Degenerate case in 2D

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1 Existence of weak solution

Consider the equation:

$$\nabla \cdot (\sigma \nabla \phi) = f \tag{1}$$

when $\sigma \in L^{1/(1-p)}(\Omega) \cap L^1(\Omega)$, for some $1 < p < \infty$ and $\dfrac{f}{\sigma} \in L^{p'}_{\sigma}(\Omega)$, then there is a weak solution in $W^{1,p}_{0,\sigma}(\Omega)$. [1]

2 Detailed

Here we have our $\sigma \sim (x^2 + y^2) = r^2$ around zeros, thus $\sigma \in L^m$ where $m > -1$, which means $\dfrac{1}{1-p} > -1$, $p > 2$. Assume $p = 2 + \varepsilon$, $\varepsilon > 0$. Then if we choose f to satisfy $\dfrac{f}{\sigma} \in L^{p'}$, then we can use the theorem to get the existence of weak solution.

3 Weak solution to Classical solution

We know if we want the weak solution to be classical, we have to use Sobolev embedding theorem somehow. Let's look at Hölder inequality,

$$\int fg \leq (\int f^{\alpha})^{\frac{1}{\alpha}} (\int g^{\beta})^{\frac{1}{\beta}} \tag{2}$$

consider $fg = |\phi|^p$, we choose

$$f^{\alpha} = r^{-2+\varepsilon}$$

to be integrable, then

$$g^{\beta} = r^{\frac{2-\varepsilon}{\alpha}\beta} |\phi|^{p\beta}$$

by some computation according to $\alpha^{-1} + \beta^{-1} = 1$, we have $\alpha = 2 - \frac{\varepsilon}{2} < 2$, $\beta = \frac{4-\varepsilon}{2-\varepsilon} > 2$. We got

$$\begin{aligned} f^{\alpha} &= r^{-2+\varepsilon} & (3) \\ g^{\beta} &= r^2 |\phi|^{p\beta} & (4) \end{aligned}$$

Recall the first theorem, if we plug in the $p\beta$ into the p in theorem of first part, we have a weak solution in $W^{1,p\beta}_{0,\sigma}$.

4 Result

by the inequality, we have $|\phi| \in L^p(\Omega)$. The same way we can get $|\nabla\phi| \in L^p$. Thus $|\phi| \in W^{1,p}$, by Sobolev embedding theorem, we have $\phi \in C^{s,q}(\Omega)$ as Hölder space, where $s + q = 1 - \dfrac{2}{p}$, thus $C^{s,q} = C^{0,1-\frac{2}{p}}$. If we take $p < \infty$, then the solution exists in $C^\alpha(\Omega)$ as a classical solution, where $\alpha = 1 - \dfrac{2}{p}$, the requirement is $\dfrac{f}{\sigma} \in L_\sigma^{p'}(\Omega)$, which is easy to satisfy.

5 Induction for 3D

We already know that in 3D case, we still have to deal with the equation

$$\sigma\Delta\phi + \nabla\sigma \cdot \nabla\phi = 0 \tag{5}$$

where σ is almost positive every where and only vanishes at finite points. Moreover, we may assume $|\sigma(x, y) - \sigma(x_j, y_j)| \sim O(|x - x_j|^2 + |y - y_j|^2)$

For the same reason, we can conclude that there is a weak solution for the problem:

$$\begin{aligned} \nabla \cdot (\sigma\nabla\phi) &= f & (6) \\ \gamma\phi &= 0 & (7) \end{aligned}$$

Where $\dfrac{f}{\sigma} \in L_\sigma^{p'}(\Omega)$. It is easy to show that σ belongs to A_p for $p > 2$ with the same reason shown in the second part. Thus we can find solution in $C^{0,\alpha}$, for $\alpha < 1$, it is obviously that the solution is bounded on Ω.

References

[1] Albo Carlos Cavalheiro, *Existence of solutions for Dirichlet problem of some degenerate quasilinear elliptic equations.* Complex Variables and Elliptic Equations Vol. 53, No. 2, February 2008, 185-194