One step reconstruction of wave speed and optical properties for PAT

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Photoacoustic Tomography(PAT)

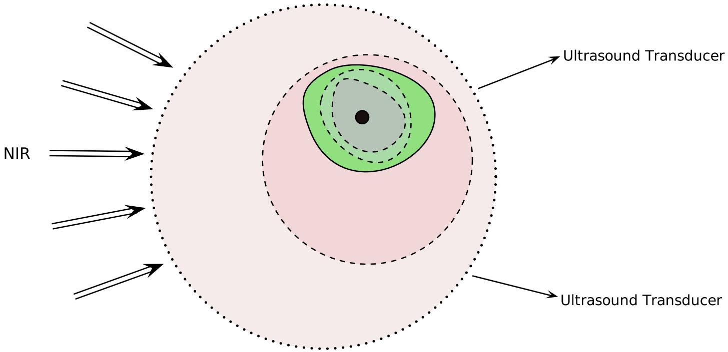

Figure : Photoacoustic Tomography: To recover scattering, absorption and photoacoustic efficiency properties of tissues from boundary measurement of acoustic signal generated with the photoacoustic effect. Two processes: propagation of NIR radiation and propagation of ultrasound. There is a (time) scale separation between the two processes.

Regular PAT Workflow

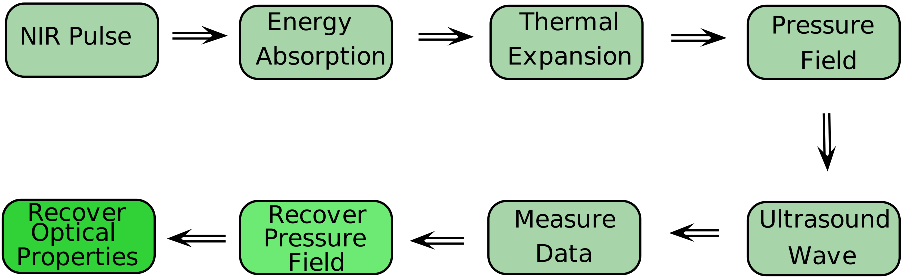

Figure : Workflow chart of both physical process during PAT and regular inversion process

Photon Diffusion Process

Let $\Omega \subset \mathbb{R}^d$ be the domain of medium. Then the equation of photon density u at position $\mathbf{x} \in \Omega$ satisfies

$$-\nabla \cdot D\nabla u(\mathbf{x}) + \sigma u(\mathbf{x}) = 0, \tag{1}$$

with boundary condition

$$\psi|_{\partial\Omega} = g(\mathbf{x}), \tag{2}$$

where $D(\mathbf{x})$ is diffusion coefficient, $\sigma(\mathbf{x})$ is absorption coefficient.

Photoacoustic Effect

The medium absorbs part of the energy of NIR photons, generate initial pressure field $H(\mathbf{x})$ through photoacoustic effect.

$$H(x) = \Gamma(\mathbf{x})\sigma(\mathbf{x})u(\mathbf{x}), \tag{3}$$

where $\Gamma(x)$ is Grüneisen coefficient, measures the efficiency of photoacoustic effect(energy $\rightarrow$ pressure).

Acoustic Wave Propagation

The initial pressure field generates acoustic wave(ultrasound),

$$\begin{aligned} \frac{1}{c^2(\mathbf{x})} p_{tt} - \Delta p &= 0, \\ p(0, \mathbf{x}) &= H(x) = \Gamma(\mathbf{x})\sigma_a(\mathbf{x})u(\mathbf{x}), \\ p_t(0, \mathbf{x}) &= 0, \end{aligned} \tag{4}$$

during the photoacoustic process, wave speed $c(\mathbf{x})$ is assumed to be unchanged. And we measure the pressure field(ultrasound signal) on the surface $\Sigma = \partial\Omega$ of domain for sufficient long time $T \gg 1$,

$$\mathcal{M}(t, \mathbf{x}) = p(t, \mathbf{x})|_{[0,T]\times\Sigma} \tag{5}$$

Two Step Reconstruction

1. The first step is solve an inverse source problem with (4) and (5). In practice, $c(\mathbf{x})$ is variable, and time reversal method works well when $c(\mathbf{x})$ is known.
2. The second step is nonlinear and there are plenty of literatures, it is known that we can reconstruct any two of $D(\mathbf{x})$, $\sigma(\mathbf{x})$ and $\Gamma(\mathbf{x})$ if the third is known. e.g., if D is given, we have a chain like

$$\mathcal{M}(t, \mathbf{x}) \to H \to (\Gamma, \sigma) \tag{6}$$

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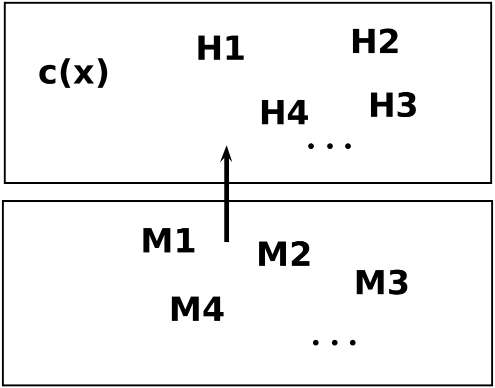
Two Step Reconstruction

It is proved that when $c(\mathbf{x})$ is unknown, the reconstruction in first step will be very unstable.

Why? Intuitively, we are using less data to recover more unknowns.



Two Step Reconstruction

No matter how many measurements are taken, the first step always requires recovery of the new H, does not reduce the number of unknowns at all. The fact is:

H is not important.

One Step Reconstruction

We combine the two process as one, try to use the measurement to directly recover the properties of interest. Since we introduced a new unknown $c(\mathbf{x})$, we need to known one more in (D, σ, Γ), e.g., D and Γ are known. And the problem turns to be a simple chain,

$$\mathcal{M}(t, \mathbf{x}) \to (c, \sigma) \tag{8}$$

One Step Reconstruction

Current main results on uniqueness and stability:

1. When $c(\mathbf{x})$ is constant and unknown, we can uniquely recover $c(x)$.
2. When $c(\mathbf{x}) = \alpha f(\mathbf{x})$ where α is unknown real number and f is known, we can uniquely recover $c(\mathbf{x})$.
3. There is no Sobolev stability estimate. (no stability involving finite order of derivatives).

In other words, it seems if $c(\mathbf{x})$ only depends on finite (one ?) variables, there is no problem on uniqueness.

One Step Reconstruction

Our results on uniqueness:

1. In 1D, use one measurement, $c(\mathbf{x})$ can be recovered uniquely.
2. In 3D, use one measurement, if $c(\mathbf{x})$ is radial (only depends on $|\mathbf{x}|$), then $c(\mathbf{x})$ can be determined uniquely.
3. In 3D, use infinite measurements, $c(\mathbf{x})$ can be determined uniquely.

In other words, if $c(\mathbf{x})$ is one dimensional, single measurement is enough.

The Idea

Use linearized model, say $c(x) = c_0 + \delta c$, $\sigma = \sigma_0 + \delta\sigma$. Then the perturbed wave equation is

$$\begin{aligned} \partial_{tt}\delta p - c_0^2 \delta p &= \frac{2\delta c}{c_0}\partial_{tt} p_0 \\ \delta p(x,0) &= \delta H \\ \partial_t \delta p(x,0) &= 0 \end{aligned} \tag{9}$$

perturbed measurement is $\delta\mathcal{M} = \delta p(\mathbb{R}^+ \times \Sigma)$.

The Idea

For example, in 1D.

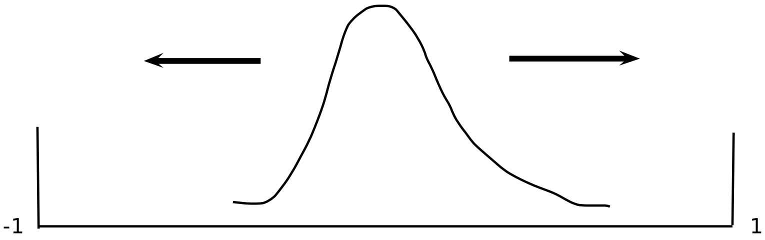


Figure : Compare the measurements on both sides from different direction of time (forward and backward).

The Idea

We somehow can cancel the term with perturbation from $\sigma(x)$ (or δH). And the remaining will only depend on perturbed $\delta c(x)$. From here to show uniqueness requires some tricks.

More important thing: boundary measurement is enough to determine both wave speed and initial condition.

Conclusion

1. Made some progress on PAT.
2. Some uniqueness and stability results, very likely to be logarithm stability.
3. In 2D, our method does not work.