# Introductory Course on Non-smooth Optimisation

## Lecture 02 - Proximal gradient descent

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**Unconstrained non-smooth optimisation**

Consider minimising

$$\min_{x \in \mathbb{R}^n} R(x),$$

where $R : \mathbb{R}^n \to ]-\infty, +\infty]$ is proper convex and lower semi-continuous.

- $\Gamma_0$: the class of proper convex and lower semi-continuous functions on $\mathbb{R}^n$.
- The set of minimisers, *i.e.*

$$\mathrm{Argmin}(R) = \{x \in \mathbb{R}^n : R(x) = \min_{x \in \mathbb{R}^n} R(x)\}$$

  is non-empty.
- $R(x)$ is non-differentiable...

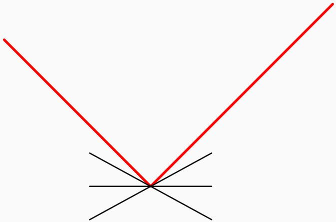
## Subdifferential

Let $R \in \Gamma_0$, the subdifferential of $R$ at $x \in \mathrm{dom}(R)$ is defined by

$$\partial R : \mathbb{R}^n \rightrightarrows \mathbb{R}^n,\ x \mapsto \left\{ g \in \mathbb{R}^n \,|\, R(y) \geq R(x) + \langle g,\ y - x \rangle,\ \ \forall y \in \mathbb{R}^n \right\}.$$

**Example** absolute value function

$$\partial |x| = \begin{cases} +1 : x > 0 \\ [-1, 1] : x = 0 \\ -1 : x < 0 \end{cases}$$



## Subdifferential

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## Convexity

Let $R \in \Gamma_0$ and $x \in \mathrm{dom}(R)$, then

- $\partial R(x) = \big\{ g \in \mathbb{R}^n : R(y) \geq R(x) + \langle g,\ y - x \rangle \big\}$.
- $\partial R(x)$ is closed and convex.

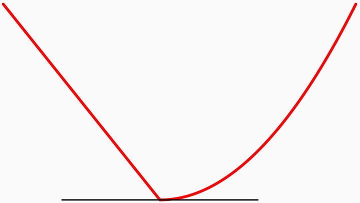
## Monotonicity

Let $R \in \Gamma_0$, then $\forall x, y \in \mathrm{dom}(R)$,

$$\langle u - v,\ x - y \rangle \geq 0,\ \ \forall u \in \partial R(x),\ v \in \partial R(y).$$

Given $x^\star \in \mathbb{R}^n$, it minimises $R(x)$ if and only if

$$0 \in \partial R(x^\star).$$



$$R(y) \geq R(x^\star) + \langle g,\, y - x \rangle \text{ holds for all } y \in \mathrm{dom}(R) \iff 0 \in \partial R(x^\star).$$

### Subgradient descent

**initial** : $x_0 \in \mathrm{dom}(R)$;

**repeat** :

1. Choose step-size $\gamma_k > 0$ and a subgradient $g_k \in \partial R(x_k)$
2. Update $x_{k+1} = x_k - \gamma_k g_k$

**until** : stopping criterion is satisfied.

**Subgradient descent**

**initial** : $x_0 \in \mathrm{dom}(R)$;

**repeat** :

1. Choose step-size $\gamma_k > 0$ and a subgradient $g_k \in \partial R(x_k)$
2. Update $x_{k+1} = x_k - \gamma_k g_k$

**until** : stopping criterion is satisfied.

Step-size rule:

- Fixed step-size: $\gamma_k$ is constant.
- Fixed length: $\gamma_k \|g_k\| = \|x_{k+1} - x_k\|$ is a constant.
- Diminishing step-size: $\gamma_k \to 0, \ \sum_i \gamma_i = +\infty$.

Assumptions:

- $R$ has minimiser $x^\star$ and finite optimal value $R(x^\star)$.
- $R$ is convex, $\mathrm{dom}(R) = \mathbb{R}^n$.
- $R$ is Lipschitz consinuout with constant $L$:

$$|R(x) - R(y)| \leq L\|x - y\|, \ \forall x, y \in \mathrm{dom}(R).$$

**NB**: the Lipschitz continuity implies $\|g\| \leq L$ for all $x \in \mathrm{dom}(R)$.

Subgradient descent is **NOT** a descent method.

**Fixed step-size $\gamma_k \equiv \gamma$**

$$R_{k,best} - R(x^\star) \le \frac{\|x_0 - x^\star\|^2}{2k\gamma} + \frac{\gamma L^2}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$.
- For large $k$, $R_{k,best}$ is approximately $\frac{\gamma L^2}{2}$ suboptimal.

Subgradient descent is **NOT** a descent method.

**Fixed step-length $\gamma_k = c/\|g_k\|$**

$$R_{k,best} - R(x^\star) \le \frac{\|x_0 - x^\star\|^2}{2kc} + \frac{cL}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$.
- For large $k$, $R_{k,best}$ is approximately $\frac{cL}{2}$ suboptimal.

Subgradient descent is **NOT** a descent method.

**Diminishing step-size** $\gamma_k \to 0,\ \sum_i \gamma_i = +\infty$:

$$R_{k,best} - R(x^\star) \leq \frac{\|x_0 - x^\star\|^2 + L^2 \sum_{i=1}^k \gamma_i^2}{\sum_{i=1}^k \gamma_i}.$$

- If $\sum_{i=1}^k \gamma_i^2 / \sum_{i=1}^k \gamma_i \to 0$, then $R_{k,best} \to R(x^\star)$.
- Choice of $\gamma_k$: $\gamma_k = c/k^q,\ q \in ]1/2, 1[$.

# Optimal step-size

**For fixed number of iterations** if $c_i = \gamma_i \|g_i\|$ and $\|x_0 - x^\star\| \leq D$,

$$R_{k,best} - R(x^\star) \leq \frac{D^2 + L^2 \sum_{i=1}^k c_i^2}{2 \sum_{i=1}^k \gamma_i / L}.$$

- For given $k$, rhs is minimised by $c_i = c = D/\sqrt{k}$.
- Hence the rate
$$R_{k,best} - R(x^\star) \leq \frac{LD}{\sqrt{k}}.$$
- Iteration complexity: reach $R_{k,best} - R(x^\star) < \epsilon$ in $O(1/\epsilon^2)$ steps.

**When $R(x^\star)$ is available** step-size

$$\gamma_k = \frac{R(x_k) - R(x^\star)}{\|g_k\|^2}.$$

Convergence rate

$$R_{k,best} - R(x^\star) \leq \frac{LD}{\sqrt{k}}.$$

**NB**: $O(1/\sqrt{k})$ is the best rate can be obtained by subgradient method.

# Remarks

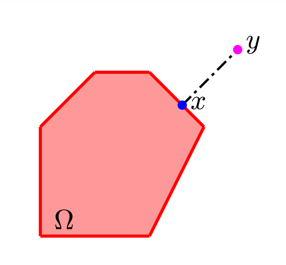
- Handles non-smooth problem
- Simple iterative scheme
- Slow convergence rate
- No clear stopping criterion

**NB**: need a better approach to handle non-smoothness...

# Outline

**Indicator function** : let $\Omega \subseteq \mathbb{R}^n$

$$\iota_\Omega(x) = \begin{cases} 0 : x \in \Omega, \\ +\infty : x \notin \Omega. \end{cases}$$




Projection of $y$ onto $\Omega$:

$$\min_{x \in \Omega} \|x - y\|.$$

**Projection**

Projection mapping onto a set is defined by

$$\mathcal{P}_\Omega(y) \stackrel{\text{def}}{=} \mathrm{argmin}_{x \in \Omega} \|x - y\|.$$

## Constrained smooth optimisation

Let $F \in C_L^1$ and $\Omega \subseteq \mathbb{R}^n$ be a closed and convex set

$$\min_{x \in \Omega} F(x).$$

## Projected gradient descent

**initial** : $x_0 \in \Omega$;

**repeat** :

1. Choose step-size $\gamma_k \in ]0, 2/L[$
2. Gradient descent $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$
3. Projection $x_{k+1} = \mathcal{P}_\Omega(x_{k+1/2})$

**until** : stopping criterion is satisfied.

As $\iota_\Omega \in \Gamma_0$, the constrained optimisation problem is equivalent to

$$\min_{x\in\mathbb{R}^n} \iota_\Omega(x) + F(x).$$

**Composite optimisation**

Consider the following optimisation problem

$$\min_{x\in\mathbb{R}^n} \big\{\Phi(x) \stackrel{\text{def}}{=} R(x) + F(x)\big\}.$$

**Assumtions**

- $F \in C_L^1$
- $R \in \Gamma_0$
- $\mathrm{Argmin}(\Phi) \neq \emptyset$

**Examples** regularised LSE, image processing...

Projection

$$\mathcal{P}_\Omega(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x\in\Omega} \|x-y\|$$
$$= \operatorname{argmin}_{x\in\mathbb{R}^n} \iota_\Omega(x) + \frac{1}{2}\|x-y\|^2.$$

Proximal mapping

$$\operatorname{prox}_R(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x\in\mathbb{R}^n} R(x) + \frac{1}{2}\|x-y\|^2.$$

### Projected gradient descent

**initial** : $x_0 \in \Omega$;

**repeat** :

1. Choose step-size $\gamma_k \in ]0, 2/L[$
2. Gradient descent $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$
3. Projection $x_{k+1} = \operatorname{prox}_{\gamma_k R}(x_{k+1/2})$

**until** : stopping criterion is satisfied.

# Interpretation

- A.K.A Forward–Backward splitting
  - Forward step: gradient descent of $F$.
  - Backward step: proximal mapping of $R$.
- Iteration in one line

$$x_{k+1} = \mathrm{prox}_{\gamma_k R}\big(x_k - \gamma_k \nabla F(x_k)\big).$$

- Definition of $\mathrm{prox}_{\gamma R}$,

$$\begin{aligned} x_{k+1} &= \mathrm{argmin}_x \Big\{ \gamma_k R(x) + \tfrac{1}{2}\|x - (x_k - \gamma_k \nabla F(x_k))\|^2 \Big\} \\ &= \mathrm{argmin}_x \Big\{ \gamma_k R(x) + \gamma_k \langle \nabla F(x_k),\, x - x_k\rangle + \tfrac{1}{2}\|x - x_k\|^2 \Big\} \\ &= \mathrm{argmin}_x \left\{ R(x) + \boxed{F(x_k) + \langle \nabla F(x_k),\, x - x_k\rangle + \frac{1}{2\gamma_k}\|x - x_k\|^2} \right\}. \end{aligned}$$

**NB**: $x_{k+1}$ minimises $R(x)$ plus the majorisation of $F(x)$ at $x_k$ if $\gamma_k \leq \frac{1}{L}$.

# Special cases

- **Gradient descent** $R = 0$

$$x_{k+1} = x_k - \gamma_k \nabla F(x_k).$$

- **Proximal point algorithm** $F = 0$

$$x_{k+1} = \mathrm{prox}_{\gamma_k R}(x_k).$$

- **Projected gradient descent** $R = \iota_\Omega$

$$x_{k+1} = \mathcal{P}_\Omega\big(x_k - \gamma_k \nabla F(x_k)\big).$$

- **ISTA: iterative shrinkage-thresholding algorithm** $R = \|x\|_1$

$$x_{k+1} = \mathcal{T}_\gamma\big(x_k - \gamma_k \nabla F(x_k)\big),$$

where

$$\big(\mathcal{T}_\gamma(y)\big)_i = \begin{cases} \mathrm{sign}(y_i) \cdot (|y_i| - \gamma) : |y_i| > \gamma, \\ 0 : |y_i| \le \gamma. \end{cases}$$

Define

$$E_\gamma(x, y) \stackrel{\text{def}}{=} R(x) + F(y) + \langle \nabla F(y), x - y \rangle + \frac{1}{2\gamma}\|x - y\|^2$$

and $y_+ \stackrel{\text{def}}{=} \text{argmin}_x E_\gamma(x, y)$.

**Lemma**

Let $y \in \mathbb{R}^n$ and $\gamma \in ]0, 2/L[$ such that

$$\Phi(y_+) \le E_\gamma(y_+, y),$$

then for any $x \in \mathbb{R}^n$,

$$\Phi(x) - \Phi(y_+) \ge \frac{1}{2\gamma}\big(\|x - y_+\|^2 - \|x - y\|^2\big).$$

**Lemma**

Given $y \in \mathbb{R}^n$ and $\gamma \in ]0, 1/L]$, then for any $x \in \mathbb{R}^n$,

$$\Phi(y_+) + \frac{1}{2\gamma}\|y_+ - x\|^2 \le \Phi(y) + \frac{1}{2\gamma}\|y - x\|^2.$$

# Convergence analysis

**NB**: proximal gradient is a descent method.

Consider $\gamma_k \equiv \gamma \in ]0, 1/L]$

- For each step
$$\Phi(x_k) - \Phi(x_{k+1}) \geq \frac{\gamma}{2}\|x_k - x_{k+1}\|^2.$$
- Regarding $\Phi(x^\star)$
$$\Phi(x_{k+1}) - \Phi(x^\star) \leq \frac{\gamma}{2}\big(\|x_k - x^\star\|^2 - \|x_{k+1} - x^\star\|^2\big).$$
- Summing up
$$\begin{aligned} k\big(\Phi(x_k) - \Phi(x^\star)\big) &\leq \textstyle\sum_{i=1}^k \big(\Phi(x_i) - \Phi(x^\star)\big) \\ &\leq \frac{\gamma}{2} \textstyle\sum_{i=1}^k \big(\|x_{i-1} - x^\star\|^2 - \|x_i - x^\star\|^2\big) \leq \frac{\gamma}{2}\|x_0 - x^\star\|^2 \end{aligned}$$
- $O(1/k)$ rate
$$\Phi(x_k) - \Phi(x^\star) \leq \frac{\gamma\|x_0 - x^\star\|^2}{2k}.$$

**NB**: not optimal and can be accelerated.

# Outline

## Proximal mapping

The proximal mapping (proximity operator) of a function $R \in \Gamma_0$ is defined by

$$\mathrm{prox}_{\gamma R}(y) \stackrel{\mathrm{def}}{=} \mathrm{argmin}_{x \in \mathbb{R}^n} \gamma R(x) + \frac{1}{2}\|x - y\|^2.$$

**Optimality condition** denote $y_+ \stackrel{\mathrm{def}}{=} \mathrm{prox}_{\gamma R}(y)$,

$$\begin{aligned} 0 \in \gamma \partial R(y_+) + y_+ - y \quad &\Longleftrightarrow \quad y \in (\mathrm{Id} + \gamma \partial R)(y_+) \\ &\Longleftrightarrow \quad y_+ = (\mathrm{Id} + \gamma \partial R)^{-1}(y). \end{aligned}$$

**Projection** $R(x) = \iota_\Omega(x)$, $\partial \iota_\Omega(x) = \mathcal{N}_\Omega(x) = \{g : \langle g,\, v - x\rangle \leq 0\}$

$$\mathcal{P}_\Omega = (\mathrm{Id} + \mathcal{N}_\Omega)^{-1}.$$

**Examples**

- Hyperplane: $\Omega = \{x : a^T x = b\}$, $a \neq 0$

$$\mathcal{P}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a.$$

- Affine subspace: $\Omega = \{x : Ax = b\}$ with $A \in \mathbb{R}^{m\times n}$, $\mathrm{rank}(A) = m < n$

$$\mathcal{P}_\Omega = x + A^T (AA^T)^{-1}(b - Ax).$$

- Half space: $\Omega = \{x : a^T x \leq b\}$, $a \neq 0$

$$\mathcal{P}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a \ \text{ if } a^T x > b \quad \text{ and } \quad x \ \text{ if } a^T x \leq b.$$

- Nonnegative orthant: $\Omega = \mathbb{R}^n_+$

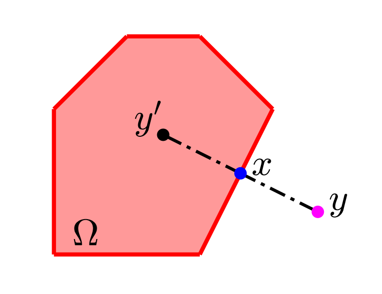
$$\mathcal{P}_\Omega = \big(\max\{0, x_i\}\big)_i.$$

**Projection** $R(x) = \iota_\Omega(x), \partial\iota_\Omega(x) = \mathcal{N}_\Omega(x) = \{g : \langle g, v - x\rangle \le 0\}$

$$\mathcal{P}_\Omega = (\mathrm{Id} + \mathcal{N}_\Omega)^{-1}.$$

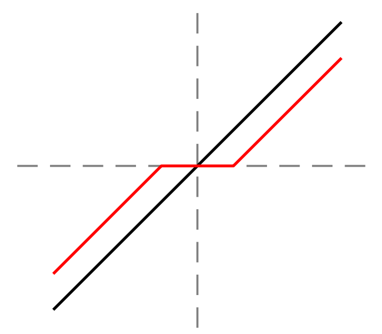
**Reflection**

$$\mathcal{R}_{\mathcal{N}_\Omega} = 2\mathcal{P}_\Omega - \mathrm{Id} = \mathcal{P}_\Omega + (\mathcal{P}_\Omega - \mathrm{Id}).$$



**Soft-threshold** : $R(x) = |x|$,

$$\mathrm{prox}_{\gamma R}(y) = \mathcal{T}_{\gamma}(y) = \begin{cases} y - \gamma : y > \gamma, \\ 0 : y \in [-\gamma, \gamma], \\ y + \gamma : y < -\gamma. \end{cases}$$



**Quadratic function** $R(x) = \frac{1}{2}x^T Ax + b^T x + c,\ A \succeq 0$

$$\mathrm{prox}_{\gamma R}(y) = (\mathrm{Id} + \gamma A)^{-1}(y - \gamma b).$$

**Euclidean norm** $R(x) = \|x\|$

$$\mathrm{prox}_{\gamma R}(y) = \begin{cases} (1 - \frac{\gamma}{\|y\|})y : \|y\| > \gamma, \\ 0 : o.w. \end{cases}$$

**Logarithmic barrier** $R(x) = -\sum_i \log(x_i)$

$$\big(\mathrm{prox}_{\gamma R}(y)\big)_i = \frac{y_i + \sqrt{y_i^2 + 4\gamma}}{2},\ \ i = 1, ..., n.$$

**Nuclear norm** $R(x) = \sum_i \sigma_i$

$$\mathrm{prox}_{\gamma R}(y) = U\mathcal{T}_\gamma(\Sigma)V^T.$$

**Quadratic perturbation** $H(x) = R(x) + \frac{\alpha}{2}\|x\|^2 + \langle x,\, u\rangle + \beta,\ \alpha \geq 0$

$$\mathrm{prox}_H = \mathrm{prox}_{R/(\alpha+1)}\Big(\frac{x-u}{\alpha+1}\Big).$$

**Translation** $H(x) = R(x - z)$

$$\mathrm{prox}_H = z + \mathrm{prox}_R(x - z).$$

**Scaling** $H(x) = R(x/\rho)$

$$\mathrm{prox}_H = \rho\,\mathrm{prox}_{R/\rho^2}\Big(\frac{x}{\rho}\Big).$$

**Reflection** $H(x) = R(-x)$

$$\mathrm{prox}_H = -\mathrm{prox}_R(-x).$$

**Composition** $H = R \circ L$ with $L$ being bijective bounded linear mapping such that $L^{-1} = L^*$,

$$\mathrm{prox}_H = L^* \circ \mathrm{prox}_R \circ L.$$

# Outline

## An inertial proximal gradient

**Initial** : $x_0 \in \mathbb{R}^n$ and $\gamma \in ]0, 2/L[$;

$$y_k = x_k + a_k(x_k - x_{k-1}),\ a_k \in [0, 1],$$
$$x_{k+1} = \mathrm{prox}_{\gamma R}(y_k - \gamma \nabla F(x_k)).$$

- Recovers inertial PPA when $F = 0$, and heavy-ball method when $R = 0$.
- Convergence via studying the Lyapunov function

$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \frac{a_k}{2\gamma}\|x_k - x_{k-1}\|^2.$$

- In general, no convergence rate.

## A general inertial proximal gradient

**Initial** : $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma \in ]0, 2/L[$;

$$\begin{aligned} y_k &= x_k + a_k(x_k - x_{k-1}), \ a_k \in [0,1], \\ z_k &= x_k + b_k(x_k - x_{k-1}), \ b_k \in [0,1], \\ x_{k+1} &= \mathrm{prox}_{\gamma R}(y_k - \gamma \nabla F(z_k)). \end{aligned}$$

- Convergence via studying the Lyapunov function

$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \frac{a_k}{2\gamma}\|x_k - x_{k-1}\|^2.$$

- In general, no convergence rate.

- Can be extend to multiple steps, *e.g.*

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots .$$

**Assumption** $R = 0,\ F = \frac{1}{2}\|Ax - f\|^2$ and $(a_k, b_k) \equiv (a, b)$.

- $A^T A$ is symmetric positive definite with $A^T A \succeq \alpha \mathrm{Id}$.
- Taylor expansion
$$x_{k+1} = y_k - \gamma \nabla^2 F(x^\star)(z_k - x^\star).$$
- Let $d_k = (x_k - x^\star, x_{k-1} - x^\star)^T$ and $H = \nabla^2 F,\ G = \mathrm{Id} - \gamma H$, then
$$d_{k+1} = \underbrace{\begin{bmatrix} (a-b)\mathrm{Id} + (1+b)G, & -(a-b)\mathrm{Id} - bG \\ \mathrm{Id}, & 0 \end{bmatrix}}_{M} d_k.$$
- Spectral radius: $\eta = \rho(G) = 1 - \gamma\alpha$ and $\rho = \rho(M)$...

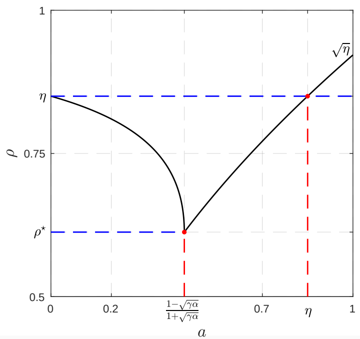
## Spectral radius $\rho$

Between $\eta$ and $\rho$,

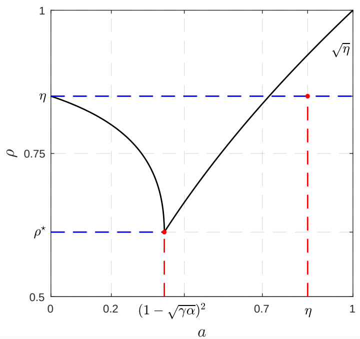
- $\eta$ and $\rho$ satisfy the relation

$$\rho^2 - \big((a-b) + (1+b)\eta\big)\rho + (a-b) + b\eta = 0.$$

- Given any $(a, b) \in [0, 1[^2$, then $\rho(M) < 1$ if, and only if $\frac{2(b-a)-1}{1+2b} < \eta$.




$b = a$




$b = 0$

## Remarks

- Given $b \in [0, 1]$, there exists optimal choice of $a \in [0, 1]$ such that

$$\rho = 1 - \sqrt{\gamma\alpha}$$

  can be obtained.

- Take $b = a$, for

$$a \in \left] \tfrac{1-\sqrt{\gamma\alpha}}{1+\sqrt{\gamma\alpha}}, 1 \right],$$

  the leading eigenvalue of $M$ is complex.

- Continue $b = a$, for

$$a \in ]\eta, 1],$$

  the inertial scheme is actually slower than the original scheme.

# Outline

**FISTA**

**Initial** : $x_0 \in \mathbb{R}^n, x_{-1} = x_0, \gamma = 1/L$ and $t_0 = 1$;

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \; a_k = \frac{t_{k-1} - 1}{t_k},$$
$$y_k = x_k + a_k(x_k - x_{k-1}),$$
$$x_{k+1} = \mathrm{prox}_{\gamma R}\big(y_k - \gamma \nabla F(y_k)\big).$$

- A special case of inertial proximal gradient descent.
- Inertial parameters

$$t_k \approx \frac{k+1}{2} \quad \text{and} \quad a_k \to 1.$$

**Nesterov** compute $\phi_k \in ]0,1[$ from equation

$$\phi_k^2 = (1-\phi_k)\phi_{k-1}^2$$

and $a_k = \frac{\phi_{k-1}(1-\phi_{k-1})}{\phi_{k-1}^2+\phi_k}$.

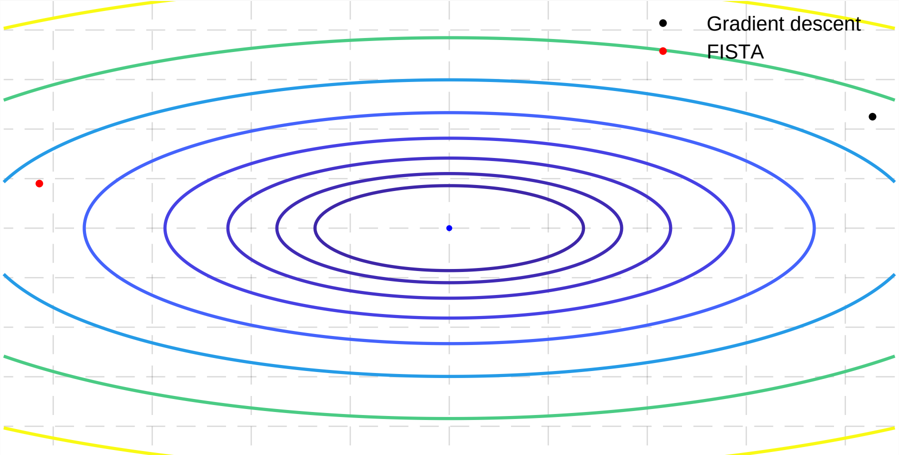
- $\phi_k$ reads
$$\phi_k = \frac{-\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}{2} = \frac{2\phi_{k-1}^2}{\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}.$$
- Let $t_k = 1/\phi_k$,
$$\frac{1}{t_k} = \frac{2}{1+\sqrt{1+4t_{k-1}^2}}.$$
- Which leads to
$$t_k = \frac{1+\sqrt{1+4t_{k-1}^2}}{2}.$$
- Moreover, $a_k = \dfrac{t_{k-1}-1}{t_k}$.

**NB**: FISTA is not a descent method.

- Denote $f_k = \Phi(x_k) - \Phi(x^\star)$ and $u_k = t_k x_k - (t_k - 1)x_{k-1} - x^\star$, then
$$\frac{2}{L} t_k^2 f_k - \frac{2}{L} t_{k+1}^2 f_{k+1} \geq \|u_{k+1}\|^2 - \|u_k\|^2.$$
- Let $c_k, d_k$ be positive sequences, if
$$c_k - c_{k+1} \geq d_{k+1} - d_k \forall k \geq 1, \text{ with} c_1 + d_1 < C, \ C > 0$$
then $c_k < C$ for all $k \geq 1$.
- $\frac{2}{L} t_k^2 f_k \leq \|x_0 - x^\star\|^2$
- $t_k \geq \frac{k+1}{2}$,
$$\Phi(x_k) - \Phi(x^\star) \leq \frac{L\|x_0 - x^\star\|^2}{2(k+1)^2}.$$

# Outline

# Oscillation of FISTA



Why FISTA oscillates

- for LSE, leading eigenvalue fo the system is complex.
- over extropolation, momentum beats gradient.

**Restarting FISTA**

**Initial** : $x_0 \in \mathbb{R}^n, x_{-1} = x_0, \gamma = 1/L$ and $t_0 = 1$;

**repeat** :

1. Run FISTA iteration
2. If $\langle y_k - x_{k+1},\, x_{k+1} - x_k \rangle > 0$: $t_k = 1, y_k = x_k$.

**until** : stopping criterion is satisfied.

# Outline

# Regression problems

**$\ell_1$-regularised least square (LASSO)**

$$\min_{x \in \mathbb{R}^n} \ \mu \|x\|_1 + \frac{1}{2}\|Ax - f\|^2.$$

**Sparse logistic regression**

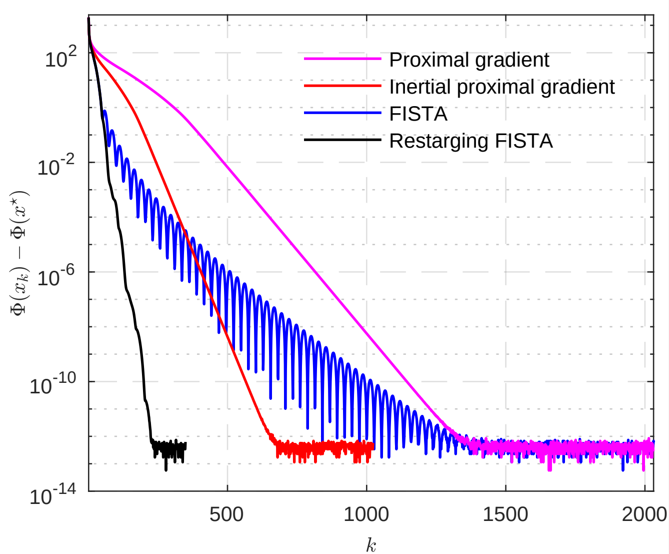
$$\min_{x \in \mathbb{R}^n} \ \mu \|x\|_1 + \frac{1}{m} \sum_{i=1}^{m} \log\big(1 + e^{-l_i h_i^T x}\big),$$

where $\mu = 10^{-2}$. The `australian` data set from LIBSVM [1] is considered.

[1] `https://www.csie.ntu.edu.tw/ cjlin/libsvmtools/datasets/`

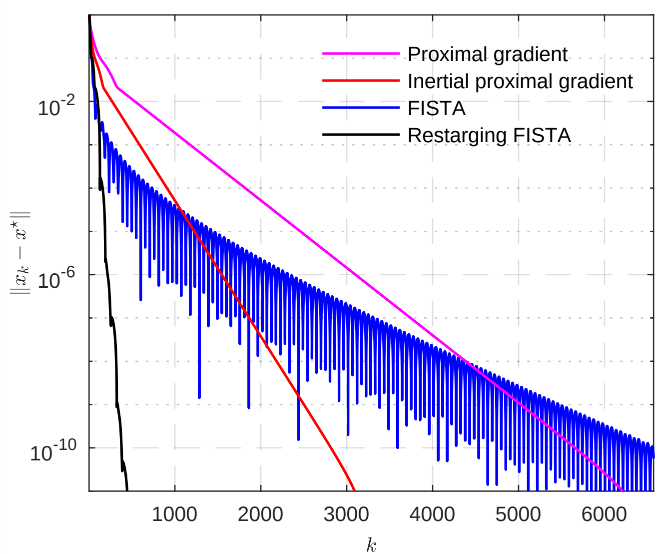
# Compared methods

- Proximal gradient descent
- Inertial proximal gradient descent
- FISTA
- Restarting FISTA



Proximal gradient
Inertial proximal gradient
FISTA
Restarging FISTA
$10^{2}$
$10^{-2}$
$10^{-6}$
$10^{-10}$
$10^{-14}$
500
1000
1500
2000
$k$
$\Phi(x_k) - \Phi(x^\star)$


LASSO




Sparse logistic regression

# Reference


- B. Polyak. “Introduction to optimization”. Optimization Software, 1987.
- Y. Nesterov. “Introductory lectures on convex optimization: A basic course”. Vol. 87. Springer Science & Business Media, 2013.
- A. Beck and M. Teboulle. “A fast iterative shrinkage-thresholding algorithm for linear inverse problems”. SIAM Journal on Imaging Sciences, 2(1):183–202, 2009.
- H. Bauschke and P. L. Combettes. “Convex analysis and monotone operator theory in Hilbert spaces”. Springer, 2011.
- B. O’Donoghue and E. J. Candés. “Adaptive restart for accelerated gradient schemes”. Foundations of Computational Mathematics, pages 1–18, 2012.