

# Computational Finance



# Monte Carlo Methods

## Brownian Motion

- We saw last week that the binomial tree implies for  $X_t \equiv \log S_t$  that

$$X_{i\delta t} = X_{(i-1)\delta t} + R_i \iff \Delta X_i = R_i, \quad (\dagger)$$

where  $R_i = \log u$  or  $R_i = \log d$ , with probabilities  $\mathbb{Q}[u]$  and  $\mathbb{Q}[d]$ .

- Equation  $(\dagger)$  is a *stochastic difference equation*.
- Its *solution*

$$X_T = \log S_0 + \sum_{i=1}^N R_i = \log S_0 + \sigma \sqrt{\delta t} (2k - N)$$

is called a *binomial process*, or in the special case with  $\mathbb{E}[R_i] = 0$ , a *random walk*.

- We also saw that if we let  $N \rightarrow \infty$  (so that  $\delta t \rightarrow 0$ ),

$$X_T - X_0 \xrightarrow{d} N(\mu T, \sigma^2 T), \quad \mu \equiv r - \frac{1}{2}\sigma^2.$$

- The argument can be repeated for every  $X_t$ ,  $t \leq T$ , showing that

$$X_t - X_0 \xrightarrow{d} N(\mu t, \sigma^2 t),$$

and that for any  $0 < t < T$ ,  $X_t - X_0$  and  $X_T - X_t$  are independent.

- As  $\delta t \rightarrow 0$ ,  $\{X_t\}_{t \geq 0}$  becomes a continuous time process: the indexing set is now given by the entire positive real line.
- This continuous time limit (with  $\mu = 0$  and  $\sigma^2 = 1$ ) is called *Brownian motion*, or *Wiener process*.
- From now on, rather than modelling in discrete time and then letting  $\delta t \rightarrow 0$ , we will directly model in continuous time, using Brownian motion as a building block.

- Definition of (standard) *Brownian Motion*: Stochastic process  $\{W_t\}_{t \geq 0}$  satisfying
  - $W_0 = 0$ ;
  - The increments  $W_t - W_s$  are independent for all  $0 \leq s < t$ ;
  - $W_t - W_s \sim N(0, t - s)$  for all  $0 \leq s \leq t$ ;
  - Continuous sample paths.
- This is standard Brownian motion, whereas  $X_t = \sigma W_t$  is Brownian motion with variance  $\sigma^2$ .
- Restriction that process start at zero may be loosened by considering  $X_t = X_0 + \sigma W_t$ .
- Brownian motion with drift:  $X_t = X_0 + \mu t + \sigma W_t$ , so that  $\mathbb{E}[X_t] = X_0 + \mu t$ ,  $\text{Var}[X_t] = \sigma^2 t$ .

- Properties of Brownian Sample Paths:

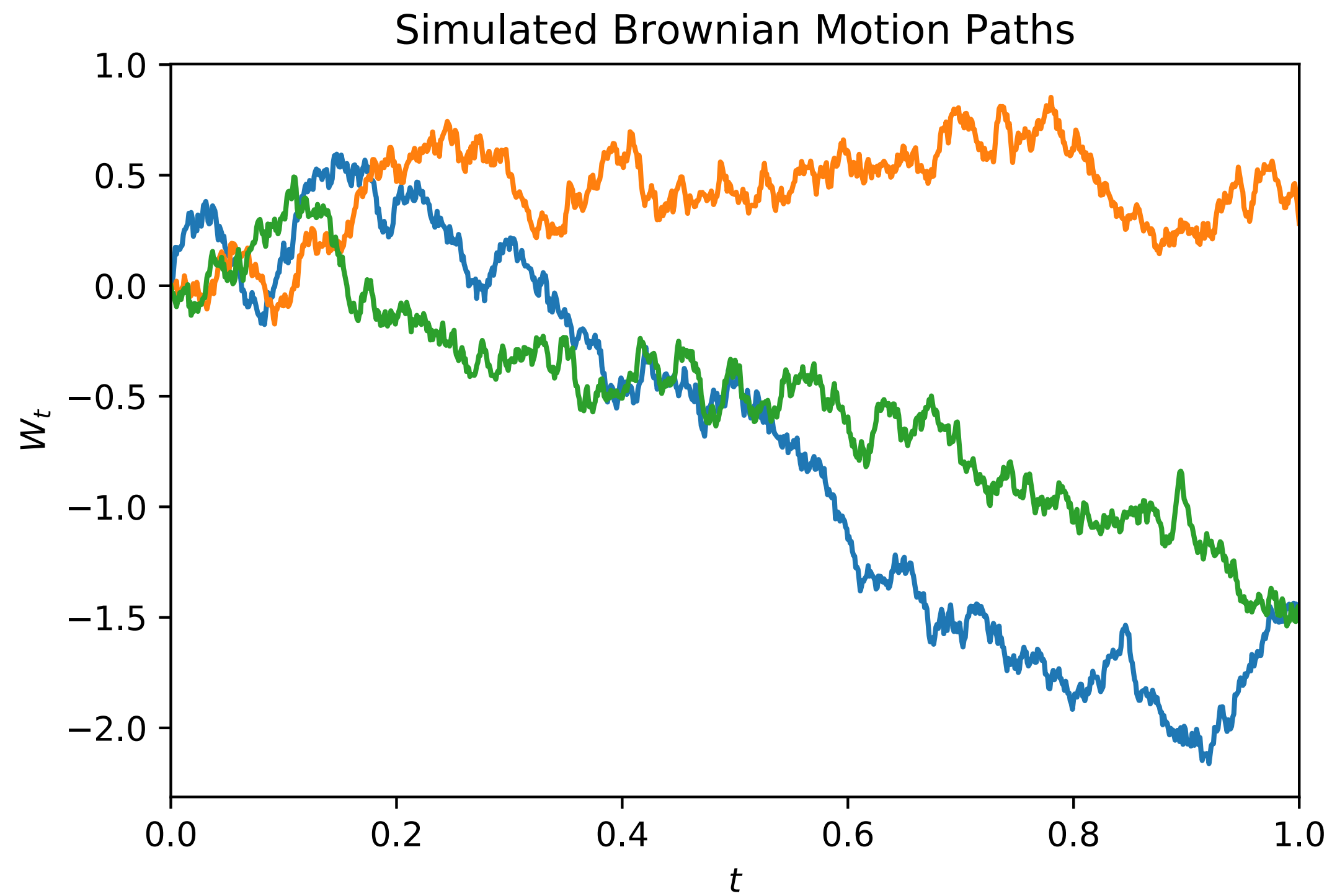
- *Continuity*: by assumption, and also  $W_{t+\delta t} - W_t \sim N(0, \delta t) \rightarrow 0$  as  $\delta t \downarrow 0$ ;

- *Nowhere differentiability*: intuitively, this is seen from

$$\frac{W_t - W_{t-\delta t}}{\delta t} \sim N\left(0, \frac{1}{\delta t}\right), \quad \frac{W_{t+\delta t} - W_t}{\delta t} \sim N\left(0, \frac{1}{\delta t}\right);$$

left and right difference quotients do not have (common) limit as  $\delta t \downarrow 0$ .

- *Self-similarity*: Zooming in on a Brownian motion yields another Brownian motion: for any  $c > 0$ ,  $X_t = \sqrt{c}W_{t/c}$  is a Brownian motion.

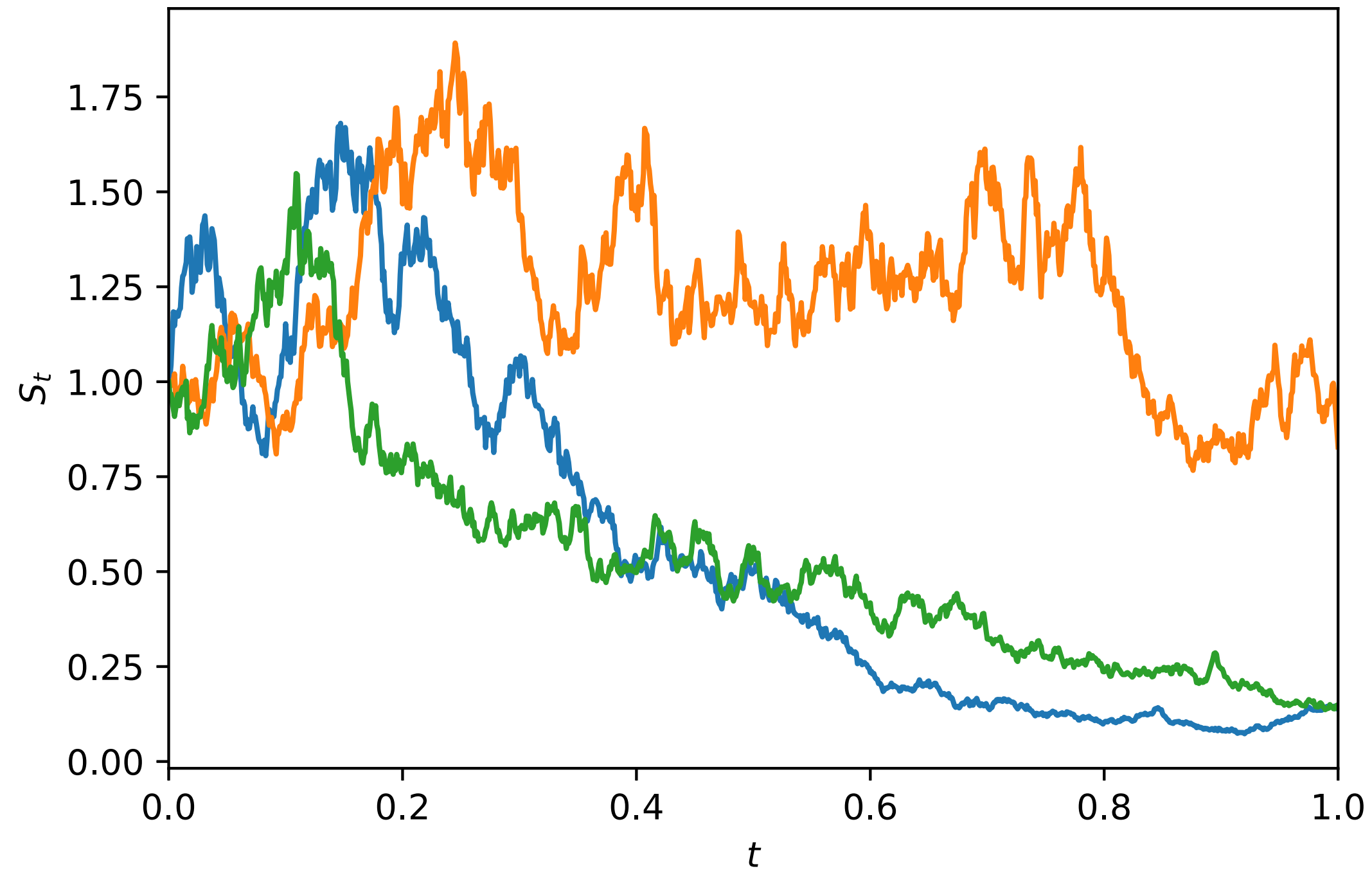


- Brownian motion itself is not a very useful model for stock prices, because it can become negative. Instead we model  $X_t \equiv \log S_t$  as a Brownian motion with drift:

$$\begin{aligned} X_t &= X_0 + \mu t + \sigma W_t, \text{ so that} \\ S_t &= \exp(X_t) \\ &= S_0 \exp(\mu t + \sigma W_t). \end{aligned}$$

- The resulting process for  $S_t$  is called *Geometric Brownian motion* (GBM).
- This implies that the log return  $\log S_t - \log S_s = X_t - X_s, s < t$ , is independent of  $X_s$ , with constant variance for fixed  $(t - s)$ .

Simulated Geometric Brownian Motion Paths





# Continuous Time Martingales

- In continuous time, a process  $\{X_t\}_{t \geq 0}$  is a *martingale* if
  - $\mathbb{E}[|X_t|] < \infty$ , for all  $t \geq 0$ ;
  - $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ , for all  $t > s \geq 0$ , where  $\mathcal{F}_t$  denotes the information on  $X_t$  up to time  $t$ .
- E.g., for Brownian motion
  - $\mathbb{E}[|W_t|] < \infty$  because  $W_t \sim N(0, t)$ ;
  - $\mathbb{E}[W_t | \mathcal{F}_s] = \mathbb{E}[W_s + (W_t - W_s) | \mathcal{F}_s] = W_s + 0$  because of independent increments.

- For Geometric Brownian motion,  $X_t = X_0 \exp(\mu t + \sigma W_t)$ , so that  $X_t = X_s \exp(\mu(t - s) + \sigma(W_t - W_s))$ . Thus

$$\begin{aligned}\mathbb{E}[X_t | \mathcal{F}_s] &= \mathbb{E} [X_s \exp(\mu(t - s) + \sigma(W_t - W_s)) | \mathcal{F}_s] \\ &= X_s \exp(\mu(t - s)) \mathbb{E} [\exp(\sigma(W_t - W_s))] \\ &= X_s \exp(\mu(t - s)) \exp\left(\frac{1}{2} \sigma^2 (t - s)\right).\end{aligned}$$

- The last line above follows because  $\mathbb{E}[\exp(z)] = \exp(\mu + \frac{1}{2} \sigma^2)$  if  $z \sim N(\mu, \sigma^2)$ . The distribution of  $\exp(z)$  is called the *lognormal*.
- Hence GBM is a martingale if and only if  $\mu = -\frac{1}{2} \sigma^2$ .

# Ito Processes

- Ito processes generalize Brownian motion with drift by allowing the drift and volatility to be time-varying and potentially stochastic.
- The trick is to describe the dynamics of a process with a *stochastic differential equation* (SDE), the continuous time equivalent of a stochastic difference equation.
- Take, for example, Brownian motion with drift,  $X_t = X_0 + \mu t + \sigma W_t$ .
- We know from calculus that

$$\int_{\tau}^t \mu ds = \mu \int_{\tau}^t ds = \mu(t - \tau).$$

- If we define  $\int_{\tau}^t dW_s = W_t - W_{\tau}$ , then we see that

$$X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma dW_s.$$

- This is often written in differential form as

$$dX_t = \mu dt + \sigma dW_t.$$

Note that this is just short hand notation for the integral form.

- An Ito process generalizes this by allowing  $\mu$  and  $\sigma$  to be time-varying and stochastic:

$$dX_t = \mu_t dt + \sigma_t dW_t. \quad (\dagger)$$

- Again, this is just short-hand for

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s,$$

where we define

$$\int_0^T \mu_s ds \equiv \lim_{n \rightarrow \infty} \sum_{i=0}^{N-1} \mu(t_i) \Delta t_{i+1}, \quad \int_0^T \sigma(t) dW_t \equiv \lim_{n \rightarrow \infty} \sum_{i=0}^{N-1} \sigma(t_i) \Delta W_{t_{i+1}},$$

$$t_i \equiv iT/N, \Delta t_{i+1} \equiv t_{i+1} - t_i, \text{ and } \Delta W_{t_{i+1}} \equiv [W_{t_{i+1}} - W_{t_i}].$$

- Remarks:
  - $X_t$  is the sum of two integrals. The first is called a *Riemann integral*, the second is an *Ito integral*.
  - **Do not** think of the integrals as an *area under the curve* like in high school. Your intuition for the Ito integral should be that we are summing infinitesimally small Brownian increments  $dW_t$ , each scaled by the instantaneous volatility  $\sigma_t$ .
  - If  $\mu_t$  and  $\sigma_t$  depend only on the *current*  $W_t$ , then  $(\dagger)$  is called a *stochastic differential equation*. Example:  $\mu(t, x) = \mu x$  and  $\sigma(t, x) = \sigma x$ , so that
 
$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$
  - The *solution* to an SDE is an equation that describes  $X_t$  in terms of just  $W_t$  (i.e.,  $X_t$  does not appear on the RHS). Often, Ito's lemma is helpful in finding it.

# Ito's Lemma

- Ito's lemma answers the question: if  $X_t$  is an Ito process with given dynamics, then what are the dynamics of a function  $f(t, X_t)$ ?
- It can be stated as follows: Let  $\{X_t\}_{t \geq 0}$  be an Ito process satisfying  $dX_t = \mu_t dt + \sigma_t dW_t$ , and consider a function  $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$  with continuous partial derivatives

$$\dot{f}(t, x) = \frac{\partial f(t, x)}{\partial t}, \quad f'(t, x) = \frac{\partial f(t, x)}{\partial x}, \quad f''(t, x) = \frac{\partial^2 f(t, x)}{\partial x^2}.$$

Then

$$df(t, X_t) = \dot{f}(t, X_t)dt + f'(t, X_t)dX_t + \frac{1}{2}f''(t, X_t)\sigma_t^2 dt.$$

- Example: Geometric Brownian Motion. Let

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad (\ddagger)$$

and  $X_t = f(S_t) = \log S_t$ . Then  $\dot{f}(S_t) = 0$ ,  $f'(S_t) = 1/S_t$ ,  $f''(S_t) = -1/S_t^2$ , and

$$\begin{aligned} dX_t &= df(S_t) = \dot{f}(S_t)dt + f'(S_t)dS_t + \frac{1}{2}f''(S_t)(S_t\sigma)^2 dt \\ &= \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(S_t\sigma)^2 dt \\ &= \frac{1}{S_t}(S_t\mu dt + S_t\sigma dW_t) - \frac{1}{2}\sigma^2 dt \\ &= \nu dt + \sigma dW_t, \quad \nu = \mu - \frac{1}{2}\sigma^2 \end{aligned}$$

- I.e.,  $(\ddagger)$  is the SDE for GBM:  $S_t = \exp(X_t) = S_0 \exp(\nu t + \sigma W_t)$ .

- Intuition (see Hull, 2012, Appendix to Ch. 13): In standard calculus, the total differential

$$df = \dot{f}(t, g(t))dt + f'(t, g(t))dg(t)$$

is the linear part of a Taylor expansion; the remaining terms are of smaller order as  $dt, dg(t) \rightarrow 0$ , so the total differential is a local linear approximation to  $f$ .

- If  $g(t) = X_t$ , an Ito process, take a 2nd order Taylor approximation:

$$\begin{aligned} \delta f \approx & \dot{f}(t, X_t)\delta t + f'(t, X_t)\delta X_t \\ & + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial t^2}(\delta t)^2 + 2\frac{\partial^2 f}{\partial t \partial X_t}(\delta t)(\delta X_t) + \frac{\partial^2 f}{\partial X_t^2}(\delta X_t)^2 \right]. \end{aligned}$$



- We have that  $\delta X_t = (X_{t+\delta t} - X_t) \approx \mu_t \delta t + \sigma_t \delta W_t \sim N(\mu_t \delta t, \sigma_t^2 \delta t)$ . Thus,  $\mathbb{E}[(\delta X_t)^2] \approx (\mu_t \delta t)^2 + \sigma_t^2 \delta t \approx \sigma_t^2 \delta t$ ; i.e., the 2nd order term is of the same order of magnitude as the 1st order term  $\delta t$ .
- It can be shown that as  $\delta t \rightarrow 0$ ,  $(\delta X_t)^2$  can be treated as non-stochastic:  $(dX_t)^2 = \sigma_t^2 dt$ . Together with  $(dt)^2 = 0$  and  $(dt)(dX_t) = 0$  this gives the result.

# Simulating Ito Processes

- Suppose we want to simulate sample paths of an Ito process described by the SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t.$$

- For pricing European claims, we only need draws for  $X_T$ , but for path-dependent options, we need the entire path  $\{X_t\}_{t \in [0, T]}$ .
- A simple way is to discretize the model, for a small time step  $\delta t$ , as

$$\delta X_t = X_{t+\delta t} - X_t \approx \mu(t, X_t)\delta t + \sigma(t, X_t)\delta W_t,$$

where  $\delta W_t \sim N(0, \delta t)$ . This is known as the *Euler scheme*.

- With  $\delta t = T/N$ , we can sample the path at  $N$  discrete times  $t_i = i\delta t$  as

$$X_{i+1} = X_i + \mu(t_i, X_i)\delta t + \sigma(t_i, X_i)\sqrt{\delta t}Z_i,$$

where the  $Z_i$  are independent standard normal random numbers and we use  $X_i$  and  $X_{i\delta t}$  interchangeably.

- In order to implement this, we need a way of drawing random samples from the normal distribution.
- Computers are deterministic machines. They cannot generate true random numbers.
- Instead, they construct sequences of pseudo-random numbers from a specified distribution that *look* random, in the sense that they pass certain statistical tests.
- E.g., NumPy's `np.random.randn(d0[, d1, ...])` constructs an array of standard normal pseudo random numbers.
- Random number generators use a *seed* value for initialization. Given the same seed, the same pseudo-random sequence will be returned.
- NumPy picks the seed automatically. To force it to use a specific seed, use `np.random.seed(n)`. Putting this line at the beginning of your Monte-Carlo program ensures that you get exactly the same results every time the program is run.

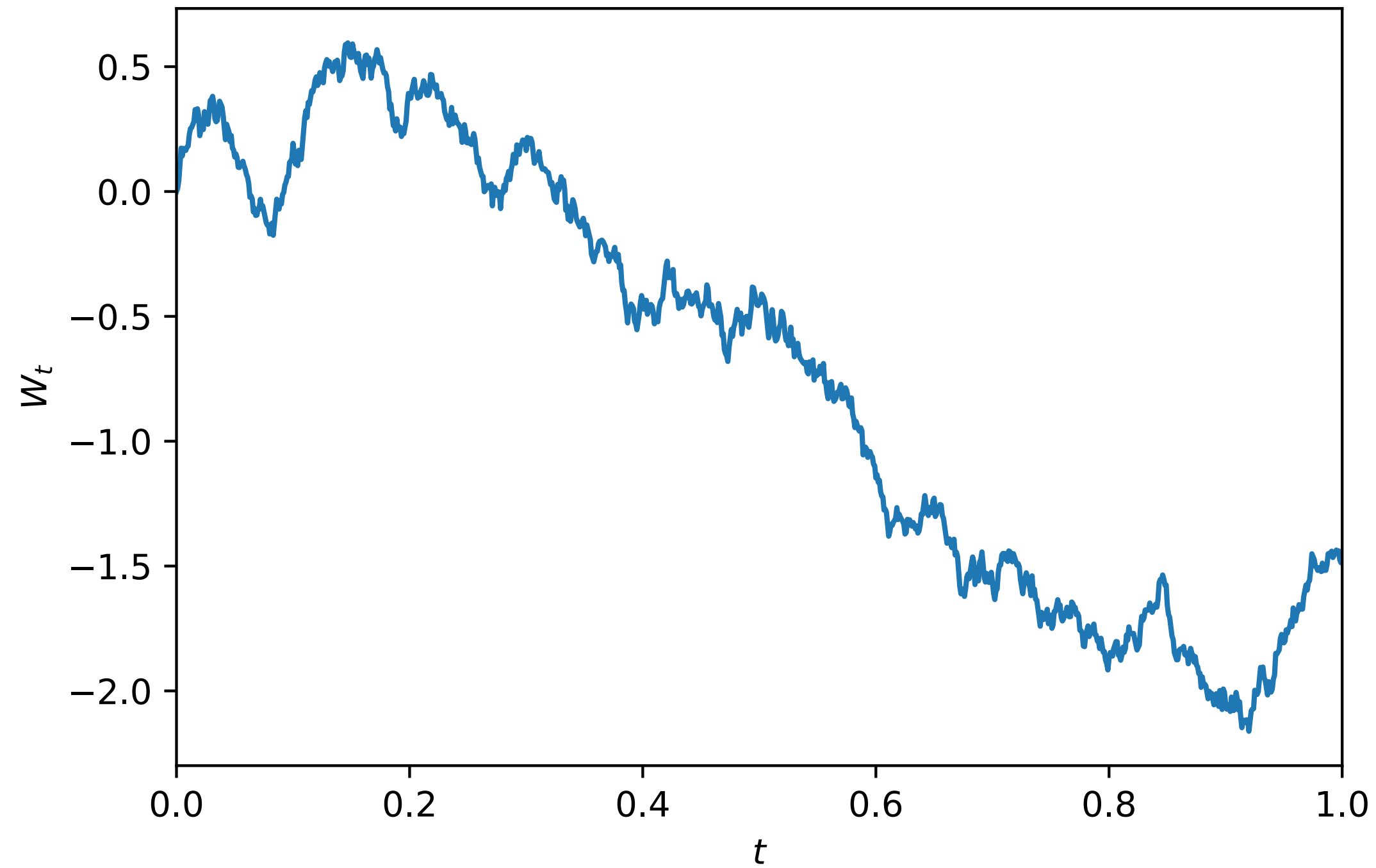
## Example 1: Simulating Brownian Motion

```
In [1]: import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
%matplotlib inline
```

```
In [2]: def bmsim(T, N, X0=0, mu=0, sigma=1):
        """Simulate a Brownian motion path.
        """
        deltaT = float(T)/N
        tvec = np.linspace(0, T, N+1)
        z = np.random.randn(N+1) #N+1 is one more than we need, actually. This way we won't have to grow dX.
        dX = mu*deltaT + sigma*np.sqrt(deltaT)*z #X[j+1]-X[j]=mu*deltaT + sigma*np.sqrt(deltaT)*z[j].
        dX[0] = 0.
        X = np.cumsum(dX)
        X += X0
        return tvec, X
```

```
In [3]: np.random.seed(0)
tvec, W = bmsim(1, 1000)
W = pd.Series(W, index=tvec)
W.plot()
plt.title('Simulated Brownian Motion Path')
plt.xlabel("$t$"); plt.ylabel("$W_t$");
plt.savefig("img/BMpath.svg"); plt.close()
```

Simulated Brownian Motion Path



## Example 2: Simulating GBM

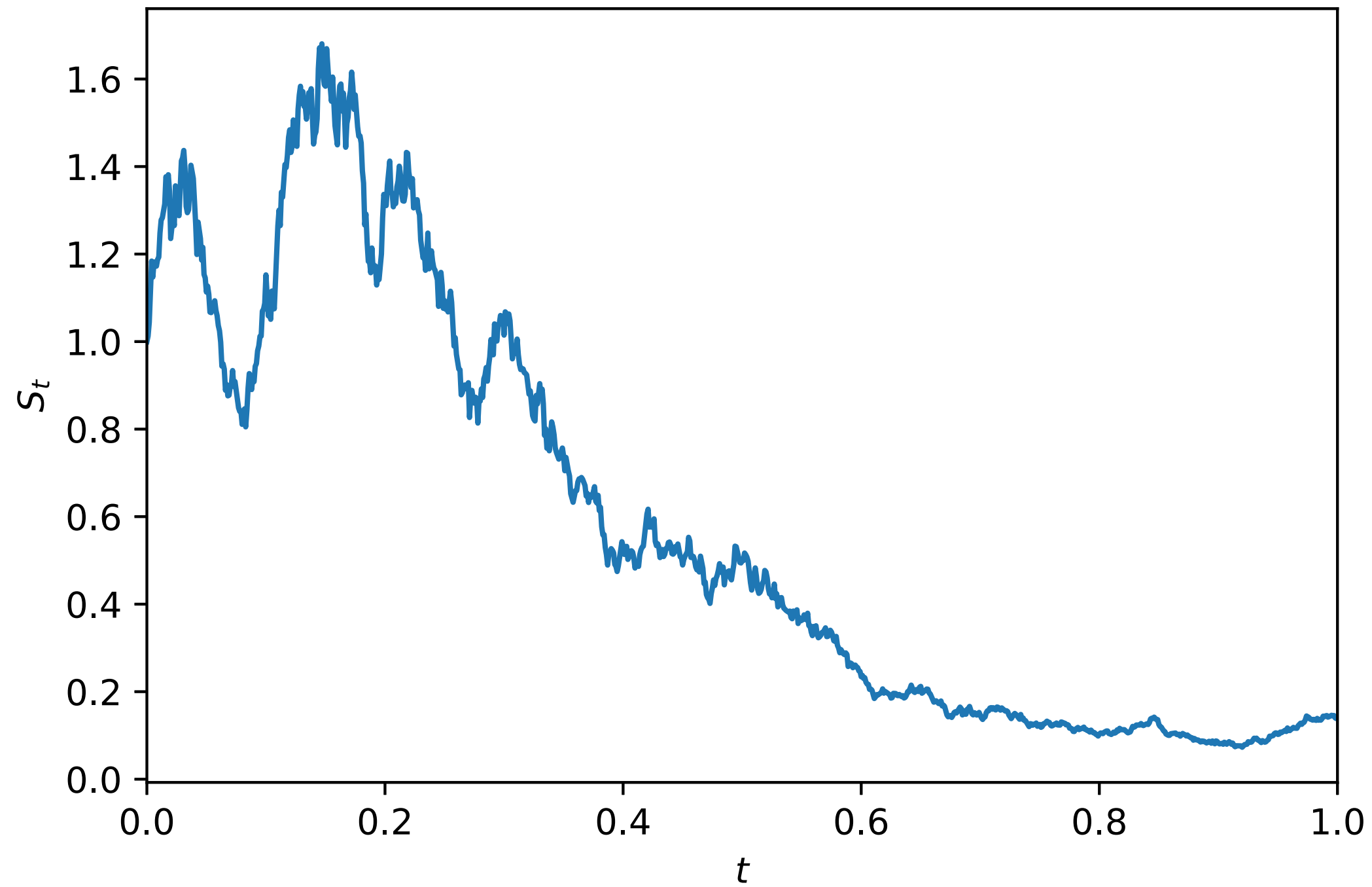
- The Euler scheme for the GBM  $dS_t = S_t\mu dt + S_t\sigma dW_t$  is

$$S_{i+1} = S_i + S_i\mu\delta t + S_i\sigma\sqrt{\delta t}Z_i.$$

```
In [4]: def gbmsim(T, N, S0=1, mu=0, sigma=1):
        """Simulate a Geometric Brownian motion path.
        """
        deltaT = float(T)/N
        tvec = np.linspace(0, T, N+1)
        z = np.random.randn(N+1) #Again one more than we need. This keeps it comparable to bmsim.
        S = np.zeros_like(z)
        S[0] = S0
        for j in xrange(0, N): #Note: we can no longer vectorize this, because S[:, j] is needed for S[:, j+1].
            S[j+1] = S[j] + mu*S[j]*deltaT + sigma*S[j]*np.sqrt(deltaT)*z[j+1]
        return tvec, S
```

```
In [5]: np.random.seed(0)
        tvec, S = gbmsim(1, 1000)
        S = pd.Series(S, index=tvec)
        S.plot()
        plt.title('Simulated Geometric Brownian Motion Path')
        plt.xlabel("$t$"); plt.ylabel("$S_t$")
        plt.savefig("img/GBMpath.svg"); plt.close()
```

Simulated Geometric Brownian Motion Path



- In the case of BM, the Euler scheme correctly reproduces the distribution of the  $W_{t_i}$ .
- This is not true in general: in Example 2 above, the Euler approximation

$$S_{i+1} = S_i + S_i\mu\delta t + S_i\sigma\sqrt{\delta t}Z_i$$

implies that the distribution of  $S_{t+\delta t} - S_t$  is normal, not log-normal as it should be.

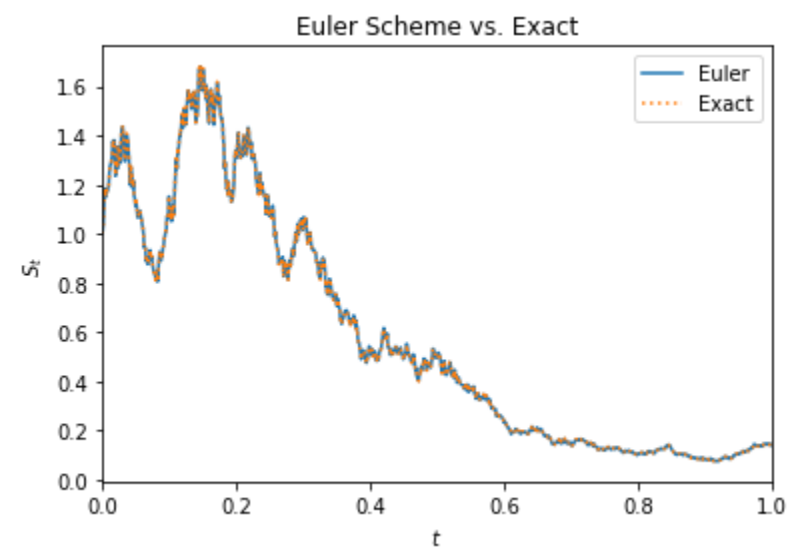
- Under mild conditions, the error introduced by discretization will disappear as  $\delta t \rightarrow 0$ .
- In the case of GBM, this error can be avoided altogether: let  $X_t \equiv \log S_t$ . By Ito's lemma,

$$dX_t = \nu dt + \sigma dW_t, \quad \nu = \mu - \frac{1}{2}\sigma^2,$$

so we can simulate  $X_t$  instead and then take the exponential.



```
In [6]: N=1000 #Try changing N to 100, then 10!
np.random.seed(0)
tvec, S1 = gbmsim(1, N)
np.random.seed(0) #Use the same seed, otherwise we'd get different paths.
tvec, X = bmsim(1, N, 0, -.5)
S2 = np.exp(X)
S1 = pd.Series(S1, index=tvec)
S2 = pd.Series(S2, index=tvec)
S1.plot()
S2.plot(linestyle=":")
plt.title("Euler Scheme vs. Exact")
plt.xlabel("$t$")
plt.ylabel("$S_t$")
plt.legend(["Euler", "Exact"]);
```



# The Black-Scholes Model

- Black and Scholes assumed the following model:
  - The stock  $\{S_t\}_{t \in [0, T]}$  follows GBM:
$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$
  - The stock pays no dividends.
  - Cash bond price  $B_t = e^{rt} \iff dB_t = rB_t dt$ ; riskless lending and borrowing at the same rate  $r$ .
  - European style derivative with price  $C_t$  and payoff  $C_T = (S_T)$ .
  - Trading may occur continuously, with no transaction costs.
  - No arbitrage opportunities.
- The problem is to find the price  $C_t, t \in [0, T]$ .

- It can be shown that the FTAP holds in continuous time as well: if the market is arbitrage free, then there exists a risk neutral measure  $\mathbb{Q}$  under which all assets earn the risk free rate (on average), and the price of a claim is the discounted expected payoff under  $\mathbb{Q}$ . If the market is complete, then  $\mathbb{Q}$  is unique. This gives us a pricing formula for general European claims:

$$C_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} [C_T | \mathcal{F}_t] .$$

- This implies that if we can simulate the stock price under the measure  $\mathbb{Q}$ , then we can price the claim by Monte Carlo simulation.

- In the BS model, it can be shown that under the risk-neutral measure  $\mathbb{Q}$ ,

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}},$$

where  $W_t^{\mathbb{Q}}$  is a  $\mathbb{Q}$ -Brownian Motion.

- Note that by Ito's Formula, the discounted stock price  $\tilde{S}_t \equiv e^{-rt} S_t =: f(t, S_t)$  satisfies

$$\begin{aligned} d\tilde{S}_t &= \dot{f}(t, S_t)dt + f'(t, S_t)dS_t + \frac{1}{2}f''(t, S_t)\sigma^2 S_t^2 dt \\ &= -re^{-rt} S_t dt + e^{-rt} dS_t + 0 \\ &= -r\tilde{S}_t dt + e^{-rt} (rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}) \\ &= \sigma \tilde{S}_t dW_t^{\mathbb{Q}}. \end{aligned}$$

- I.e.,  $\tilde{S}_t$  is an Ito process without drift, and thus a martingale. This is the reason that  $\mathbb{Q}$  is also called the equivalent martingale measure.

- We can extend the BS model by assuming that the underlying pays a continuous dividend at rate  $\delta$  (realistic only for indices, not individual stocks). Then a position of 1 share generates an instantaneous dividend stream  $\delta S_t dt$ , in addition to the capital gains  $dS_t$ .

- Note that only the holder of the underlying receives the dividend.

- The pricing formula remains the same, but now the risk-neutral dynamics of  $S_t$  are

$$dS_t = (r - \delta)S_t dt + \sigma S_t dW_t^{\mathbb{Q}}.$$

- The expected growth rate of the underlying under  $\mathbb{Q}$  is  $r - \delta$ , so the expected return from holding it (capital gains plus dividend yield) is  $r$ .

- The price of a call is now

$$C_t = e^{-\delta(T-t)} S_t \Phi(d_1) - e^{-r(T-t)} K \Phi(d_2),$$

where

$$d_{1,2} = \frac{\log(S_t/K) + [(r - \delta) \pm \frac{1}{2}\sigma^2](T - t)}{\sigma\sqrt{T - t}}.$$

# Monte Carlo Pricing

- The goal in Monte Carlo simulation is to obtain an estimate of

$$\theta \equiv \mathbb{E}[X],$$

for some random variable  $X$  with finite expectation. The assumption is that we have a means of sampling from the distribution of  $X$ , but no closed-form expression for  $\theta$ .

- Suppose we have a sample  $\{X_i\}_{i \in \{1, \dots, n\}}$  of *independent* draws for  $X$ , and let

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

- The sample average  $\bar{X}_n$  is an *unbiased estimator* of  $\theta$ :  $\mathbb{E}[\bar{X}_n] = \theta$ .
- The *weak law of large numbers* states that

$$\bar{X}_n \xrightarrow{p} \theta,$$

where the arrow denotes *convergence in probability*; i.e., as the sample size grows, the  $\bar{X}_n$  becomes a better and better estimate of  $\theta$ .

- Thus, our strategy is to use a computer to draw  $n$  (pseudo) random numbers  $X_i$  from the distribution of  $X$ , and then estimate  $\theta$  as the sample mean of the  $X_i$ .
- $n$  is called the number of *replications*.
- For finite  $n$ , the sample average will be an approximation to  $\theta$ .
- It is usually desirable to have an estimate of the accuracy of this approximation. Such an estimate can be obtained from the *central limit theorem* (CLT), which states that

$$\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \sigma^2),$$

provided that  $\sigma^2$ , the variance of  $X$ , is finite. The arrow denotes convergence in distribution; this implies that for large  $n$ ,  $\bar{X}_n$  has approximately a normal distribution.

- Of course  $\sigma^2$  is unknown, but we can estimate it as

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (\bar{X}_n - X_i)^2.$$

- A 95% confidence interval (CI) is an interval  $[c_l, c_u]$  such that

$$\mathbb{P}[c_l \leq \theta \leq c_u] = 0.95.$$

- The CLT implies that, in the limit as  $n \rightarrow \infty$ ,

$$\mathbb{P}[-1.96\sigma \leq \sqrt{n}(\bar{X}_n - \theta) \leq 1.96\sigma] = 0.95 \Leftrightarrow$$

$$\mathbb{P}[\bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}} \leq \theta \leq \bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}] = 0.95.$$

Hence  $c_l = \bar{X}_n - 1.96\frac{\sigma}{\sqrt{n}}$  and  $c_u = \bar{X}_n + 1.96\frac{\sigma}{\sqrt{n}}$  is an asymptotically valid CI.

- Note that  $c_l$  and  $c_u$  are random variables; we should interpret this as "before the experiment is performed, there is a 95% chance that a CI computed according to this formula will contain  $\theta$ ". After performing the experiment, this statement is not valid anymore; the interval is now fixed, and contains  $\theta$  with probability either 0 or 1.
- The unknown parameter  $\sigma$  can be consistently estimated by  $\sqrt{\hat{\sigma}^2}$ .



# Application: Asian Options

- The payoff of Asian options depends on the *average* price of the underlying,  $\bar{S}_T$ . Types:
  - Average price Asian call with payoff  $(\bar{S}_T - K)^+$ ;
  - Average price Asian put with payoff  $(K - \bar{S}_T)^+$ ;
  - Average strike Asian call with payoff  $(S_T - \bar{S}_T)^+$ ;
  - Average strike Asian put with payoff  $(\bar{S}_T - S_T)^+$ .
- It is important to specify how the average is computed: the continuous, arithmetic, and geometric averages are, respectively,

$$\frac{1}{T} \int_0^T S_t dt, \quad \frac{1}{N} \sum_{i=1}^N S_{t_i} \quad \text{and} \quad \left( \prod_{i=1}^N S_{t_i} \right)^{1/N},$$

where the  $t_i$  are a set of  $N$  specified dates.

- Exact Black-Scholes type pricing formulas for Asian options only exist in special cases (e.g., the geometric average Asian call, see next week), so we rely on Monte Carlo simulation.

- Our pricing formula

$$C_t = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} [C_T | \mathcal{F}_t]$$

is exactly in the required form.

- As an example, consider pricing an arithmetic average price call with payoff

$$C_T = (\bar{S}_T - K)^+, \quad \text{where} \quad \bar{S}_T = \frac{1}{N} \sum_{i=1}^N S_{t_i},$$

which cannot be priced analytically.

- The payoff is path-dependent, so we need to simulate the entire asset price path, not just  $S_T$ .

```
In [7]: from scipy.stats import norm
def asianmc(S0, K, T, r, sigma, delta, N, numsim=10000):
    """Monte Carlo price of an arithmetic average Asian call.
    """
    X0 = np.log(S0)
    nu = r - delta - .5 * sigma ** 2
    payoffs = np.zeros(numsim)
    for j in xrange(numsim):
        _, X = bmsim(T, N, X0, nu, sigma) #Convention: underscore holds value to be discarded.
        S = np.exp(X)
        payoffs[j] = max(S[1:].mean() - K, 0.)
    g = np.exp(-r * T) * payoffs
    C = g.mean(); s = g.std()
    zq = norm.ppf(0.975)
    Cl = C - zq / np.sqrt(numsim) * s
    Cu = C + zq / np.sqrt(numsim) * s
    return C, Cl, Cu
```

```
In [8]: S0 = 11; K = 10; T = 3/12.; r = 0.02; sigma = .3; delta = 0.01; N = 10
np.random.seed(0)
C0, Cl, Cu = asianmc(S0, K, T, r, sigma, delta, N); C0, Cl, Cu
```

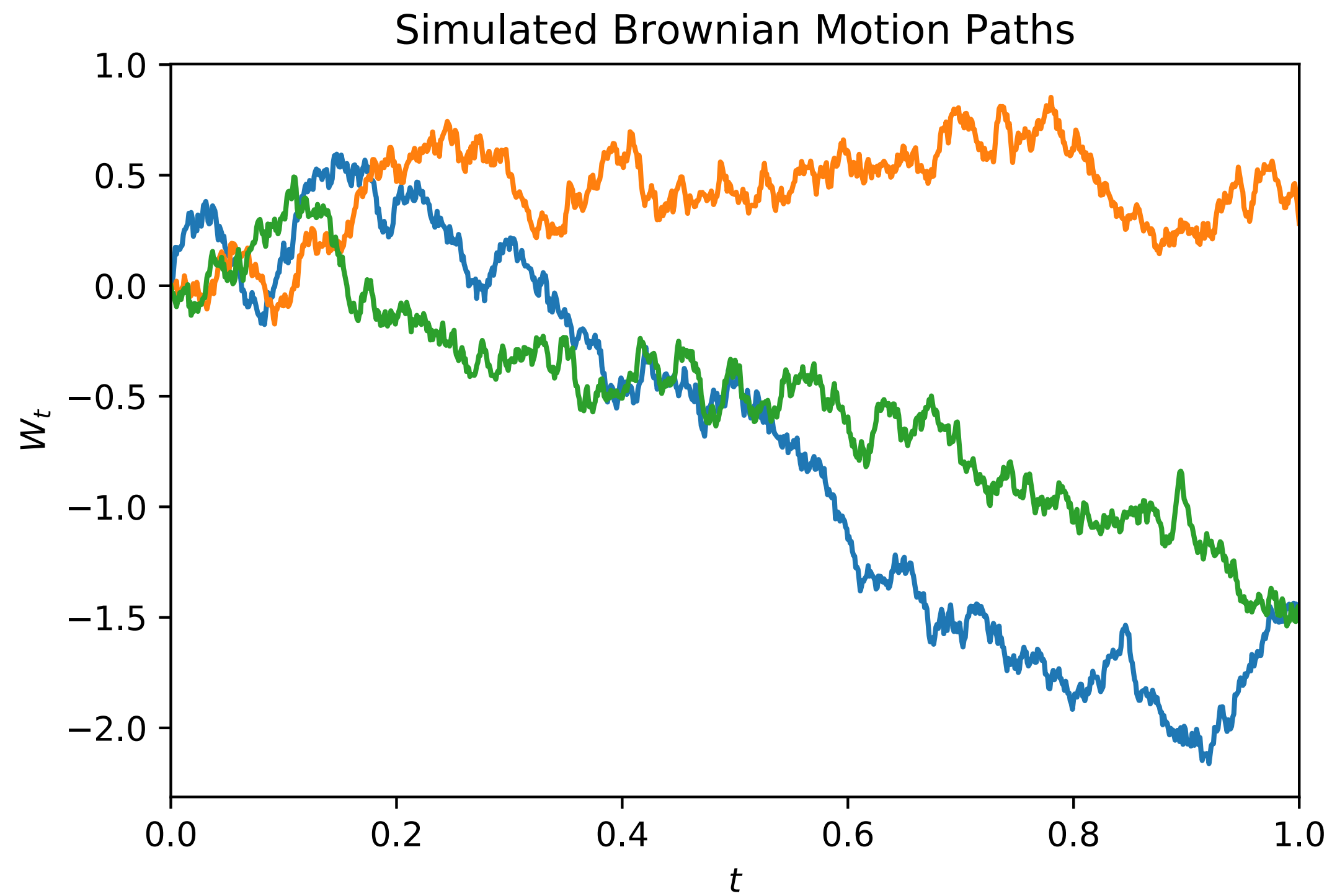
```
Out[8]: (1.0927262054551385, 1.0747653929130998, 1.1106870179971773)
```

# Code Optimization

- Our code for pricing the Asian option is likely inefficient, because it contains a loop.
- The code can be 'vectorized' to speed it up.
- First step: simulate a bunch of Brownian paths in one shot.
- The resulting code is actually almost identical:

```
In [9]: def bmsim_vec(T, N, X0=0, mu=0, sigma=1, numsim=1): #Note new input: numsim, the number of paths.
        """Simulate `numsim` Brownian motion paths.
        """
        deltaT = float(T)/N
        tvec = np.linspace(0, T, N+1)
        z = np.random.randn(numsim, N+1)  #(N+1)->(numsim, N+1)
        dX = mu*deltaT + sigma*np.sqrt(deltaT)*z
        dX[:, 0] = 0. #dX[0]->dX[:, 0]
        X = np.cumsum(dX, axis=1) #cumsum(dX)->cumsum(dX, axis=1)
        X += X0
        return tvec, X
```

```
In [10]: np.random.seed(0)
        tvec, W = bmsim_vec(1, 1000, numsim=3)
        W = pd.DataFrame(W.transpose(), index=tvec)
        W.plot().legend().remove()
        plt.title('Simulated Brownian Motion Paths')
        plt.xlabel("$t$"); plt.ylabel("$W_t$");
        plt.savefig("img/BMpaths.svg"); plt.close()
```



- Here is the vectorized code for the Asian option:

```
In [11]: def asianmc_vec(S0, K, T, r, sigma, delta, N, numsim=10000):  
        """Monte Carlo price of an arithmetic average Asian call.  
        """  
        X0 = np.log(S0)  
        nu = r - delta - .5 * sigma ** 2  
        #simulate all paths at once:  
        _, X = bmsim_vec(T, N, X0, nu, sigma, numsim)  
        S = np.exp(X)  
        payoffs = np.maximum(S[:, 1:].mean(axis=1) - K, 0.) #S[1:]->S[:, 1:], max->maximum, mean()->mean(axis=1)  
        g = np.exp(-r * T) * payoffs  
        C = g.mean(); s = g.std()  
        zq = norm.ppf(0.975)  
        Cl = C - zq / np.sqrt(numsim) * s  
        Cu = C + zq / np.sqrt(numsim) * s  
        return C, Cl, Cu
```

- Let's see if it works:

```
In [12]: np.random.seed(0)
         C0_vec, _, _ = asianmc_vec(S0, K, T, r, sigma, delta, N)
         np.allclose(C0_vec, C0)
```

Out[12]: True

- And time it:

```
In [13]: %timeit asianmc(S0, K, T, r, sigma, delta, N)
```

1 loop, best of 3: 371 ms per loop

```
In [14]: %timeit asianmc_vec(S0, K, T, r, sigma, delta, N)
```

100 loops, best of 3: 6.05 ms per loop



- Our code for the Euler scheme can likewise be adjusted to compute many paths in one shot.
- We're still stuck with the loop over  $t$  though, which cannot be vectorized because  $S_{i+1}$  depends on  $S_i$ .
- We'll use Numba's JIT compiler to speed it up further.

```

In [15]: from numba import jit
         @jit(nopython=True)
         def gbmsim_vec(T, N, S0=1, mu=0, sigma=1, numsim=1, seed=0):
             """Simulate `numsim` Geometric Brownian motion paths.
             """
             deltaT = float(T)/N
             tvec = np.linspace(0, T, N+1)
             np.random.seed(seed) #Note: with jit-compiled functions, the RNG must be seeded INSIDE the compiled code.
             z = np.random.randn(numsim, N+1)
             S = np.zeros_like(z)
             S[:, 0] = S0
             for j in xrange(0, N):
                 S[:, j+1]=S[:, j] + mu*S[:, j]*deltaT + sigma*S[:, j]*np.sqrt(deltaT)*z[:, j+1]
             return tvec, S

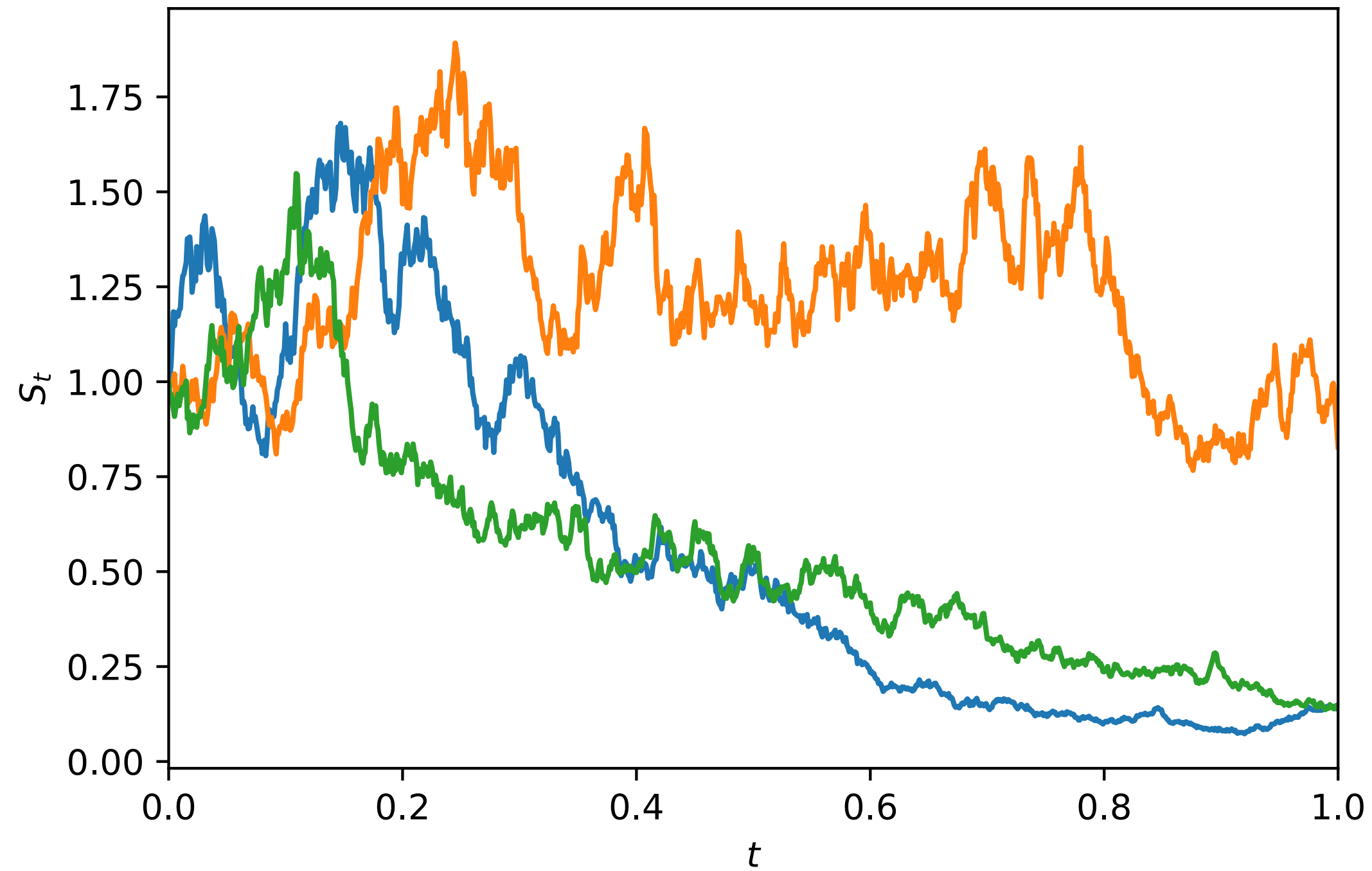
```

```

In [16]: tvec, S = gbmsim_vec(1, 1000, numsim=3, seed=0)
         S = pd.DataFrame(S.transpose(), index=tvec)
         S.plot().legend().remove()
         plt.title('Simulated Geometric Brownian Motion Paths')
         plt.xlabel("$t$"); plt.ylabel("$S_t$")
         plt.savefig("img/GBMpaths.svg"); plt.close()

```

Simulated Geometric Brownian Motion Paths



- The compiled code produces the same results:

```
In [17]: np.random.seed(0)
_, S1 = gbmsim(1, 1000)
_, S2 = gbmsim_vec(1, 1000, seed=0)
np.allclose(S1, S2)
```

Out[17]: True

- But it is quite a bit faster:

```
In [18]: %%timeit #Cell magic (for timing the entire cell).
for k in xrange(10): #10 paths.
    gbmsim(1, 1000)
```

10 loops, best of 3: 22.9 ms per loop

```
In [19]: %%timeit gbmsim_vec(1, 1000, numsim=10)
```

The slowest run took 545.83 times longer than the fastest. This could mean that an intermediate result is being cached.

1 loop, best of 3: 570  $\mu$ s per loop