Problems

Suppose that $X(t)$ is a Poisson process such that $E[X(9)] = 6$.

- Find the mean and the variance of $X(8)$;
- Find $P[X(2) \leq 3]$;
- Find $P[X(4) \leq 5 | X(2) \leq 3]$.

Solution

$$E[X(t)] = \lambda t$$

$$E[X(9)] = 6 \Rightarrow \lambda = 2/3$$

- $E[X(8)] = 8\lambda = 16/3$
- $X(2)$ is Poisson distributed with parameter $2\lambda = 4/3$. Therefore,

$$P[X(2) \leq 3] = \exp(-2\lambda) \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}$$

- The random variables $Z = X(2)$ and $W = X(4) - X(2)$ are independent and Poisson distributed with parameter 2λ. Hence,

$$P[Z = k] = \exp(-2\lambda) \frac{(2\lambda)^k}{k!}$$

$$\begin{aligned}
P[Z=k, W=m] &= P[Z=k] \cdot P[W=m|Z=k] \\
&= P[Z=k] \cdot P[W=m] \\
&= \exp(-4\lambda)\frac{(2\lambda)^k}{k!}\frac{(2\lambda)^m}{m!}
\end{aligned}$$

Now, we have

$$P[X(4) \leq 5 | X(2) \leq 3] = \frac{P[Z \leq 5, W \leq 5-z]}{P[Z \leq 3]}$$

where

$$\begin{aligned}
P[Z \leq 3] &= \sum_{k=0}^{3} P[Z=k] \\
P[Z \leq 5, W \leq 5-z] &= \sum_{k=0}^{3} \sum_{m=0}^{5-k} P[Z=k, W=m]
\end{aligned}$$