_k, we have

$$< \nabla_{E_i} E_j \,,\, E_k >= 0$$

at any point y on γ_i. Therefore,

$$\begin{aligned} E_i\big(< \nabla_{E_i} E_j \,,\, E_k > \big) &= 0 \\ < \nabla_{E_i} \nabla_{E_i} E_j \,,\, E_k > + < \nabla_{E_i} E_j \,,\, \nabla_{E_i} E_k > &= 0 \\ < \nabla_{E_i} \nabla_{E_i} E_j \,,\, E_k > &= 0 \end{aligned}$$

at y. Hence, $\nabla_{E_i} \nabla_{E_i} E_j = \mathbf{0}$ at x.

As an implication of properties (1) and (2), at the point x,

$$\begin{aligned} a_{i_1 \cdots i_p, j} &= da_{i_1 \cdots i_p}(e_j) + a_{i_1 \cdots j_r \cdots i_p} \Gamma^{i_r}_{j j_r} \\ &= da_{i_1 \cdots i_p}(e_j)\,, \end{aligned}$$

$$\begin{aligned} a_{i_1 \cdots i_p, jj} &= da_{i_1 \cdots i_p, j}(e_j) + a_{i_1 \cdots i_p, s}\, \omega_{sj}(e_j) + \sum a_{i_1 \cdots j_r \cdots i_p, j}\, \omega_{j_r i_r}(e_j) \\ &= da_{i_1 \cdots i_p, j}(e_j)\,. \end{aligned}$$

So we have $(a_{i_1 \cdots i_p})_j = da_{i_1 \cdots i_p}(e_j) = a_{i_1 \cdots i_p, j}$ at x.

$$\begin{aligned} (a_{i_1 \cdots i_p})_{jj} &= \big(d(a_{i_1 \cdots i_p})_j\big)(e_j) + a_{i_1 \cdots i_p, k}\, \omega_{kj}(e_j) \\ &= d\Big(a_{i_1 \cdots i_p, j} - a_{i_1 \cdots j_r \cdots i_p} \Gamma^{i_r}_{j j_r}\Big)(e_j) \\ &= d(a_{i_1 \cdots i_p, j})(e_j) - da_{i_1 \cdots j_r \cdots i_p}(e_j)\, \Gamma^{i_r}_{j j_r} - a_{i_1 \cdots j_r \cdots i_p}\, d\Gamma^{i_r}_{j j_r}(e_j) \end{aligned}$$

Note that

$$\begin{aligned} d\Gamma^{i_r}_{j j_r}(e_j) &= e_j\big(< \nabla_{e_j} e_{j_r} \,,\, e_{i_r} > \big) \\ &= < \nabla_{e_j} \nabla_{e_j} e_{j_r} \,,\, e_{i_r} > + < \nabla_{e_j} e_{j_r} \,,\, \nabla_{e_j} e_{i_r} > \\ &= 0 \quad \text{at } x\,. \end{aligned}$$

Therefore, $(a_{i_1 \cdots i_p})_{jj} = a_{i_1 \cdots i_p, jj}$ at x. Finally, we may justify the Bochner formula as follows.

$$
\begin{aligned}
\Delta|\omega|^2 &= 2 \sum a^2_{i_1 \cdots i_p, j} + 2 \sum a_{i_1 \cdots i_p}\, a_{i_1 \cdots i_p, jj} \\
&= 2\,|\nabla\omega|^2 + 2 < \nabla^*\nabla\omega\,,\, \omega > \\
&= 2\,|\nabla\omega|^2 + 2 < \Delta\omega\,,\, \omega > + 2 < E(\omega)\,,\, \omega >
\end{aligned}
$$

-END-