Sophex: Cheng-Yau gradient estimate

Yimin Zhong

1 Introduction

Assume that $\boldsymbol{M}$ is a noncompact Riemann manifold and $f : B(R) \to \mathbb{R}$ harmonic and positive if the Ricci curvature on $B(R)$ has a lower bound $\mathrm{Ric}(\boldsymbol{M}) \geq -(n-1)K$ for some $K \geq 0$, then

$$\sup_{B(R/2)} |\nabla \log f| \leq (n-1)K + \frac{C}{R} \tag{1}$$

We are going to prove the case $\boldsymbol{M} = \mathbb{R}^n$, i.e, $\mathrm{Ric}(\boldsymbol{M}) = 0$. Set $h = \log f$.

2 Proof

Consider and ϕ defined on $\mathbb{R}$ which satisfies that ϕ is supported on $B(R)$.
Set $G = \phi^2 |\nabla h|^2$, then

1. G is non-negative on $B(R)$ and $G = 0$ on $\partial B(R)$. Thus G attains its maximum in $B(R)$, say at x_0. Then

$$|\nabla G(x_0)| = 0 \quad , \quad \Delta G(x_0) \leq 0 \tag{2}$$

2. Consider function h,we have

$$\Delta h = -|\nabla h|^2 \tag{3}$$

$$\frac{1}{2}\Delta |\nabla h|^2 = \sum_{i,j} |h_{ij}|^2 - \langle \nabla h, \nabla |\nabla h|^2 \rangle \tag{4}$$

3. Choose orthogonal frame $\{e_i\}$, s.t.
   - $h_\alpha = 0$, whenever $\alpha \neq 0$.
   - $|h_1| = |\nabla h|$, which means $e_1 = \dfrac{\nabla h}{|\nabla h|}$.
4. Because of the special frame, we obtain that
   - $\langle \nabla h, \nabla |\nabla h|^2 \rangle = 2h_i h_j h_{ij} = 2h_1^2 h_{11} = 2|\nabla h|^2 h_{11}$.
   - $|\nabla |\nabla h|^2|^2 = 4\sum_{i,j} |h_{ij} h_j|^2 = 4\sum_i |h_{1i}|^2 |\nabla h|^2$.

$$
\begin{aligned}
\sum_{i,j}|h_{ij}|^2 &\geq |h_{11}|^2 + \sum_{\alpha\geq 2}|h_{\alpha\alpha}|^2 + 2\sum_{\alpha\geq 2}|h_{1\alpha}|^2 \\
&\geq |h_{11}|^2 + 2\sum_{\alpha\geq 2}|h_{1\alpha}|^2 + \frac{1}{n-1}|\sum_{\alpha\geq 2}h_{\alpha\alpha}|^2 \\
&= |h_{11}|^2 + 2\sum_{\alpha\geq 2}|h_{1\alpha}|^2 + \frac{1}{n-1}|\Delta h - h_{11}|^2 \\
&= |h_{11}|^2 + 2\sum_{\alpha\geq 2}|h_{1\alpha}|^2 + \frac{1}{n-1}||\nabla h|^2 + h_{11}|^2 \\
&\geq \frac{n}{n-1}|h_{11}|^2 + 2\sum_{\alpha\geq 2}|h_{1\alpha}|^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}h_{11}|\nabla h|^2
\end{aligned}
\tag{5}
$$

Here we used Cauchy-Schwarz Inequality and (3).

5. Because $n \geq 2$, we obtain:

$$
\begin{aligned}
\sum_{i,j}|h_{ij}|^2 &\geq \frac{n}{n-1}|h_{11}|^2 + \frac{n}{n-1}\sum_{\alpha\geq 2}|h_{1\alpha}|^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}h_{11}|\nabla h|^2 \\
&= \frac{n}{4(n-1)}\frac{1}{|\nabla h|^2}|\nabla|\nabla h|^2|^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}h_{11}|\nabla h|^2
\end{aligned}
\tag{6}
$$

6. Thus by (4),

$$
\begin{aligned}
\frac{1}{2}\Delta|\nabla h|^2 &\geq \frac{n}{4(n-1)}\frac{1}{|\nabla h|^2}|\nabla|\nabla h|^2|^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}h_{11}|\nabla h|^2 - 2|\nabla h|^2 h_{11} \\
&= \frac{n}{4(n-1)}\frac{1}{|\nabla h|^2}|\nabla|\nabla h|^2|^2 + \frac{1}{n-1}|\nabla h|^4 - \frac{2n-4}{n-1}h_{11}|\nabla h|^2
\end{aligned}
\tag{7}
$$

7. Substitute $|\nabla h|^2 = \dfrac{G}{\phi^2}$ in the last inequality:

$$
\frac{1}{2}\Delta(\frac{G}{\phi^2}) \geq \frac{n}{4(n-1)}\frac{\phi^2}{G}|\nabla\frac{G}{\phi^2}|^2 + \frac{1}{n-1}|\frac{G}{\phi^2}|^2 - \frac{n-2}{n-1}\langle\nabla\frac{G}{\phi^2}, \nabla h\rangle \tag{8}
$$

8. Before we continue:

$$
\begin{aligned}
\Delta(\phi^2\cdot\frac{G}{\phi^2}) &= \nabla\cdot[\nabla(\phi^2\cdot\frac{G}{\phi^2})] \\
&= \nabla\cdot[(\nabla\phi^2)\frac{G}{\phi^2} + \phi^2\nabla(\frac{G}{\phi^2})] \\
&= (\Delta\phi^2)\frac{G}{\phi^2} + 2\nabla\phi^2\cdot\nabla(\frac{G}{\phi^2}) + \Delta(\frac{G}{\phi^2})\phi^2
\end{aligned}
\tag{9}
$$

9. Therefore we multiply (8) with ϕ^2. Then by (9),

$$\begin{aligned}\frac{1}{2}\Delta(\frac{G}{\phi^2})\phi^2 &= \frac{1}{2}\Delta G - \frac{1}{2}\Delta\phi^2\cdot\frac{G}{\phi^2} - \nabla\phi^2\cdot\nabla(\frac{G}{\phi^2})\\ &\geq \frac{n}{4(n-1)}\frac{\phi^4}{G}|\nabla(\frac{G}{\phi^2})|^2 + \frac{1}{n-1}\frac{G^2}{\phi^2} - \frac{n-2}{n-1}\phi^2\langle\nabla(\frac{G}{\phi^2}),\nabla h\rangle\end{aligned} \tag{10}$$

10. Let's put this at the point x_0. We have:
    - $|\nabla(\frac{G}{\phi^2})|^2 = |(\nabla G)\frac{1}{\phi^2} + G\nabla(\frac{1}{\phi^2})|^2 = G^2|\nabla(\frac{1}{\phi^2})|^2 = 4G^2\frac{|\nabla\phi|^2}{\phi^6}$
    - $\phi^2\langle\nabla\phi^2, -\frac{2\nabla\phi}{\phi^3}\cdot G\rangle = \phi^2\langle 2\phi\nabla\phi, -\frac{2\nabla\phi}{\phi^3}\cdot G\rangle = -4G\langle\nabla\phi,\nabla\phi\rangle = -4G|\nabla\phi|^2$

11. Then at x_0,

$$\begin{aligned}0 \geq \frac{1}{2}\Delta G \geq \frac{n}{4(n-1)}\frac{\phi^4}{G}\cdot G^2\frac{4|\nabla\phi|^2}{\phi^2} + \frac{1}{n-1}\frac{G^2}{\phi^2} - \frac{n-2}{n-1}\phi^2\langle\nabla(\frac{G}{\phi^2}),\nabla h\rangle\\ + \frac{1}{2}\Delta\phi^2\cdot\frac{G}{\phi^2} + \nabla\phi^2\cdot\nabla(\frac{G}{\phi^2})\end{aligned} \tag{11}$$

12. Multiply (11) with ϕ^2:

$$\begin{aligned}0 \geq \frac{n}{n-1}G|\nabla\phi|^2 + \frac{1}{n-1}G^2 + \frac{n-2}{n-1}\phi^4\langle G\frac{2\nabla\phi}{\phi^3},\nabla h\rangle\\ + \frac{1}{2}\Delta\phi^2\cdot G + \phi^2\nabla\phi^2\cdot\nabla(\frac{G}{\phi^2})\end{aligned} \tag{12}$$

which is:

$$\begin{aligned}0 &\geq \frac{n}{n-1}G|\nabla\phi|^2 + \frac{1}{n-1}G^2 + \frac{1}{2}\Delta\phi^2\cdot G - 4G|\nabla\phi|^2 + \frac{2n-4}{n-1}G\phi\langle\nabla\phi,\nabla h\rangle\\ &\geq \frac{n}{n-1}G|\nabla\phi|^2 + \frac{1}{n-1}G^2 + \frac{1}{2}\Delta\phi^2\cdot G - 4G|\nabla\phi|^2 - \frac{2n-4}{n-1}G\phi|\nabla\phi||\nabla h|\\ &= \frac{n}{n-1}G|\nabla\phi|^2 + \frac{1}{n-1}G^2 + \frac{1}{2}\Delta\phi^2\cdot G - 4G|\nabla\phi|^2 - \frac{2n-4}{n-1}G^{3/2}|\nabla\phi|\end{aligned} \tag{13}$$

which means:

$$0 \geq \frac{n}{n-1}|\nabla\phi|^2 + \frac{1}{n-1}G + \frac{1}{2}\Delta\phi^2 - \frac{2n-4}{n-1}G^{1/2}|\nabla\phi| \tag{14}$$

And we can simplify it to:

$$(2n-3)|\nabla\phi|^2 - (n-1)\phi\Delta\phi + (2n-4)|\nabla\phi|G^{1/2} \geq G \tag{15}$$

for any cutoff function ϕ at x_0.

13. Consider $\phi = R^2 - \rho^2$, where $\rho = \sqrt{\sum_j x_j^2}$, then

    - $|\nabla \rho| = 1$
    - $\Delta \rho^2 = 2n$

    Take it into (15),

$$4(2n-3)\rho^2 + 2n(n-1)(R^2 - \rho^2) + 2(2n-4)\rho G^{1/2} \geq G \tag{16}$$

Thus

$$4(2n-3)R^2 + 2n(n-1)R^2 + 2(2n-4)RG^{1/2} \geq G \tag{17}$$

Then

$$G^{1/2} \leq \{2(n-2) + \sqrt{6n^2 - 10n + 4}\}R \tag{18}$$

Because x_0 is the maximum point, then

$$\sup_{B(R)} G^{1/2} \leq C(n) \cdot R$$

restrict this to $B_{R/2}$, then we have:

$$\sup_{B(R/2)} (R^2 - \rho^2)|\nabla h| \leq C(n) \cdot R$$

therefore

$$\frac{3}{4} R^2 \sup_{B(R/2)} |\nabla h| \leq C(n) \cdot R$$

i.e,

$$\sup_{B(R/2)} |\nabla h| \leq \frac{4}{3} C(n) \cdot R$$

2.1 Remark

- This estimate is sharp, when we come to the example of linear functions.
- We can replace the cutoff function ϕ by other cutoff functions. For example, we may choose exponential function to be cutoff function, we can obtain better estimate when x is large.

3 Application

1. If f is positive and harmonic in $\mathbb{R}^n$, then f is a constant.
2. What if when the domain is $\mathbb{R}^n - \{0\}$?
3. When $\dfrac{f(x)}{|x|} \to 0$, then f is a constant.
4. Prove Harnack Inequality.