Recovering point sources in unknown environment with differential data

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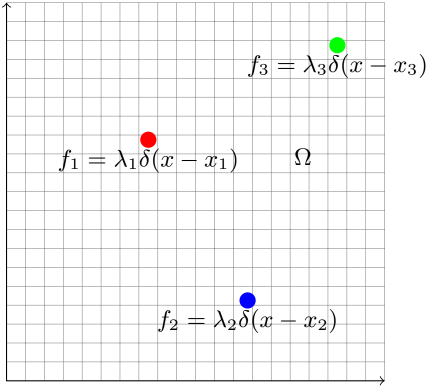
Introduction

The field u satisfies

$$\Delta u(x) + k^2(1+n(x))u(x) = f(x) \quad \text{in } \Omega \tag{1}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

k is the given frequency, $n(x)$ is refractive index, supported in compact domain Ω and $f(x)$ is source function as superposition of point sources.



source function $f(x) = \sum_{j=1}^{m} \lambda_j \delta(x - x_j)$

$u = 0$ on $\partial\Omega$

differential data: $\dfrac{\partial u}{\partial n}$ on $\partial\Omega$

Forward source problem

Problem (Forward source problem)

We assume source $f(x)$ is the superposition of some point sources, find a function $u \in H^1(\Omega)$ such that

$$B(u, v) = \langle f, v \rangle \tag{3}$$

where sesquilinear form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ is defined as:

$$B(u, v) = \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle (1 + n)u, v \rangle_{L^2(\Omega)} - \langle \Lambda \gamma u, \gamma v \rangle_{L^2(\partial\Omega)} \tag{4}$$

where $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is the trace operator, and $\Lambda : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the Dirichlet to Neumann operator.

Here we assume 0 is not an eigenvalue of operator $\Delta + k^2(1 + n(x))$ and we can obtain well-posedness from the forward problem by utilizing Resiz-Fredholm theory.

Inverse source problem

Problem (Inverse source problem)

Let's assume the unknown source function f is the superposition of several point sources and u be the solution to the forward source problem for some given frequency $k \in \mathcal{K}$, provided the Cauchy data on the boundary $\partial\Omega$, finding the source f.

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- Multi-frequency

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- Unknown heterogeneous media

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  - Stenfanov P and Uhlmann G, IOP, 2009
- Point sources
  - Alves C, Kress R and Serranho P, IOP, 2009

Scheme

Our Objective

- To recover both the refractive index $n(x)$ and source function $f(x)$ using differential data
- To study stability of recovering $f(x)$ with respect to perturbation of $n(x)$

We are going to formulate our scheme as two steps:

- Reconstruct refractive index $n(x)$ by adding test sources and measuring differential data
- Reconstruct source function $f(x)$ from the reconstructed $\hat{n}(x)$

Recovering refractive index

The solution field u satisfies:

$$\Delta u + k^2(1+n)u = f \tag{5}$$

If we put some test sources g in our domain $\Omega \subset \mathbb{R}^n$, then the modified field $\tilde{u}$ should satisfy:

$$\Delta \tilde{u} + k^2(1+n)\tilde{u} = f + g \tag{6}$$

Subtract (5) from (6), taking $\omega = \tilde{u} - u$:

$$\Delta \omega + k^2(1+n)\omega = g \tag{7}$$

with data as $\Lambda\gamma\omega = \Lambda\gamma\tilde{u} - \Lambda\gamma u$ on the boundary of the domain Ω.

Recovering refractive index

If we choose g carefully, we can modify (7) as:

$$\Delta\hat{\omega} + k^2(1+n)\hat{\omega} = 0 \quad \text{in } \Omega \tag{8}$$

$$\hat{\omega} = h \quad \text{on } \partial\Omega \tag{9}$$

where h is the modified boundary condition corresponding to the test source function, multi-frequency $k \in \mathcal{K}$, where $\mathcal{K}$ is an admissible set, and measurements are $\frac{\partial\hat{\omega}}{\partial n} = \Lambda\gamma\hat{\omega}$ on the boundary.

Recovering refractive index

Theorem (Uniqueness)

If Dirichlet to Neumann map $\Lambda : h(x) \to \dfrac{\partial h(x)}{\partial n}$ *associated with refractive index* $n(x)$ *is provided for each* $h(x) \in C^1(\mathbb{R}^n)$*, then* $n(x)$ *can be recovered uniquely.*

Recovering refractive index

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Unfortunately this problem is extremely ill-posed.

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In practice we use finite number of test source functions $\{g_i\}_{i=1}^{|\mathcal{I}|}$.

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Qustion

Can we get a good recovery of $n(x)$ by using a few test source functions only?

Stability

Theorem (Stability estimation,Nagayasu-Uhlmann-Wang,IOP,2013)

Suppose $n_1(x), n_2(x)$ *are two refractive indices associated with Dirichlet to Neumann maps* Λ_1, Λ_2 *respectively. And assume that for* $s > \frac{n}{2} + 1$, $\|n_l(x)\|_{H^s} \leq M$, $\mathrm{supp}(n_1 - n_2) \subset \Omega \subset \mathbb{R}^n$,*there exists a constant* $C(n, s, \Omega), C_1(n, s, \Omega)$, *such that if* $k^2 \geq \frac{1}{C_1 M}$, $\|\Lambda_1 - \Lambda_2\| \leq e^{-1}$,

$$\|n_1 - n_2\|_{H^{-s}} \leq \frac{C}{k^2} \exp(Ck^2)\|\Lambda_1 - \Lambda_2\| + C(k^2 - \log(\|\Lambda_1 - \Lambda_2\|))^{n-2s}$$

We can see there is an exponential instability here. And we can see there is an increasing stability by increasing frequencies k.

Algorithm

We formulate the approach as the following minimization problem:

$$\min_n J(n)$$

where,

$$J(n) = \sum_{k\in\mathcal{K}} \sum_{\mathbf{d}_j} \int_{\partial\Omega} \|\Lambda\gamma\hat{\omega}_j - \Lambda h_j\|^2 \mathrm{d}\sigma + \alpha \int_{\Omega} \|\nabla n\|^2 \mathrm{d}x$$

where Λh_j is the differential data for test source $g_j = \mathbf{e}^{ikx\cdot\mathbf{d}_j}$, α is regularization parameter.

Algorithm 1 pseudocode for recovering $n(x)$

1: select initial $n = n_0(x)$
2: **while** $J(n) \geq \epsilon$ **do**
3: update n by using Newton's method and linear search
4: **for** $j = 1$ to $|\mathcal{I}|$ **do**
5: solve $\hat{\omega}_j$ of Helmholtz equation (8) associated with the updated n for each $\mathbf{d}_j$
6: **end for**
7: compute $J(n)$
8: **end while**

Numerical results

Recovering refractive index: Smooth Ex.1

For $n(x)$ is chosen as:

$$n(x) = \begin{cases} 0.5 + 0.5\cos(2\pi|x|/R) & \text{if } |x| \leq 0.4 \\ 0 & \text{if } |x| > 0.4 \end{cases} \tag{10}$$

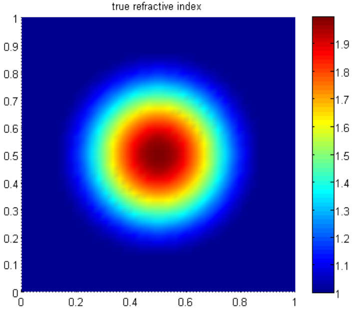

Figure : Ex.1 Exact index

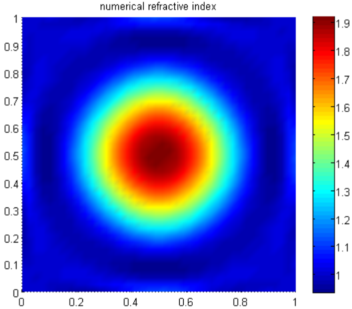

Figure : Ex.1 Reconstruction

Numerical results

Recovering refractive index: Smooth Ex.1

Error of the reconstruction in relative L^2 norm is 2.7%, and in relative L^∞ norm is 6.1%.

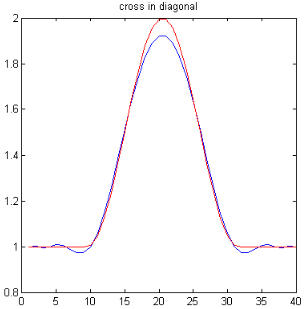

Figure : Ex.1 Cross section on diagonal

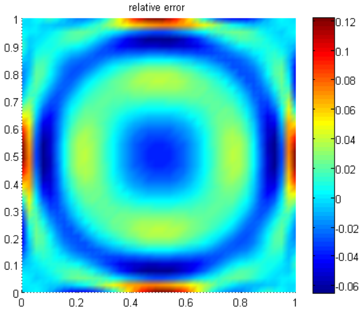

Figure : Ex.1 Relative error

Numerical results
Recovering refractive index: Smooth Ex.2

For $n(x)$ is chosen as:

$$n(x) = \begin{cases} 0.5 + 0.5\cos(2\pi|x|/R) & \text{if } |x| \le 0.45 \\ 0 & \text{if } |x| > 0.45 \end{cases} \tag{11}$$

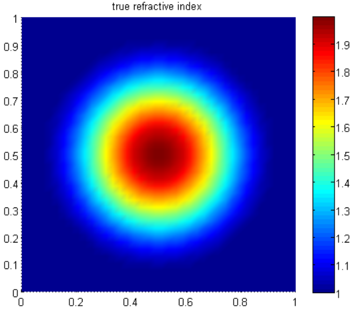

Figure : Ex.2 Exact index

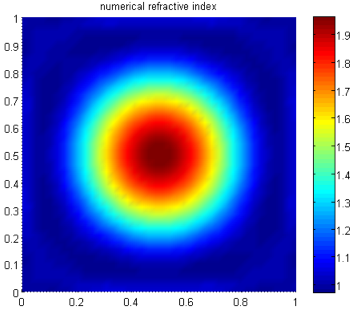

Figure : Ex.2 Reconstruction

Numerical results

Recovering refractive index: Smooth Ex.2

Error of the reconstruction in relative L^2 norm is 1.5%, and in relative L^∞ norm is 2.6%.

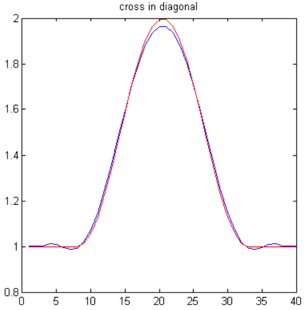

Figure : Ex.2 Cross section on diagonal

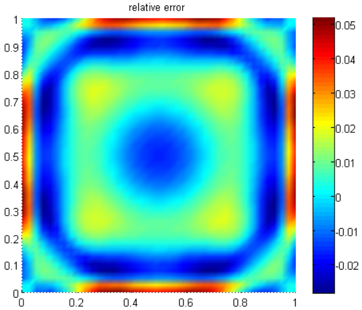

Figure : Ex.2 Relative error

Numerical results

Recovering refractive index: Smooth but bumpy Ex.3

$n(x)$ is chosen to have two bumps on diagonal with the same shape as the previous examples.

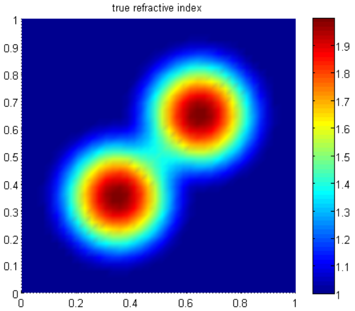

Figure : Ex.3 Exact index

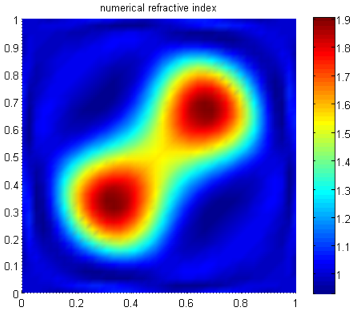

Figure : Ex.3 Reconstruction

Numerical results

Recovering refractive index: Smooth but bumpy Ex.3

Error of the reconstruction in relative L^2 norm is 3.94%, and in relative L^∞ norm is 7.32%.

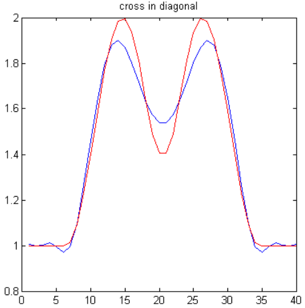

Figure : Ex.3 Cross section on diagonal

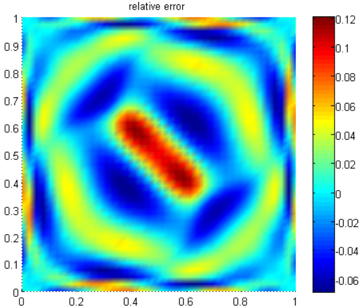

Figure : Ex.3 Relative error

Numerical results

Recovering refractive index: Smooth but bumpy Ex.4

$n(x)$ is chosen to have 4 bumps on 4 corners with the same shape as the previous examples.

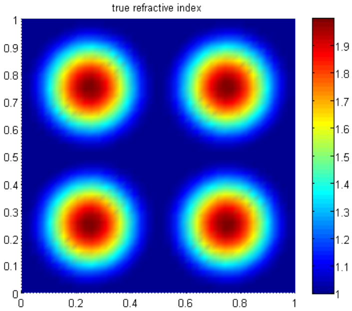

Figure : Ex.4 Exact index

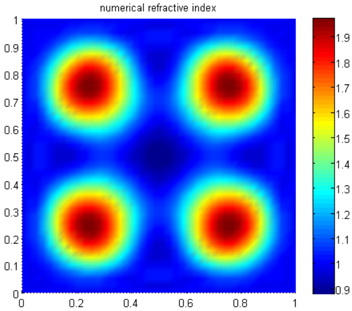

Figure : Ex.4 Reconstruction

Numerical results

Recovering refractive index: Smooth but bumpy Ex.4

Error of the reconstruction in relative L^2 norm is 2.0%, and in relative L^∞ norm is 6.0%.

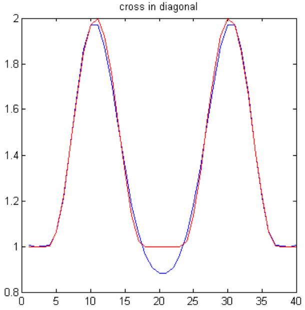

Figure : Ex.4 Cross section on diagonal

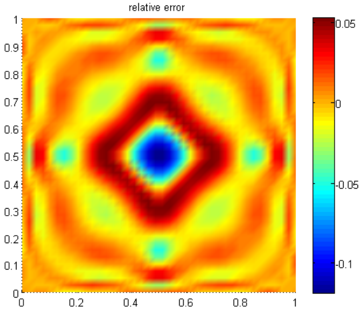

Figure : Ex.4 Relative error

Numerical results
Recovering refractive index: Non-smooth Ex.5

$n(x)$ is chosen to be as

$$n(x) = \begin{cases} 0.12 & \text{if } |x-0.25|^2 + |y-0.5|^2 \le 0.1 \\ 0.8 & \text{if } 0.5 \le x \le 0.75, 0.25 \le y \le 0.75 \\ 0 & \text{else cases} \end{cases} \tag{12}$$

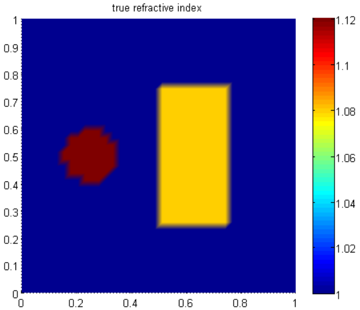

Figure : Ex.5 Exact index

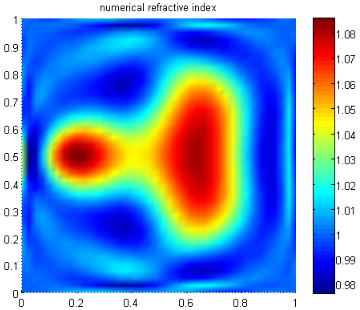

Figure : Ex.5 Reconstruction

Numerical results

Recovering refractive index: Non-smooth Ex.5

Error of the reconstruction in relative L^2 norm is 1.91%, and in relative L^∞ norm is 6.81%.

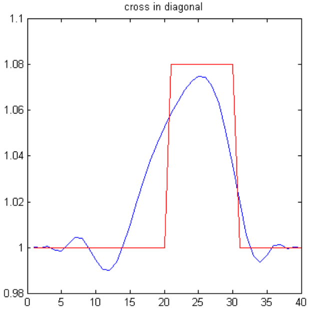

Figure : Ex.5 Cross section on diagonal

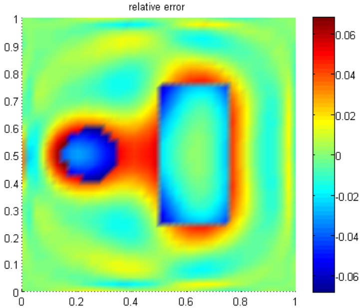

Figure : Ex.5 Relative error

Recovering source

After we have reconstructed the refractive index $\hat{n}(x)$, we then shall use the reconstructed refractive index $\hat{n}$ to recover the source.

$$\begin{aligned} \Delta u + k^2(1+\hat{n})u &= f \qquad \text{in } \Omega \qquad (13) \\ u &= 0 \qquad \text{on } \partial\Omega \qquad (14) \end{aligned}$$

Measurement is the differential data $\Lambda\gamma u$ on $\partial\Omega$.

Stability

Theorem (Hölder Stability for recovering location in $\mathbb{R}^3$)

Suppose u_l, for $l = 1, 2$ be the solutions to the problem associated with sources $f_l = \sum_{j=1}^{s} P_j^l \delta(x - x_j^l)$ and refractive index $n_l(x)$ respectively. Assume the intensity is bounded below, and the locations of point sources are not too close. The there exists a permutation π of $\{1, 2, \dots, s\}$ such that

$$\max_j \|\mathcal{S}(x_j^1) - \mathcal{S}(x_{\pi(j)}^2)\| \leq C \left(\frac{\sqrt{|\partial\Omega|}\mathrm{diam}(\Omega)^{2s-1}}{\rho\eta^{s-1}} \|n_1 - n_2\|_{L^2(\partial\Omega)} \right)^{\frac{1}{s}} \tag{15}$$

where $\rho = \min_j(|P_j^1|, |P_j^2|)$, $\eta = \min_l \min_{k \neq j} \mathrm{dist}(x_k^l, x_j^l)$ and $C = C(k, \Omega, \{x_j^l\})$ is a constant. $\mathcal{S}$ is a projection operator to a manifold.

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Question

How about the stability of intensities?

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$$\max_j \|\mathcal{S}(x_j^1) - \mathcal{S}(x_{\pi(j)}^2)\| \leq C \left(\frac{\sqrt{|\partial\Omega|}\,\mathrm{diam}(\Omega)^{2s-1}}{\rho\eta^{s-1}} \|n_1 - n_2\|_{L^2(\partial\Omega)} \right)^{\frac{1}{s}} \tag{15}$$

where $\rho = \min_j(|P_j^1|, |P_j^2|)$*,* $\eta = \min_l \min_{k \neq j} \mathrm{dist}(x_k^l, x_j^l)$ *and* $C = C(k, \Omega, \{x_j^l\})$ *is a constant.* $\mathcal{S}$ *is a projection operator to a manifold.*

Question

How about the stability of intensities?

Question

How about 2D?

Algorithm

We again formulate our inverse problem as a minimization problem:

$$\min_{f} K(f)$$

where $f(x)$ is superposition of point sources and,

$$K(f) = \int_{\partial\Omega} \|\Lambda\gamma u - h\|^2 \mathrm{d}\sigma$$

Algorithm 2 pseudocode for recovering source $f(x)$

select initial $f = \sum_{j=1}^{s} P_j \delta(x - x_j)$
while $K(f) \geq \epsilon$ **do**
  for $j = 1$ to s **do**
    update x_j and P_j by using Newton's method and linear search
  end for
  solve Helmholtz equation (13) associated with the updated x_j and P_j and compute $K(f)$
end while

Numerical results
Recovering source function

To be convenient in computing, we didn't choose $\delta(x)$ as our source function, we choose another function D(x) to approximate $\delta(x)$.

$$D(x_0, y_0, h_0) = h_0 \exp\left(\frac{-(x - x_0)^2 - (y - y_0)^2}{\varepsilon^2}\right) \tag{16}$$

And here we choose ε as $\dfrac{1}{2}$ of the mesh size. h_0 is the intensity, (x_0, y_0) is the location of the source.

Numerical results

Recovering source function Ex 1. one point source

Consider there is only one point source in the field with the following smooth numerical reconstructed refractive index.

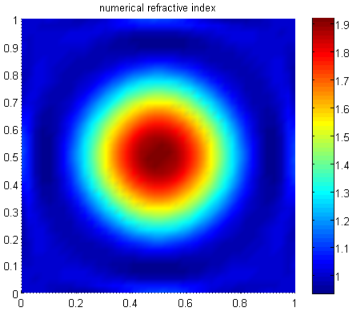

Figure : Ex.1 Numerical refractive index

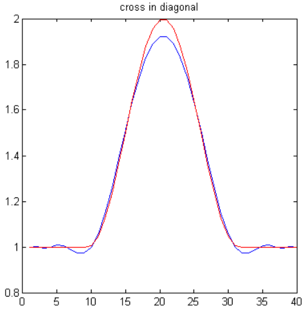

Figure : Ex.1 Numerical refractive index

Numerical results

Recovering source function Ex 1. one point source

The absolute error on location is $(2.556 \times 10^{-5}, 4.343 \times 10^{-5})$, relative error on intensity is 2.42×10^{-5}.

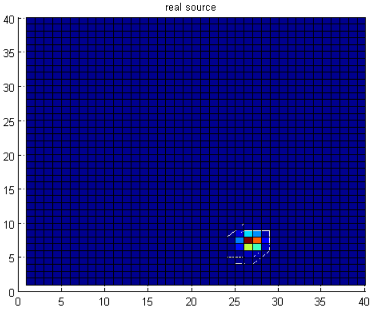

Figure : Ex.1 Exact source location

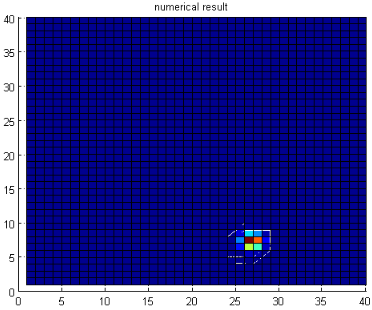

Figure : Ex.1 Numerical source location

Numerical results

Recovering source function Ex 2. one point source

Consider there is only one point source with following bumpy refractive index.

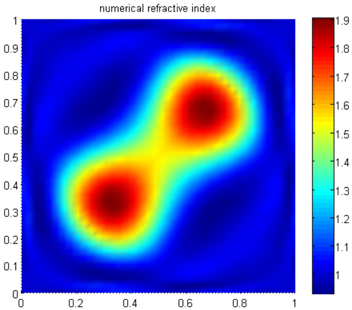

Figure : Ex.2 Numerical refractive index

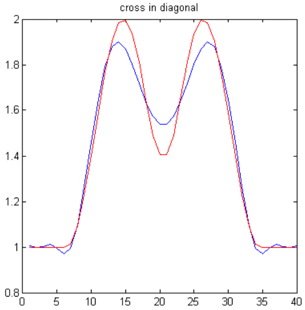

Figure : Ex.2 Numerical refractive index

Numerical results

Recovering source function Ex 2. one point source

The absolute error on location is $(1.465 \times 10^{-5}, 5.460 \times 10^{-7})$, relative error on intensity is 8.59×10^{-4}.

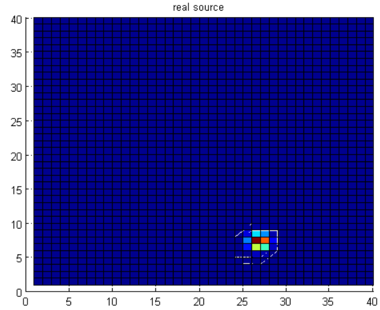

Figure : Ex.2 Exact source location

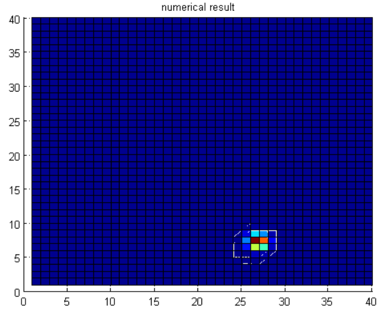

Figure : Ex.2 Numerical source location

Numerical results

Recovering source function Ex 3. two point sources

Consider there are two point sources with different intensities associated with following 4-cornered bumpy refractive index.

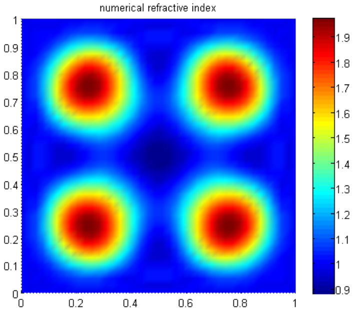

Figure : Ex.3 Numerical refractive index

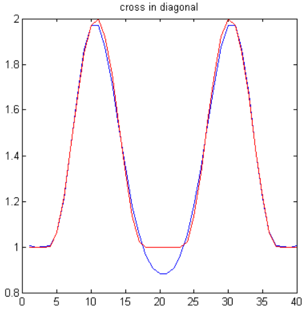

Figure : Ex.3 Numerical refractive index

Numerical results

Recovering source function Ex 3. two point sources

The absolute error on location is $(2.700 \times 10^{-5}, 3.381 \times 10^{-6})$, relative error on intensity is 2.32×10^{-3}.

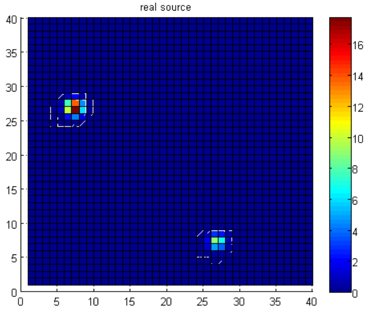

Figure : Ex.3 Exact source location

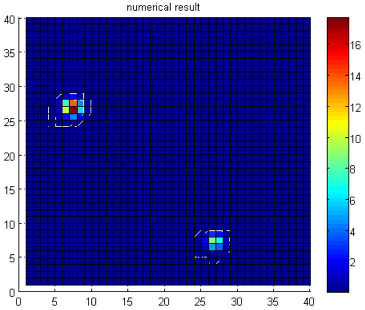

Figure : Ex.3 Numerical source location

Numerical results

Recovering source function Ex 4. two point sources

Consider there are two points sources with different intensities associated with following non-smooth refractive index.

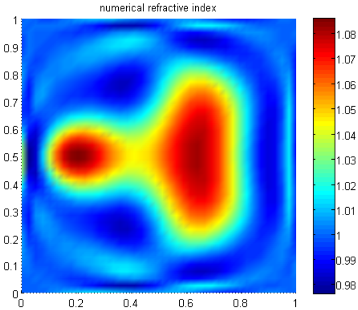

Figure : Ex.4 Numerical refractive index

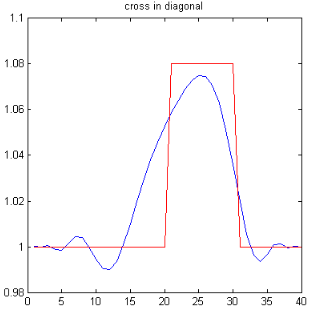

Figure : Ex.4 Numerical refractive index

Numerical results

Recovering source function Ex 4. two point sources

The absolute error on location is $(1.474 \times 10^{-5}, 8.793 \times 10^{-6})$, relative error on intensity is 1.49×10^{-4}.

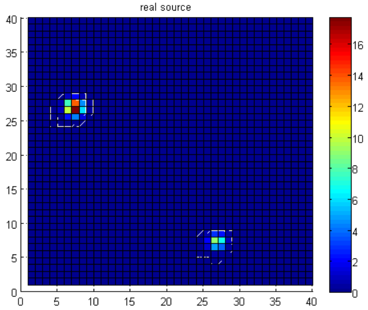

Figure : Ex.4 Exact source location

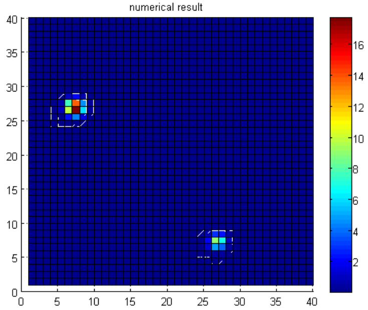

Figure : Ex.4 Numerical source location

Summary

- Add test sources and measure differential data

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- Reconstruction for both refractive index $n(x)$ and source $f(x)$

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- Add test sources and measure differential data
- Reconstruction for both refractive index $n(x)$ and source $f(x)$
- Stability of recovering source with respect to perturbation of $n(x)$

Thanks.