# Introductory Course on Non-smooth Optimisation

Lecture 05

Peaceman–Rachford, Douglas–Rachford splitting

# Outline

# Outline

# Sum of two operators

## Problem

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in A(x) + B(x).$$

## Assumptions

- $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are maximal monotone
- $\operatorname{zer}(A + B) \neq \emptyset$

# Outline

# Peaceman–Rachford splitting

**Peaceman–Rachford splitting**

Let $z_0 \in \mathbb{R}^n$, $\gamma > 0$:

$$\begin{aligned} x_k &= \mathcal{J}_{\gamma B}(z_k) \\ y_k &= \mathcal{J}_{\gamma A}(2x_k - z_k) \\ z_{k+1} &= z_k + 2(y_k - x_k) \end{aligned}$$

- dates back to 1950s for solving numerical PDEs
- the resolvents of $A, B$ are evaluated separately

# How to derive

- given $x^\star \in \operatorname{zer}(A+B)$, there exists $z^\star \in \mathbb{R}^n$ such that

$$\begin{cases} z^\star - x^\star \in \gamma A(x^\star) \\ x^\star - z^\star \in \gamma B(x^\star) \end{cases} \implies \begin{cases} \qquad z^\star \in x^\star + \gamma A(x^\star) \\ 2x^\star - z^\star \in x^\star + \gamma B(x^\star) \end{cases}$$

- apply the resolvent

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star) \\ x^\star = \mathcal{J}_{\gamma B}(2x^\star - z^\star) \end{cases}$$

- equivalent formulation

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star) \\ z^\star = z^\star + 2\big(\mathcal{J}_{\gamma B}(2x^\star - z^\star) - x^\star\big) \end{cases}$$

- iteration

$$\begin{cases} \quad x_k = \mathcal{J}_{\gamma A}(z_k) \\ z_{k+1} = z_k + 2\big(\mathcal{J}_{\gamma B}(2x_k - z_k) - x_k\big) \end{cases}$$

# Fixed-point characterisartion

**Fixed-point formulation** Recall reflection operator $\mathcal{R}_{\gamma A} = 2\mathcal{J}_{\gamma A} - \mathrm{Id}$.

- $y_k = \mathcal{J}_{\gamma A}(2x_k - z_k) = \mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k)$
- For $z_k$,

$$\begin{aligned} z_{k+1} &= z_k + 2(y_k - x_k) \\ &= z_k + 2\big(\mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - \mathcal{J}_{\gamma B}(z_k)\big) \\ &= 2\mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) \\ &= (2\mathcal{J}_{\gamma A} - \mathrm{Id}) \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) \end{aligned}$$

**Property**

- $\mathcal{R}_{\gamma A} = 2\mathcal{J}_{\gamma A} - \mathrm{Id}, \mathcal{R}_{\gamma B} = 2\mathcal{J}_{\gamma B} - \mathrm{Id}$ are non-expansive
- $\mathcal{T}_{\mathrm{PR}} = \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B}$ is non-expansive

**NB**: Cannot guarantee convergence in general

# Convergence

- Uniform monotonicity: $\phi : \mathbb{R}_+ \to [0, +\infty]$ is increasing and vanishes only at 0

$$\langle u - v, x - y \rangle \geq \phi(\|x - y\|), \ (x, u), (y, v) \in \mathrm{gra}(B)$$

- If $B$ is uniformly monotone, then $\mathrm{zer}(A + B) = \{x^\star\}$ and $\mathrm{fix}(\mathcal{T}_{\mathrm{PR}}) \neq \emptyset$. Moreover

$$\langle x - y, \mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y) \rangle \geq \|\mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y)\|^2 + \gamma\phi(\|\mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y)\|)$$

- Let $z^\star \in \mathrm{fix}(\mathcal{T}_{\mathrm{PR}})$, then $x^\star = \mathcal{J}_{\gamma A}(z^\star)$, and

$$\begin{aligned}
&\|z_{k+1} - z^\star\|^2 \\
&= \|\mathcal{R}_{\gamma A}\mathcal{R}_{\gamma B}(z_k) - \mathcal{R}_{\gamma A}\mathcal{R}_{\gamma B}(z^\star)\|^2 \\
&\leq \|(2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z^\star)\|^2 \\
&= \|z_k - z^\star\|^2 - 4\langle z_k - z^\star, \mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star) \rangle + 4\|\mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star)\|^2 \\
&\leq \|z_k - z^\star\|^2 - 4\gamma\phi(\|\mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star)\|)
\end{aligned}$$

- $\phi(\|z_k - z^\star\|) \to 0$ and $\|z_k - z^\star\| \to 0$.

# Outline

# Douglas–Rachford splitting

To overcome the problem of Peaceman–Rachford splitting.

**Douglas–Rachford splitting**

Let $z_0 \in \mathbb{R}^n$, $\gamma > 0$, $\lambda \in ]0, 2[$:

$$\begin{aligned} x_k &= \mathcal{J}_{\gamma B}(z_k) \\ y_k &= \mathcal{J}_{\gamma A}(2x_k - z_k) \\ z_{k+1} &= z_k + \lambda(y_k - x_k) \end{aligned}$$

- ADMM is closely related with Douglas–Rachford (next week)

# How to derive

- given $x^\star \in \mathrm{zer}(A+B)$, there exists $z^\star \in \mathbb{R}^n$ such that
$$\begin{cases} z^\star - x^\star \in \gamma A(x^\star) \\ x^\star - z^\star \in \gamma B(x^\star) \end{cases} \implies \begin{cases} z^\star \in x^\star + \gamma A(x^\star) \\ 2x^\star - z^\star \in x^\star + \gamma B(x^\star) \end{cases}$$
- apply the resolvent
$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star) \\ x^\star = \mathcal{J}_{\gamma B}(2x^\star - z^\star) \end{cases}$$
- equivalent formulation
$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star) \\ z^\star = z^\star + \left(\mathcal{J}_{\gamma B}(2x^\star - z^\star) - x^\star\right) \end{cases}$$
- iteration
$$\begin{cases} x_k = \mathcal{J}_{\gamma A}(z_k) \\ z_{k+1} = z_k + \left(\mathcal{J}_{\gamma B}(2x_k - z_k) - x_k\right) \end{cases}$$

# Fixed-point characterisartion

**Fixed-point formulation** Same as PR, $y_k = \mathcal{J}_{\gamma A} \circ \mathcal{R}_{\gamma B}(z_k)$

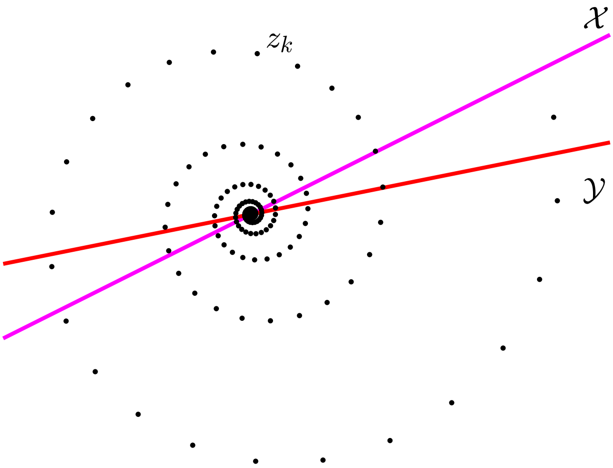
$$
\begin{aligned}
z_{k+1} &= (1-\lambda)z_k + \lambda\big(z_k + (y_k - x_k)\big) \\
&= (1-\lambda)z_k + \lambda\big(\tfrac{1}{2}z_k + \tfrac{1}{2}(z_k + 2(y_k - x_k))\big) \\
&= (1-\lambda)z_k + \lambda\tfrac{1}{2}(\mathrm{Id} + \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B})(z_k)
\end{aligned}
$$

**Property**

- $\mathcal{T}_{\mathrm{DR}} = \frac{1}{2}(\mathrm{Id} + \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B})$ is firmly non-expansive
- $\mathcal{T}_{\mathrm{DR}}^{\lambda} = (1-\lambda)\mathrm{Id} + \lambda\mathcal{T}_{\mathrm{DR}}$ is $\frac{\lambda}{2}$-averaged non-expansive
- Peaceman–Rachford is the limiting case of Douglas–Rachford, $\lambda = 2$

**NB**: guaranteed convergence if $\lambda(2-\lambda) > 0$

# Convergence rate



# Convergence rate

- Let $\mathcal{X}, \mathcal{Y}$ be two subspaces

$$\mathcal{X} = \{x : Ax = 0\},\ \mathcal{Y} = \{x : Bx = 0\}$$

  and assume

$$1 \leq p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \leq q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \leq n - 1.$$

- Projection onto subspace

$$\mathrm{proj}_{\mathcal{X}}(x) = x - A^T(AA^T)^{-1}Ax$$

- Define diagonal matrices

$$c = \mathrm{diag}\big(\cos(\theta_1), \cdots, \cos(\theta_p)\big)$$
$$s = \mathrm{diag}\big(\sin(\theta_1), \cdots, \sin(\theta_p)\big)$$

## Convergence rate

- Suppose $p+q<n$, then there exists orthogonal matrix $U$ such that

$$\mathrm{proj}_{\mathcal{X}} = U \left[\begin{array}{cc|cc} \mathrm{Id}_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

and

$$\mathrm{proj}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ cs & c^2 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

## Convergence rate

- For the composition

$$\mathrm{proj}_{\mathcal{X}} \circ \mathrm{proj}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

  and

$$\mathrm{proj}_{\mathcal{X}^\perp} \circ \mathrm{proj}_{\mathcal{Y}^\perp} = U \left[\begin{array}{cc|cc} 0_p & 0 & 0 & 0 \\ -cs & c^2 & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-p-q} \end{array}\right] U^*$$

# Convergence rate

- Fixed-point operator

$$
\begin{aligned}
\mathcal{T}_{\text{DR}} &= \text{proj}_{\mathcal{X}} \circ \text{proj}_{\mathcal{Y}} + \text{proj}_{\mathcal{X}^\perp} \circ \text{proj}_{\mathcal{Y}^\perp} \\
&= U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ -cs & c^2 & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & \text{Id}_{n-p-q} \end{array}\right] U^*
\end{aligned}
$$

- Consider relaxation

$$
\begin{aligned}
\mathcal{T}_{\text{DR}}^{\lambda} &= (1-\lambda)\text{Id} + \lambda \mathcal{T}_{\text{DR}} \\
&= U \left[\begin{array}{cc|cc} \text{Id}_p - \lambda s^2 & \lambda cs & 0 & 0 \\ -\lambda cs & \text{Id}_p - \lambda s^2 & 0 & 0 \\ \hline 0 & 0 & (1-\lambda)\text{Id}_{q-p} & 0 \\ 0 & 0 & 0 & \text{Id}_{n-p-q} \end{array}\right] U^*
\end{aligned}
$$

# Convergence rate

- Eigenvalues

$$\sigma(\mathcal{T}_{\text{DR}}^{\lambda}) = \begin{cases} \{1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i) | i=1,...,p\} \cup \{1\} : q = p \\ \{1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i) | i=1,...,p\} \cup \{1\} \cup \{1-\lambda\} : q > p \end{cases}$$

- Complex eigenvalues

$$|1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i)| = \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2}$$

  and

$$1 \geq \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2} \geq |1-\lambda|$$

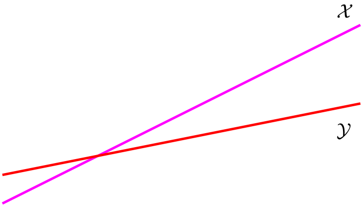
- $\lim_{k\to+\infty} \mathcal{T}_{\text{DR}}^k = \mathcal{T}_{\text{DR}}^{\infty}$ and $z_k - z^\star = (\mathcal{T}_{\text{DR}} - \mathcal{T}_{\text{DR}}^{\infty})(z_{k-1} - z^\star)$
- Spectral radius, minimises at $\lambda = 1$

$$\rho(\mathcal{T}_{\text{DR}} - \mathcal{T}_{\text{DR}}^{\infty}) = \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2}$$

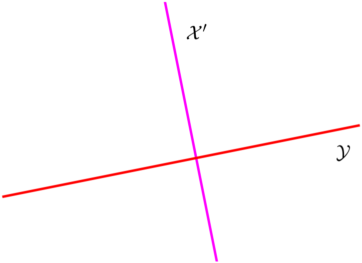
- $\widetilde{\mathcal{T}}_{\text{DR}} = \mathcal{T}_{\text{DR}} - \mathcal{T}_{\text{DR}}^{\infty}$

$$\begin{aligned}\|z_k - z^\star\| = \|\widetilde{\mathcal{T}}_{\text{DR}} z_{k-1} - \widetilde{\mathcal{T}}_{\text{DR}} z^\star\| = \dots = \|\widetilde{\mathcal{T}}_{\text{DR}}^{\,k}(z_0 - z^\star)\| \\ \leq C\big(\rho(\widetilde{\mathcal{T}}_{\text{DR}})\big)^k \|z_0 - z^\star\|\end{aligned}$$

# Optimal metric for DR




$\mathcal{X}$ and $\mathcal{Y}$




$\mathcal{X}'$ and $\mathcal{Y}$

**Optimal metrix** A invertable operation which makes the Friedrichs angle between $\mathcal{X}'$ and $\mathcal{Y}$ the largest, *e.g.* $\frac{\pi}{2}$...

# Outline

# More than two operators

**Problem** $s \in \mathbb{N}_+$ and $s \geq 2$

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in \sum_i A_i(x).$$

**Assumptions**

- for each $i = 1, ..., s$, $A_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone
- $\text{zer}(\sum_i A_i) \neq \emptyset$

# Product space

- Let $\boldsymbol{\mathcal{H}} = \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s \text{ times}}$ endowed with the scalar inner-product and norm

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{\mathcal{H}},\ \langle\!\langle \boldsymbol{x}, \boldsymbol{y} \rangle\!\rangle = \sum_{i=1}^{s} \langle x_i,\, y_i \rangle,\ |\!|\!| \boldsymbol{x} |\!|\!| = \sqrt{\sum_{i=1}^{s} \|x_i\|^2}.$$

- Let

$$\boldsymbol{\mathcal{S}} = \{\boldsymbol{x} = (x_i)_i \in \boldsymbol{\mathcal{H}} : x_1 = \cdots = x_s\}$$

and its orthogonal complement

$$\boldsymbol{\mathcal{S}}^\perp = \{\boldsymbol{x} = (x_i)_i \in \boldsymbol{\mathcal{H}} : \sum_{i=1}^{s} x_i = 0\}.$$

# Equivalent formulation

Define $\boldsymbol{A}$ by

$$\boldsymbol{A}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \to A_1(x_1) \times \cdots \times A_s(x_s).$$

## Lifted problem

$$\text{Find } \ \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \ \text{ such that } \ 0 \in \boldsymbol{A}(\boldsymbol{x}) + \mathcal{N}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}).$$

- the resolvent of $\boldsymbol{A}$ is seperable, *i.e.* $\mathcal{J}_{\gamma \boldsymbol{A}} = (\mathcal{J}_{\gamma A_i})_i$
- define the canonical isometry,

$$\boldsymbol{C} : \mathbb{R}^n \to \boldsymbol{\mathcal{S}},\ x \mapsto (x, \cdots, x),$$

then $\mathrm{proj}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{z}) = \boldsymbol{C}(\frac{1}{m} \sum_{i=1}^{s} z_i)$.

# Outline

# Problem

**DR in product space** for $\boldsymbol{x}^\star \in \mathcal{S}$, $\exists -\boldsymbol{v} \in \mathcal{S}$ such that

$$-\boldsymbol{v} \in \mathcal{S}^\perp = \mathcal{N}_{\mathcal{S}}(\boldsymbol{x}^\star) \quad \text{and} \quad \boldsymbol{v} \in \boldsymbol{A}(\boldsymbol{x}^\star)$$

**Problem** $V$ is a close subspace

$$\text{Find } x \in V \text{ and } v \in V^\perp \text{ such that } v \in A(x).$$

**Assumptions**

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone
- admits at least one solution

# Partial inverse

**Partial inverse**

Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued and $V \subseteq \mathbb{R}^n$ be a closed subspace. The partial inverse of $A$ respect to $V$ is the operator $A_V : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ define by

$$\operatorname{gra}(A_V) = \left\{ \left( \operatorname{proj}_V(x) + \operatorname{proj}_{V^\perp}(u), \operatorname{proj}_{V^\perp}(x) + \operatorname{proj}_V(u) \right) : (x, u) \in \operatorname{gra}(A) \right\}.$$

**Example** Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, then $A_{\mathbb{R}^n} = A$ and $A_{\{0\}} = A^{-1}$.

# Spingarn's method of partial inverses

An application of Proximal Point Algorithm.

**Spingarn**

Let $x_0 \in V$, $u_0 \in V^{\perp}$:

$$y_k = \mathcal{J}_A(x_k + u_k)$$

$$v_k = x_k + u_k - y_k$$

$$(x_{k+1}, u_{k+1}) = \left(\mathrm{proj}_V(y_k), \mathrm{proj}_{V^{\perp}}(v_k)\right)$$

# Fixed-point characterisation

- define mapping

  $$L : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n : (x, u) \to \big(\mathrm{proj}_V(x) + \mathrm{proj}_{V^\perp}(u), \mathrm{proj}_{V^\perp}(x) + \mathrm{proj}_V(u)\big)$$

-
  $$\begin{aligned} p = \mathcal{J}_{A_V}(x) \;&\Leftrightarrow\; (p, x - p) \in \mathrm{gra}(A_V) \\ &\Leftrightarrow\; L(p, x - p) \in L\big(\mathrm{gra}(A_V)\big) = \mathrm{gra}(A) \\ &\Leftrightarrow\; \big(\mathrm{proj}_V(p) + \mathrm{proj}_{V^\perp}(x - p), \mathrm{proj}_V(x - p) + \mathrm{proj}_{V^\perp}(p)\big) \in \mathrm{gra}(A) \end{aligned}$$

- let $q = \mathrm{proj}_V(p) + \mathrm{proj}_{V^\perp}(x - p)$

  $$\begin{aligned} p = \mathcal{J}_{A_V}(x) \;&\Leftrightarrow\; x - q = \mathrm{proj}_V(x - p) + \mathrm{proj}_{V^\perp} p \in A(q) \\ &\Leftrightarrow\; q = \mathcal{J}_A(x) \end{aligned}$$

# Fixed-point characterisation

- let $z_k = x_k + u_k$, since $x_k \in V$ and $u_k \in V^\perp$

$$\begin{aligned}
&\mathrm{proj}_V(z_{k+1}) + \mathrm{proj}_{V^\perp}(z_k - z_{k+1}) \\
&= x_{k+1} + \mathrm{proj}_{V^\perp}(u_k) - u_{k+1} \\
&= \mathrm{proj}_V(y_k) + \mathrm{proj}_{V^\perp}(v_k - x_k + y_k) - \mathrm{proj}_{V^\perp}(v_k) \\
&= \mathrm{proj}_V(y_k) + \mathrm{proj}_{V^\perp}(v_k) + \mathrm{proj}_{V^\perp}(y_k) - \mathrm{proj}_{V^\perp}(v_k)
\end{aligned}$$

- $z_{k+1} = \mathcal{J}_A(z_k)$

# Outline

# Inertial DR splitting

**An inertial DR splitting**

**Initial** : $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0$;

$$y_k = z_k + a_{0,k}(z_k - z_{k-1}) + a_{1,k}(z_{k-1} - z_{k-2}) + \cdots ,$$

$$z_{k+1} = \mathcal{T}_{\mathrm{DR}}(y_k)$$

- relaxation can be applied

# Outline

# Example: basis pursuit

## Basis pursuit

$$\min_{x\in\mathbb{R}^n} \quad \|x\|_1$$
$$\text{such that} \quad Ax = b$$

- $A : \mathbb{R}^n \to \mathbb{R}^m$ with $m << n$
- $b \in \mathrm{Img}(A)$

# Example: image inpainting

## Image inpainting

$$\min_{X \in \mathbb{R}^{n \times n}} \quad \|WX\|_1$$
$$\text{such that} \quad \mathrm{proj}_\Omega(X) = \bar{X}$$

- $W$: total variation operator, orthonomal basis, redundant wavelet frame
- Observation constraint

$$\left(\mathrm{proj}_\Omega(X)\right)_{i,j} = \begin{cases} \bar{X}_{i,j} : (i,j) \in \Omega \\ 0 : (i,j) \notin \Omega \end{cases}$$

- Painting reconstruction in museum

# Example: matrix completion

## Matrix completion

$$\begin{aligned} \min_{X \in \mathbb{R}^{n \times n}} \quad & \|X\|_* \\ \text{such that} \quad & \mathrm{proj}_\Omega(X) = \bar{X} \end{aligned}$$

- Observation constraint

$$\left(\mathrm{proj}_\Omega(X)\right)_{i,j} = \begin{cases} \bar{X}_{i,j} : (i,j) \in \Omega \\ 0 : (i,j) \notin \Omega \end{cases}$$

- Netflix prize, recommendation system

# Example: variation ineuality

## Variation ineuality

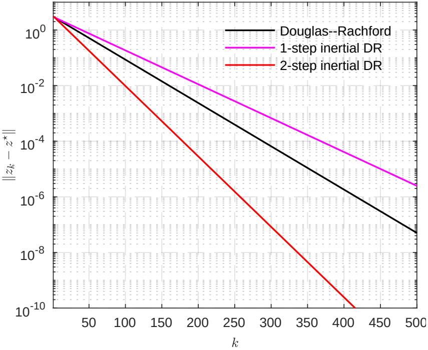
$$\text{Find } x \in \mathbb{R}^n \ \text{ such that } \quad \exists u \in A(x),\ \forall y \in \mathbb{R}^n:\ \langle x - y,\, u\rangle + R(x) \le R(y).$$

- $R \in \Gamma_0$
- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone

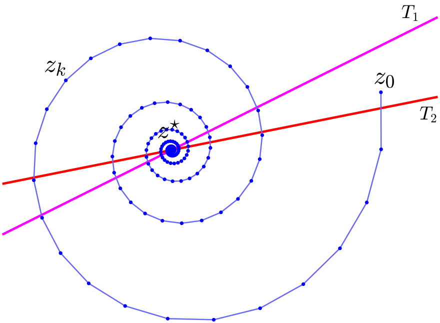
**Example** Let $R, J \in \Gamma_0$, and $x^\star \in \mathrm{Argmin}(R + J)$, then $\exists u \in \partial J(x^\star)$ s.t. $-u \in \partial R(x^\star)$ and

$$\begin{aligned} & \langle y - x^\star,\, -u\rangle + R(x^\star) \le R(y) \\ \iff\ & \langle x^\star - y,\, u\rangle + R(x^\star) \le R(y) \end{aligned}$$

# Numerical experiment




Comparison




Tracjectory

# Reference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