Numerical Seminar
Existence and stability estimate of H_p^1 solution of radiative transport system

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Disclaimation. *Without further notification, we take $\Omega \times D$ as default space.*

1 Introduction

In this section, we are going to review the essential results about radiative transport equation in $\mathbb{R}^d$.

1.1 Space H_p^1, trace and extension

Let D be connected closed convex bounded set in $\mathbb{R}^d$, Lipschitz boundary as ∂D, d as the diameter of D. $\Omega = \mathbb{S}^{d-1}$ as unit sphere in $\mathbb{R}^d$. We introduce H_p^1 norm and some other useful norms as following,

Definition 1.1. Let $1 \le p \le \infty$, norm in $H_p^1(\Omega \times D)$ is defined as

$$\|\psi\|_{H_p^1} = \|(s, \nabla)\psi\|_{L^p} + \|\psi\|_{L^p} \tag{1}$$

also, we introduce trace norm

$$\|\psi\|_{L^p(\Gamma)} = \left[\|\psi\|^p_{L^p(\Gamma_+)} + \|\psi\|^p_{L^p(\Gamma_-)}\right]^{1/p} \tag{2}$$

$$\|\psi\|_{\widetilde{L}^p(\Gamma)} = \left[\|\psi\|^p_{\widetilde{L}^p(\Gamma_+)} + \|\psi\|^p_{\widetilde{L}^p(\Gamma_-)}\right]^{1/p} \tag{3}$$

where

$$\|\psi\|_{L^p(\Gamma_\pm)} = \left[\int_\Omega ds \int_{\pi_s} \sum_i |\psi(s, Q + \xi_{i,\pm}(s,Q)s)|^p dQ\right]^{1/p} \tag{4}$$

$$\|\psi\|_{\widetilde{L}^p(\Gamma_\pm)} = \left[\int_\Omega ds \int_{\pi_s} \sum_i (\xi_{i,+} - \xi_{i,-})|\psi(s, Q + \xi_{i,\pm}(s,Q)s)|^p dQ\right]^{1/p} \tag{5}$$

and it is known that,

Lemma 1.1. *H_p^1 is complete and $C^{(0,1)}$ is dense in it.*

Lemma 1.2. *Let $\psi \in H_p^1$, then*

$$\|\psi\|_{\widetilde{L}^p(\Gamma)} \le c\|\psi\|_{H_p^1} \tag{6}$$

Lemma 1.3. *Consider internal domain $D' \subset D$ and $\operatorname{dist}(\partial D, D') \ge \delta > 0$, $\Gamma' = \Omega \times \partial D'$, then if $\psi \in H_p^1$, then $\gamma\psi|_{\Gamma'} \in L^p(\Gamma')$ and*

$$\|\psi\|_{L^p(\Gamma')} \le c\|\psi\|_{H_p^1} \tag{7}$$

Lemma 1.4. *There exists an extension $\widetilde{\psi}$ such that $\widetilde{\psi}|_\Gamma = \psi|_\Gamma$ and $\widetilde{\psi} \in H_p^1(\Omega \times \mathbb{R}^d)$, $\iff$ $\gamma\psi \in L^p(\Gamma)$, and moreover,*

$$\|\psi\|_{H_p^1} + \|\psi\|_{L^p(\Gamma)} \sim \|\widetilde{\psi}\|_{H_p^1(\Omega\times\mathbb{R}^d)} \tag{8}$$

1.2 Radiative transport equation

The equation is integro-differential as

$$(s, \nabla)\psi + \sigma_t\psi = \sigma_s \fint_\Omega \Theta(s, s')\psi(s', x)ds' + f \tag{9}$$

$$\psi|_{\Gamma_-} = g(s, x) \tag{10}$$

where Θ is symmetric w.r.t (s, s'). Usually we require the radiative process to be physical, namely

$$\left\|\frac{\sigma_s}{\sigma_t}\right\|_{L^\infty(D)} \le k_0 = \text{const} < 1$$

Remark. It is worth to notice that even scattering coefficient σ_s is negative, the requirement is still valid. Transport coefficient permits the solution decays along any direction, negative scattering(collision) operator actually behaves as absorption from neighborhood which somehow compensates the decay. Under above condition, the decay won't be fully compensated by neighborhood's contribution.

Definition 1.2. Define scattering operator S as

$$S\psi = \sigma_s \fint_\Omega \Theta(s, s')\psi(s', x)ds' \tag{11}$$

and introduce non-scattering operator $L : H_p^1 \to L^p$,

$$L\psi = (s, \nabla)u + \sigma_t u \tag{12}$$

The equation is formulated as

$$L\psi = S\psi + f \tag{13}$$

with incoming boundary condition $\psi|_{\Gamma_-} = g$.

Lemma 1.5. *The following norms are equivalent to H_p^1 norm,*

1. $[v]_{1,p} = \|v\|_{\widetilde{L}^p(\Gamma_-)} + \|Lv\|_{L^p}$
2. $[v]_{2,p} = \|v\|_{\widetilde{L}^p(\Gamma_-)} + \|(L-S)v\|_{L^p}$

It is trivial to conclude uniqueness of solution from $[v]_{2,p} \sim \|v\|_{H_p^1}$.

Remark. Follow the proof of lemma 1.5, we are able to relax the condition on absorption and scattering coefficients to

$$\|\frac{\sigma_s}{\sigma_t}\| \leq 1 \tag{14}$$

actually, we only need k_0 to satisfy

$$k_0 < \frac{1}{1 - \exp(-\sigma_0 d)} \tag{15}$$

even the absorption coefficient is negative, we are still able to expect a unique solution if the domain size is well controlled. When $\sigma_a \equiv 0$, physically there is no photons absorbed, incoming and outgoing will be equal, it is called *Conservative Milne Problem*, and we already showed this equation has unique solution.

Lemma 1.6. *The solution will satisfy maximum principle, when $f = 0$*

$$\|\psi\|_\infty \leq \|g\|_\infty \tag{16}$$

1.3 Radiative transport system in L^∞ (multigroup)

Disclaimation. *Without further notification, from now on, assume vanishing incoming boundary condition.*

The multigroup radiative transport equation satisfies

$$L_i\psi_i = \sum_j S_{ij}\psi_j + f_i \tag{17}$$

Let's assume L_i is associated with transport coefficient σ_i, S_{ij} is associated with scattering coefficient σ_{ij} and kernel K_{ij}, without any confusion.

Lemma 1.7. *The solution of non-scattering transport equation*

$$L_\sigma u = h(x, v) \tag{18}$$

admits unique solution

$$u(x, v) = \int_0^{\tau(x,v)} \exp\left(-\int_0^s \sigma(x - \xi v, v) d\xi\right) h(x - sv, v) ds \tag{19}$$

Definition 1.3. Define function $E(x, v, s)$ as following,

$$E(x, v, s) = \exp\left(-\int_0^s \sigma(x - \xi v, v) d\xi\right) \tag{20}$$

then we can show

$$\psi_i(x, v) = \int_0^{\tau(x,v)} E_i(x, v, s) \left(\sum_j S_{ij}\psi_j + f_i\right)(x - sv, v) ds \tag{21}$$

Definition 1.4. If $\psi = (\psi_1, \dots, \psi_N)$ satisfies

$$\psi_i \geq \int_0^{\tau(x,v)} E_i(x, v, s) \left(\sum_j S_{ij}\psi_j + f_i\right)(x - sv, v) ds \tag{22}$$

we call ψ an upper solution, and if

$$\psi_i \leq \int_0^{\tau(x,v)} E_i(x, v, s) \left(\sum_j S_{ij}\psi_j + f_i\right)(x - sv, v) ds \tag{23}$$

we call ψ an lower solution. It is obvious that a lower and upper solution is indeed a solution.

Definition 1.5. Define operator $Tz = (Tz_1, \dots, Tz_N)$,

$$Tz_i = \int_0^{\tau(x,v)} E_i(x, v, s) \left(\sum_{j=1}^{i} S_{ij}Tz_j + \sum_{j=i}^{N} S_{ij}z_j + f_i\right)(x - sv, v) ds \tag{24}$$

Lemma 1.8. *If $\sigma_i > 0$ and $\sigma_{ij} \geq 0$, then for any upper solution $u = \bar{u}_0$, sequence*

$$u, Tu, T^2u, \dots$$

is non-increasing. Also, it is similar to show that lower solution sequence is non-decreasing.

Regarding the source term f_i, we can assume $f_i \geq 0$ due to the linearity, otherwise, we decompose $f_i = f_i^+ - f_i^-$ naturally.

Definition 1.6. Define non-negative *maximal*(resp. *minimal*) solution u^* w.r.t to upper solution $\bar{u}$, if for any solution $0 \le u \le \bar{u}$, we have

$$u \le u^* \le \bar{u} \tag{25}$$

or (for minimal)

$$u_* \le u \le \bar{u} \tag{26}$$

Lemma 1.9. *Suppose there is an non-negative upper solution $\bar{u}$, then the minimal sequence $\{\underline{u}^k\}$ generated by $\underline{u}^0 = 0$ will converge to a minimal solution u_*. And maximal sequence $\{\bar{u}_k\}$ generated by $\bar{u}_0 = \bar{u}$ will converge to a maximal solution u^*.*

Remark. By using dominating theorem, the limits are solutions. Since the a non-negative solution is also an negative upper solution. We conclude that a solution exists *if and only if* an upper solution exists.

Theorem 1.10 (Existence of upper solution)**.** *If*

$$\sum_{j=1}^{N} \sigma_{ij} < \underline{\sigma}^i = \inf_x \sigma_i,$$

then there exists an upper solution. This condition can be interpreted as a variation of scattering should not exceed transport effect.

Proof. The proof is constructive. Assume $f(x,v) = \sup_i f_i(x,v)$ and

$$0 < \delta = \inf_i 1 - \frac{\sum_{j=1}^N \sigma_{ij}}{\underline{\sigma}^i} \tag{27}$$

We consider

$$\psi_i = \frac{1}{\delta} f(x,v) \int_0^{\tau(x,v)} \exp\left(-\int_0^s \sigma_i(x - \xi v, v) d\xi\right) ds \tag{28}$$

then

$$\sum_{j=1}^{N} S_{ij}\psi_j = \frac{1}{\delta} \sum_{j=1}^{N} f \sigma_{ij} \fint K_{ij}(v,v') \int_0^{\tau} \exp\left(-\int_0^s \sigma_i(x-\xi v, v) d\xi\right) ds dv'$$

since

$$\int_0^{\tau} E_i(x,v,s) ds < \frac{1}{\underline{\sigma}^i}$$

$$\sum_{j=1}^{N} S_{ij}\psi_j + f_i \le \frac{f}{\delta} \sum_j \sigma_{ij} \sup_v \int_0^{\tau} E_i(x,v,s) ds + f_i \le \frac{f}{\delta} \tag{29}$$

Therefore $\psi = (\psi_i)_{i=1,\dots,N}$ is an upper solution. $\square$

2 Radiative Transport system in H_p^1

To be general, we consider multigroup transport equation

$$v \cdot \nabla U(x,v) + \Sigma U(x,v) = SU(x,v) + Q(x,v) \tag{30}$$

where Σ is absorption coefficient matrix, physically we assume all entries in Σ are positive. S is scattering operator matrix,

$$S_{ij}u_j = \sigma_{ij}(x) \fint_{\mathbb{S}^{d-1}} K_{ij}(v,v')u_j(x,v')dv' \tag{31}$$

Here we still assume vanishing incoming boundary condition.

2.1 Lax-Milgram framework

Disclaimation. *In following context, we always assume the influx is zero, which automatically forms a subspace with induced norm.*

We consider inner product for functions $u \in H_p^1$ and $v \in H_q^1$ with *zero influx*(Banach space) as

$$\langle u, v\rangle_H = \langle u, v\rangle_L + \langle s \cdot \nabla u, s \cdot \nabla v\rangle_L \tag{32}$$

where

$$\langle f, g\rangle_L = \int_\Omega \int_{\mathbb{S}^{d-1}} f(v',x)g(v',x)dv'dx \tag{33}$$

By divergence theorem, taking $\mathbb{S} = \mathbb{S}^{d-1}$ for short,

$$\langle f, v' \cdot \nabla g\rangle_L + \langle g, v' \cdot \nabla f\rangle_L = \int_{\mathbb{S}} \int_{\partial\Omega} fg(\mathbf{n} \cdot v')\mathrm{d}S\mathrm{d}v' \tag{34}$$

We can formulate bilinear form on $f = (f_1, \ldots, f_N)$ and $\psi = (\psi_1, \ldots, \psi_N)$ from

$$\langle v \cdot \nabla\psi + (\Sigma - K)\psi, f\rangle \tag{35}$$

using divergence theorem,

$$\langle v \cdot \nabla f, g\rangle = -\langle f, v \cdot \nabla g\rangle + \int_{\mathbb{S}} \int_{\partial\Omega} fg\{(\mathbf{n} \cdot v')_+ - (\mathbf{n} \cdot v')_-\}\mathrm{d}S\mathrm{d}v' \tag{36}$$

The bilinear form, using vanishing incoming boundary condition,

$$B(f,\psi) = -\langle \psi, v \cdot \nabla f\rangle + \int_{\mathbb{S}} \int_{\partial\Omega} f\psi(\mathbf{n} \cdot v')_+\mathrm{d}S\mathrm{d}v' + \langle (\Sigma - K)\psi, f\rangle \tag{37}$$

$$F(\psi) = \langle Q, \psi\rangle \tag{38}$$

In order to show existence of solution in H_p^1, we seek solution in larger space $\widetilde{H}_0^1$.

Definition 2.1. Define norm in $\tilde{H}_p^0$ as

$$\|\psi\|_{p,\widetilde{H}} = \|\psi\|_p + \|\psi\|_{p,\Gamma} \tag{39}$$

In our setting, the influx is vanished, it is known that

Lemma 2.1. *If $\psi \in H_p^1$, and $\psi \in L^p(\Gamma_-)$, then*

$$\|\psi\|_{p,\Gamma_+} \leq c(\|\psi\|_{H_p^1} + \|\psi\|_{p,\Gamma_-}) \tag{40}$$

especially when $\psi|_{\Gamma_-} = 0$, then

$$\|\psi\|_{p,\Gamma} \leq c\|\psi\|_{p,H} \tag{41}$$

and in this case

$$\|\psi\|_{p,\widetilde{H}} \leq c\|\psi\|_{p,H} \tag{42}$$

Lemma 2.2. *$B(f,\psi)$ on $H_p^1 \times \left(\widetilde{H}_p^0 \cap H_p^1\right)$ is bounded and weak coercive if $(\Sigma - K)$ is uniformly elliptic.*

Proof.

$$B(f,\psi) \leq |\langle \psi, v \cdot \nabla f \rangle| + |\int_{\mathbb{S}} \int_{\partial\Omega} f\psi(\mathbf{n} \cdot v')_+ \mathrm{d}S\mathrm{d}v'| + |\langle (\Sigma - K)\psi, f \rangle| \tag{43}$$

using following estimate,

$$\begin{aligned} |\langle \psi, v \cdot \nabla f \rangle| &\leq \sum_{i=1}^{N} |\langle \psi_i, v \cdot \nabla f_i \rangle| &(44)\\ &\leq \sum_i \|\psi_i\|_p \|v \cdot \nabla f_i\|_q \leq \sum_i \|\psi_i\|_{p,\widetilde{H}} \|f_i\|_{q,H} &(45) \end{aligned}$$

$$\begin{aligned} |\int_{\mathbb{S}} \int_{\partial\Omega} f\psi(\mathbf{n} \cdot v')_+ \mathrm{d}S\mathrm{d}v'| &\leq \sum_i |\int_{\mathbb{S}} \int_{\partial\Omega} f_i\psi_i(\mathbf{n} \cdot v')_+| &(46)\\ &\leq \sum_i \|\psi_i(\mathbf{n} \cdot v')_+^{1/p}\|_{p,\Gamma} \|f_i(\mathbf{n} \cdot v')_+^{1/q}\|_{q,\Gamma} &(47)\\ &\leq \sum_i \|\psi_i\|_{p,\Gamma} \|f_i\|_{q,\Gamma} &(48)\\ &\leq \sum_i \|\psi\|_{p,\widetilde{H}} \|f_i\|_{q,H} &(49) \end{aligned}$$

$$|\langle (\Sigma - K)\psi, f \rangle| \leq \sum_i |\langle (\Sigma_i - K_i)\psi, f \rangle| \leq \sum_i \|(\Sigma_i - K_i)\psi\|_p \|f\|_q \tag{50}$$

while when $p \geq 1$,

$$\|(\Sigma_i - K_i)\psi\|_p \leq \|\Sigma_i\psi\|_p + \|K_i\psi\|_p \leq \|\Sigma\|_\infty\|\psi\|_p + c\|\psi\|_p \tag{51}$$

since

$$\begin{aligned}
\|K_i\psi\|_p &\leq \sum_j \|K_{ij}\psi_j\|_p = \sum_j \|\sigma_{ij} \fint K_{ij}(v, v')\psi_j(x, v')dv'\|_p && (52)\\
&\leq CN\|\sigma\|_\infty\|K\|_\infty\|\psi_j\|_p && (53)
\end{aligned}$$

the boundedness is proved.

Regard the coercive side, due to fact that $C^{(0,1)}$ is dense in any H_p^1, for $\psi \in \widetilde{H}_p^0 \cap H_p^1$, we can find $f_\epsilon \in C^{(0,1)}$ such that

$$\|f_\epsilon - \psi\|_{p,H} \leq \epsilon \tag{54}$$

Therefore by divergence theorem,

$$\langle f_\epsilon, v \cdot \nabla\psi\rangle + \langle \psi, v \cdot \nabla f_\epsilon\rangle = \int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v)\mathrm{d}S\mathrm{d}v \tag{55}$$

which yields

$$2\langle \psi, v \cdot \nabla f_\epsilon\rangle = \int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v)\mathrm{d}S\mathrm{d}v + T(f_\epsilon, \psi) \tag{56}$$

where $T(f_\epsilon, \psi) = \langle \psi - f_\epsilon, v \cdot \nabla\psi\rangle + \langle \psi, v \cdot \nabla(\psi - f_\epsilon)\rangle$. Noticing that $\psi = 0$ on influx boundary condition,

$$\begin{aligned}
B(f_\epsilon, \psi) &= -\langle \psi, v \cdot \nabla f_\epsilon\rangle \\
&\quad + \int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v')_+\mathrm{d}S\mathrm{d}v' + \langle(\Sigma - K)\psi, f_\epsilon\rangle && (57)\\
&= -\frac{1}{2}\int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v)\mathrm{d}S\mathrm{d}v - \frac{1}{2}T(f_\epsilon, \psi) \\
&\quad + \int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v')_+\mathrm{d}S\mathrm{d}v' + \langle(\Sigma - K)\psi, f_\epsilon\rangle && (58)\\
&= \frac{1}{2}\int_{\mathbb{S}}\int_{\partial\Omega} f_\epsilon\psi(\mathbf{n} \cdot v)_+\mathrm{d}S\mathrm{d}v - \frac{1}{2}T(f_\epsilon, \psi) \\
&\quad + \langle(\Sigma - K)\psi, f_\epsilon\rangle && (59)\\
&\geq \frac{1}{2}\|f_\epsilon\|_{q,\Gamma}\|\psi\|_{p,\Gamma} + \kappa\|\psi\|_p\|f_\epsilon\|_q - C\epsilon(\|f\|_{p,H} + \|f_\epsilon\|_{q,\widetilde{H}}) \\
&\geq c\|\psi\|^2_{p,\widetilde{H}} - C\epsilon\|f_\epsilon\|_{q,H} \quad (p \geq q) && (60)
\end{aligned}$$

Remark. If the domain is bounded, we can use if $q \geq p$, then

$$\|f\|_p \leq C\|f\|_q \tag{61}$$

Remark. If $p = q = 2$, then it will be sitting in the standard Lax-Milgram framework. By applying Babuška-Lax-Milgram theorem, we can conclude a unique solution in $H_p^1 \cap \widetilde{H}_p^0$ from the weak form.

□