Stability for inverse point source problem

Yimin Zhong

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1 Problem 1

Consider problem, without special notation, assume γ and Λ as their usual definitions. The domain $\Omega \subset \mathbb{R}^3$.

$$\begin{aligned} \Delta u + k^2 n(x) u &= F & (1) \\ \gamma u &= f & (2) \\ \Lambda \gamma u &= g & (3) \end{aligned}$$

where the source $F(x) = \sum_{j=1}^m P_j \delta(x - x_j)$, with $P_j \in \mathbb{R}$, $x_j \in \Omega$, for convenience, $n(x)$ is known as a smooth real function with compact support on Ω.

1.1 Stability of recovering location, respect to Cauchy data

What is the stability of recovering the location of x_j. Suppose the number of point sources is known as m. Or we formulate the stability argument as following statement.

Statement 1.1 *If u_l for $l = 1, 2$ be the solutions of equation 1 associated with Cauchy data (f^l, g^l) and sources $F^l = \sum_{j=1}^m P_j^l \delta(x - x_j^l)$, if we have $\|(f^1, g^1) - (f^2, g^2)\| \le \epsilon$, can we find a permutation π of $\{1, 2, \ldots, m\}$ such that*

$$\max_{j \ge 1} \|x_j^1 - x_{\pi(j)}^2\| \le \epsilon'$$

and as $\epsilon \to 0$, $\epsilon' \to 0$ as well.

1.2 Construction with perturbation

Since the speed is known $n(x)$, thus we set space $M = \{v \in H^s, \Delta v + k^2 n v = 0\}$, for any solution u satisfies 1, we have

$$\begin{aligned} \int v\Delta u + k^2 nuvdx &= \sum_{j=1}^m P_j v(x_j) & (4) \\ \int u\Delta v + k^2 nvudx &= 0 & (5) \end{aligned}$$

Thus by Green's formula, we have

$$\int_{\Gamma}(vg - f\frac{\partial v}{\partial n})d\sigma = \sum_{j=1}^{m} P_j v(x_j) \tag{6}$$

And in order to find out the distance between points x_j^l, we are going to generate a function which sits in M and has zeros x_j^l.

Remark Now our problem is to find a differentiable function ϕ such that $\phi \in M$ and $\phi(\xi_j) = 0$, where $j = 1, 2, \cdots, m$.

1.3 Existence by induction

Suppose φ satisfies that $\Delta\varphi + k^2 n\varphi = 0$, and $\varphi(\xi_j) = 0$. Here $j = 1, \cdots, m$, where $m \geq 0$. We are going to find $u = \varphi\phi$ such that $\Delta u + k^2 nu = 0$, where $\phi(\zeta) = 0, \zeta \neq \xi_j$, then

$$0 = \Delta(\varphi\phi) + k^2 n\varphi\phi = \phi\Delta\varphi + 2\nabla\varphi \cdot \nabla\phi + \varphi\Delta\phi + k^2 n\varphi\phi = \varphi\Delta\phi + 2\nabla\varphi \cdot \nabla\phi \tag{7}$$

We set out to find ϕ such that $\Delta\phi = 0$, and $\nabla\phi \cdot \nabla\varphi = 0$.

1.3.1 Moving frame

We consider local coordinate $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where $\mathbf{e}_1 = \dfrac{\nabla\varphi}{|\nabla\varphi|}$, when $\nabla\varphi \neq \mathbf{0}$, this defines a coordinate change $\mathbf{u} = \mathbf{u}(\mathbf{r})$. By vector analysis,

$$\Delta\phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u_1}) + \frac{\partial}{\partial u_2}(\frac{h_1 h_3}{h_2}\frac{\partial\phi}{\partial u_2}) + \frac{\partial}{\partial u_3}(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial u_3})\right] \tag{8}$$

where $h_1 = h_2 = h_3 = 1$ and $\dfrac{\partial\phi}{\partial u_1} = 0$, which simplifies the problem as

$$\Delta\phi = \left[\left(\frac{\partial}{\partial u_2}\right)^2 + \left(\frac{\partial}{\partial u_3}\right)^2\right]\phi = 0 \tag{9}$$

Any solution ϕ to this function will admit a solution to our original problem by setting $\widetilde{\phi} = \phi(u) - \phi(u(\zeta))$. Here we can always select $\phi = u_2 + iu_3$.

1.3.2 Existence

We have seen that if $m = 0$, we can always find a solution to the Helmholtz equation, then by induction, we can find the solution $\phi \in M$, such that $\phi(\xi_j) = 0$, for $j = 1, \cdots, m$.

Initially we take a Φ_0, satisfies Helmholtz equation, then

$$\Delta\Phi_0 + k^2 n\Phi_0 = 0 \tag{10}$$

And $\nabla\Phi_0 \neq \mathbf{0}$ in Ω. Apply the moving local coordinate, and take $\phi = u_2 + iu_3$.

- For ξ_1, we take $\Phi_1(\xi) = \Phi_0(\xi)\,(\phi(\xi) - \phi(\xi_1))$, then $\langle \nabla\Phi_0, \nabla\phi\rangle = 0$ and $\Delta\phi = 0$.
- For ξ_{k+1}, we take $\Phi_{k+1}(\xi) = \Phi_k(\xi)\,(\phi(\xi) - \phi(\xi_{k+1}))$, we can see that

$$\begin{aligned} \nabla\phi &= [0, 1, i]^t \\ \nabla\Phi_k &= \left[\frac{\partial\Phi_k}{\partial u_1}, \frac{\partial\Phi_k}{\partial u_2}, i\frac{\partial\Phi_k}{\partial u_2}\right]^t \end{aligned} \tag{11}$$

which means $\langle \nabla\Phi_k, \nabla\phi\rangle = 0$, $\Delta\phi = 0$. And the last thing is to check $\nabla\Phi_k \neq \mathbf{0}$, since

$$\nabla\Phi_k = \left[\frac{\partial\Phi_k}{\partial u_1}, \frac{\partial\Phi_k}{\partial u_2}, i\frac{\partial\Phi_k}{\partial u_2}\right]^t \tag{12}$$

$$\frac{\partial\Phi_k}{\partial u_1} = \frac{\partial\Phi_0}{\partial u_1}\prod_{j=1}^{k}(\phi(\xi) - \phi(\xi_j)) \tag{13}$$

$$\frac{\partial\Phi_k}{\partial u_2} = \Phi_0\sum_{j=1}^{k}\prod_{l=1,l\neq j}^{k}(\phi(\xi) - \phi(\xi_l)) \tag{14}$$

Since every root is simple, we know $\nabla\Phi_k$ is never going to be zero vector.
- Thus we take

$$\Phi(\xi) = \Phi_m(\xi) = \Phi_0(\xi)\prod_{j=1}^{m}(\phi(\xi) - \phi(\xi_j)) \tag{15}$$

which satisfies the Helmholtz equation and vanishes at ξ_j, $j = 1, \cdots, m$.

Definition 1.2 *We define projection map $\mathcal{S} : \mathbb{R}^3 \to \mathbb{C}$, $\mathcal{S}(\xi) = \phi(\xi)$, then our generated function is*

$$\Phi = \Phi_0\prod_{j=1}^{m}(\mathcal{S}(\xi) - \mathcal{S}(\xi_j)) \tag{16}$$

Definition 1.3 *We define function ϕ_l such that $\phi_l \in M$ and ϕ_l vanishes at x_j^1 and x_j^2 except x_l^2.*

1.4 Main proof

$$\int_\Gamma (g^1\phi_l - f^1\frac{\partial\phi_l}{\partial n})d\sigma = \sum_{j=1}^{m}P_j^1\phi_l(x_j^1) = 0 \tag{17}$$

$$\int_\Gamma (g^2\phi_l - f^2\frac{\partial\phi_l}{\partial n})d\sigma = \sum_{j=1}^{m}P_j^2\phi_l(x_j^2) = P_l^2\phi_l(x_l^2) \tag{18}$$

Subtract the above one from the below one,

$$
\begin{aligned}
& |\int_\Gamma ((g^2-g^1)\phi_l - (f^2-f^1)\frac{\partial \phi_l}{\partial n})d\sigma| && (19)\\
= \ & |P_l^2 \phi_l(x_l^2)| && (20)\\
= \ & |P_l^2 \prod_{j=1}^m (\mathcal{S}(x_l^2)-\mathcal{S}(x_j^1))\prod_{i\neq l}(\mathcal{S}(x_l^2)-\mathcal{S}(x_i^2))\Phi_0(x_l^2)| && (21)\\
\geq \ & C|c\eta_l^m \xi^{m-1}| && (22)\\
\geq \ & C|c\eta_l^m \xi^{m-1}| && (23)
\end{aligned}
$$

where $c = \min_l P_l^2$,$\eta_l = \min_j |\mathcal{S}(x_l^2) - \mathcal{S}(x_j^1)|$, $\xi = \min_{i\neq l} |\mathcal{S}(x_l^2) - \mathcal{S}(x_i^2)|$

While by Cauchy-Schwarz inequality.

$$
\begin{aligned}
& |\int_\Gamma ((g^2-g^1)\phi_l - (f^2-f^1)\frac{\partial \phi_l}{\partial n})d\sigma| && (24)\\
\leq \ & (\|g^2-g^1\|\|\phi_l\| + \|f^2-f^1\|\|\frac{\partial \phi_l}{\partial n}\|) && (25)
\end{aligned}
$$

And $\|\phi_l\|_{L^2(\Gamma)} \leq C\sqrt{|\Gamma|}(\mathrm{diam}(\Omega))^{2m-1}$, $\|\frac{\partial \phi_l}{\partial n}\|_{L^2(\Gamma)} \leq C_1\sqrt{|\Gamma|}(\mathrm{diam}(\Omega))^{2m-1}$.

For each l, we have

$$|c\eta_l^m(\xi)^{m-1}e^{-kT}| \leq C\{|g^2-g^1\| + \|f^2-f^1\|\}\sqrt{|\Gamma|}(\mathrm{diam}(\Omega))^{2m-1} \tag{26}$$

Thus,

$$\max_l \eta_l \leq \{\frac{C\sqrt{|\Gamma|}(\mathrm{diam}(\Omega))^{2m-1}}{c\xi^{m-1}}\{\|g^2-g^1\| + \|f^2-f^1\|\}\}^{\frac{1}{m}} \tag{27}$$

1.5 Stability of intensity, respect to Cauchy data

We set out to find a two-step stability for intensities. Suppose we have reconstructed the locations, and intensities are remained unknown. The problem is formulated as following.

Statement 1.4 *Suppose our sources are $F^1 = \sum_{j=1}^m P_j^1 \delta(x-x_j^1)$ and $F^2 = \sum_{j=1}^m P_j^2 \delta(x-x_j^2)$ associated with Cauchy data (f^l, g^l) and with the knowledge of the maximal error on local locations of point sources, we are going to find out the stability on intensities w.r.t perturbation on Cauchy data.*

Take

$$H_l = \prod_{i\neq l}(\mathcal{S}(\xi)-\mathcal{S}(x_i^1))\prod_{j\neq l}(\mathcal{S}(\xi)-\mathcal{S}(x_j^2)) \tag{28}$$

then according our construction, there is a $\Phi_0 : \mathbb{R}^3 \to \mathbb{C}$ such that $u_l = H_l\Phi_0$ satisfies the Helmholtz equation:

$$\Delta u_l + k^2 n u_l = 0 \tag{29}$$

Similarly, we have

$$\int_\Gamma (g^1 u_l - f^1 \frac{\partial u_l}{\partial n}) d\sigma = \sum_{j=1}^m P_j^1 u_l(x_j^1) = P_l^1 u_l(x_l^1) \tag{30}$$

$$\int_\Gamma (g^2 u_l - f^2 \frac{\partial u_l}{\partial n}) d\sigma = \sum_{j=1}^m P_j^2 u_l(x_j^2) = P_l^2 u_l(x_l^2) \tag{31}$$

Therefore

$$P_l^1 u_l(x_l^1) - P_l^2 u_l(x_l^2) = \int_\Gamma ((g^1 - g^2) u_l - (f^1 - f^2) \frac{\partial u_l}{\partial n}) d\sigma \tag{32}$$

The RHS can be bounded by Hölder estimation.

$$\left| \int_\Gamma ((g^1 - g^2) u_l - (f^1 - f^2) \frac{\partial u_l}{\partial n}) d\sigma \right| \leq C\Big(\|g^1 - g^2\| + \|f^1 - f^2\| \Big) \sqrt{|\Gamma|} (\text{diam}(\Omega))^{2m-2} \tag{33}$$

And LHS can be written as

$$\begin{aligned} |P_l^1 u_l(x_l^1) - P_l^2 u_l(x_l^2)| &= |(P_l^1 - P_l^2) u_l(x_l^1) + P_l^2 (u_l(x_l^1) - u_l(x_l^2))| && (34) \\ &\geq \Big| |(P_l^1 - P_l^2) u_l(x_l^1)| - |P_l^2 (u_l(x_l^1) - u_l(x_l^2))| \Big| && (35) \\ &\geq |(P_l^1 - P_l^2) u_l(x_l^1)| - |P_l^2 (u_l(x_l^1) - u_l(x_l^2))| && (36) \end{aligned}$$

Observe 33 and 36, we can see

$$\begin{aligned} & |P_l^1 - P_l^2| |u_l(x_l^1)| \\ \leq \quad & C\Big(\|g^1 - g^2\| + \|f^1 - f^2\| \Big) \sqrt{|\Gamma|} (\text{diam}(\Omega))^{2m-2} + |P_l^2 (u_l(x_l^1) - u_l(x_l^2))| \\ \leq \quad & C\Big(\|g^1 - g^2\| + \|f^1 - f^2\| \Big) \sqrt{|\Gamma|} (\text{diam}(\Omega))^{2m-2} + \beta M \eta \end{aligned}$$

where $\beta = \max_{j=1}^m |P_l^2|$, $M = \max_\Omega |\nabla u|$, $\eta = \max_l \eta_l$ is defined in 27. And

$$|P_l^1 - P_l^2| |u_l(x_l^1)| \geq C |P_l^1 - P_l^2| \xi^{m-1} \gamma_l^{m-1} \tag{37}$$

where $\xi = \min_{j \neq i} |\mathcal{S}(x_i^1) - \mathcal{S}(x_j^1)|$,$\gamma_l = \min_{j \neq l} |\mathcal{S}(x_l^1) - \mathcal{S}(x_j^2)|$.

Thus

$$|P_l^2 - P_l^1| \leq \frac{C\Big(\|g^1 - g^2\| + \|f^1 - f^2\| \Big) \sqrt{|\Gamma|} (\text{diam}(\Omega))^{2m-2} + \beta M \eta}{C \xi^{m-1} \gamma_l^{m-1}} \tag{38}$$

Remark Here we can see that $\gamma_l = \min_{j \neq l} |\mathcal{S}(x_l^1) - \mathcal{S}(x_j^2)| \geq \min_{j=1}^m |\mathcal{S}(x_l^1) - \mathcal{S}(x_j^2)| = \eta_l$

2 Problem 2

2.1 Stability, respect to $n(x)$

We expect the stability as the following informal statement.

Statement 2.1 *Consider fixed Cauchy data (f, g) on the boundary, and refractive index n_l, $l = 1, 2$, the associated point sources are $F^l = \sum_{j=1}^m P_j^l \delta(x - x_j^l)$. If there is small difference(under some norm) on n^1, n^2, we need to find out if there is small difference between the sources.*

$$\Delta u_1 + k^2 n_1 u_1 = \sum_{j=1}^{m} P_j^1 \delta(x - x_j^1) \tag{39}$$

$$\Delta u_2 + k^2 n_2 u_2 = \sum_{j=1}^{m} P_j^2 \delta(x - x_j^2) \tag{40}$$

If we consider any ϕ such that

$$\Delta \phi + k^2 n_1 \phi = 0 \tag{41}$$

Then for u_1,

$$\int_\Gamma \left(\phi g - f \frac{\partial \phi}{\partial n} \right) d\sigma = \sum_{j=1}^{m} P_j^1 \phi(x_j^1) \tag{42}$$

and for u_2,

$$\int_\Gamma \left(\phi g - f \frac{\partial \phi}{\partial n} \right) d\sigma = \sum_{j=1}^{m} P_j^2 \phi(x_j^2) + \int_\Omega k^2 (n_1 - n_2) u_2 \phi \tag{43}$$

subtract the above one from the below one,

$$\sum_{j=1}^{m} P_j^2 \phi(x_j^2) + \int_\Omega k^2 (n_1 - n_2) u_2 \phi = \sum_{j=1}^{m} P_j^1 \phi(x_j^1) \tag{44}$$

- We choose $\phi = \phi_l$ to vanish at x_j^1 and x_j^2 except x_l^2, then we can get a similar result on location's stability.
- Suppose we have been aware of the approximated locations, we take u_l to vanish at x_j^1 and x_j^2 except x_l^1 and x_l^2. Then we can get a similar result about the stability of intensity.