Laplacian Operator (I)

1. Laplacian of function

Let (M, g) be an n-dimensional Riemannian manifold. Let ∇ be the Riemannian connection of g. Suppose $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal frame on M, and $\{\omega_1, \omega_2, \cdots, \omega_n\}$ is its dual coframe. Define connection forms ω_{ij} in $\Omega^1(M)$ by

$$\omega_{ij}(e_k) = < \nabla_{e_k} e_i, e_j > = \Gamma^j_{ki}\,. \tag{1}$$

Note that $\omega_{ij} = -\omega_{ji}$. Because $[e_i, e_j] \neq 0$, in general Γ^j_{ki} is different from Γ^j_{ik}.

The Cartan structural equations are:

$$d\omega_i = \omega_{ij} \wedge \omega_j \quad \text{and} \quad d\omega_{ij} = \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \tag{2}$$

For the first identity of (2), we have

$$\begin{aligned}
d\omega_i(e_j, e_k) &= \frac{1}{2}\Big(e_j\big(\omega_i(e_k)\big) - e_k\big(\omega_i(e_j)\big) - \omega_i\big([e_j, e_k]\big)\Big) \\
&= -\frac{1}{2}\big(\Gamma^i_{jk} - \Gamma^i_{kj}\big) \\
&= \frac{1}{2}\big(\Gamma^k_{ji} - \Gamma^j_{ki}\big)\,, \\
\big(\omega_{is} \wedge \omega_s\big)(e_j, e_k) &= \frac{1}{2}\Big(\omega_{is}(e_j)\,\omega_s(e_k) - \omega_{is}(e_k)\,\omega_s(e_j)\Big) \\
&= \frac{1}{2}\big(\Gamma^k_{ji} - \Gamma^j_{ki}\big).
\end{aligned}$$

In the second identity of (2), we let

$$\Omega_{ij} = \frac{1}{2} R_{ijkl}\, \omega_k \wedge \omega_l \tag{3}$$

with $R_{ijkl} = < \nabla_{e_i}\nabla_{e_j} e_k - \nabla_{e_j}\nabla_{e_i} e_k - \nabla_{[e_i,e_j]} e_k, e_l >$.

$$\begin{aligned}
\Omega_{ij}(e_r, e_s) &= \frac{1}{4} R_{ijrs} - \frac{1}{4} R_{ijsr} \\
&= \frac{1}{2} R_{ijrs} \\
&= \frac{1}{2} g\Big(\nabla_{e_i}\nabla_{e_j} e_r - \nabla_{e_j}\nabla_{e_i} e_r - \nabla_{[e_i,e_j]} e_r\,,\, e_s\Big) \\
&= \frac{1}{2} g\Big(\nabla_{e_i}(\Gamma^p_{jr}\, e_p) - \nabla_{e_j}(\Gamma^p_{ir}\, e_p) - \Gamma^p_{ij}\,\Gamma^q_{pr}\, e_q + \Gamma^p_{ji}\,\Gamma^q_{pr}\, e_q\,,\, e_s\Big) \\
&= \frac{1}{2} g\Big(e_i(\Gamma^p_{jr})\, e_p + \Gamma^p_{jr}\,\Gamma^q_{ip}\, e_q - e_j(\Gamma^p_{ir})\, e_p - \Gamma^p_{ir}\,\Gamma^q_{jp}\, e_q - \Gamma^p_{ij}\,\Gamma^q_{pr}\, e_q + \Gamma^p_{ji}\,\Gamma^q_{pr}\, e_q\,,\, e_s\Big) \\
&= \frac{1}{2}\Big(e_i(\Gamma^s_{jr}) - e_j(\Gamma^s_{ir}) + \Gamma^p_{jr}\,\Gamma^s_{ip} - \Gamma^p_{ir}\,\Gamma^s_{jp} - \Gamma^p_{ij}\,\Gamma^s_{pr} + \Gamma^p_{ji}\,\Gamma^s_{pr}\Big)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
\big(d\omega_{ij} - \omega_{ik}\wedge\omega_{kj}\big)(e_r, e_s) \;&=\; \frac{1}{2}\Big(e_r(\Gamma^j_{si}) - e_s(\Gamma^j_{ri}) - (\Gamma^p_{rs} - \Gamma^p_{sr})\,\Gamma^j_{pi} - \Gamma^k_{ri}\,\Gamma^j_{sk} + \Gamma^k_{si}\,\Gamma^j_{rk}\Big)\\
&=\; \frac{1}{2}\Big(e_r(\Gamma^j_{si}) - e_s(\Gamma^j_{ri}) + \Gamma^k_{si}\,\Gamma^j_{rk} - \Gamma^k_{ri}\,\Gamma^j_{sk} - \Gamma^p_{rs}\,\Gamma^j_{pi} + \Gamma^p_{sr}\,\Gamma^j_{pi}\Big)\\
&=\; \frac{1}{2}\,R_{rsij}\\
&=\; \frac{1}{2}\,R_{ijrs}\,.
\end{aligned}
$$

Therefore, the second part of (2) is justified.

Let f be a smooth function on M. The first covariant derivative of f is defined by its gradient,

$$\nabla f \;=\; df(e_i)\,e_i \;=\; f_i\,e_i.$$

The second derivative of f is given by

$$
\begin{aligned}
\nabla^2 f \;&=\; \nabla(\nabla f) \;=\; \nabla(f_i\,e_i)\\
&=\; df_i\otimes e_i + f_i\,\nabla e_i\\
&=\; df_i\otimes e_i + f_i\,\omega_{ij}\otimes e_j\\
&=\; \big(df_i + f_j\,\omega_{ji}\big)\otimes e_i
\end{aligned}
$$

If we write $\nabla^2 f \;=\; f_{ik}\,\omega_k\otimes e_i$. then

$$f_{ik}\,\omega_k \;=\; df_i + f_j\,\omega_{ji} \tag{4}$$

Using the fact that $d^2 f = 0$, we obtain

$$
\begin{aligned}
d\big(f_i\,\omega_i\big) \;&=\; 0\\
df_i\wedge\omega_i + f_i\,d\omega_i \;&=\; 0\\
\big(df_j + f_i\,\omega_{ij}\big)\wedge\omega_j \;&=\; 0\\
f_{jk}\,\omega_k\wedge\omega_j \;&=\; 0.
\end{aligned}
$$

Therefore, for every j and k, $f_{jk} = f_{kj}$. The Hessian of f is a (0,2)-tensor defined by

$$\mathrm{Hess}(f) \;=\; f_{ij}\,\omega_j\otimes\omega_i.$$

The Laplacian of f is the trace of the Hessian of f, i.e.

$$\Delta f \;=\; \sum_{j=1}^{n} f_{jj} \tag{5}$$

Explicitly,

$$
\begin{aligned}
f_{jj} &= df_j(e_j) + f_k\,\omega_{kj}(e_j)\\
&= e_j\big(e_j(f)\big) + f_k\,\Gamma^j_{jk}\\
&= e_j\big(e_j(f)\big) - f_k\,\Gamma^k_{jj}\\
\Longrightarrow\quad \Delta f &= \sum_{j=1}^{n} e_j\big(e_j(f)\big) - f_k\,\Gamma^k_{jj}\,.
\end{aligned}
$$

2. Area functional

From here we let N^n be a submanifold of (M^m, g) with $n < m$. Suppose $\{e_1, e_2, \cdots, e_m\}$ is an orthonormal frame on M while $\{e_1, e_2, \cdots, e_n\}$ forms an orthonormal frame on N.

Let ∇ be the Riemannian conenction of g on M. It projects to be the Riemannian connection ∇^N on N. The second fundamental form of N is defined by

$$
\mathrm{II}(X, Y) \,=\, (\nabla_X Y)^{\perp} \tag{6}
$$

Here $Z^{\perp}$ denotes the orthogonal component of Z to TN.

$$
\begin{aligned}
\mathrm{II}(e_i, e_j) &= (\nabla_{e_i} e_j)^{\perp}\\
&= \Big(\sum_{k=1}^{n}\Gamma^k_{ij}\,e_k + \sum_{p=n+1}^{m}\Gamma^p_{ij}\,e_p\Big)^{\perp}\\
&= \sum_{p=n+1}^{m}\Gamma^p_{ij}\,e_p
\end{aligned}
$$

The mean curvature vector is the trace of II over TN.

$$
\mathbf{H} \,=\, \mathrm{tr}(\mathrm{II}) \,=\, \sum_{p=n+1}^{m}\sum_{j=1}^{n}\Gamma^p_{jj}\,e_p \tag{7}
$$

We are going to explain the geometric meaning of the mean curvature vector $\mathbf{H}$.

Let $\phi_t : N \to M$ be a variation of N into M where $-\epsilon < t < \epsilon$. Fix a point p on N. Let $\mathbf{x} = \big(x_1, x_2, \cdots, x_n\big)$ be the normal coordinates at p over a neighborhood U with $\mathbf{x}(\mathbf{0}) = p$. Composited with the normal coordinates, the variation ϕ_t is regarded as

$$
\phi_t = \phi_t(\mathbf{x}) \,:\, U \to M\,.
$$

Suppose ϕ_0 is the prescribed embedding of N into M, and $T = d\phi_t\big(\dfrac{\partial}{\partial t}\big)$ is always transverse to N.

The metric coefficients on U pulled back by the embedding ϕ_t are given by

$$g_{ij}(\mathbf{x},t) \;=\; g\Big(d\phi_t(\frac{\partial}{\partial x_i})\,,\, d\phi_t(\frac{\partial}{\partial x_j})\Big)\,.$$

Moreover, we let

$$G(\mathbf{x},t) \;=\; \det\big(g_{ij}(\mathbf{x},t)\big)\,.$$

By definition we have $g_{ij}(\mathbf{0},0)=\delta_{ij}$. In the following, we let

$$E_j(\mathbf{x},t) \;=\; d\phi_t\Big(\frac{\partial}{\partial x_j}\Big)\,.$$

Given a fixed t, the Christoffel symbols of $g_{ij}(\mathbf{x},t)$'s are defined by

$$\nabla_{E_i}E_j \;=\; \Gamma^k_{ij}(\mathbf{x},t)\,E_k.$$

Similarly, we have $\Gamma^k_{ij}(\mathbf{0},0)=0$ for every i,j,k.

The area functional of ϕ_t over U is given by

$$A_t(U) \;=\; \int_U \sqrt{G(\mathbf{x},t)}\,d\mathbf{x}\,.$$

At the point p,

$$\frac{\partial\sqrt{G}}{\partial t}(\mathbf{0},0) \;=\; \frac{1}{2\sqrt{G(\mathbf{0},0)}}\,\frac{\partial G}{\partial t}(\mathbf{0},0) \;=\; \frac{1}{2}\,\frac{\partial G}{\partial t}(\mathbf{0},0)\,.$$

By the first row expansion of $G(\mathbf{x},t)$, we have

$$G(\mathbf{x},t) \;=\; \sum_{j=1}^{n} g_{1j}(\mathbf{x},t)\,c_{1j}(\mathbf{x},t)\,,$$

where c_{ij} is the cofactor at the (i,j)-entry. Therefore,

$$\begin{aligned}
\frac{\partial G}{\partial t}(\mathbf{0},0) \;&=\; \sum_{j=1}^{n} c_{1j}(\mathbf{0},0)\,\frac{\partial g_{1j}}{\partial t}(\mathbf{0},0) \;+\; \sum_{j=1}^{n} g_{1j}(\mathbf{0},0)\,\frac{\partial c_{1j}}{\partial t}(\mathbf{0},0)\\
&=\; \frac{\partial g_{11}}{\partial t}(\mathbf{0},0) \;+\; \frac{\partial c_{11}}{\partial t}(\mathbf{0},0)
\end{aligned}$$

Since $c_{11}(\mathbf{x},t)$ is the determinant of the minor matrix of $[g_{ij}(\mathbf{x},t)]$ at the $(1,1)$ entry, we may carry out first row expansion on the minor matrix again. We conclude that

$$\frac{\partial G}{\partial t}(\mathbf{0},0) \;=\; \sum_{j=1}^{n} \frac{\partial g_{jj}}{\partial t}(\mathbf{0},0)\,. \tag{8}$$

$$\begin{aligned}
\frac{\partial G}{\partial t}(\mathbf{0},0) &= \sum_{j=1}^{n} \frac{\partial}{\partial t}\Big|_{t=0} \Big(g\Big(E_j(\mathbf{0},t)\,,\, E_j(\mathbf{0},t)\Big)\Big) \\
&= \sum_{j=1}^{n} T\Big(g(E_j, E_j)\Big) \\
&= \sum_{j=1}^{n} 2\, g\Big((\nabla_T E_j)(\mathbf{0},0), E_j(\mathbf{0},0)\Big)
\end{aligned}$$

Since $[T, E_j] = 0$, we have $\nabla_T E_j = \nabla_{E_j} T$. At $(\mathbf{x}, 0)$, let

$$T = T^{tan} + T^{\perp}$$

on N where T^{tan} is the tangential component and $T^{\perp}$ is the normal component. Therefore,

$$\begin{aligned}
\frac{\partial G}{\partial t}(\mathbf{0},0) &= \sum_{j=1}^{n} 2\, g\Big((\nabla_{E_j} T)(\mathbf{0},0)\,,\, E_j(\mathbf{0},0)\Big) \\
&= \sum_{j=1}^{n} 2\, g\Big((\nabla_{E_j} T^{tan})(\mathbf{0},0)\,,\, E_j(\mathbf{0},0)\Big) \,+\, 2\, g\Big((\nabla_{E_j} T^{\perp})(\mathbf{0},0)\,,\, E_j(\mathbf{0},0)\Big)
\end{aligned}$$

At the point $\mathbf{x} = \mathbf{0}$ and $t = 0$,

$$\operatorname{div}(T^{tan})(\mathbf{0},0) \,=\, \sum_{i,j=1}^{n} g\Big(\nabla_{E_i} T^{tan}\,,\, E_j\Big)\, g^{ij}(\mathbf{0},0) \,=\, \sum_{j=1}^{n} g\Big(\nabla_{E_j} T^{tan}\,,\, E_j\Big)\,.$$

It leads to

$$\begin{aligned}
\frac{\partial G}{\partial t}(\mathbf{0},0) &= 2 \operatorname{div}(T^{tan})(\mathbf{0},0) \,+\, 2\, \sum_{j=1}^{n} g\Big((\nabla_{E_j} T^{\perp})(\mathbf{0},0)\,,\, E_j(\mathbf{0},0)\Big) \\
&= 2 \operatorname{div}(T^{tan})(\mathbf{0},0) \,+\, 2\, \sum_{j=1}^{n} E_j\Big(g(T^{\perp}, E_j)\Big) - g\big(T^{\perp}, \nabla_{E_j} E_j\big)
\end{aligned}$$

$T^{\perp}$ is always orthogonal to the tangent vector E_j on N, so $g(T^{\perp}, E_j) = 0$ at $(\mathbf{x}, 0)$. Hence,

$$\frac{\partial G}{\partial t}(\mathbf{0},0) = 2 \operatorname{div}(T^{tan})(\mathbf{0},0) \,-\, 2\, \sum_{j=1}^{n} g\Big(T^{\perp}(\mathbf{0},0), (\nabla_{E_j} E_j)(\mathbf{0},0)\Big)$$

In terms of the basis $\{E_1, E_2, \cdots, E_n\}$, the mean curvature vector at $(\mathbf{0}, 0)$ is found by

$$\mathbf{H}(\mathbf{0},0) \,=\, \sum_{i,j=1}^{n} \mathrm{II}(E_i, E_j)\, g^{ij}(\mathbf{0},0) \,=\, \sum_{j=1}^{n} \mathrm{II}(E_j, E_j) \,=\, \sum_{j=1}^{n} (\nabla_{E_j} E_j)^{\perp}(\mathbf{0},0)\,.$$

Since $T^{\perp}$ is normal to N, we have

$$\frac{\partial G}{\partial t}(\mathbf{0},0) \,=\, 2 \operatorname{div}(T^{tan})(\mathbf{0},0) \,-\, 2\, g\big(T^{\perp}(\mathbf{0},0)\,,\, \mathbf{H}(\mathbf{0},0)\big)\,.$$

Since the point p on N is arbitrary, we conclude that

$$\frac{\partial G}{\partial t}(p,0) \;=\; 2\,\mathrm{div}(T^{tan})(p,0) \;-\; 2\,g\big(T^{\perp}(p,0)\,,\,\mathbf{H}(p,0)\big) \tag{9}$$

at any p on N. Back to the area functional A_t of ϕ_t, it gives

$$\frac{d}{dt}dA_t\Big|_{(p,0)} \;=\; \Big(\mathrm{div}(T^{tan}) \;-\; g(T^{\perp},\mathbf{H})\Big)\,dA_0(p,0)$$

Suppose the vector field T is compactly supported on N. Then, we integrate both sides on N.

$$\begin{aligned}
A_0'(N) &= \int_N \frac{dA_t}{dt}\Big|_{t=0} \\
&= \int_N \mathrm{div}(T^{tan})\,dA_0 \;-\; \int_N g(T^{\perp},\mathbf{H})\,dA_0 \\
&= -\int_N g(T^{\perp},\mathbf{H})\,dA
\end{aligned}$$

The mean curvature vector $\mathbf{H}$ is always perpendicular to the direction of the variation, T, so $A'(0) = 0$ for any direction T if and only if $\mathbf{H} = 0$ on N.

3. Embedding in $\mathbb{R}^N$

Let $(x_1, x_2, \cdots, x_N)$ be the Cartesian coordinates on $\mathbb{R}^N$. Let M be an n-dimensional submanifold of $\mathbb{R}^N$. Let

$$f : U \to \mathbb{R}^N; \quad (x_1,\cdots,x_N) \;=\; f(u_1,\cdots,u_n)$$

be a conformal parametrization on M. Denote the second fundamental form and the mean curvature vector on $f(U)$ by II and $\mathbf{H}$ respectively. Let

$$E_j \;=\; \frac{\partial f}{\partial u_j} \quad \text{for } j = 1,\cdots,n\,.$$

Choose normal vectors $E_{n+1}, E_{n+2}, \cdots, E_N$ so that $\mathcal{B} \;=\; \big\{E_1, E_2, \cdots, E_N\big\}$ is a local frame for $\mathbb{R}^N$ on $f(U)$. We also assume that $g(E_i, E_j) \;=\; \lambda\,\delta_{ij}$ for any i, j. If we replace E_j's by

$$e_j \;=\; \frac{1}{\sqrt{\lambda}}\,E_j$$

for every j, then we get to an orthonormal frame on $\mathbb{R}^N$. Let $\tilde{g}$ and $\tilde{\nabla}$ be the Euclidean metric and its Riemannian connection on $\mathbb{R}^N$.

Given a real-valued function ϕ on $f(U)$,

$$\begin{aligned}
\tilde{\nabla}^2\phi &= \tilde{\nabla}\big(e_J(\phi)\,e_J\big) \\
&= e_I\,e_J(\phi)\,e^I \otimes e_J \;+\; e_J(\phi)\,e^I \otimes \tilde{\nabla}_{e_I} e_J \\
&= e_I\,e_J(\phi)\,e^I \otimes e_J \;+\; e_J(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_I} e_J\,,\,e_K\big)\,e^I \otimes e_K \\
&= \Big(e_I\,e_J(\phi) \;+\; e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_I} e_K\,,\,e_J\big)\Big)\,e^I \otimes e_J\,.
\end{aligned}$$

Therefore, we have

$$\mathrm{Hess}_{\mathbb{R}^N}(\phi) \,=\, \Big(e_I\, e_J(\phi) \,+\, e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_I} e_K\,,\, e_J\big)\Big)\, e^I \otimes e^J\,.$$

We separate the index $I = 1, 2, \cdots, N$ to two parts: $i = 1, \cdots, n$ and $\alpha = n+1, \cdots, N$.

$$\begin{aligned}\mathrm{Hess}_{\mathbb{R}^N}(\phi) \;&=\; e_i\, e_j(\phi)\, e^i \otimes e^j \,+\, e_i\, e_\beta(\phi)\, e^i \otimes e^\beta \,+\, e_\alpha\, e_j(\phi)\, e^\alpha \otimes e^j \,+\, e_\alpha\, e_\beta(\phi)\, e^\alpha \otimes e^\beta \\ &\quad + e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_i} e_K\,,\, e_j\big)\, e^i \otimes e^j \,+\, e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_\alpha} e_K\,,\, e_j\big)\, e^\alpha \otimes e^j \\ &\quad + e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_i} e_K\,,\, e_\beta\big)\, e^i \otimes e^\beta \,+\, e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_\alpha} e_K\,,\, e_\beta\big)\, e^\alpha \otimes e^\beta\end{aligned}$$

Denote the restriction of $\mathrm{Hess}_{\mathbb{R}^N}(\phi)$ to TM by $\mathrm{Hess}^*_{\mathbb{R}^N}(\phi)$. We have

$$\begin{aligned}\mathrm{Hess}^*_{\mathbb{R}^N}(\phi) \;&=\; e_i\, e_j(\phi)\, e^i \otimes e^j \,+\, e_K(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_i} e_K\,,\, e_j\big)\, e^i \otimes e^j \\ &=\; e_i\, e_j(\phi)\, e^i \otimes e^j \,+\, e_k(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_i} e_k\,,\, e_j\big)\, e^i \otimes e^j \,+\, e_\alpha(\phi)\,\tilde{g}\big(\tilde{\nabla}_{e_i} e_\alpha\,,\, e_j\big)\, e^i \otimes e^j \\ &=\; \mathrm{Hess}_M(\phi) \,-\, e_\alpha(\phi)\,\tilde{g}\big(\mathrm{II}(e_i, e_j)\,,\, e_\alpha\big)\, e^i \otimes e^j\,.\end{aligned}$$

Now we put ϕ to be the k-th component function of f, i.e. $\phi = x_k$. Note that $\mathrm{Hess}_{\mathbb{R}^N}(x_k) = 0$ since every x_k has vanishing second derivatives on $\mathbb{R}^N$.

$$\begin{aligned}\mathrm{Hess}_M(x_k)(e_i, e_j) \;&=\; \tilde{g}\big(\mathrm{II}(e_i, e_j)\,,\, (\tilde{\nabla} x_k)^\perp\big) \\ &=\; \tilde{g}\big(\mathrm{II}(e_i, e_j)\,,\, \tilde{\nabla} x_k\big) \\ &=\; \tilde{g}\Big(\mathrm{II}(e_i, e_j)\,,\, \frac{\partial}{\partial x_k}\Big) \\ &=\; k\text{-th component of } \mathrm{II}(e_i, e_j)\,.\end{aligned}$$

Therefore, letting $\mathrm{Hess}_M(f) \,=\, \big(\mathrm{Hess}_M(x_1), \cdots, \mathrm{Hess}_M(x_N)\big)$, we have

$$\mathrm{Hess}_M(f) \,=\, \mathrm{II}\,. \tag{10}$$

Take the trace on both sides.

$$\Delta_M(f) \,=\, \mathbf{H}\,. \tag{11}$$

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