Homework 2

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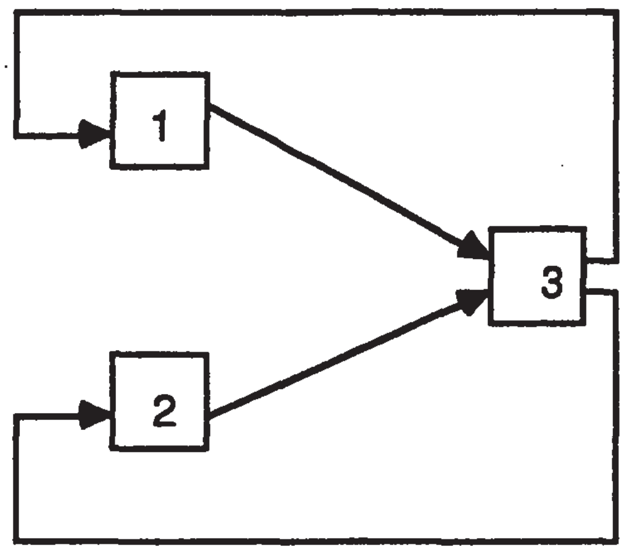
Problem 2.1 Customers arrive at a fast-food restaurant at a rate of five per minute and wait to receive their order for an average of 5 minutes. Customers eat in the restaurant with probability 0.5 and carry out their order without eating with probability 0.5. A meal requires an average of 20 minutes. What is the average number of customers in the restaurant?

Solution: A customer that carries out the order (eats in the restaurant) stays for 5 mins (25 mins). Therefore the average customer time in the system is $T = 0.5 \times 5 + 0.5 \times 25 = 15$. By Little's Theorem the average number in the system is $N = \lambda T = 5 \times 15 = 75$.

Problem 2.2 Two communication nodes 1 and 2 send files to another node 3. Files from 1 and 2 require on the average R_1 and R_2 time units for transmission, respectively. Node 3 processes a file of node $i (i = 1, 2)$ in an average of P_i time units and then requests another file from either node 1 or node 2 (the rule of choice is left unspecified). If λ_i is the throughput of node i in files sent per unit time, what is the region of all feasible throughput pairs (λ_i, λ_2) for this system?



Solution: We represent the system as shown in the figure. The number of files in the entire system is exactly one at all times. The average number in node i is $\lambda_i R_i$ and the average number in node 3 is $\lambda_1 P_1 + \lambda_2 P_2$. Therefore the throughput pairs (λ_1, λ_2) must satisfy (in addition to nonnegativity) the constraint

$$\lambda_1(R_1 + P_1) + \lambda_2(R_2 + P_2) = 1.$$

Problem 2.3 The average time T a car spends in a certain traffic system is related to the average number of cars N in the system by a relation of the form

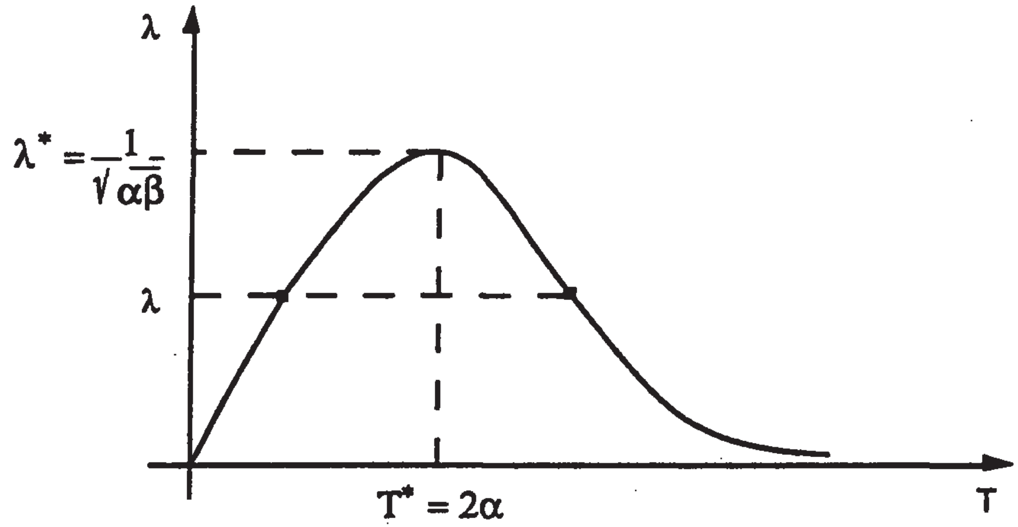
$$T = \alpha + \beta N^2,$$

where $\alpha > 0$, $\beta > 0$ are given scalars. What is the maximal car arrival rate λ that the system can sustain?

Solution: If λ is the throughput of the system, Little's theorem gives $N = \lambda T$, so from the relation $T = \alpha + \beta N^2$ we obtain $T = \alpha + \beta\lambda^2 T^2$ or

$$\lambda = \sqrt{\frac{T - \alpha}{\beta T^2}}$$

This relation between λ and T is plotted below.



The maximum value of λ is attained for the value T^* for which the derivative of $\frac{T-\alpha}{\beta T^2}$ is zero.

$$\frac{1}{\beta T^2} - \frac{2(T-\alpha)}{\beta T^3} = 0.$$

This yields $T^* = 2\alpha$. We obtain the corresponding maximal throughput value $\lambda^* = \frac{1}{2\sqrt{\alpha\beta}}$.

Problem 2.4 An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean 30 minutes. The first student arrives on time, but the second student arrives 5 minutes late. What is the expected time between the arrival of the first student and the departure of the second student?

Solution: The expected time in question equals

$$\begin{aligned} E[\text{Time}] &= (5 + E[\text{stay of 2nd student}]) \times P(\text{1st stays less or equal to 5 minutes}) \\ &\quad +(E[\text{stay of 1st}|\text{stay of 1st} \geq 5] + E[\text{stay of 2nd}]) \\ &\quad \times P(\text{1st stays more than 5 minutes}). \end{aligned}$$

We have $E[\text{stay of 2nd student}] = 30$, and, using the memoryless property of the exponential distribution,

$$E[\text{stay of 1st}|\text{stay of lst} \geq 5] = 5 + E[\text{stay of 1st}] = 35.$$

Also

$$P(\text{1st student stays less or equal to 5 minutes}) = 1 - e^{-5/30}$$

$$P(\text{1st student stays more than 5 minutes}) = e^{-5/30}.$$

By substitution we obtain

$$E[\text{Time}] = (5+30) \times (1 - e^{-5/30}) + (35+30) \times e^{-5/30} = 35 + 30 \times e^{-5/30} = 60.394.$$