Exact Non-Iterative SimRank

Given two undirected graphs F and G and an initial similarity matrix S_0, SimRank propagates the similarity via the recurrence:

$$S = cF_{adj}^{\top} S G_{adj} + S_0$$

where c is a damping or decay factor between 0 and 1. Here the adj matrices are the column-normalized adjacency matrices of the graphs.

The recurrence can be rewritten as an infinite sum, i.e.,

$$S = \sum_{k=0}^{\infty} c^k F_{adj}^{\top k} S_0 G_{adj}^k$$

Because F and G are undirected, the adjacency matrices are symmetric which implies that they are diagonalizable. It follows that the column-normalized versions are also diagonalizable, because the normalization can be written as a product of the original symmetric matrix and a positive-definite diagonal matrix (assuming no orphaned nodes).

We eigendecompose the matrices as follows:

$$\begin{aligned} F_{adj}^{\top} &= P_F D_F P_F^{-1} \\ G_{adj} &= P_G D_G P_G^{-1} \end{aligned}$$

Solving,

$$\begin{aligned} S &= \sum_{k=0}^{\infty} c^k F_{adj}^{\top k} S_0 G_{adj}^k \\ &= \sum_{k=0}^{\infty} c^k (P_F D_F P_F^{-1})^k S_0 (P_G D_G P_G^{-1})^k \\ &= \sum_{k=0}^{\infty} c^k P_F D_F^k P_F^{-1} S_0 P_G D_G^k P_G^{-1} \\ &= P_F \left(\sum_{k=0}^{\infty} c^k D_F^k P_F^{-1} S_0 P_G D_G^k \right) P_G^{-1} \\ &= P_F \left(\sum_{k=0}^{\infty} (c \cdot eigs_F \cdot eigs_G^{\top})^k \odot (P_F^{-1} S_0 P_G) \right) P_G^{-1} \end{aligned}$$

where $eigs_F$ and $eigs_G$ are column vectors corresponding to the diagonals of D_F and D_G, respectively. Note that, because the adjacency matrices are column-normalized and $0 < c < 1$, we have $|c\lambda_F \lambda_G| < 1$ for any eigenvalue λ_F of F_{adj} and λ_G of G_{adj}.

Thus, the last equation contains the sum of an infinite geometric series, where $c \cdot eigs_F \cdot eigs_G^\top$ is the common ratio. Rewriting one last time, we arrive at the final form:

$$
\begin{aligned}
R &= c \cdot eigs_F \cdot eigs_G^\top \\
A &= P_F^{-1} S_0 P_G \\
S &= P_F(A \oslash (1 - R))P_G^{-1}
\end{aligned}
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