

The **Autocorrelation Function (ACF)** and **Partial Autocorrelation Function (PACF)** are important tools in time series analysis for identifying relationships between observations at different lags. Let's break down their formulas and provide a basic demonstration of how they work.

1. Autocorrelation Function (ACF)

Formula: The ACF at lag k is calculated as:

$$\rho(k) = \frac{\sum_{t=k+1}^n (Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^n (Y_t - \bar{Y})^2}$$

Where:

- Y_t is the value of the time series at time t .
- \bar{Y} is the mean of the time series.
- n is the total number of observations.
- k is the lag for which you want to calculate the autocorrelation.

Demonstration: Suppose we have a simple time series: $[2, 3, 5, 7, 11]$.

1. Calculate the mean:

$$\bar{Y} = \frac{2 + 3 + 5 + 7 + 11}{5} = 5.6$$

2. Calculate ACF for lag $k = 1$:

$$\begin{aligned} \rho(1) &= \frac{(3 - 5.6)(2 - 5.6) + (5 - 5.6)(3 - 5.6) + (7 - 5.6)(5 - 5.6) + (11 - 5.6)(7 - 5.6)}{(2 - 5.6)^2 + (3 - 5.6)^2 + (5 - 5.6)^2 + (7 - 5.6)^2 + (11 - 5.6)^2} \\ &= \frac{(-2.6)(-3.6) + (-0.6)(-2.6) + (1.4)(-0.6) + (5.4)(1.4)}{(-3.6)^2 + (-2.6)^2 + (-0.6)^2 + (1.4)^2 + (5.4)^2} \\ &= \frac{9.36 + 1.56 - 0.84 + 7.56}{12.96 + 6.76 + 0.36 + 1.96 + 29.16} \\ &= \frac{17.64}{51.2} \approx 0.344 \end{aligned}$$

2. Partial Autocorrelation Function (PACF)

Formula: The PACF at lag k can be calculated using the following formula:

$$\phi(k) = \rho(k) - \sum_{j=1}^{k-1} \phi(j)\rho(k-j)$$

Where:

- $\phi(k)$ is the PACF at lag k .

- $\rho(k)$ is the ACF at lag k .
- $\phi(j)$ are the partial autocorrelations at lags j .

Demonstration: Let's continue with our earlier example and compute PACF for $k = 1$.

1. Since PACF for lag 1 is simply equal to ACF for lag 1:

$$\phi(1) = \rho(1) \approx 0.344$$

2. Calculate PACF for lag $k = 2$: Using our previous calculation of $\rho(1)$, we calculate $\rho(2)$:

- Suppose $\rho(2) = 0.2$ (for demonstration).

Then, we calculate PACF:

$$\begin{aligned}\phi(2) &= \rho(2) - \phi(1) \cdot \rho(1) \\ &= 0.2 - (0.344 \cdot 0.344) \\ &= 0.2 - 0.118336 \approx 0.081664\end{aligned}$$

Summary

- **ACF** measures the correlation between observations at different lags.
- **PACF** measures the correlation between observations at a given lag after removing the effects of shorter lags.

In practice, ACF and PACF are typically calculated using statistical software or libraries (e.g., `statsmodels` in Python or `acf` and `pacf` functions in R) because the calculations can be cumbersome for larger datasets.