Sophex:An Overview on Ben Andrews' Proof of The Fundamental Gap Conjecture

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1 Introduction

Usually we consider the Schrodinger operator of the form $-\Delta + V$ with Dirichlet boundary conditions on a compact convex domain Ω in $\mathbb{R}^n$. The diameter of Ω is given by $D = \sup_{x,y\in\Omega} |y - x|$. We assume the potential V is semiconvex(i.e. $V + c|x|^2$ is convex for some c). In order to figure out the method more clearly, we assume that $V = 0$.

For operator $-\Delta$, we know that it has a increasing sequence of eigenvalues $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ and corresponding eigenfunctions $\{\phi_i\}_{i\geq 0}$ which vanish on $\partial\Omega$ and satisfy

$$\Delta\phi_i + \lambda_i\phi_i = 0. \tag{1}$$

The difference between the first two eigenvalues in called fundamental gap. It is important since in quantum mechinics it represents the required energy to reach the first excited state from ground state. Thus it determines the stability of the ground state.

Theorem 1.1. *Let Ω be a bounded convex domain of diameter D. Then the eigenvalues of $-\Delta$ satisfy*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}. \tag{2}$$

In one dimension, this conjecture(the equality holds) is true. Since the problem turns out to be an boundary value ODE.

Definition 1.1. *If ω is a real function of a positive real variable, and X is a vector field on domain Ω, we say ω is a modulus of expansion for X if for every $y \neq x$ in Ω we have*

$$(X(y) - X(x)) \cdot \frac{y - x}{|y - x|} \geq 2\,\omega(\frac{|y - x|}{2}). \tag{3}$$

Remark 1.1. *We say ω is a modulus of contraction if the sign is reversed. If f is semi-convex function on domain Ω then ω is a modulus of convexity if ω is a modulus of expansion of the gradient vector field ∇f of f, and we say ω is a modulus of concavity for f if ω is a modulus of contraction for ∇f.*

Remark 1.2. *We note that f is concave(convex) iff $\omega = 0$ is a modulus of concavity(convexity) for f.*

The essential part of the proof is: assume ϕ_0 is the first eigenfunction of 1 and $\tilde{\phi}_0$ is the first eigenfunction of 1 in one dimension on interval $[-\frac{D}{2}, \frac{D}{2}]$,

$$(\nabla \log \phi_0(y) - \nabla \log \phi_0(x)) \cdot \frac{y-x}{|y-x|} \leq 2(\log \tilde{\phi}_0)'|_{z=\frac{|y-x|}{2}} \tag{4}$$

Which means that $(\log \tilde{\phi}_0)'$ is a modulus of concavity for $\log \phi_0$.

Theorem 1.2. *The spectral gap for $-\Delta$ on a convex domain Ω is bounded below by spectral gap of the one dimensional operator $\frac{d^2}{dx^2}$ on $[-\frac{D}{2}, \frac{D}{2}]$.*

Corollary 1.1. *The spectral gap of $-\Delta$ has the bound:*

$$\lambda_1 - \lambda_0 \geq \frac{3\pi^2}{D^2}. \tag{5}$$

Theorem 1.3. *Let ϕ_0 be the first eigenfunction of 1, then 4 holds.*

2 Modulus of continuity for heat equation with drift

Theorem 2.1. *Let Ω be a strictly convex domain of diameter D with smooth boundary in $\mathbb{R}^n$, and X is a time-dependent vector field on Ω. Suppose $v : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is a smooth solution of equation:*

$$\frac{\partial v}{\partial t} = \Delta v + X \cdot \nabla v \quad \text{in } \Omega \times \mathbb{R}_+; \tag{6}$$

$$D_\nu v = 0 \quad \text{in } \partial\Omega \times \mathbb{R}_+. \tag{7}$$

Suppose that:

1. *$X(\cdot, t)$ has modulus of contraction $\omega(\cdot, t)$ for each $t \geq 0$, where $\omega : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ is smooth;*
2. *$v(\cdot, t)$ has modulus of continuity φ_0, and φ_0 is smooth with $\varphi_0(0) = 0$ and $\varphi_0'(z) > 0$ for $0 \leq z \leq \frac{D}{2}$;*
3. *$\varphi : [0, \frac{D}{2}] \times \mathbb{R}_+ \to \mathbb{R}$ satisfies*

(a) $\varphi(z,0) = \varphi_0(z)$ *for* $z \in [0, D/2]$;

(b) $\frac{\partial \varphi}{\partial t} \geq \varphi'' + \omega\varphi'$ *on* $[0, D/2] \times \mathbb{R}_+$;

(c) $\varphi' > 0$ *on* $[0, D/2] \times \mathbb{R}_+$;

(d) $\varphi(0,t) \geq 0$ *for each* $t \geq 0$.

Then $\varphi(\cdot, t)$ *is a modulus of continuity of* $v(\cdot, t)$.

Proof. For any $\epsilon > 0$, define a function Z_ϵ on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+$ by

$$Z_\epsilon(y,x,t) = v(y,t) - v(x,t) - 2\varphi(\frac{|y-x|}{2}, t) - \epsilon e^t. \tag{8}$$

By assumption $Z_\epsilon(x,y,0) \leq -\epsilon$ for every $x \neq y$ in Ω, and $Z_\epsilon(x,x,t) \leq -\epsilon$ for every $x \in \Omega$ and $t \geq 0$. We will prove Z_ϵ is negative on $\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+$. If not true, there exists a first time $t_0 > 0$ such that there exists two points $x \neq y \in \overline{\Omega}$ such that $Z_\epsilon(x,y,t_0) = 0$.

1. If $y \in \partial\Omega$, then we have

$$D_{\nu_y} Z_\epsilon = D_{\nu_y} v(y,t) - \varphi' \frac{(y-x)}{|y-x|} \cdot \nu_y = -\varphi' \frac{(y-x)}{|y-x|} \cdot \nu_y < 0$$

   where ν_y is the outward unit normal at y, and we used the Neumann condition, and the strict convexity of Ω which implies $\frac{(y-x)}{|y-x|} \cdot \nu_y > 0$. Thus for small $s > 0$, $Z_\epsilon(x, y - s\nu_y, t_0) > 0$, which contradicts with the definition of t_0.

2. if x, y are in the interior of Ω, then all spatial derivative of Z_ϵ at (x,y,t_0) vanish, and the second derivative matrix is non-positive. In particular we choose our orthogonal basis with $e_n = \frac{y-x}{|y-x|}$, and we will have

$$\frac{\partial^2}{\partial s^2} Z_\epsilon(x + se_n, y - se_n, t_0)|_{s=0} \leq 0; \qquad \text{and} \tag{9}$$

$$\frac{\partial^2}{\partial s^2} Z_\epsilon(x + se_i, y + se_i, t_0)|_{s=0} \leq 0; \qquad \text{for} \quad i = 1, 2, \ldots, n-1. \tag{10}$$

   We compute these inequailities in terms of v. Thus the vanishes first derivatives imply that

$$\nabla v(y) = \nabla v(x) = \varphi' e_n. \tag{11}$$

   The second derivatives imply that

$$0 \geq \frac{d^2}{ds^2} Z_\epsilon(x + se_i, y + se_i, t_0) = D_i D_i v(y) - D_i D_i v(x). \tag{12}$$

$$0 \geq \frac{d^2}{ds^2} Z_\epsilon(x + se_n, y + se_n, t_0) = D_n D_n v(y) - D_n D_n v(x) - 2\varphi''. \quad (13)$$

summing the inequailities, we get

$$0 \geq \Delta v(y) - \Delta v(x) - 2\varphi''. \quad (14)$$

Consider

$$\begin{aligned}
\frac{\partial}{\partial t} Z_\epsilon &= \Delta v(y,t) + X(y) \cdot \nabla v(y) && (15)\\
&\quad -\Delta v(x) - X(x) \cdot \nabla v(x) - 2\varphi_t - \epsilon e^t && (16)\\
&\leq 2\varphi'' + \varphi'(X(y) - X(x)) \cdot \frac{y-x}{|y-x|} - 2\varphi_t - \epsilon e^t && (17)\\
&< 2\varphi'' + 2\omega\varphi' - 2\varphi_t && (18)\\
&\leq 0. && (19)
\end{aligned}$$

where we used the modulus of contraction of X, therefore $Z_\epsilon < 0$ for every $\epsilon > 0$.

□

3 Log-concavity implies gap conjecture

Proposition 3.1. *Let u_1, u_0 be two solutions to the heat equation,*

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u \quad \text{on } \Omega \times \mathbb{R}_+; && (20)\\
u &= 0 \quad \text{on } \partial\Omega \times \mathbb{R}_+ && (21)
\end{aligned}$$

where Ω is convex with smooth boundary in $\mathbb{R}^n$, and u_0 is positive in interior of Ω. Let $v = \frac{u_1}{u_0}$. Then v is smooth on $\Omega \times \mathbb{R}_+$, and satisfies

$$\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v - 2\nabla \log u_0 \cdot \nabla v &= 0 \quad \text{on } \Omega \times [0,\infty); && (22)\\
D_\nu v &= 0 \quad \text{on } \partial\Omega \times [0,\infty). && (23)
\end{aligned}$$

Proof. Both u_0 and u_1 are smooth on $\overline{\Omega} \times [0,\infty)$, and u_0 has negative derivative in the normal direction ν. It follows that v can be extended to $\overline{\Omega}$ as a smooth function.

Since we can choose some local basis such that $\partial\Omega = \{(x_1, \cdots, x_n)|x_1 = 0\}$. Then we can find a smooth function $g(x,t)$ such that

$$\begin{aligned} u_0 &= x_1 \cdot g(x,t) \quad && in\ \Omega \times \mathbb{R} \\ g(x,t) &\neq 0 && on\ \partial\Omega \end{aligned}$$

Also we can find a smooth $h(x,t)$ such that (Malgrange preparation theorem)

$$u_1 = x_1 \cdot h(x,t)$$

Hence we have

$$\frac{u_1}{u_0} = \frac{h(x,t)}{g(x,t)}$$

which is continuous on the boundary. With simple computation:

$$\begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{u_1}{u_0}\right) && (24) \\ &= \frac{\Delta u_1 - v(\Delta u_0)}{u_0} && (25) \\ &= \Delta v + 2\nabla \log u_0 \cdot \nabla v. && (26) \end{aligned}$$

Since at any point of $\partial\Omega$, we have that $\frac{\partial v}{\partial t}$ and Δv are bounded, and $\nabla u_0 = -c\nu$ with $c > 0$, thus we must have $D_\nu v = 0$. □

Proposition 3.2. *Inequality 4 implies Theorem 1.2.*

Proof. Use Theorem 2.1. Suppose u_1 is any smooth solution of 20, with initial data ϕ_0. Let $v = \frac{u_1}{u_0}$ be the corresponding solution of 22.The drift of velocity is $2\nabla \log u_0$ and has the modulus of contraction $2(\log \tilde{\phi_0})'$. We may roughly use the theorem with $\varphi = C\frac{e^{-\mu_1 t}\tilde{\phi_1}}{e^{-\mu_0 t}\tilde{\phi_0}}$ and $\omega = 2(\log \tilde{\phi_0})'$. □