Introductory Course on Non-smooth Optimisation

Lecture 00 - Introduction of introduction

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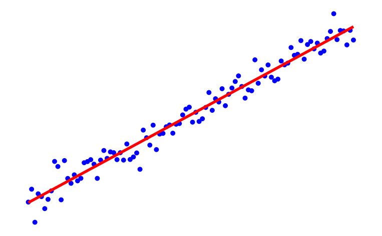
Example: least square estimation



Given cluster of points $(h_i, v_i)_{i=1,\ldots,m}$, find a line $v = ah + b$ such that it minimises

$$\sum_{i=1}^{m} \|ah_i + b - v_i\|^2.$$

Let

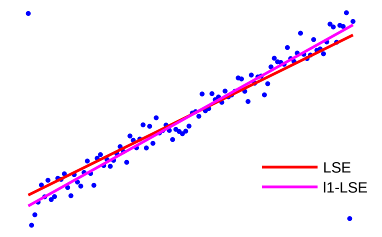
$$A = \begin{bmatrix} \vdots & \vdots \\ h_i & 1 \\ \vdots & \vdots \end{bmatrix}, \quad x = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} \vdots \\ v_i \\ \vdots \end{pmatrix},$$

then the above problem is equivalent to

$$\min_{x \in \mathbb{R}^2} \|Ax - w\|^2.$$

Closed form solution if $A^T A$ has full rank: $x^\star = (A^T A)^{-1} A^T w$.

Example: least square with outliers



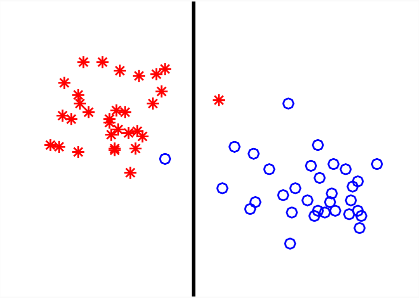
LSE is not robust to outliers. To deal with outliers which are sparse, consider

$$\min_{x \in \mathbb{R}^2} \|Ax - w\|_1.$$

Challenge: non-smooth, and no closed form solution.

Solvers: ADMM (alternating direction methods of multipliers)...

Example: sparse logistic regression



Given two cluster of points $(z_i, y_i) \in \mathbb{R}^n \times \{\pm 1\}, i = 1, ..., m$, find a separation hyperplane via

$$\min_{(b,x)\in\mathbb{R}\times\mathbb{R}^n} \mu\|x\|_1 + \frac{1}{m}\sum_{i=1}^m f(\langle x,\ z_i\rangle + b, y_i),$$

where $f(u_i, y_i) = \log(1 + e^{-u_i y_i})$.

Requirement: x is sparse, that is x has few non-zero elements.

Example: image deblur

x_{ob}

w

recovered x

How to blur an image

$$w = H * x_{\mathrm{ob}} + \omega,$$

where $H \in \mathbb{R}^{m \times n}$ is blur kernel, $\omega \in \mathbb{R}^m$ is additive noise.

How to deblur? Sharp edges are the most important part of images.

$$\min_{x \in \mathbb{R}^{m \times n}} \mu \|\nabla x\|_1 + \frac{1}{2}\|H * x - w\|^2.$$

$\|\nabla x\|$ promotes sharp edges: out of the solutions of the LSE $\frac{1}{2}\|H * x - w\|^2$, finding the one x which as “proper” sharp edges...

w $\qquad$ x_l $\qquad$ x_s

Forward mixture model,

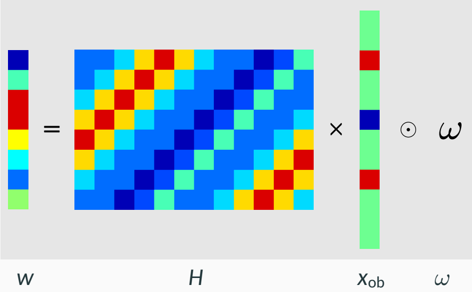
$$w = x_{\mathrm{ob},l} + x_{\mathrm{ob},s} + \omega,$$

$x_{\mathrm{ob},l} \in \mathbb{R}^{m\times n}$ is low-rank, $x_{\mathrm{ob},s} \in \mathbb{R}^{m\times n}$ is sparse and $\omega \in \mathbb{R}^{m\times n}$ is noise.

How to decompose w into low-rank part plus sparse part?

$$\min_{x_l, x_s \in \mathbb{R}^{m\times n}} \mu_1 \|x_s\|_1 + \mu_2 \|x_l\|_* + \frac{1}{2}\|x_s + x_l - w\|_F^2.$$

$\|x_l\|_* = \sum_i \sigma_i$, where $(\sigma_i)_{i=1,\ldots,\mathrm{rank}(x_l)}$ are the singular values of x_l.



Forward model:

$$w = (Hx_{\mathrm{ob}}) \odot \omega.$$

Goal: recover x_{ob}

Challenge: ill-posed

Hope: prior knowledge of x_{ob}

- Regularisation: promoting low-complexity structure to the solution...
- Examples:

Sparsity	ℓ_1-norm, $\ell_{1,2}$-norm, ℓ_p-norm, ℓ_0 pseudo-norm
Analysis sparsity	total variation, wavelet, dictionary...
Low-rank	nuclear norm, rank function
Constraints	simplex, non-negativity...
Nerual networks	CNN...

Optimisation problem

Least square

$$\min_{x\in\mathbb{R}^n} \|Ax - w\|^2.$$

Least square with outliers

$$\min_{x\in\mathbb{R}^n} \|Ax - w\|_1.$$

Sparse logistic regression

$$\min_{x\in\mathbb{R}^n} \mu\|x\|_1 + \frac{1}{m}\textstyle\sum_{i=1}^m f(\langle x,\, z_i\rangle + b, y_i).$$

Image deblur

$$\min_{x\in\mathbb{R}^{m\times n}} \mu\|\nabla x\|_1 + \frac{1}{2}\|H * x - w\|^2.$$

Principal component pursuit

$$\min_{x_\mathrm{l}, x_\mathrm{s}\in\mathbb{R}^{m\times n}} \mu_1\|x_\mathrm{s}\|_1 + \mu_2\|x_\mathrm{l}\|_* + \frac{1}{2}\|w - x_\mathrm{l} - x_\mathrm{s}\|^2.$$

Challenges

Optimisation problems

$$\min_{x\in\mathbb{R}^n} \; F(x) + R(x),$$

$$\text{subject to} \;\; x \in \Omega.$$

Example: $F = \frac{1}{2}\|Ax - w\|^2$, $R = \|x\|_1$ and $\Omega = \{x | f_i(x) \leq b_i, i = 1, ..., m\}$.

- No closed form solution: iterative scheme
- Non-smooth: difficult to evaluate
- Non-linear: linearisation or approximation
- Non-convex: no global minimiser guarantee
- Composite: need proper numerical schemes
- High dimension: computational demanding
- Others: hardware limitation *e.g.* mobile devices

Goals

- recognise/formulate problems as optimisation problems
- characterise the property of solution
- familiar with and able to apply first-order methods

Contents

- convex analysis and set-valued analysis
- first-order methods
- examples and applications

NB: rigorous mathematical proofs will not be the focus of this course...

First-order methods

Brief history

- Origins from numerical PDE dates back to 1950s
- Received attention since 1970s
- Tremendous development since new century...

Applications

- Before 1990, mostly in operation research (*e.g.* linear programming)
- Since 1990, becomes ubiquitous in signal/image processing, inverse problems, data science, statistics, machine learning, game theory...

First-order methods

First-order methods (FoM)...

- F Gradient descent, Heavy-ball
- R Proximal Point Algorithm (PPA), inertial PPA
- $F + R$ Forward–Backward splitting (FB), inertial FB, Nesterov/FISTA
  $F = \frac{1}{m}\sum_i f_i$: stochastic gradient methods
- $R_1 + R_2$ Douglas–Rachford splitting
- $F + R(\mathcal{W}\cdot)$ Class of Primal–Dual splitting
  Alternating Direction Method of Multipliers (ADMM)
- $F + \sum_{i=1}^{r} R_i$ Three-operator splitting ($r = 2$)
  Forward–Douglas–Rachford ($r = 2, R_2 = \iota_{\mathcal{V}}(\cdot)$)
  Generalized Forward–Backward splitting ($r \geq 2$)
- $\cdots$

Schedule

- Convex optimisation: 12 lectures
- Non-convex optimisation: 2 lectures
- Stochastic optimisation: 2 lectures

Projects

- Convex optimisation: about 4
- Non-convex optimisation: 1
- Stochastic optimisation: 1

Only 1 or 2 projects will be mandatory, and they will be done in groups.

Programming language: MATLAB, Python.

References

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