Introductory Course on Non-smooth Optimisation

Lecture 04

Backward–Backward splitting

Outline

1. Forward–Backward splitting revisit
2. MAP continue
3. Backward–Backward splitting
4. Numerical experiments

Monotone inclusion pronblem

Problem

Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be β-cocoercive for some $\beta > 0$, $s > 1$ be a positive integer, such that for each $i \in \{1, ..., s\}$: $A_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone. Consider the problem

$$\text{Find } \ x \in \mathbb{R}^n \ \textit{ such that } \ 0 \in B(x) + \sum_{i=1}^{s} A_i(x). \tag{0.1}$$

- A_i can be composed with linear mapping, *e.g.* $L^* \circ A \circ L$
- Even if the resolvent of B and each A_i is simple, the resolvent of $B + \sum_i A_i$ in most cases is not solvable
- Use the properties of operators and structure of problem to derive operator splitting schemes

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Monotone inclusion problem

Problem (Monotone inclusion)

$$\text{Find } x \in \mathbb{R}^n \ \textit{such that} \ 0 \in A(x) + B(x)$$

Assumptions:

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone
- $B : \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoersive
- $\mathrm{zer}(A + B) \neq \emptyset$

Characterisation of solution: $\gamma > 0$

$$x^\star - \gamma B(x^\star) \in x^\star + \gamma A(x^\star) \quad \Leftrightarrow \quad x^\star = \mathcal{J}_{\gamma A} \circ (\mathrm{Id} - \gamma B)(x^\star).$$

Example:

$$\min_{x \in \mathbb{R}^n} R(x) + F(x),$$

with $R \in \Gamma_0$ and $F \in C_L^1$.

Forward–Backward splitting

Fixed-point operator: $\gamma \in]0, 2\beta[$

$$\mathcal{T}_{\text{FB}} = \mathcal{J}_{\gamma A} \circ (\text{Id} - \gamma B)$$

- $\mathcal{J}_{\gamma A}$ is firmly non-expansive
- $\text{Id} - \gamma B$ is $\frac{\gamma}{2\beta}$-averaged non-expansive
- $\mathcal{T}_{\text{FB}}$ is $\frac{2\beta}{4\beta-\gamma}$-averaged non-expansive
- $\text{fix}(\mathcal{T}_{\text{FB}}) = \text{zer}(A + B)$

Forward–Backward splitting

Let $\gamma \in]0, 2\beta[,\ \lambda_k \in [0, \frac{4\beta-\gamma}{2\beta}]$:

$$x_{k+1} = (1 - \lambda_k)x_k + \lambda_k \mathcal{T}_{\text{FB}}(x_k)$$

- Special case of Krasnosel'skiĭ-Mann iteration
- Recovers proximal point algorithm when $B = 0$

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Method of alternating projection

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ be closed convex and non-empty, such that $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$

$$\min_{x \in \mathbb{R}^n} \ \iota_{\mathcal{X}}(x) + \iota_{\mathcal{Y}}(x).$$

Method of alternating projection (MAP)

Let $x_0 \in \mathcal{X}$:

$$y_{k+1} = \mathrm{proj}_{\mathcal{Y}}(x_k)$$

$$x_{k+1} = \mathrm{proj}_{\mathcal{X}}(y_{k+1})$$

Fixed-point operator: $x_{k+1} = \mathcal{T}_{\text{MAP}}(x_k)$,

$$\mathcal{T}_{\text{MAP}} \stackrel{\text{def}}{=} \mathrm{proj}_{\mathcal{X}} \circ \mathrm{proj}_{\mathcal{Y}}$$

- $\mathrm{proj}_{\mathcal{X}}, \mathrm{proj}_{\mathcal{Y}}$ are firmly non-expansive
- $\mathcal{T}_{\text{MAP}}$ is $\frac{2}{3}$-averaged non-expansive
- $\mathrm{fix}(\mathcal{T}_{\text{MAP}}) = \mathcal{X} \cap \mathcal{Y}$

Derive MAP

- Feasibility problem is equivalent to

$$\min_{x,y\in\mathbb{R}^n} \ \iota_{\mathcal{X}}(x) + \frac{1}{2}\|x-y\|^2 + \iota_{\mathcal{Y}}(y).$$

- Optimality condition

$$\begin{aligned} 0 &\in \mathcal{N}_{\mathcal{Y}}(y^\star) + y^\star - x^\star \\ 0 &\in \mathcal{N}_{\mathcal{X}}(x^\star) + x^\star - y^\star. \end{aligned}$$

- Fixed-point characterisation

$$\begin{aligned} y^\star &= \mathrm{proj}_{\mathcal{Y}}(x^\star) \\ x^\star &= \mathrm{proj}_{\mathcal{X}}(y^\star). \end{aligned}$$

- Fixed-point iteration

$$\begin{aligned} y_{k+1} &= \mathrm{proj}_{\mathcal{Y}}(x_k) \\ x_{k+1} &= \mathrm{proj}_{\mathcal{X}}(y_{k+1}). \end{aligned}$$

Example: SDP feasibility

SDP feasibility

Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0 \quad \text{and} \quad \mathrm{Tr}(A_i X) = b_i,\ i = 1, ..., m.$$

Two sets and projection:

- $\mathcal{X} = \mathcal{S}^n_+$ is the positive semidefinite cone. Let $Y_k = \sum_{i=1}^n \sigma_i u_i u_i^T$ be the SVD of Y_k, then

$$\mathrm{proj}_{\mathcal{X}}(Y_k) = \sum_{i=1}^n \max\{0, \sigma_i\} u_i u_i^T.$$

- $\mathcal{Y}$ is the affine set in $\mathcal{S}^n$ define by the linear inequalities,

$$\mathrm{proj}_{\mathcal{Y}}(X_k) = X_k - \sum_{i=1}^m u_i A_i,$$

  where u_i are found from the normal equations

$$Gu = \big(\mathrm{Tr}(A_i X_k) - b_i, \cdots, \mathrm{Tr}(A_i X_k) - b_m\big),\ G_{i,j} = \mathrm{Tr}(A_i A_j).$$

Convergence rate

Let $\mathcal{X}, \mathcal{Y}$ be two subspaces, and assume

$$1 \leq p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \leq q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \leq n - 1.$$

Principal angles The principal angles $\theta_k \in [0, \frac{\pi}{2}]$, $k = 1, \ldots, p$ between $\mathcal{X}$ and $\mathcal{Y}$ are defined by, with $u_0 = v_0 \stackrel{\text{def}}{=} 0$, and

$$\cos(\theta_k) \stackrel{\text{def}}{=} \langle u_k,\ v_k \rangle = \max \langle u,\ v \rangle \quad \text{s.t.} \quad u \in \mathcal{X}, v \in \mathcal{Y}, \|u\| = 1, \|v\| = 1,$$
$$\langle u,\ u_i \rangle = \langle v,\ v_i \rangle = 0,\ i = 0, \cdots, k-1.$$

Friedrichs angle The Friedrichs angle $\theta_F \in]0, \frac{\pi}{2}]$ between $\mathcal{X}$ and $\mathcal{Y}$ is

$$\cos\big(\theta_F(\mathcal{X}, \mathcal{Y})\big) \stackrel{\text{def}}{=} \max \langle u,\ v \rangle \quad \text{s.t.} \quad u \in \mathcal{X} \cap (\mathcal{X} \cap \mathcal{Y})^{\perp}, \|u\| = 1,$$
$$v \in \mathcal{Y} \cap (\mathcal{X} \cap \mathcal{Y})^{\perp}, \|v\| = 1.$$

Lemma

The Friedrichs angle is θ_{d+1} where $d \stackrel{\text{def}}{=} \dim(\mathcal{X} \cap \mathcal{Y})$. Moreover, $\theta_F(\mathcal{X}, \mathcal{Y}) > 0$.

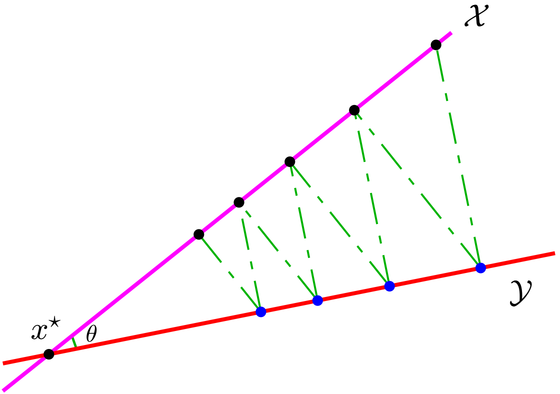
Convergence rate

Example $\mathcal{X}, \mathcal{Y}$ are defined by

$$\mathcal{X} = \{x : Ax = 0\}, \ \mathcal{Y} = \{x : Bx = 0\}$$

Projection onto subspace

$$\mathrm{proj}_{\mathcal{X}}(x) = x - A^T(AA^T)^{-1}Ax$$



Convergence rate

- Define diagonal matrices

$$c = \operatorname{diag}\big(\cos(\theta_1), \cdots, \cos(\theta_p)\big)$$
$$s = \operatorname{diag}\big(\sin(\theta_1), \cdots, \sin(\theta_p)\big)$$

- Suppose $p + q < n$, then there exists orthogonal matrix U such that

$$\mathrm{proj}_{\mathcal{X}} = U \left[\begin{array}{cc|cc} \mathrm{Id}_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

$$\mathrm{proj}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ -cs & c^2 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*,$$

Convergence rate

- Fixed-point operator

$$
\begin{aligned}
\mathcal{T}_{\text{MAP}} &= \text{proj}_{\mathcal{X}} \circ \text{proj}_{\mathcal{Y}} \\
&= U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array} \right] U^*
\end{aligned}
$$

- Consider relaxation

$$
\begin{aligned}
\mathcal{T}_{\text{MAP}}^{\lambda} &= (1-\lambda)\text{Id} + \lambda \mathcal{T}_{\text{MAP}} \\
&= U \left[\begin{array}{cc|cc} (1-\lambda)\text{Id}_p + \lambda c^2 & \lambda cs & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & (1-\lambda)\text{Id}_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array} \right] U^*
\end{aligned}
$$

Convergence rate

- Eigenvalues

$$\sigma(\mathcal{T}^{\lambda}_{\text{MAP}}) = \left\{1 - \lambda \sin^2(\theta_i) | i = 1, ..., p\right\} \cup \{1 - \lambda\}$$

- Spectral radius

$$\rho(\mathcal{T}^{\lambda}_{\text{MAP}}) = \max\left\{1 - \lambda \sin^2(\theta_F), |1 - \lambda|\right\}$$

- No relaxation

$$\rho(\mathcal{T}_{\text{MAP}}) = \cos^2(\theta_F)$$

- Convergence rate, $C > 0$ is some constant

$$\begin{aligned} \|x_k - x^\star\| &= \|\mathcal{T}_{\text{MAP}} x_{k-1} - \mathcal{T}_{\text{MAP}} x^\star\| \\ &= ... \\ &= \|\mathcal{T}^k_{\text{MAP}}(x_0 - x^\star)\| \\ &\leq C \|\mathcal{T}_{\text{MAP}}\|^k \|x_0 - x^\star\| \end{aligned}$$

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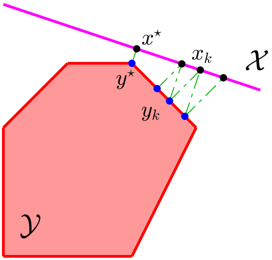
Best pair problem

When the problem is feasible, MAP returns $x_k, y_k \to x^\star \in \mathcal{X} \cap \mathcal{Y}$.



Best pair problem

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ be closed and convex, such that

$$\mathcal{X} \cap \mathcal{Y} = \emptyset.$$

Consider finding two points in $\mathcal{X}$ and $\mathcal{Y}$ such that they are the closest, that is

$$\min_{x \in \mathcal{X}, y \in \mathcal{Y}} \|x - y\|.$$

- MAP can be applied and

$$(x_k, y_k) \to (x^\star, y^\star)$$

  where $(x^\star, y^\star)$ is a best pair.
- When $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, $x^\star = y^\star$.

Backward–Backward splitting

Consider

$$\text{Find } x, y \in \mathbb{R}^n \text{ such that } 0 \in A(x) + B(y),$$

where

- $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are maximal monotone
- The set of solition is non-empty

There exists $x^\star, y^\star \in \mathbb{R}^n$ and $\gamma > 0$ such that

$$y^\star - x^\star \in \gamma A(x^\star)$$

$$x^\star - y^\star \in \gamma B(y^\star).$$

Backward–Backward splitting

Let $x_0 \in \mathbb{R}^n$, $\gamma > 0$:

$$y_{k+1} = \mathcal{J}_{\gamma B}(x_k)$$

$$x_{k+1} = \mathcal{J}_{\gamma A}(y_{k+1})$$

Regularised monotone inclusion

Yosida approximation

$$ {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id} - \mathcal{J}_{\gamma A}) $$

which is γ-cocoercive.

Regularised monotone inclusion

$$ \text{Find } \ x \in \mathbb{R}^n \ \text{ such that } \ 0 \in A(x) + {}^{\gamma}B(x) $$

Forward–Backward splitting $\tau \in]0, 2\gamma]$

$$ x_{k+1} = \mathcal{J}_{\tau A} \circ (\mathrm{Id} - \tau\, {}^{\gamma}B)(x_k) $$

BB as special case of FB let $\tau = \gamma$

$$ \begin{aligned} x_{k+1} &= \mathcal{J}_{\gamma A} \circ (\mathrm{Id} - \gamma\, {}^{\gamma}B)(x_k) \\ &= \mathcal{J}_{\gamma A} \circ \big(\mathrm{Id} - \gamma \tfrac{1}{\gamma}(\mathrm{Id} - \mathcal{J}_{\gamma B})\big)(x_k) \\ &= \mathcal{J}_{\gamma A} \circ \mathcal{J}_{\gamma B}(x_k) \end{aligned} $$

Inertial BB splitting

An inertial Backward–Backward splitting

Initial: $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0,\ \tau \in]0, 2\gamma]$;

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots,$$

$$x_{k+1} = \mathcal{J}_{\gamma A} \circ \mathcal{J}_{\gamma B}(y_k),\ \lambda_k \in [0, 1].$$

An inertial BB splitting based on Yosida approximation

Initial: $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0$;

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots,$$

$$z_k = x_k + b_{0,k}(x_k - x_{k-1}) + b_{1,k}(x_{k-1} - x_{k-2}) + \cdots,$$

$$x_{k+1} = \mathcal{J}_{\tau A} \circ \big(y_k - \tau^{\gamma} B(z_k)\big),\ \lambda_k \in [0, 1].$$

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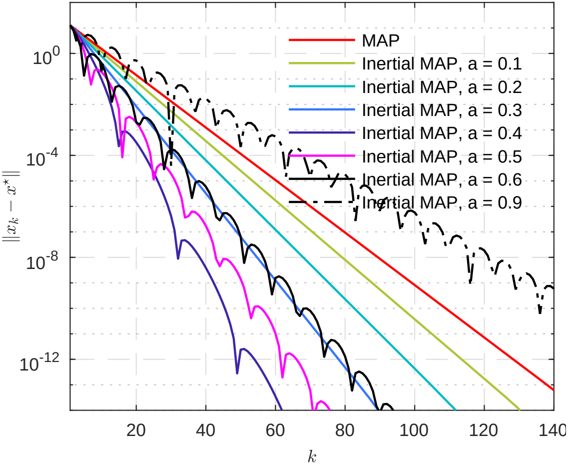
Numerical experiment

Feasibility problem for two subspaces:

$$A = [-4/5, 1] \quad \text{and} \quad B = [-1/5, 1]$$



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