Trajectory of Alternating Direction Method of Multipliers and Adaptive Acceleration

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Alternating Direction Method of Multipliers (ADMM)

Constrained and composite optimisation problem:

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} R(x) + J(y) \quad \text{such that} \quad Ax + By = b \qquad (\mathcal{P})$$

under basic assumptions

- R, J are proper, convex, lower semi-continuous functions.
- $A : \mathbb{R}^n \to \mathbb{R}^p$ and $B : \mathbb{R}^m \to \mathbb{R}^p$ are injective linear operators.
- $\mathrm{ri}\,(\mathrm{dom}(R) \cap \mathrm{dom}(J)) \neq \emptyset$ and the set of minimizers is non-empty.

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Question: *How should one accelerate the convergence of ADMM?*

The use of inertial

Given a fixed point sequence $z_{k+1} = \mathcal{F}(z_k)$, accelerate by
$$\bar{z}_{k+1} = z_k + \alpha_k(z_k - z_{k-1}), \quad \alpha_k > 0$$
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[Heavy-Ball/Nesterov accelerated gradient/FISTA]

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The performance of inertial-ADMM in general is less clear.

Our contributions

1. We study the local trajectory of a sequence generated by ADMM under the framework of partial smoothness.

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Based on this trajectory analysis:

2. We obtain insight into when inertial will work and fail.

3. We develop an acceleration scheme with local acceleration rates.

The ADMM iterates

Augmented Lagrangian: For $\gamma > 0$ and Lagrangian multiplier $\psi \in \mathbb{R}^p$

$$\mathcal{L}(x, y, \psi) \stackrel{\text{def.}}{=} R(x) + J(y) + \langle \psi, Ax + By - b \rangle + \frac{\gamma}{2}\|Ax + By - b\|_2^2.$$

The ADMM iterations:

$$x_k = \mathrm{argmin}_{x \in \mathbb{R}^n} R(x) + \frac{\gamma}{2}\|Ax + By_{k-1} - b + \frac{1}{\gamma}\psi_{k-1}\|^2$$

$$y_k = \mathrm{argmin}_{y \in \mathbb{R}^m} J(y) + \frac{\gamma}{2}\|Ax_k + By - b + \frac{1}{\gamma}\psi_{k-1}\|^2$$

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Define $z_k \overset{\text{def.}}{=} \psi_{k-1} + \gamma Ax_k$.

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$$\psi_k = z_k + \gamma(By_k - b)$$

Then, $z_k = \mathcal{F}(z_{k-1})$ for some fixed point operator $\mathcal{F}$ [†].

[†] Due to the equivalence between ADMM and Douglas-Rachford splitting [Gabay '83]

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We will analyse the behaviour of $\{z_k\}_k$.

R is partly smooth at x relative to a set $\mathcal{M} \ni x$ if $\partial R(x) \neq \emptyset$ and

Smoothness:
  $\mathcal{M}$ is a C^2-manifold, $R|_{\mathcal{M}}$ is C^2 near x.
Sharpness:
  Tangent space $\mathcal{T}_{\mathcal{M}}(x)$ is $\mathrm{par}\left(\partial R(x)\right)^{\perp}$.
Continuity:
  ∂R is continuous along $\mathcal{M}$ near x.
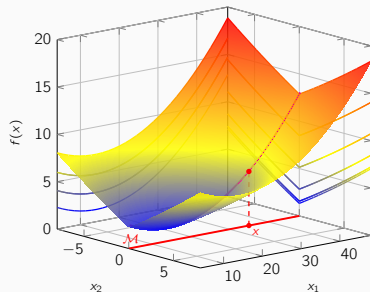
$\mathrm{par}(C)$: sub-space parallel to C, where C is a non-empty convex set.

Examples: $\ell_1, \ell_{1,2}, \ell_{\infty}$-norm, Nuclear norm, Total variation.

Partial smoothness

If R, J are partly smooth, then under property **non-degeneracy conditions** around the fixed points:

Local linearisation [Liang, Fadili & Peyré '16]:
There exists $K \in \mathbb{N}$ and a matrix M_{ADMM} such that for all $k \geqslant K$,
$$z_k - z^\star = M_{\text{ADMM}}(z_{k-1} - z^\star) + o(\|z_{k-1} - z^\star\|).$$

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The behaviour of z_k is eventually **regular**.

Partial smoothness and sequence trajectory

Let $v_k \stackrel{\text{def.}}{=} z_k - z_{k-1}$ and let $\theta_k = \angle(v_k, v_{k-1})$.

Two non-smooth terms

R and J are locally polyhedral around $x^\star$ and $y^\star$.
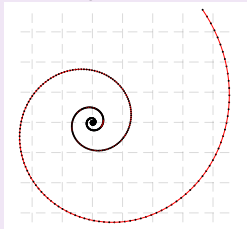
Spiral trajectory:

$$\cos(\theta_k) = \cos(\alpha) + \mathcal{O}(\eta^{2k})$$

with $\eta < 1, \alpha > 0$.

M_{ADMM} has **complex** eigenvalues

At least one smooth term

A is a full rank square matrix and R is locally $\mathcal{C}^2$ around $x^\star$.
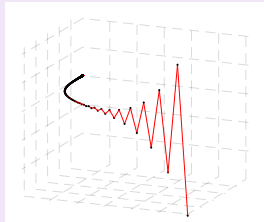
Straight line trajectory:

$\cos(\theta_k) \to 1$ when

$$\gamma > \|(A^\top A)^{-\frac{1}{2}} \nabla^2 R(x^\star)(A^\top A)^{-\frac{1}{2}}\|.$$

M_{ADMM} has all **real** eigenvalues

One inertial-ADMM iteration:
$$x_k = \operatorname{argmin}_{x\in\mathbb{R}^n} R(x) + \frac{\gamma}{2}\|Ax - \frac{1}{\gamma}(\bar{z}_{k-1} - 2\psi_{k-1})\|^2,$$
$$z_k = \psi_{k-1} + \gamma Ax_k,$$
$$\bar{z}_k = z_k + a_k(z_k - z_{k-1}),$$
$$y_k = \operatorname{argmin}_{y\in\mathbb{R}^m} J(y) + \frac{\gamma}{2}\|By + \frac{1}{\gamma}(\bar{z}_k - \gamma b)\|^2,$$
$$\psi_k = \bar{z}_k + \gamma(By_k - b).$$

Intuition: inertial-ADMM accelerates if z_k is moving along a straight path...

Failure of inertial-ADMM

Find $z \in T_1 \cap T_2$. Solve using ADMM

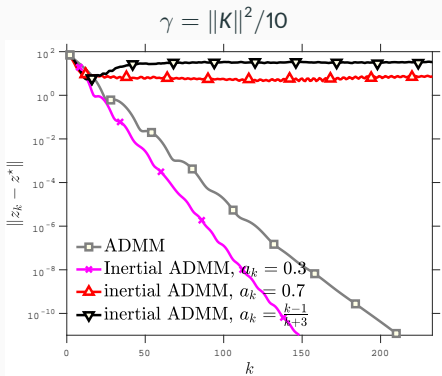$$\min_{x,y} \iota_{T_1}(x) + \iota_{T_2}(y) \quad \text{such that} \quad x - y = 0.$$

Consider $z_k \overset{\text{def.}}{=} \psi_{k-1} + \gamma x_k$. **Standard ADMM:**

Find $z \in T_1 \cap T_2$. Solve using ADMM

$$\min_{x,y} \iota_{T_1}(x) + \iota_{T_2}(y) \quad \text{such that} \quad x - y = 0.$$

Consider $z_k \stackrel{\text{def.}}{=} \psi_{k-1} + \gamma x_k$. **Inertial ADMM with $a = 0.2$:**

LASSO example:

$$\min_{x,y\in\mathbb{R}^n} \mu\|x\|_1 + \frac{1}{2}\|Ky - f\|_2^2 \quad \text{such that} \quad x - y = 0.$$
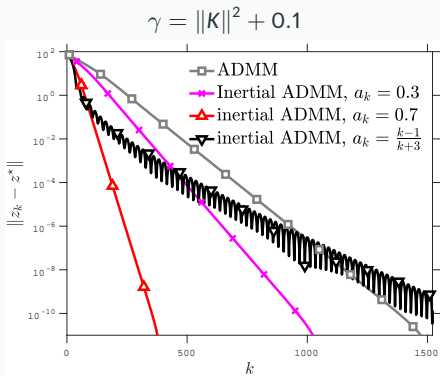
Eventual trajectory:

- Straight line when $\gamma > \|K\|^2$
- M_{ADMM} may have complex leading eigenvalue if $\gamma \leqslant \|K\|^2$.

Adaptive acceleration for ADMM (A^3DMM)

Goal: Given past points $\{z_{k-j}\}_{j=0}^q$, predict z_{k+1}.

Idea: Define $v_j \overset{\text{def.}}{=} z_j - z_{j-1}$,

- Fit the past directions $v_{k-1}, \ldots, v_{k-q}$ to the latest direction v_k:
$$c_k \overset{\text{def.}}{=} \operatorname{argmin}_{c \in \mathbb{R}^q} \|V_{k-1}c - v_k\|^2,$$
where $V_{k-1} = [v_{k-1}, \ldots, v_{k-q}] \in \mathbb{R}^{n \times q}$.

- If $V_k c_k \approx v_{k+1}$, then $\bar{z}_{k,1} \overset{\text{def.}}{=} z_k + V_k c_k \approx z_{k+1}$

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Repeat s times to predict z_{k+s}.

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Define: $H(c_k) \overset{\text{def.}}{=} \left[\ c_k\ \middle|\ \dfrac{\text{Id}_{q-1}}{0_{1,q-1}}\ \right]$ and $\mathcal{E}_{s,q,k} = V_k \left(\sum_{j=1}^s H(c_k)^j\right)_{(:,1)}$

The *s*-**step extrapolation** is $\bar{z}_{k,s} = z_k + \mathcal{E}_{s,q,k}$.

A^{3}DMM

Initial: Let $s \geqslant 1$, $q \geqslant 1$, $\bar{q} = q + 1$. Let $\bar{z}_0 = z_0 \in \mathbb{R}^p$ and $V_0 = 0_{p \times q}$.

Repeat: For $k \geqslant 1$

$$y_k = \mathrm{argmin}_{y \in \mathbb{R}^m} J(y) + \tfrac{\gamma}{2} \| By + \tfrac{1}{\gamma} \left(\bar{z}_{k-1} - \gamma b \right) \|^2$$

$$\psi_k = \bar{z}_{k-1} + \gamma(By_k - b)$$

$$x_k = \mathrm{argmin}_{x \in \mathbb{R}^n} R(x) + \tfrac{\gamma}{2} \| Ax - \tfrac{1}{\gamma} \left(\bar{z}_{k-1} - 2\psi_k \right) \|^2$$

$$z_k = \psi_k + \gamma A x_k$$

$$v_k = z_k - z_{k-1} \quad \text{and} \quad V_k = [v_k, V_k(:, 1 : q - 1)]$$

If $\mathrm{mod}(k, \bar{q}) = 0$: Compute coefficients c_k and let $C_k \stackrel{\text{def.}}{=} H(c_k)$

$\quad$ If $\rho(C_k) < 1$: $\quad \bar{z}_k = z_k + a_k \mathcal{E}_{s,q,k}$; $\quad$ else: $\quad \bar{z}_k = z_k$.

If $\mathrm{mod}(k, \bar{q}) \neq 0$: $\bar{z}_k = z_k$.

Remarks

Global convergence is guaranteed for appropriate choice of a_k.

Related to vector extrapolation techniques from the 1960's.

[Aitken '27, Wynn '62, Andersen '65...]

Local acceleration depends on the coefficients fitting error
$\varepsilon_k \stackrel{\text{def.}}{=} \min_c \|V_{k-1}c - v_k\|$.

- If M_{ADMM} is diagonalisable, then $\varepsilon_k = \mathcal{O}(|\lambda_{\bar{q}}|^k)$ where $\lambda_{\bar{q}}$ is the $\bar{q}^{th}$ largest eigenvalue.
- Guaranteed local acceleration for $q = 2$ if R and J are polyhedral.

Remarks

Implementation:

- Typically set $q \leqslant 10$.
- Extra memory cost of $p \times (q + 1)$ (storing V_k).
- Extra computation cost of $q^2 p$ every $(q + 2)$ iterations.
- One could also extrapolate $\{x_k, y_k\}$ simultaneously. But this would require extra storage of past directions.

Basis pursuit type problem with $\Omega \overset{\text{def.}}{=} \{x \in \mathbb{R}^n \; ; \; Kx = f\}$:

$$\min_{x,y \in \mathbb{R}^n} R(x) + \iota_\Omega(y) \quad \text{such that} \quad x - y = 0.$$
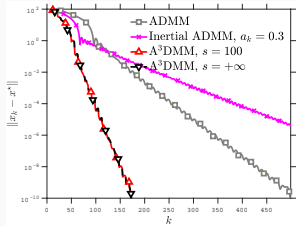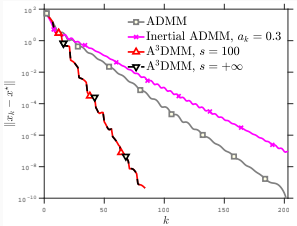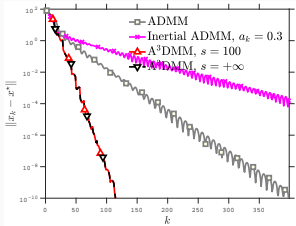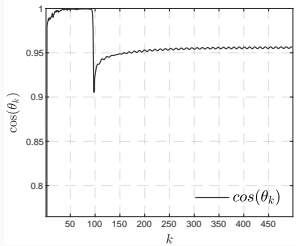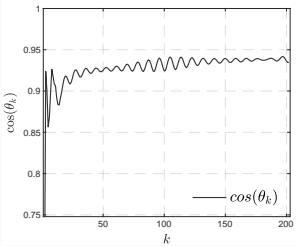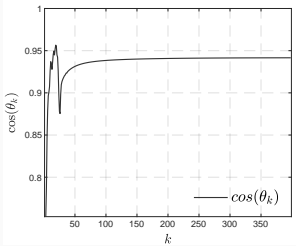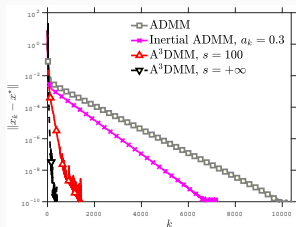

ℓ_1-norm

$\ell_{1,2}$-norm

Nuclear norm

ℓ_1-norm
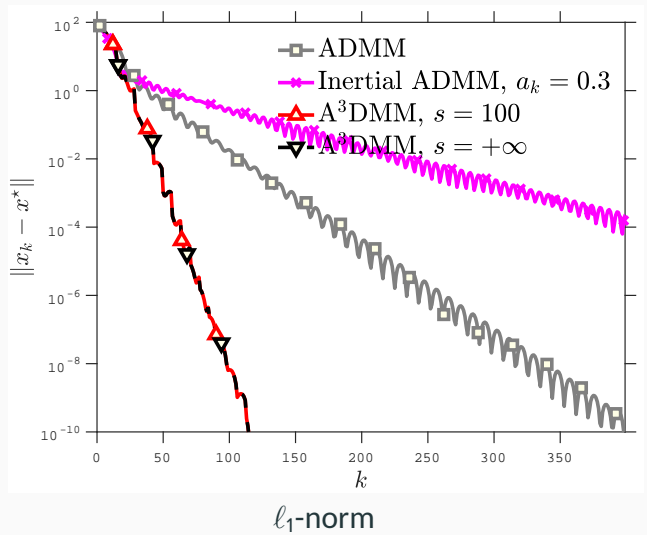
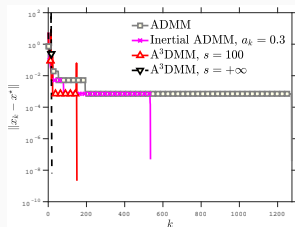Inertial ADMM is **slower** than ADMM as eventual trajectory is a spiral.
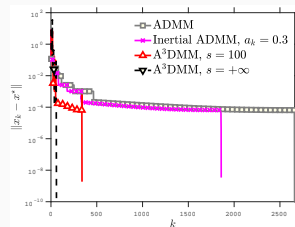
Consider the LASSO problem

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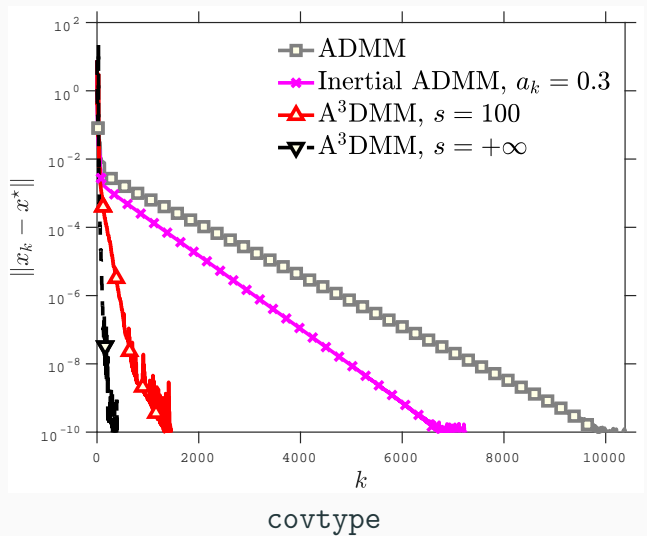

covtype
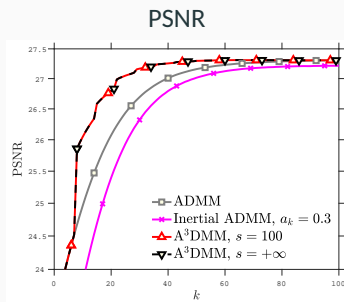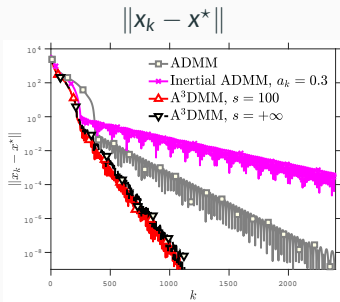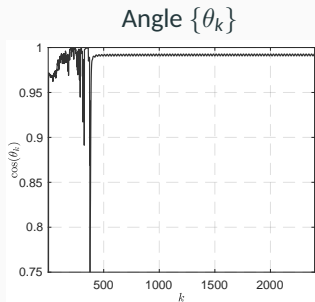

ijcnn1

phishing

covtype

Inertial ADMM does accelerate, but A³DMM is significantly faster.

Experiment: Total variation based image inpainting

Let $\Omega \stackrel{\text{def.}}{=} \{x \in \mathbb{R}^{n \times n} ; P_{\mathcal{D}}(x) = f\}$, $P_{\mathcal{D}}$ randomly sets 50% pixels to zero and consider

$$\min_{x \in \mathbb{R}^{n \times n}} \|y\|_1 + \iota_{\Omega}(x) \quad \text{such that} \quad \nabla x - y = 0.$$

Angle $\{\theta_k\}$ $\quad \|x_k - x^\star\|$ $\quad$ PSNR

- Both functions are polyhedral, trajectory is a spiral.
- Inertial ADMM is **slower** than ADMM.

Experiment: Total variation based image inpainting

Original image

ADMM, PSNR = 26.6935

Inertial ADMM, PSNR = 26.3203

Corrupted image

A^3DMM $s = 100$, PSNR = 27.1668

A^3DMM $s = +\infty$, PSNR = 27.1667

Summary of contributions

Trajectory of ADMM For sequence $\{z_k\}_{k \in \mathbb{N}}$

- When both R and J are locally polyhedral around the fixed point, $\{z_k\}_{k \in \mathbb{N}}$ eventually moves along a **spiral**.
- When at least one of R or J is smooth, the trajectory of $\{z_k\}_{k \in \mathbb{N}}$ depends on γ and can be either a spiral or a **straight line**.

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An adaptive acceleration for ADMM

- The different trajectory behaviour of ADMM can lead to the **failure** of the inertial technique.
- We propose an acceleration strategy based on the idea of following the sequence trajectory.

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Poster: East Exhibition Hall B+C #115!