READING 71

PRICING AND VALUATION OF FUTURES CONTRACTS

MODULE 71.1: FUTURES VALUATION

Video covering this content is available online.

LOS 71.a: Compare the value and price of forward and futures contracts.

While the *price* of a forward contract is constant over its life when no mark-to-market gains or losses are paid, its *value* will fluctuate with changes in the value of the underlying. The payment at settlement of the forward reflects the difference between the (unchanged) forward price and the spot price of the underlying.

The price and value of a futures contract *both* change when daily mark-to-market gains and losses are settled. Consider a futures contract on 100 ounces of gold at $1,870 purchased on Day 0. The following illustrates the changes in contract price and value with daily mark-to-market payments.

Day 0	Price = settlement price of 1,870	MTM value = 0
Day 1	Settlement price = 1,875	MTM value = $500
	$500 addition to margin account	
	New futures price = 1,875	MTM value = 0
Day 2	Settlement price = 1,855	MTM value = −$2,000
	$2,000 deduction from margin account	
	New futures price = 1,855	MTM value = 0

The change in the futures price to the settlement price each day returns its value to zero. Prices of forward contracts for which mark-to-market gains and losses are settled daily will also be adjusted to the settlement price.

Interest rate futures contracts are available on many market reference rates. We may view these as exchange-traded equivalents to forward rate agreements. One key difference is that interest rate futures are quoted on a price basis. For a market reference rate from time *A* to time *B*, an interest rate futures price is stated as follows:

$$\text{futures price} = 100 - (100 \times \text{MRR}_{A,\ B-A})$$

For example, if the futures price for a 6-month rate six months from now is 97, then $MRR_{6m,\ 6m}$ = 3%.

Like other futures contracts, interest rate futures are subject to daily mark-to-market. The **basis point value (BPV)** of an interest rate futures contract is defined as:

BPV = notional principal × period × 0.01%

If the contract in our example is based on notional principal of €1,000,000, its BPV is €1,000,000 × (0.0001 / 2) = €50. This means a one basis point change in the MRR will change the futures contract value by €50.

LOS 71.b: Explain why forward and futures prices differ.

For pricing, the most important distinction between futures and forwards is that with futures, mark-to market gains and losses are paid each day. Gains above initial margin can be withdrawn from a futures account and losses that reduce margin deposits below their maintenance level require payments into the account. Forwards most often have no mark-to-market cash flows, with gains or losses settled at contract expiration. Forwards typically do not require or provide funds in response to fluctuations in value during their lives.

If interest rates are constant or uncorrelated with futures prices over time, the prices of futures and forwards are the same. A positive correlation between interest rates and the futures price means that (for a long position) daily settlement provides funds (excess margin) when rates are high and they can earn more interest, and requires funds (margin deposits) when rates are low and opportunity cost of deposited funds is less. Because of this, futures are theoretically more attractive than forwards when interest rates and futures prices are positively correlated, and less attractive than forwards when interest rates and futures prices are negatively correlated.

Because of the short maturity of most forwards and the availability of funds at near risk-free rates, differences between equivalent forwards and futures are not observed in practice. Additionally, derivative dealers in some markets with central clearing are required to post margin and may require derivative investors to post mark-to-market margin payments as well.

A separate issue arises for interest rate forwards and futures settlement payments. Recall that the payoff on an interest rate forward is the present value (at the beginning of the forward period) of any interest savings (at the end of the forward period) from the difference between the realized MRR and the forward MRR. Because the realized MRR is the discount rate for calculating the payment for a given amount of future interest savings, the payment for an increase in the MRR will be less than the payment for an equal decrease in the MRR, as the following example will illustrate.

Consider a \$1 million interest rate future on a 6-month MRR priced at 97.50 (an MRR of 2.5%) that settles six months from now. Each basis point change in the (annualized) MRR will change the value of the contract by 0.0001 × 6/12 × \$1 million = \$50. If the MRR at settlement is either 2.51% or 2.49%, the payoff on the future at the end of one year is either \$50 higher or \$50 lower than when the MRR at settlement is 2.5%.

Compare this result with the payoffs for an otherwise equivalent forward, F_{6m6m}, priced at 2.5%.

If the MRR at settlement is 2.51%, the long receives 50/(1 + 0.0251/2) = $49.3803.

If the MRR at settlement is 2.49%, the long must pay 50/(1 + 0.0249/2) = $49.3852.

The value of the forwards exhibit convexity. An increase in rates decreases the forward's value by less than a decrease in the interest rate increases the forward's value, just as we saw with bonds. Also just as with bonds, the convexity effect for the value of forwards increases for longer periods. The convexity of forwards is termed **convexity bias** and forwards and futures prices can be significantly different for longer-term interest rates.

MODULE QUIZ 71.1

1. For a forward contract on an asset that has no costs or benefits from holding it to have zero value at initiation, the arbitrage-free forward price must equal the:
   A. expected future spot price.
   B. future value of the current spot price.
   C. present value of the expected future spot price.
2. For a futures contract to be more attractive than an otherwise equivalent forward contract, interest rates must be:
   A. uncorrelated with futures prices.
   B. positively correlated with futures prices.
   C. negatively correlated with futures prices.

KEY CONCEPTS

LOS 71.a

For a forward contract on which no mark-to-market gains or losses are paid, the forward price is constant over its life, but the contract's value will fluctuate with changes in the value of the underlying.

For a futures contract, the price and value both change when daily mark-to-market gains and losses are settled. The change in the futures price to the settlement price each day returns its value to zero.

Unlike forward rate agreements, interest rate futures are quoted on a price basis:

$$\text{futures price} = 100 - (100 \times MRR_{A,\ B-A})$$

LOS 71.b

Because gains and losses on futures contracts are settled daily, prices of forwards and futures that have the same terms may be different if interest rates are correlated with futures prices. Futures are more valuable than forwards when interest rates and futures prices are positively correlated and less valuable when they are negatively correlated. If interest rates are constant or uncorrelated with futures prices, the prices of futures and forwards are the same.

Convexity bias can result in price differences between interest rate futures contracts and otherwise equivalent forward rate agreements.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 71.1

1. **B** For an asset with no holding costs or benefits, the forward price must equal the future value of the current spot price, compounded at the risk-free rate over the term of the forward contract, for the contract to have a value of zero at initiation. Otherwise an arbitrage opportunity would exist. (LOS 71.a)

2. **B** If interest rates are positively correlated with futures prices, interest earned on cash from daily settlement gains on futures contracts will be greater than the opportunity cost of interest on daily settlement losses, and a futures contract is more attractive than an otherwise equivalent forward contract that does not feature daily settlement. (LOS 71.b)

READING 72

PRICING AND VALUATION OF INTEREST RATES AND OTHER SWAPS

MODULE 72.1: SWAP VALUATION

Video covering this content is available online.

LOS 72.a: Describe how swap contracts are similar to but different from a series of forward contracts.

In a simple interest-rate swap, one party pays a floating rate and the other pays a fixed rate on a notional principal amount. Consider a 1-year swap with quarterly payments, one party paying a fixed rate and the other a floating rate equal to a 90-day market reference rate (MRR). At each payment date the difference between the swap fixed rate and the MRR is paid to the party that owes the least, that is, a net payment is made from one party to the other.

We can separate these payments into a known payment and three unknown payments that are equivalent to the payments on three forward rate agreements (FRAs). Let MRR_n represent the floating rate payment (based on the 90-day MRR) owed at the end of quarter *n* and *F* be the fixed payment owed at the end of each quarter. We can represent the swap payment to be received by the fixed-rate payer at the end of period *n* as $MRR_n - F$. We can replicate each of these payments to (or from) the fixed-rate payer in the swap with a forward contract, specifically a long position in an FRA with a contract rate equal to the swap fixed rate and a settlement value based on the 90-day MRR.

We illustrate this separation below for a 1-year fixed-for-floating swap with a fixed rate of *F* and floating-rate payments for period *n* of MRR_n. Note that if the fixed rate and MRR are quoted as annual rates, the payments will be $(MRR_n - F)$ times one-fourth of the notional principal.

First payment (90 days from now) = $MRR_1 - F$ which is known at time zero because the payment 90 days from now is based on the 90-day MRR at time zero and the swap fixed rate, *F*, both of which are known at the initiation of the swap.

Second payment (180 days from now) is equivalent to a long position in an FRA with contract rate *F* that settles in 180 days and pays $MRR_2 - F$.

Third payment (270 days from now) is equivalent to a long position in an FRA with contract rate *F* that settles in 270 days and pays $MRR_3 - F$.

Fourth payment (360 days from now) is equivalent to a long position in an FRA with contract rate *F* that settles in 360 days and pays $MRR_4 - F$.

Note that a forward on a 90-day MRR that settles 90 days from now, based on the 90-day MRR at that time, actually pays the present value of the difference between the fixed rate *F* and the 90-day MRR 90 days from now (times the notional principal amount). Thus, the forwards in our example actually pay on days 90, 180, and 270. However, the amounts paid are equivalent to the differences between the fixed-rate payment and floating-rate payment that are due when interest is actually due on days 180, 270, and 360, which are the amounts we used in the example.

Therefore, we can describe an interest-rate swap as equivalent to a series of forward contracts, specifically FRAs, each with a forward contract rate equal to the swap fixed rate. However, there is one important difference. Because the forward contract rates are all equal in the FRAs that are equivalent to the swap, these would not be zero-value forward contracts at the initiation of the swap. Recall that forward contracts are based on a contract rate for which the value of the forward contract at initiation is zero. There is no reason to suspect that the swap fixed rate results in a zero value forward contract for each of the future dates. Instead, a swap is most likely to consist of some forwards with positive values and some forwards with negative values. The sum of their values will equal zero at initiation.

Finding the swap fixed rate that gives the swap a zero value at initiation, which is also known as the **par swap rate**, is not difficult if we follow our principle of no-arbitrage pricing. The fixed rate payer in a swap can replicate that derivative position by borrowing at a fixed rate and lending the proceeds at a variable (floating) rate. For the swap in our example, borrowing at the fixed rate *F* and lending the proceeds at the 90-day MRR will produce the same cash flows as the swap. At each date, the payment due on the fixed-rate loan is F_n and the interest received on lending at the floating rate is MRR_n.

LOS 72.b: Contrast the value and price of swaps.

As with FRAs, the *price* of a swap is the fixed rate of interest specified in the swap contract (the par swap rate) and the *value* depends on how expected future floating rates change over time. At initiation, a swap has zero value because the present value of the fixed-rate payments equals the present value of the expected floating-rate payments.

We can solve for the no-arbitrage fixed rate, termed the **par swap rate**, from the following equality:

$$\frac{MRR_1}{1+S_1}+\frac{MRR_2}{(1+S_2)^2}+\frac{MRR_3}{(1+S_3)^3}+\frac{MRR_4}{(1+S_4)^4}=$$
$$\frac{F}{1+S_1}+\frac{F}{(1+S_2)^2}+\frac{F}{(1+S_3)^3}+\frac{F}{(1+S_4)^4}$$

where S_1 through S_4 are the current effective spot rates for 90, 180, 270, and 360 days, MRR_1 through MRR_4 are the forward 90-day rates implied by the spot rates, and *F* is the fixed rate payment.

Given the current spot rates (S_1 to S_4), we can calculate the implied (expected) forward rates (MRRs), and then solve for *F*, the fixed rate that will give the swap a value of zero.

An increase in expected future 90-day rates will produce an increase the value of the fixed-rate payer position in a swap, while a decrease in expected rates will decrease the value of that position. At any point in time, the value of the fixed-rate payer side of a swap will be the present value of the expected future floating-rate payments, minus the present value of the future fixed-rate payments. This calculation is based on the spot rates and implied future 90-day rates at that point in time and can be used for any required mark-to-market payments.

MODULE QUIZ 72.1

1. Which of the following is *most* similar to the floating-rate receiver position in a fixed-for-floating interest-rate swap?
   A. Buying a fixed-rate bond and a floating-rate note.
   B. Buying a floating-rate note and issuing a fixed-rate bond.
   C. Issuing a floating-rate note and buying a fixed-rate bond.
2. The price of a fixed-for-floating interest-rate swap:
   A. is specified in the swap contract.
   B. is paid at initiation by the floating-rate receiver.
   C. may increase or decrease during the life of the swap contract.

KEY CONCEPTS

LOS 72.a

In a simple interest-rate swap, one party pays a floating rate and the other pays a fixed rate on a notional principal amount. The first payment is known at initiation and the rest of the payments are unknown. The unknown payments are equivalent to the payments on FRAs. The par swap rate is the fixed rate at which the sum of the present values of these FRAs equals zero.

LOS 72.b

The price of a swap is the fixed rate of interest specified in the swap contract. The value depends on how expected future floating rates change over time. An increase in expected future short-term future rates will increase the value of the fixed-rate payer position in a swap, and a decrease in expected future rates will decrease the value of the fixed-rate payer position.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 72.1

1. **B** The floating-rate receiver (fixed-rate payer) in a fixed-for-floating interest-rate swap has a position similar to issuing a fixed-coupon bond and buying a floating-rate note. (LOS 72.a)

2. **A** The price of a fixed-for-floating interest-rate swap is defined as the fixed rate specified in the swap contract. Typically a swap will be priced such that it has a value of zero at initiation and neither party pays the other to enter the swap. (LOS 72.b)

READING 73

PRICING AND VALUATION OF OPTIONS

MODULE 73.1: OPTION VALUATION

Video covering this content is available online.

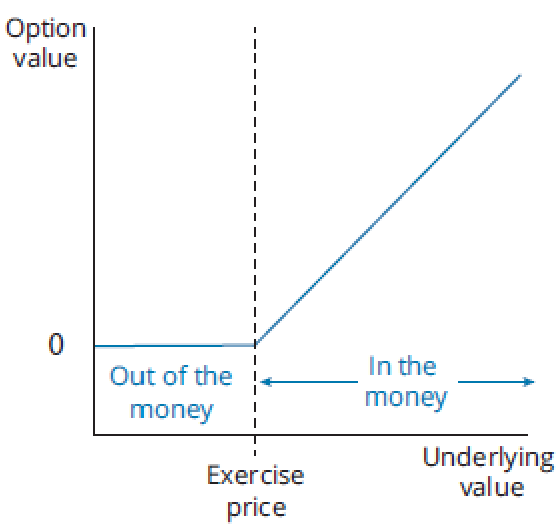
LOS 73.a: Explain the exercise value, moneyness, and time value of an option.

Moneyness refers to whether an option is *in the money* or *out of the money*. If immediate exercise of the option would generate a positive payoff, it is in the money. If immediate exercise would result in a loss (negative payoff), it is out of the money. When the current asset price equals the exercise price, exercise will generate neither a gain nor loss, and the option is *at the money*.

The following describes the conditions for a **call option** to be in, out of, or at the money. *S* is the price of the underlying asset and *X* is the exercise price of the option.

- *In-the-money call options*. If S – X > 0, a call option is in the money. S – X is the amount of the payoff a call holder would receive from immediate exercise, buying a share for *X* and selling it in the market for a greater price *S*.
- *Out-of-the-money call options.* If S – X < 0, a call option is out of the money.
- *At-the-money call options*. If S = X, a call option is said to be at the money.

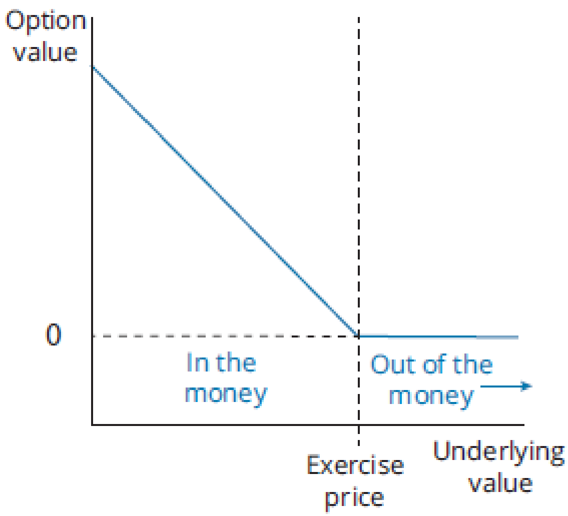
Figure 73.1: Call Option Moneyness



The following describes the conditions for a **put option** to be in, out of, or at the money.

- *In-the-money put options*. If X − S > 0, a put option is in the money. X − S is the amount of the payoff from immediate exercise, buying a share for *S* and exercising the put to receive *X* for the share.
- *Out-of-the-money put options*. When the stock's price is greater than the exercise price, a put option is said to be out of the money. If X − S < 0, a put option is out of the money.
- *At-the-money put options*. If S = X, a put option is said to be at the money.

Figure 73.2: Put Option Moneyness



EXAMPLE: Moneyness

Consider a July 40 call and a July 40 put, both on a stock that is currently selling for $37/share. Calculate how much these options are in or out of the money.

PROFESSOR'S NOTE

A July 40 call is a call option with an exercise price of $40 and an expiration date in July.

Answer:

The call is $3 out of the money because S − X = −$3.00. The put is $3 in the money because X − S = $3.00.

We define the **exercise value** (or **intrinsic value**) of an option as the maximum of zero and the amount that the option is in the money. That is, the exercise value is the amount an option is in the money, if it is in the money, or zero if the option is at or out of the money. The exercise value is the value of the option if exercised immediately.

Prior to expiration, an option has **time value** in addition to its exercise value. The time value of an option is the amount by which the **option premium** (price) exceeds the exercise value and is sometimes called the *speculative value* of the option. This relationship can be written as:

option premium = exercise value + time value

At any point during the life of an option, its value will typically be greater than its exercise value. This is because there is some probability that the underlying asset price will change in an amount that gives the option a positive payoff at expiration greater than the current exercise value. Recall that an option's exercise value (to a buyer) is the amount of the payoff at expiration and has a lower bound of zero.

When an option reaches expiration, there is no time remaining and the time value is zero. This means the value at expiration is either zero, if the option is at or out of the money, or its exercise value, if it is in the money.

LOS 73.b: Contrast the use of arbitrage and replication concepts in pricing forward commitments and contingent claims.

To model forward commitments, we used no-arbitrage pricing based on an initial value of zero to both parties. With options, however, the initial values of options are positive; the buyer pays a premium (the option price) to the writer (seller). Another difference is that where forward commitments have essentially unlimited gains or losses for both parties (except to the extent that prices are constrained by zero), options are one-sided: Potential losses for the buyer, and potential gains for the writer, are limited to the premium paid. For these reasons, the no-arbitrage approach we use for pricing contingent claims is different from the model we use for forward commitments.

The following is some terminology that we will use to determine the minimum and maximum values for European options:

S_t = price of the underlying stock at time t
X = exercise price of the option
$T-t$ = time to expiration
c_t = price of a European call at any time t prior to expiration at time = T
p_t = price of a European put at any time t prior to expiration at time = T
Rf = risk-free rate

Upper Bound for Call Options

The maximum value of a European call option at any time *t* is the time-*t* share price of the underlying stock. This makes sense because no one would pay more for the right to buy an asset than the asset's market value. It would be cheaper simply to buy the underlying asset. At time *t* = 0, the upper boundary condition for European call options is $c_0 \leq S_0$, and at any time *t* during a European call option's life, the upper boundary condition is $c_t \leq S_t$.

Upper Bound for Put Options

Logically the value of a put option cannot be more than its exercise price. This would be its exercise value if the underlying stock price goes to zero. However, because European puts cannot be exercised prior to expiration, their maximum value is the *present value* of the exercise price discounted at the risk-free rate. Even if the stock price goes to zero and is expected to stay at zero, the put buyer will not receive the intrinsic value, *X*, until the expiration date.

At time $t = 0$, the upper boundary condition can be expressed for European put options as:

$$p_0 \leq X\ (1 + Rf)^{-T}$$

At any time *t* during a European put option's life, the upper boundary condition is:

$$p_t \leq X\ (1 + Rf)^{-(T-t)}$$

Lower Bounds for Options

Theoretically, no option will sell for less than its intrinsic value and no option can take on a negative value. For European options, however, the lower bound is not so obvious because these options are not exercisable immediately.

To determine the lower bounds for European options, we can examine the value of a portfolio in which the option is combined with a long or short position in the stock and a pure discount bond.

For a *European call option*, construct the following portfolio:

- A long at-the-money European call option with exercise price *X,* expiring at time *T*.
- A long discount bond priced to yield the risk-free rate that pays *X* at option expiration.
- A short position in one share of the underlying stock priced at $S_0 = X$.

The current value of this portfolio is $c_0 - S_0 + X(1 + Rf)^{-T}$.

At expiration time *T*, this portfolio will pay $c_T - S_T + X$. That is, we will collect $c_T = \max[0, S_T - X]$ on the call option, pay S_T to cover our short stock position, and collect *X* from the maturing bond.

- If $S_T \geq X$, the call is in-the-money, and the portfolio will have a zero payoff because the call pays $S_T - X$, the bond pays $+X$, and we pay $-S_T$ to cover our short position. That is, the time $t = T$ payoff is: $S_T - X + X - S_T = 0$.
- If $S_T < X$, the call is out-of-the-money, and the portfolio has a positive payoff equal to $X - S_T$ because the call value, c_T, is zero, we collect *X* on the bond, and pay $-S_T$ to cover the short position. So, the time $t = T$ payoff is: $0 + X - S_T = X - S_T$.

No matter whether the option expires in-the-money, at-the-money, or out-of-the-money, the portfolio value will be equal to or greater than zero. We will never have to make a payment.

To prevent arbitrage, any portfolio that has no possibility of a negative payoff cannot have a negative value. Thus, we can state the value of the portfolio *at time* $t = 0$ as:

$$c_0 - S_0 + X(1 + Rf)^{-T} \geq 0$$

which allows us to conclude that:

$$c_0 \geq S_0 - X(1 + Rf)^{-T}$$

Combining this result with the earlier minimum on the call value of zero, we can write:

$$c_0 \geq \text{Max}[0, S_0 - X(1 + Rf)^{-T}]$$

Note that $X(1 + Rf)^{-T}$ is the present value of a pure discount bond with a face value of X.

For a *European put option* we can derive the minimum value by forming the following portfolio at time $t = 0$:

- A long at-the-money European put option with exercise price X, expiring at T.
- A short position on a risk-free bond priced at $X(1 + Rf)^{-T}$, equivalent to borrowing $X(1 + Rf)^{-T}$.
- A long position in a share of the underlying stock priced at S_0.

At expiration time T, this portfolio will pay $p_T + S_T - X$. That is, we will collect $p_T = \text{Max}[0, X - S_T]$ on the put option, receive S_T from the stock, and pay X on the bond (loan).

- If $S_T > X$, the payoff will equal: $p_T + S_T - X = S_T - X$.
- If $S_T \leq X$, the payoff will be zero.

Again, a no-arbitrage argument can be made that the portfolio value must be zero or greater, because there are no negative payoffs to the portfolio.

At time $t = 0$, this condition can be written as:

$$p_0 + S_0 - X(1 + Rf)^{-T} \geq 0$$

and rearranged to state the minimum value for a European put option at time $t = 0$ as:

$$p_0 \geq X(1 + Rf)^{-T} - S_0$$

We have now established the minimum bound on the price of a European put option as:

$$p_0 \geq \text{Max}[0, X(1 + Rf)^{-T} - S_0]$$

Figure 73.3 summarizes what we now know regarding the boundary prices for European options at any time t prior to expiration at time t = T.

Figure 73.3: Lower and Upper Bounds for Options

Option	Minimum Value	Maximum Value
European call	$c_t \geq \text{Max}[0, S_t - X(1 + Rf)^{-(T-t)}]$	S_t
European put	$p_t \geq \text{Max}[0, X(1 + Rf)^{-(T-t)} - S_t]$	$X(1 + Rf)^{-(T-t)}$

PROFESSOR'S NOTE

For the exam, know the price limits in Figure 73.3. You will not be asked to derive them, but you may be expected to use them.

LOS 73.c: Identify the factors that determine the value of an option and describe how each factor affects the value of an option.

There are six factors that determine option prices.

1. **Price of the underlying asset.** For call options, the higher the price of the underlying, the greater its exercise value and the higher the value of the option. Conversely, the lower the price of the underlying, the less its exercise value and the lower the value of the call option. In general, call option values increase when the value of the underlying asset increases.

   For put options this relationship is reversed. An increase in the price of the underlying reduces the value of a put option.

2. **The exercise price.** A higher exercise price decreases the values of call options and a lower exercise price increases the values of call options.

   A higher exercise price increases the values of put options and a lower exercise price decreases the values of put options.

3. **The risk-free rate of interest.** An increase in the risk-free rate will increase call option values, and a decrease in the risk-free rate will decrease call option values.

   An increase in the risk-free rate will decrease put option values, and a decrease in the risk-free rate will increase put option values.

PROFESSOR'S NOTE

One way to remember the effects of changes in the risk-free rate is to think about present values of the payments for calls and puts. These statements are strictly true only for in-the-money options, but it's a way to remember the relationships. The holder of a call option will pay in the future to exercise a call option and the present value of that payment is lower when the risk-free rate is higher, so a higher risk-free rate increases a call option's value. The holder of a put option will receive a payment in the future when the put is exercised and an increase in the risk-free rate decreases the present value of this payment, so a higher risk-free rate decreases a put option's value.

4. **Volatility of the underlying.** Volatility is what makes options valuable. If there were no volatility in the price of the underlying asset (its price remained constant), options would always be equal to their exercise values and time or speculative value would be zero. An increase in the volatility of the price of the underlying asset increases the values of both put and call options and a decrease in volatility of the price of the underlying decreases both put values and call values.

5. **Time to expiration.** Because volatility is expressed per unit of time, longer time to expiration effectively increases expected volatility and increases the value of a call

option. Less time to expiration decreases the time value of a call option so that at expiration its value is simply its exercise value.

For most put options, longer time to expiration will increase option values for the same reasons. For some European put options, however, extending the time to expiration can decrease the value of the put. In general, the deeper a put option is in the money, the higher the risk-free rate, and the longer the current time to expiration, the more likely that extending the option's time to expiration will decrease its value.

To understand this possibility consider a put option at $20 on a stock with a value that has decreased to $1. The exercise value of the put is $19 so the upside is very limited, the downside (if the price of the underlying subsequently increases) is significant, and because no payment will be received until the expiration date, the current option value reflects the present value of any expected payment. Extending the time to expiration would decrease that present value. While overall we expect a longer time to expiration to increase the value of a European put option, in the case of a deep in-the-money put, a longer time to expiration could decrease its value.

6. **Costs and benefits of holding the asset.** If there are benefits of holding the underlying asset (dividend or interest payments on securities or a convenience yield on commodities), call values are decreased and put values are increased. The reason for this is most easily understood by considering cash benefits. When a stock pays a dividend, or a bond pays interest, this reduces the value of the asset. Decreases in the value of the underlying asset decrease call values and increase put values.

    Positive storage costs make it more costly to hold an asset. We can think of this as making a call option more valuable because call holders can have long exposure to the asset without paying the costs of actually owning the asset. Puts, on the other hand, are less valuable when storage costs are higher.

MODULE QUIZ 73.1

1. The price of an out-of-the-money option is:
   A. less than its time value.
   B. equal to its time value.
   C. greater than its time value.
2. The lower bound for the value of a European put option is:
   A. Max(0, S − X).
   B. Max[0, $X(1 + Rf)^{-(T-t)}$ − S].
   C. Max[0, S − $X(1 + Rf)^{-(T-t)}$].
3. A decrease in the risk-free rate of interest will:
   A. increase put and call option prices.
   B. decrease put option prices and increase call option prices.
   C. increase put option prices and decrease call option prices.

KEY CONCEPTS

LOS 73.a

If immediate exercise of an option would generate a positive payoff, the option is in the money. If immediate exercise would result in a negative payoff, the option is out of the money. An option's exercise value is the greater of zero or the amount it is in the money. Time value is the amount by which an option's price is greater than its exercise value. Time value is zero at expiration.

LOS 73.b

The approach for pricing contingent claims is different from the model for forward commitments because contingent claims have one-sided payoffs and values at initiation that are not equal to zero. A replication model for European options is based on the value of a portfolio in which the option is combined with a pure discount bond and a long or short position in the underlying.

LOS 73.c

Factors that determine the value of an option:

Increase in:	Effect on Call Option Values	Effect on Put Option Values
Price of underlying asset	Increase	Decrease
Exercise price	Decrease	Increase
Risk-free rate	Increase	Decrease
Volatility of underlying asset	Increase	Increase
Time to expiration	Increase	Increase, except some European puts
Costs of holding underlying asset	Increase	Decrease
Benefits of holding underlying asset	Decrease	Increase

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 73.1

1. **B** Because an out-of-the-money option has an exercise value of zero, its price is its time value. (LOS 73.a)

2. **B** The lower bound for a European put ranges from zero to the present value of the exercise price less the current stock price, where the exercise price is discounted at the risk-free rate. (LOS 73.b)

3. **C** A decrease in the risk-free rate will decrease call option values and increase put option values. (LOS 73.c)

READING 74

OPTION REPLICATION USING PUT–CALL PARITY

MODULE 74.1: PUT-CALL PARITY

Video covering this content is available online.

LOS 74.a: Explain put–call parity for European options.

Our derivation of **put-call parity** for European options is based on the payoffs of two portfolio combinations: a fiduciary call and a protective put.

A *fiduciary call* is a combination of a call with exercise price *X* and a pure-discount, riskless bond that pays *X* at maturity (option expiration). The payoff for a fiduciary call at expiration is *X* when the call is out of the money, and X + (S − X) = S when the call is in the money.

A *protective put* is a share of stock together with a put option on the stock. The expiration date payoff for a protective put is (X − S) + S = X when the put is in the money, and *S* when the put is out of the money.

PROFESSOR'S NOTE

When working with put-call parity, it is important to note that the exercise prices on the put and the call and the face value of the riskless bond are all equal to *X*.

If at expiration *S* is greater than or equal to *X*:

- The protective put pays *S* on the stock while the put expires worthless, so the payoff is *S*.
- The fiduciary call pays *X* on the bond portion while the call pays (S − X), so the payoff is X + (S − X) = S.

If at expiration *X* is greater than *S*:

- The protective put pays *S* on the stock while the put pays (X − S), so the payoff is S + (X − S) = X.
- The fiduciary call pays *X* on the bond portion while the call expires worthless, so the payoff is *X*.

In either case, the payoff on a protective put is the same as the payoff on a fiduciary call. Our no-arbitrage condition holds that portfolios with identical payoffs regardless of future conditions must sell for the same price to prevent arbitrage. We can express the put-call parity relationship as:

$$c + X(1 + Rf)^{-T} = S + p$$

Equivalencies for each of the individual securities in the put-call parity relationship can be expressed as:

$$S = c - p + X(1 + Rf)^{-T}$$
$$p = c - S + X(1 + Rf)^{-T}$$
$$c = S + p - X(1 + Rf)^{-T}$$
$$X(1 + Rf)^{-T} = S + p - c$$

Note that the options must be European style and the puts and calls must have the same exercise price and time to expiration for these relations to hold.

The single securities on the left-hand side of the equations all have exactly the same payoffs as the portfolios on the right-hand side. The portfolios on the right-hand side are the **synthetic** equivalents of the securities on the left. For example, to synthetically produce the payoff for a long position in a share of stock, use the following relationship:

$$S = c - p + X(1 + Rf)^{-T}$$

This means that the payoff on a long stock can be synthetically created with a long call, a short put, and a long position in a risk-free discount bond.

The other securities in the put-call parity relationship can be constructed in a similar manner.

PROFESSOR'S NOTE
After expressing the put-call parity relationship in terms of the security you want to synthetically create, the sign on the individual securities will indicate whether you need a long position (+ sign) or a short position (− sign) in the respective securities.

EXAMPLE: Call option valuation using put-call parity

Suppose that the current stock price is $52 and the risk-free rate is 5%. You have found a quote for a 3-month put option with an exercise price of $50. The put price is $1.50, but due to light trading in the call options, there was not a listed quote for the 3-month, $50 call. Estimate the price of the 3-month call option.

Answer:

Rearranging put-call parity, we find that the call price is:

$$\text{call} = \text{put} + \text{stock} - \text{present value (X)}$$

$$\text{call} = \$1.50 + \$52 - \frac{\$50}{1.05^{0.25}} = \$4.11$$

This means that if a 3-month, $50 call is available, it should be priced at (within transaction costs of) $4.11 per share.

LOS 74.b: Explain put–call *forward* parity for European options.

Put-call-forward parity is derived with a forward contract rather than the underlying asset itself. Consider a forward contract on an asset at time *T* with a contract price of $F_0(T)$. At contract initiation the forward contract has zero value. At time *T*, when the forward contract settles, the long must purchase the asset for $F_0(T)$. The purchase (at time = 0) of a pure discount bond that will pay $F_0(T)$ at maturity (time = T) will cost $F_0(T)(1 + Rf)^{-T}$.

By purchasing such a pure discount bond and simultaneously taking a long position in the forward contract, an investor has created a synthetic asset. At time = T the proceeds of the bond are just sufficient to purchase the asset as required by the long forward position. Because there is no cost to enter into the forward contract, the total cost of the synthetic asset is the present value of the forward price, $F_0(T)(1 + Rf)^{-T}$.

The put-call-forward parity relationship is derived by substituting the synthetic asset for the underlying asset in the put-call parity relationship. Substituting $F_0(T)(1 + Rf)^{-T}$ for the asset price S_0 in $S + p = c + X(1 + Rf)^{-T}$ gives us:

$$F_0(T)(1 + Rf)^{-T} + p_0 = c_0 + X(1 + Rf)^{-T}$$

which is put-call-forward parity at time 0, the initiation of the forward contract, based on the principle of no arbitrage. By rearranging the terms, put-call-forward parity can also be expressed as:

$$p_0 - c_0 = [X - F_0(T)](1 + Rf)^{-T}$$

Application of Options Theory to Corporate Finance

We can view the claims of a firm's equity holders and debt holders as a call option and a put option, respectively. Consider a firm that has a value of V_t at time = t and has issued debt in the form of a zero-coupon bond that will pay D at time = T. At time = T, if $V_T > D$ the equity holders receive $V_T - D$ and if $V_T < D$, the firm is insolvent and equity holders receive nothing. The payoff to the equity holders at time = T can be written as $\text{Max}(0, V_T - D)$ which is equivalent to a call option with the firm value as the underlying and an exercise price of D.

At time = T, if $V_T > D$ the debt holders receive D and if $V_T < D$, the firm is insolvent and debt holders receive V_T. The payoff to the debt holders at time = T can be written as $\text{Min}(V_T, D)$. This is equivalent to a portfolio that is long a risk-free bond that pays D at *t* = T, and short (has sold) a put option on the value of the firm, V_T, with an exercise price of D. If $V_T > D$ the portfolio pays D and the put expires worthless, and if $V_T < D$ the portfolio pays $D - (D - V_T) = V_T$ and the debtholders effectively pay $D - V_T$ on the short put position.

MODULE QUIZ 74.1

1. The put-call parity relationship for European options must hold because a protective put will have the same payoff as a(n):
   A. covered call.
   B. fiduciary call.
   C. uncovered call.
2. The put-call-forward parity relationship *least likely* includes:
   A. a risk-free bond.
   B. call and put options.
   C. the underlying asset.

KEY CONCEPTS

LOS 74.a

A fiduciary call (a call option and a risk-free zero-coupon bond that pays the strike price *X* at expiration) and a protective put (a share of stock and a put at *X*) have the same payoffs at expiration, so arbitrage will force these positions to have equal prices: $c + X(1 + Rf)^{-T} = S + p$. This establishes put-call parity for European options.

Based on the put-call parity relation, a synthetic security (stock, bond, call, or put) can be created by combining long and short positions in the other three securities.

$$c = S + p - X(1 + Rf)^{-T}$$
$$p = c - S + X(1 + Rf)^{-T}$$
$$S = c - p + X(1 + Rf)^{-T}$$
$$X(1 + Rf)^{-T} = S + p - c$$

LOS 74.b

Because we can replicate the payoff on an asset by lending the present value of the forward price at the risk-free rate and taking a long position in a forward, we can write put-call-forward parity as:

$$c_0 + X(1 + Rf)^{-T} = F_0(T)(1 + Rf)^{-T} + p_0$$

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 74.1

1. **B** Given call and put options on the same underlying asset with the same exercise price and expiration date, a protective put (underlying asset plus a put option) will have the same payoff as a fiduciary call (call option plus a risk-free bond that will pay the exercise price on the expiration date) regardless of the underlying asset price on the expiration date. (LOS 74.a)

2. **C** The put-call-forward parity relationship is $F_0(T)(1 + RFR)^{-T} + p_0 = c_0 + X(1 + Rf)^{-T}$, where $X(1 + Rf)^{-T}$ is a risk-free bond that pays the exercise price on the

expiration date, and $F_0(T)$ is the forward price of the underlying asset. (LOS 74.b)

READING 75

VALUING A DERIVATIVE USING A ONE-PERIOD BINOMIAL MODEL

MODULE 75.1: BINOMIAL MODEL FOR OPTION VALUES

LOS 75.a: Explain how to value a derivative using a one-period binomial model.

Video covering this content is available online.

Recall from Quantitative Methods that a **binomial model** is based on the idea that, over the next period, some value will change to one of two possible values (binomial). To construct a one-period binomial model for pricing an option, we need:

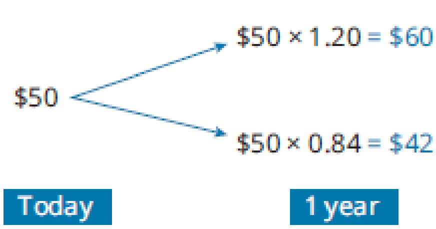
- A value for the underlying at the beginning of the period.
- An exercise price for the option. The exercise price can be different from the value of the underlying. We assume the option expires one period from now.
- Returns that will result from an up-move and a down-move in the value of the underlying over one period.
- The risk-free rate over the period.

For now we do not need to consider the probabilities of an up-move or a down-move. Later in this reading we will examine one-period binomial models with risk-neutral probabilities.

One-Period Binomial Model for a Call Option

As an example, we can model a call option with an exercise price of $55 on a stock that is currently valued (S_0) at $50. Let us assume that in one period the stock's value will either increase (S_1^u) to $60 or decrease ($S_1^d$) to $42. We state the return from an up-move (R^u) as $60 / $50 = 1.20, and the return from a down-move (R^d) as $42 / $50 = 0.84.

Figure 75.1: One-Period Binomial Tree



The call option will be in the money after an up-move or out of the money after a down-move. Its value at expiration after an up-move, c_1^u, is Max(0, $60 – $55) = $5. Its value after a down-move, c_1^d, is Max(0, $42 – $55) = 0.

Now we can use no-arbitrage pricing to determine the initial value of the call option (c_0). We do this by creating a portfolio of the option and the underlying stock, such that the portfolio will have the same value following either an up-move (V_1^u) or a down-move (V_1^d) in the stock. To value a call option, we create a portfolio that is *short* a call (i.e., the call writer) and *long* a number of shares of the stock that we will denote as *h*. We must solve for the *h* that results in $V_1^u = V_1^d$.

- The initial value of our portfolio, V_0, is $hS_0 - c_0$ (remember we are short the call option).
- The portfolio value after an up-move, V_1^u, is $hS_1^u - c_1^u$.
- The portfolio value after a down-move, 9.

In our example, $V_1^u = h(\$60) - \$5$, and $V_1^d = h(\$42) - 0$. Setting $V_1^u = V_1^d$ solving for *h*, we get:

$$h(\$60) - \$5 = h(\$42)$$

$$h(\$60) - h(\$42) = \$5$$

$$h = \$5 \,/\, (\$60 - \$42) = 0.278$$

This result, the number of shares of the underlying we would buy for each call option we would write, is known as the **hedge ratio** for this option.

With $V_1^u = V_1^d$, the value of the portfolio after one period is known with certainty. This means we can say that either V_1^u or V_1^d must equal V_0 compounded at the risk-free rate for one period. In this example, $V_1^d = 0.278(\$42) = \$11.68$, or $V_1^u = 0.278(\$60) - \$5 = \$11.68$. Let us assume the risk-free rate over one period is 3%. Then $V_0 = \$11.68 / 1.03 = \$11.34$.

Now we can solve for the value of the call option, c_0. Recall that $V_0 = hS_0 - c_0$, so $c_0 = hS_0 - V_0$. Here, $c_0 = 0.278(\$50) - \$11.34 = \$2.56$.

PROFESSOR'S NOTE

When we "create a portfolio" to construct this model, we do not assume any cash has changed hands to buy the underlying or sell the call. We simply assume a portfolio exists that is made up of these positions.

One-Period Binomial Model for a Put Option

We can model a put option using the same technique. The key difference is that the model portfolio consists of *long* positions in both a put and the underlying stock.

As an example, using the same assumptions that the stock is currently valued at $50, and in one period its value will either increase to $60 (for a return of 1.20) or decrease to $42 (for a return of 0.84), we can value a put option with an exercise price of $48.

The put option will be in the money after a down-move or out of the money after an up-move. Its value at expiration after a down-move, p_1^d, is Max(0, $48 – $60) = 0.

Again we want to find the hedge ratio *h,* the number of shares at which our portfolio has the same value after an up-move or a down-move, $V_1^u = V_1^d$.

- The initial value of our portfolio, V_0, is $hS_0 + p_0$ (remember we are long the put).
- The portfolio value after an up-move, V_1^u, is $h S_1^u + p_1^u$.
- The portfolio value after a down-move, V_1^d, is $h S_1^d + p_1^d$.

Setting $V_1^u = V_1^d$, we get:

$$h(\$60) = h(\$42) + \$6$$

$$h(\$60) - h(\$42) = \$6$$

$$h = \$6 / (\$60 - \$42) = 0.333$$

As before, we can determine the value of the portfolio one period from now using either V_1^u or V_1^d because they will be equal:

$$V_1^u = 0.333(\$60) = \$20, \text{ or } V_1^d = 0.333(\$42) + \$6 = \$20$$

Discounting this value to t = 0 at the risk-free rate of 3% gives us:

$$V_0 = \$20 / 1.03 = \$19.42$$

Finally we can solve for p_0, the value of the put option:

$$V_0 = hS_0 + p_0, \text{ so } p_0 = V_0 - hS_0$$

$$p_0 = \$19.42 - 0.333(\$50) = \$2.75$$

LOS 75.b: Describe the concept of risk neutrality in derivatives pricing.

Another approach to constructing a one-period binomial model involves risk-neutral probabilities of an up-move or a down-move. Consider a share of stock currently priced at \$30. The size of the possible price changes, and the probabilities of these changes occurring, are as follows:

$$R^u = \text{up-move factor} = 1.15$$

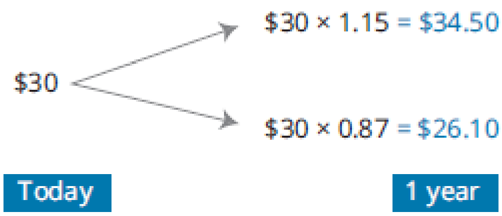
$$R^d = \text{down-move factor} = \frac{1}{R^u} = \frac{1}{1.15} = 0.87$$

$$\pi_U = \text{risk-neutral probability of an up-move} = 0.715$$

$$\pi_D = \text{risk-neutral probability of a down-move} = 1 - \pi_U = 1 - 0.715 = 0.285$$

Note that the down-move factor is the reciprocal of the up-move factor, and the probability of an up-move is one minus the probability of a down-move. The one-period binomial tree for the stock is shown in Figure 75.2. The beginning stock value of \$30 is to the left, and to the right are the two possible end-of-period stock values, 30 × 1.15 = \$34.50 and 30 × 0.87 = \$26.10.

Figure 75.2: One-Period Binomial Tree



The risk-neutral probabilities of an up-move and a down-move are calculated from the sizes of the moves and the risk-free rate:

$$\pi_U = \text{risk-neutral probability of an up-move} = \frac{1 + Rf - R^d}{R^u - R^d}$$

$$\pi_D = \text{risk-neutral probability of a down-move} = 1 - \pi_U$$

where:

Rf = risk-free rate

R^u = size of an up-move

R^d = size of a down-move

PROFESSOR'S NOTE

These two probabilities are not the actual probabilities of the up- and down-moves. They are risk-neutral pseudo probabilities. The calculation of risk-neutral probabilities does not appear to be required for the Level I exam.

We can calculate the value of an option on the stock by:

- Calculating the payoffs of the option at expiration for the up-move and down-move prices.
- Calculating the expected payoff of the option in one year as the (risk-neutral) probability-weighted average of the up-move and down-move payoffs.
- Calculating the PV of the expected payoff by discounting at the risk-free rate.

EXAMPLE: **Calculating call option value with risk-neutral probabilities**

Use the binomial tree in Figure 75.2 to calculate the value today of a 1-year call option on a stock with an exercise price of $30. Assume the risk-free rate is 7%, the current value of the stock is $30, and the up-move factor is 1.15.

Answer:

First, we need to calculate the down-move factor and risk-neutral the probabilities of the up- and down-moves:

$$R^d = \text{size of down-move} = \frac{1}{R^u} = \frac{1}{1.15} = 0.87$$

$$\pi_U = \text{risk-neutral probability of an up-move} = \frac{1 + 0.07 - 0.87}{1.15 - 0.87} = 0.715$$

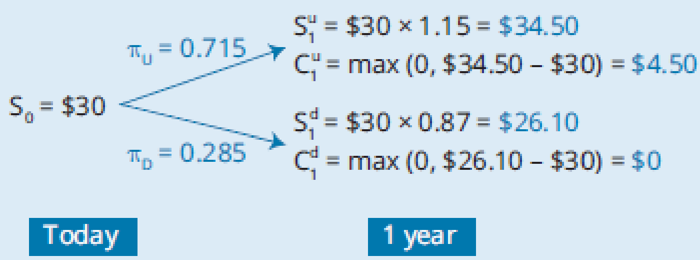
$$\pi_D = \text{risk-neutral probability of a down-move} = 1 - 0.715 - 0.285$$

Next, determine the payoffs on the option in each state. If the stock moves up to $34.50, a call option with an exercise price of $30 will pay $4.50. If the stock moves down to $26.10, the call option will expire worthless. The option payoffs are illustrated in the following figure.

Let the stock values for the up-move and down-move be S_1^u and S_1^d and for the call values, c_1^u and c_1^d.

One-Period Call Option With X = $30

The expected value of the option in one period is:



E(call option value in 1 year) = ($4.50 × 0.715) + ($0 × 0.285) = $3.22

The value of the option today, discounted at the risk-free rate of 7%, is:

$$C_0 = \frac{\$3.22}{1.07} = \$3.01$$

We can use the same basic framework to value a one-period put option. The only difference is that the payoff to the put option will be different from the call payoffs.

EXAMPLE: Valuing a one-period put option on a stock

Use the information in the previous example to calculate the value of a put option on the stock with an exercise price of $30.

Answer:

If the stock moves up to $34.50, a put option with an exercise price of $30 will expire worthless. If the stock moves down to $26.10, the put option will be worth $3.90.

The risk-neutral probabilities are 0.715 and 0.285 for an up- and down-move, respectively. The expected value of the put option in one period is:

E(put option value in 1 year) = ($0 × 0.715) + ($3.90 × 0.285) = $1.11

The value of the option today, discounted at the risk-free rate of 7%, is:

$$p_0 = \frac{\$1.11}{1.07} = \$1.04$$

In practice, we would construct a binomial model with many short periods and have many possible outcomes at expiration. However, the one-period model is sufficient to illustrate the concept and method.

Note that the actual probabilities of an up-move and a down-move do not enter directly into our calculation of option value. The size of the up-move and down-move, along with the risk-free rate, determines the risk-neutral probabilities we use to calculate the expected payoff at option expiration. Remember, the risk-neutral probabilities come from constructing a hedge that creates a certain payoff. Because their calculation is based on an arbitrage relationship, we can discount the expected payoff based on risk-neutral probabilities, using the risk-free rate.

MODULE QUIZ 75.1

1. To construct a one-period binomial model for valuing an option, are probabilities of an up-move or a down-move in the underlying price required?
   A. No.
   B. Yes, but they can be calculated from the returns on an up-move and a down-move.
   C. Yes, the model requires estimates for the actual probabilities of an up-move and a down-move.
2. In a one-period binomial model based on risk neutrality, the value of an option is *best* described as the present value of:
   A. a probability-weighted average of two possible outcomes.
   B. a probability-weighted average of a chosen number of possible outcomes.
   C. one of two possible outcomes based on a chosen size of increase or decrease.
3. A one-period binomial model for option pricing uses risk-neutral probabilities because:
   A. the model is based on a no-arbitrage relationship.
   B. they are unbiased estimators of the actual probabilities.
   C. the buyer can let an out-of-the-money option expire unexercised.

KEY CONCEPTS

LOS 75.a

A one-period binomial model for pricing an option requires the underlying asset's value at the beginning of the period, an exercise price for the option, the asset prices that will result from an up-move and a down-move, and the risk-free rate.

A portfolio of the underlying asset hedged with a position in an option can be created such that the portfolio has the same value for both an up-move and a down-move. Because the portfolio's value at the end of the period is certain, that value must be the portfolio's initial value compounded at the risk-free rate. The number of units of the underlying required to construct such portfolios is the hedge ratio.

LOS 75.b

To determine the value of an option using the concept of risk neutrality, we calculate its payoffs for both an up-move and a down-move, calculate the expected payoff as a weighted average using the risk-neutral probabilities of an up-move and a down-move, and discount this expected payoff for one period at the risk-free rate.

ANSWER KEY FOR MODULE QUIZZES

Module Quiz 75.1

1. **A** A one-period binomial model can be constructed based on replication and no-arbitrage pricing, without regard to the probabilities of an up-move or a down-move. (LOS 75.a)
2. **A** In a one-period binomial model based on risk-neutral probabilities, the value of an option is the present value of a probability-weighted average of two possible option payoffs at the end of a single period, during which the price of the underlying asset is assumed to move either up or down to specific values. (LOS 75.b)
3. **A** Because a one-period binomial model is based on a no-arbitrage relationship, we can discount the expected payoff at the risk-free rate. (LOS 75.b)

TOPIC QUIZ: DERIVATIVES

You have now finished the Derivatives topic section. Please log into your Schweser online dashboard and take the Topic Quiz on this section. The Topic Quiz provides immediate feedback on how effective your study has been for this material. Questions are more exam-like than typical Module Quiz or QBank questions; a score of less than 70% indicates that your study likely needs improvement. These tests are best taken timed; allow 1.5 minutes per question.

APPENDIX

Rates, Returns, and Yields

A **holding period return (HPR)**, or holding period yield (HPY), can be for a period of any length and is simply the percentage increase in value over the period, which is calculated as:

HPR = ending value / beginning value – 1

1. If an investor puts $2,000 into an account and 565 days later it has grown in value to $2,700, the 565-day HPY is 2,700 / 2,000 – 1 = 35%.
2. If an investor buys a share of stock for $20/share, receives a $0.40 dividend, and sells the shares after nine months, the nine-month HPY is (22 + 0.40) / 20 – 1 = 12%.

An HPR for a given period is also the **effective yield** for that period.

An **effective annual yield** is the HPR for a one-year investment or the HPY for a different period converted to its annual equivalent yield.

3. If the six-month HPR is 2%, the effective annual yield is $1.02^2 - 1 = 4.040\%$.
4. If the 125-day HPR is 1.5%, the effective annual yield is $1.015^{365/125} - 1 = 4.443\%$.
5. If the two-year HPR (two-year effective rate) is 9%, the effective annual yield is $1.09^{1/2} - 1 = 4.4031\%$.

Compounding Frequency

Sometimes the “rate” on an investment is expressed as a **simple annual rate** (or *stated rate*)—the annual rate with no compounding of returns. The number of compounding periods per year is called the **periodicity** of the rate. For a periodicity of one, the stated rate and the effective annual rate are the same. When the periodicity is greater than one (more than one compounding period per year), the effective annual rate is the effective rate for the sub-periods, compounded for the number of sub-periods.

6. A bank CD has a stated annual rate of 6% with annual compounding (periodicity of 1); the effective annual rate is 6% and a $1,000 investment will return $1,000(1.06) = $1,060 at the end of one year.
7. A bank CD has a stated annual rate of 6% with semiannual compounding (periodicity of 2); the effective annual rate is $(1 + 0.06 / 2)^2 = 1.03^2 - 1 = 6.09\%$ and a $1,000 investment will return $1,000 (1.0609) = $1,060.90 at the end of one year.
8. A bank CD has a stated annual rate of 6% with quarterly compounding (periodicity of 4); the effective annual rate is $(1 + 0.06 / 4)^4 = 1.015^4 - 1 = 6.136\%$ and a $1,000 investment will return $1,000(1.06136) = $1,061.36 at the end of one year.

Note that increasing compounding frequency increases the effective annual yield for any given stated rate. In the limit, as compounding periods get shorter (more frequent), compounding is *continuous.* A stated rate of *r* %, with continuous compounding, results in an effective annual return of $e^r - 1$.

9. A bank CD has a stated annual rate of 6%, continuously compounded; its effective annual yield is $e^{0.06} - 1 = 6.184\%$ and a \$1,000 investment will return \$1,000(1.06184) = \$1,061.84 at the end of one year.

Bond Quotations and Terminology

The **coupon rate** on a bond is the total cash coupon payments made over one year as a percentage of face value.

10. A bond with a face value of \$1,000 that pays a coupon of \$50 once each year (an annual-pay bond) has a coupon rate of 50 / 1,000 = 5% and we say it has a periodicity of 1.
11. A bond with a face value of \$1,000 that pays a coupon of \$25 twice each year (a semiannual-pay bond) has a coupon rate of (25 + 25) / 1,000 = 5% and we say it has a periodicity of 2.
12. A bond with a face value of \$1,000 that pays a coupon of \$12.50(1.25%) four times each year (a quarterly-pay bond) has a coupon rate of (12.50 + 12.50 + 12.50 + 12.50) / 1,000 = 5% and we say it has a periodicity of 4.

The **current yield** on a bond is the coupon rate divided by the bond price as a percentage of face value or, alternatively, the sum of the coupon payments for one year divided by the bond price.

13. A bond with a stated coupon rate of 5% that is selling at 98.54% of face value has a current yield of 5 / 98.54 = 5.074%.
14. A bond that is trading at \$1,058 and makes annual coupon payments that sum to \$50 has a current yield of 50 / 1,058 = 4.726%.

The **yield to maturity** (YTM) of a bond, on an *annual basis*, is the effective annual yield and is used for bonds that pay an annual coupon. For bonds that pay coupons semiannually, we often quote the YTM on a *semiannual basis*, that is, two times the effective semiannual yield. To compare the yields of two bonds, we must calculate their YTMs on the same basis.

15. A bond with a YTM of 5% on a semiannual basis has a YTM on an annual basis (effective annual yield) of $(1 + 0.05 / 2)^2 - 1 = 5.0625\%$.
16. A bond with a YTM of 5% on an annual basis has a YTM on a semiannual basis of $(1.05^{1/2} - 1) \times 2 = 4.939\%$.

Internal Rate of Return (IRR)

The internal rate of return is the discount rate that makes the PV of a series of cash flows equal to zero. This calculation must be done with a financial calculator. We can use the IRR for calculating the return on a capital project, the YTM on a bond, and the money weighted rate of return for a portfolio.

17. For the YTM of an annual-pay bond (YTM on an annual basis) on a coupon date with *N* years remaining until maturity, we calculate the annual IRR that satisfies:

$$-\text{bond price} + \frac{\text{coupon 1}}{1+\text{IRR}} + \frac{\text{coupon 2}}{(1+\text{IRR})^2} + \ldots. + \frac{\text{coupon N} + \text{face value}}{(1+\text{IRR})^N} = 0$$

18. For the YTM of a semiannual-pay bond on a coupon date with *N* years remaining until maturity, we calculate the IRR that satisfies:

$$-\text{bond price} + \frac{\text{coupon 1}}{1+\frac{\text{IRR}}{2}} + \frac{\text{coupon 2}}{\left(1+\frac{\text{IRR}}{2}\right)^2} + \ldots. + \frac{\text{coupon 2N} + \text{face value}}{\left(1+\frac{\text{IRR}}{2}\right)^{2N}} = 0$$

After solving for IRR / 2, which is the IRR for semiannual periods, we must multiply it by 2 to get the bond's YTM on a semiannual basis.

Money Market Securities

For some money market securities, such as U.S. T-bills, price quotations are given on a bond discount (or simply discount) basis. The discount yield is the percentage discount from face value of a T-bill, annualized based on a 360-day year, and is therefore not an effective yield but simply an annualized discount from face value.

19. A T-bill that will pay $1,000 at maturity in 180 days is selling for $984, a discount of 1 – 984 / 1,000 = 1.6%. The annualized discount is 1.6% × 360 / 180 = 3.2%.
20. A 120-day T-bill is quoted at a discount yield of 2.83%, its price is [1 – (0.0283 × 120 / 360)] × 1,000 = $990.57. Its 120-day *holding period return* is 1,000 / 990.57 – 1 = 0.952%. Its *effective annual yield* is $(1{,}000 / 990.57)^{365/120} - 1 = 2.924\%$.
21. HPY on a 30-day loan at a quoted market reference rate of 1.8% is 0.018 × 30 / 360 = 0.15% so the interest on a $10,000 loan is 10,000 × 0.0015 = $15.

A related yield is the **money market yield (MMY)**, which is HPY annualized based on a 360-day year.

22. A 120-day discount security with a maturity value of $1,000 that is priced at $995 has a money market yield of (1,000 / 995 – 1) × 360 / 120 = 1.5075%.

Spot and Forward Rates

Forward rates are rates for a loan to be made in a future period. They are quoted based on the period of the loan. For loans of one year, we write 1y1y for a 1-year loan to be made one year from today and 2y1y for a 1-year loan to be made two years from today.

Spot rates are discount rates for single payments to be made in the future (such as for zero-coupon bonds).

23. Given a 3-year spot rate expressed as a compound annual rate (S_3) of 2%, a 3-year bond that makes a single payment of $1,000 in three years has a current value of $1{,}000 / (1 + 0.02)^3$ = $942.32.

An *N*-year spot rate is the geometric mean of the individual annual forward rates:

$$S_N = [(1 + S_1)(1 + 1y1y)(1 + 2y1y)\ldots(1 + Ny1y)]^{1/N} - 1$$

and the annualized forward rate for M – N periods, *N* periods from now is:

$$Ny(M - N)y = \left[\frac{(1 + S_M)^M}{(1 + S_N)^N}\right]^{\frac{1}{M - N}} - 1$$

24. Given S_5 = 2.4% and S_7 = 2.6%, 5y2y = $[(1.026)^7 / (1.024)^5]^{1/2} - 1$ = 3.1017%, which is approximately equal to (7 × 2.6% – 5 × 2.4%) / 2 = 3.1%.

Par yields reflect the coupon rate that a hypothetical bond at each maturity would need to have to be priced at par, given a specific spot curve.

25. With spot rates of 1%, 2%, and 3%, a 3-year annual par bond will have a payment that will satisfy the following:

$$\frac{PMT}{1.01} + \frac{PMT}{(1.02)^2} + \frac{PMT + 100}{(1.03)^3} = 100$$

The payment is 2.96 and the par bond coupon rate is 2.96%.

FORMULAS

duration gap = Macaulay duration − investment horizon

Modified duration (annual-pay bond):

ModDur = MacDur / (1 + YTM)

Modified duration (semiannual-pay bond):

$$\text{ModDur}_{\text{SEMI}} = \text{MacDur}_{\text{SEMI}} / (1 + \text{YTM} / 2)$$

$$\text{approximate modified duration} = \frac{V_{-} - V_{+}}{2V_0 \Delta \text{YTM}}$$

$$\text{approximate convexity} = \frac{V_{-} + V_{+} - 2V_0}{(\Delta \text{YTM})^2 V_0}$$

where:

V_{-} = price of the bond if YTM is *decreased* by ΔYTM

V_{+} = price of the bond if the YTM is *increased* by ΔYTM

V_0 = current price of the bond

$$\text{portfolio duration} = W_1 D_1 + W_2 D_2 + \ldots + W_N D_N$$

where:

W_i = full price of bond *i* divided by the total value of the portfolio

D_i = duration of bond *i*

N = number of bonds in the portfolio

$$\text{effective duration} = \frac{V_{-} - V_{+}}{2V_0 \Delta \text{Curve}}$$

$$\text{effective convexity} = \frac{V_{-} + V_{+} - 2V_0}{(\Delta \text{Curve})^2 V_0}$$

expected loss = probability of default × loss given default

$$\text{debt service coverage ratio} = \frac{\text{net operating income}}{\text{debt service}}$$

$$\text{loan-to-value ratio} = \frac{\text{current mortgage amount}}{\text{current appraised value}}$$

no-arbitrage forward price: $F_0(T) = S_0 (1 + Rf)^T$

payoff to long forward at expiration = $S_T - F_0(T)$

value of forward at time *t*: $V_t(T) = [S_t + PV_t(\text{costs}) - PV_t(\text{benefit})] - F_0(T)(1 + Rf)^{-(T-t)}$

exercise value of a call = Max[0, S − X]

exercise value of a put = Max[0, X − S]

option value = exercise value + time value put-call parity: $c + X(1 + Rf)^{-T} = S + p$

put-call-forward parity: $F_0\ (T)(1 + Rf\ )^{-T} + p_0 = c_0 + X(1 + Rf)^{-T}$

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