Introductory Course on Non-smooth Optimisation

Lecture 09 - Non-convex optimisation

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Forward observation

$$b = A\mathring{x},$$

- $\mathring{x} \in \mathbb{R}^n$ is sparse.
- $A : \mathbb{R}^n \to \mathbb{R}^m$ with $m << n$.

Compressed sensing

$$\min_{x\in\mathbb{R}^n} \|x\|_0 \quad \text{s.t.} \quad Ax = b.$$

NB: NP-hard problem.

Two-phase segmentation Given an image I, which consists of foreground and background, segment the foreground. Ideally,

$$I = f_C + b_{\Omega \setminus C}.$$

Mumford–Shah model

$$E(u, C) = \int_\Omega (u - I)^2 \mathrm{d}x + \lambda \left(\int_{\Omega \setminus C} \|\nabla u\|^2 \mathrm{d}x + \alpha |C| \right),$$

where $|C| = \mathrm{peri}(C)$.

Principal component pursuit

Forward mixture model

$$w = \mathring{x} + \mathring{y} + \epsilon,$$

where $\mathring{x} \in \mathbb{R}^{m\times n}$ is κ-sparse, $\mathring{y} \in \mathbb{R}^{m\times n}$ is σ-low-rank and ϵ is noise.

Non-convex PCP

$$\begin{aligned} \min_{x,y\in\mathbb{R}^{m\times n}} \quad & \frac{1}{2}\|x + y - w\|^2 \\ \text{s.t.} \quad & \|x\|_0 \leq \kappa \quad \text{and} \quad \mathrm{rank}(y) \leq \sigma. \end{aligned}$$

Each layer of NNs is convex

- Linear operation, *e.g.* convolution.
- Non-linear activation function, *e.g.* rectifier $\max\{x, 0\}$.

The composition of convex functions is not necessarily convex...

- Neural networks are universal function approximators.
- Hence need to approximate non-convex functions.
- Cannot approximate non-convex functions with convex functions.

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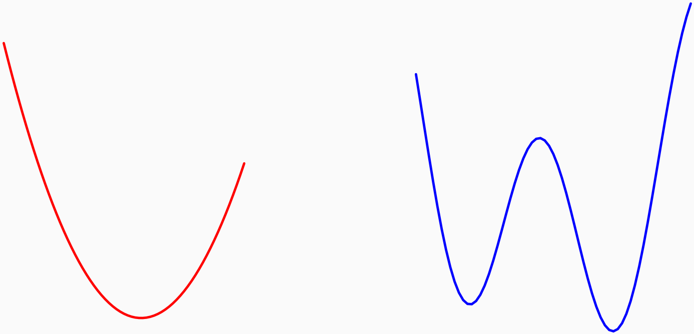
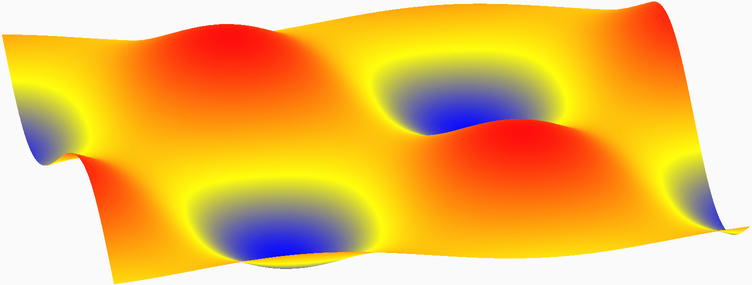
Non-convex problem

Any problem that is not convex/concave is non-convex...



Challenges

- Potentially many local minima.
- Saddle points.
- Very flat regions.
- Widely varying curvature.
- NP-hard.



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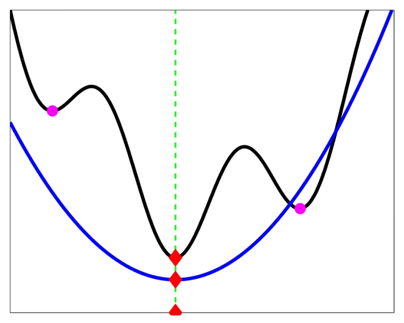
Convex relaxation

Non-convex optimisation problem

$$\min_x \; E(x).$$

Convex optimisation problem

$$\min_x \; F(x).$$

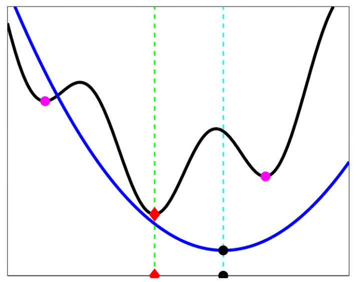


What if

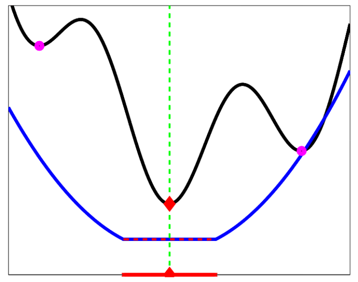
$$\mathrm{Argmin}(F) \subseteq \mathrm{Argmin}(E),$$

- Subtle and case-dependent.
- Somehow, finding F is almost equivalent to solving E.

Convex relaxation



Loose relaxation



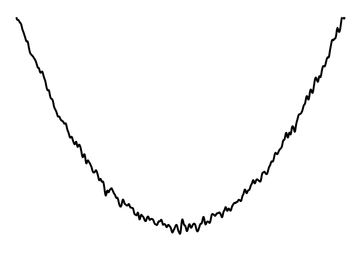
Ideal relaxation

- In practice, it is easier to obtain

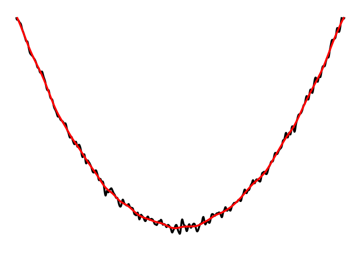
$$\mathrm{Argmin}(E) \subseteq \mathrm{Argmin}(F).$$

- Loose relaxation will **work** if two global minima are close enough.
- Ideal relaxation will **fail** if $\mathrm{Argmin}(F)$ is too large.

For certain problems, non-convexity can be treated as noise...



Original function



Convolution

- Symmetric boundary condition for the convolution.
- Almost convex problem after convolution.

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Smooth problem

Let $F \in C_L^1$.

- Gradient descent
$$x_{k+1} = x_k - \gamma \nabla F(x_k).$$
- Descent property
$$F(x_k) - F(x_{k+1}) \geq \gamma(1 - \tfrac{\gamma L}{2}) \|\nabla F(x_k)\|^2.$$
- Let $\gamma \in]0, 2/L[$,
$$\gamma(1 - \tfrac{\gamma L}{2}) \textstyle\sum_{i=0}^{k} \|\nabla F(x_i)\|^2 \leq F(x_0) - F(x_{k+1}) \leq F(x_0) - F(x^\star).$$
- $F(x^\star) > -\infty$, rhs is a positive constant.
- for lhs, let $k \to +\infty$,
$$\lim_{k \to +\infty} \|\nabla F(x_k)\|^2 = 0.$$

NB: for smooth case, a critical point is guarantee. For non-smooth problem...

Semi-algebraic set

A semi-algebraic subset of $\mathbb{R}^n$ is a finite union of sets of the form

$$\big\{x \in \mathbb{R}^n : f_i(x) = 0,\ g_j(x) \leq 0,\ i \in I, j \in J\big\}$$

where I, J are finite and $f_i, g_j : \mathbb{R}^n \to \mathbb{R}$ are real polynomial functions.

- Stability under finite $\cap, \cup$ and complementation.

Semi-algebraic set

A function or a mapping is semi-algebraic if its graph is a semi-algebraic set.

- Same definition for real-extended function or multivalued mappings.

Tarski-Seidenberg

The image of a semi-algebraic set by a linear projection is semi-algebraic.

- The closure of a semi-algebraic set A is semi-algebraic.

Example

- The graph of the derivative of a semi-algebraic function is semi-algebraic.
- Let A be a semi-algebraic subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^p$ semi-algebraic. Then $f(A)$ is semi-algebraic.
- $g(x) = \max\{F(x, y) : y \in S\}$ is semi-algebraic if F and S are semi-algebraic.
- Other examples

$$\min_x \frac{1}{2}\|Ax - b\|^2 + \mu\|x\|_p : p \text{ is rational},$$
$$\min_X \frac{1}{2}\|AX - B\|^2 + \mu\mathrm{rank}(X).$$

Convex subdifferential $R \in \Gamma_0(\mathbb{R}^n)$

$$\partial R(x) = \big\{ g : R(x') \geq R(x) + \langle g, x' - x \rangle, \forall x' \in \mathbb{R}^n \big\}.$$

Fréchet subdifferential

Given $x \in \mathrm{dom}(R)$, the Fréchet subdifferential $\hat{\partial} R(x)$ of R at x is the set of vectors v such that

$$\liminf_{x' \to x,\, x' \neq x} \frac{1}{\|x - x'\|} \big(R(x') - R(x) - \langle v, x' - x \rangle \big) \geq 0.$$

- If $x \notin \mathrm{dom}(R)$, then $\hat{\partial} R(x) = \emptyset$.

Limiting subdifferential

The limiting-subdifferential (or simply subdifferential) of R at x, written as $\partial R(x)$, reads

$$\partial R(x) \stackrel{\text{def}}{=} \{ v \in \mathbb{R}^n : \exists x_k \to x, R(x_k) \to R(x), v_k \in \hat{\partial} R(x_k) \to v \}.$$

- $\hat{\partial} R$ is convex and ∂R is closed.

Minimal norm subgradient

$$\|\partial R(x)\|_{-} = \min\{\|v\| : v \in \partial R(x)\}.$$

Critical points

- Fermat's rule: if x is a minimiser of R, then $0 \in \partial R(x)$.
- Conversely when $0 \in \partial R(x)$, the point x is called a critical point.

- When R is convex, any minimiser is a global minimiser.
- When R is non-convex
  - Local minima.
  - Local maxima.
  - Saddle point.

Sharpness

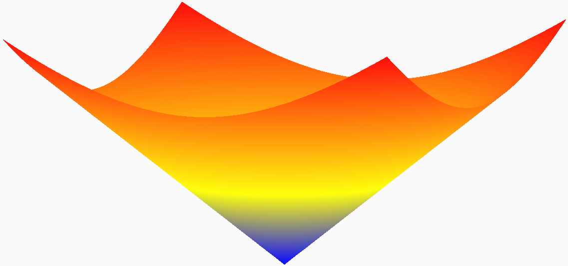
Function $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called sharp on the slice

$$[a < R < b] \stackrel{\text{def}}{=} \big\{x \in \mathbb{R}^n : a < f(x) < b\big\}.$$

If there exists $\alpha > 0$ such that

$$\|\partial R(x)\|_- \geq \alpha, \ \forall x \in [a < R < b].$$

- Norms, *e.g.* $R(x) = \|x\|$.



Łojasiewicz inequality

Let $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper lower semi-continuous, and moreover continuous along its domain. Then R is said to have Łojasiewicz property if: for any critical point $\bar{x}$, there exist $C, \epsilon > 0$ and $\theta \in [0, 1[$ such that

$$|R(x) - R(\bar{x})|^{\theta} \leq C\|v\|, \ \forall x \in \mathbb{B}_{\bar{x}}(\epsilon), \, v \in \partial R(x).$$

- By convention, let $0^0 = 0$.

Property

Suppose that R has Łojasiewicz property.

- If S is a connected subset of the set of critical points of R, that is $0 \in \partial R(x)$ for all $x \in S$, then R is constant on S.
- If in addition S is a compact set, then there exist $C, \epsilon > 0$ and $\theta \in [0, 1[$ such that

$$\forall x \in \mathbb{R}^n, \, \mathrm{dist}(x, S) \leq \epsilon, \, \forall v \in \partial R(x) : |R(x) - R(\bar{x})|^{\theta} \leq C\|v\|.$$

Proximal point algorithm

Let $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and lower semi-continuous. From arbitrary $x_0 \in \mathbb{R}^n$,

$$x_{k+1} \in \mathrm{argmin}_x\, \gamma R(x) + \frac{1}{2}\|x - x_k\|^2.$$

Assumption

- R is proper, that is

  $$\inf_{x \in \mathbb{R}^n} R(x) > -\infty.$$

  This implies

  $$\mathrm{argmin}_x\, \gamma R(x) + \frac{1}{2}\|x - x_k\|^2$$

  is non-empty and compact.
- The restriction of R to its domain is a continuous function.
- R has the Łojasiewicz property.

Property

Let $\{x_k\}_{k\in\mathbb{N}}$ be the sequence generated by non-convex PPA and $\omega(x_k)$ the set of its limiting points. Then

- Sequence $\{R(x_k)\}_{k\in\mathbb{N}}$ is decreasing.
- $\sum_k \|x_k - x_{k+1}\|^2 < +\infty$.
- If R satisfies assumption 2, then $\omega(x_k) \subset \mathrm{crit}(R)$.

If moreover, $\{x_k\}_{k\in\mathbb{N}}$ is bounded

- $\omega(x_k)$ is a non-empty compact set, and

$$\mathrm{dist}\big(x_k, \omega(x_k)\big) \to 0.$$

- If R satisfies assumption 2, then R is finite and constant on $\omega(x_k)$.

Convergence of PPA

Suppose the sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by non-convex PPA is bounded, then

$$\textstyle\sum_k \|x_k - x_{k+1}\| < +\infty,$$

and the whole sequence converges to some critical point $\bar{x} \in \mathrm{crit}(R)$.

- From definition of x_{k+1}: $R(x_{k+1}) + \frac{1}{2\gamma}\|x_k - x_{k+1}\|^2 \leq R(x_k)$.
- Consider $g(s) = s^{1-\theta}, s > 0$: $\nabla g(s) = (1-\theta)s^{-\theta}$

$$\begin{aligned} g\big(R(x_k)\big) - g\big(R(x_{k+1})\big) &\geq (1-\theta)\big(R(x_{k+1})\big)^{-\theta}\big(R(x_k) - R(x_{k+1})\big) \\ &\geq (1-\theta)\big(R(x_k)\big)^{-\theta}\frac{1}{2\gamma}\|x_k - x_{k+1}\|^2. \end{aligned}$$

- WLOG, assume $R(\bar{x}) = 0$ for $\bar{x} \in \omega(x_k)$. Let $v_k \in \partial R(x_k)$, then for all k large enough

$$0 < R(x_k)^{\theta} \leq C\|v_k\| = \frac{C}{\gamma}\|x_k - x_{k-1}\|.$$

- There exists $M > 0$

$$\frac{\|x_k - x_{k+1}\|^2}{\|x_k - x_{k-1}\|} \leq M\big(R(x_k)^{1-\theta} - R(x_{k+1})^{1-\theta}\big).$$

Convergence of PPA

Suppose the sequence $\{x_k\}_{k\in\mathbb{N}}$ generated by non-convex PPA is bounded, then

$$\textstyle\sum_k \|x_k - x_{k+1}\| < +\infty,$$

and the whole sequence converges to some critical point $\bar{x} \in \mathrm{crit}(R)$.

- Take $r \in]0, 1[$, if $\|x_k - x_{k+1}\| \geq r\|x_k - x_{k-1}\|$, then

$$\|x_k - x_{k+1}\| \leq \tfrac{M}{r}\big(R(x_k)^{1-\theta} - R(x_{k+1})^{1-\theta}\big).$$

- For all k large enough

$$\|x_k - x_{k+1}\| \leq r\|x_k - x_{k-1}\| + \tfrac{M}{r}\big(R(x_k)^{1-\theta} - R(x_{k+1})^{1-\theta}\big).$$

- There exists some $K > 0$, such that for $k \geq K$

$$\textstyle\sum_{i=K}^{k} \|x_i - x_{i+1}\| \leq \frac{r}{1-r}\|x_K - x_{K-1}\| + \frac{M}{r(1-r)}\big(R(x_K)^{1-\theta} - R(x_{k+1})^{1-\theta}\big).$$

- $R(x)$ is bounded from below. Take $k \to +\infty$...

Convergence rate

Suppose the convergence of the non-convex PPA is true. Denote θ the Łojasiewicz exponent of x_∞. The following statements hold

- If $\theta = 0$, then $\{x_k\}_{k\in\mathbb{N}}$ converges in finite number of steps.
- If $\theta \in]0, 1/2]$, then there exists $\eta \in]0, 1[$ such that
$$\|x_k - x_\infty\| = O(\eta^k).$$
- If $\theta \in]1/2, 1[$, then
$$\|x_k - x_\infty\| = O(k^{-\frac{1-\theta}{2\theta-1}}).$$

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Let $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper l.s.c. For a, b such that $-\infty < a < b < +\infty$,

$$[a < R < b] \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n : a < R(x) < b\}.$$

Kurdyka-Łojasiewicz inequality

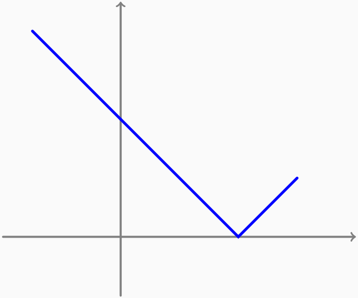
R is said to have the KL property at $\bar{x} \in \mathrm{dom}(R)$ if there exists $\eta \in]0, +\infty]$, a neighbourhood U of $\bar{x}$ and a continuous **concave** function $\varphi : [0, \eta[\to \mathbb{R}_+$ such that

- $\varphi(0) = 0$.
- φ is C^1 on $]0, \eta[$.
- for all $s \in]0, \eta[$, $\varphi'(s) > 0$.
- for all $x \in U \cap [R(\bar{x}) < R < R(\bar{x}) + \eta]$, the KL inequality holds

$$\varphi'\big(R(x) - R(\bar{x})\big)\mathrm{dist}\big(0, \partial R(x)\big) \geq 1.$$

- Proper lsc functions which satisfy KL at each point of $\mathrm{dom}(\partial R)$ are called KL functions.
- Proper l.s.c. functions are KL at non-critical points.
- Typical KL functions are the class of semi-algebraic functions.

$$\|\nabla F(x)\| \geq 0$$



$$\|\nabla(\varphi \circ F)(x)\| \geq 1$$

- When $R(\bar{x}) = 0$, then the condition becomes
$$\|\partial(\varphi \circ F)(x)\|_{-} \geq 1.$$
- φ is called a desingularising function for R, *i.e.* sharp up to reparameterization via φ.

Let Φ be proper and lower semi-continuous. Suppose a sequence $\{x_k\}_{k\in\mathbb{N}}$ is generated such that the following conditions are satisfied.

Conditions

Let $c, d > 0$ be some constants

A.1 **Sufficient decrease conditions** For each $k \in \mathbb{N}$,

$$\Phi(x_{k+1}) + c\|x_{k+1} - x_k\|^2 \leq \Phi(x_k).$$

A.2 **Relative error condition** For each $k \in \mathbb{N}$, there exists $g_{k+1} \in \partial\Phi(x_{k+1})$ such that

$$\|g_{k+1}\| \leq d\|x_{k+1} - x_k\|.$$

A.3 **Continuity condition** There exists a subsequence $\{x_{k_j}\}_{j\in\mathbb{N}}$ and $\bar{x}$ such that

$$x_{k_j} \to \bar{x}, \quad \Phi(x_{k_j}) \to \Phi(\bar{x}).$$

Convergence

Let $\Phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and l.s.c. and KL at some $\bar{x} \in \mathbb{R}^n$. Let U, η and φ be in the KL property. Let $\delta, \rho > 0$ be such that $\mathbb{B}_{\bar{x}}(\delta) \subset U$ with $\rho \in]0, \delta[$. Consider a sequence $\{x_k\}_{k\in\mathbb{N}}$ which satisfies (**A.1**)-(**A.2**). Suppose moreover

$$\Phi(\bar{x}) < \Phi(x_0) < \Phi(\bar{x}) + \eta,$$

$$\|x_0 - \bar{x}\| + 2\sqrt{\tfrac{\Phi(x_0)-\Phi(\bar{x})}{c}} + \tfrac{d}{c}\varphi(\Phi(x_0) - \Phi(\bar{x})) < \rho,$$

and

$$\forall k \in \mathbb{N},\ x_k \in \mathbb{B}_{\bar{x}}(\rho) \Rightarrow x_{k+1} \in \mathbb{B}_{\bar{x}}(\delta) \ \text{with} \ \Phi(x_{k+1}) \geq \Phi(\bar{x}).$$

Then the sequence $\{x_k\}_{k\in\mathbb{N}}$ satisfies

$$\begin{aligned}
&\forall k \in \mathbb{N},\ x_k \in \mathbb{B}_{\bar{x}}(\delta), \\
&\textstyle\sum_k \|x_k - x_{k+1}\| < +\infty, \\
&\Phi(x_k) \to \Phi(\bar{x}).
\end{aligned}$$

and converges to a point $x^\star \in \mathbb{B}_{\bar{x}}(\delta)$ such that $\Phi(x^\star) \leq \Phi(\bar{x})$.

If moreover, (**A.3**) is true, then $x^\star$ is a critical point and $\Phi(x^\star) = \Phi(\bar{x})$.

Convergence

- Condition (**A.1**) implies that $\{\Phi(x_k)\}_{k\in\mathbb{N}}$ is non-increasing, and for all $k \in \mathbb{N}$
$$\|x_{k+1} - x_k\| \leq \sqrt{\tfrac{\Phi(x_k)-\Phi(x_{k+1})}{c}}.$$
- Condition (**A.2**) and KL inequality
$$\varphi'\big(\Phi(x_k) - \Phi(\bar{x})\big) \geq \frac{1}{\|g_k\|} \geq \frac{1}{d\|x_k - x_{k-1}\|}.$$
- Since φ is concave,
$$\begin{aligned}\varphi\big(\Phi(x_k) - \Phi(\bar{x})\big) - \varphi\big(\Phi(x_k) - \Phi(\bar{x})\big) &\geq \varphi'\big(\Phi(x_k) - \Phi(\bar{x})\big)\big(\Phi(x_k) - \Phi(x_{k+1})\big) \\ &\geq \varphi'\big(\Phi(x_k) - \Phi(\bar{x})\big)c\|x_k - x_{k+1}\|^2.\end{aligned}$$
- Combining the above two yields
$$\frac{\|x_k - x_{k+1}\|^2}{\|x_k - x_{k-1}\|} \leq \frac{d}{c}\Big(\varphi\big(\Phi(x_k) - \Phi(\bar{x})\big) - \varphi\big(\Phi(x_k) - \Phi(\bar{x})\big)\Big).$$
- Apply the inequality $2\sqrt{xy} \leq x + y$,
$$2\|x_k - x_{k+1}\| \leq \|x_k - x_{k-1}\| + \frac{d}{c}\big(\varphi(\Phi(x_k) - \Phi(\bar{x})) - \varphi(\Phi(x_{k+1}) - \Phi(\bar{x}))\big).$$

Convergence

- Continue with (**A.1**),

$$\|x_1 - x_0\| \leq \sqrt{\tfrac{\Phi(x_0)-\Phi(x_1)}{c}} \leq \sqrt{\tfrac{\Phi(x_0)-\Phi(\bar{x})}{c}}.$$

- Then

$$\|x_1 - \bar{x}\| \leq \|x_1 - x_0\| + \|x_0 - \bar{x}\| \leq \|x_1 - x_0\| + \sqrt{\tfrac{\Phi(x_0)-\Phi(\bar{x})}{c}} \leq \rho.$$

- By induction, we can show that for all $k \in \mathbb{N}$

$$x_k \in \mathbb{B}_{\bar{x}}(\rho) \quad \text{and}$$

$$\textstyle\sum_{i=1}^{k} \|x_{i+1} - x_i\| + \|x_{k+1} - x_k\| \leq \|x_1 - x_0\| + \frac{d}{c}\big(\varphi(\Phi(x_1) - \Phi(\bar{x})) - \varphi(\Phi(x_{k+1}) - \Phi(\bar{x}))\big).$$

- The above directly implies

$$\textstyle\sum_{k} \|x_k - x_{k+1}\| \leq \|x_1 - x_0\| + \frac{d}{c}\varphi(\Phi(x_1) - \Phi(\bar{x})) < +\infty.$$

- Hence, there exists $x^\star \in \omega(x_k)$

$$x_k \to x^\star, \;\; g_k \to 0, \;\; \Phi(x_k) \to v \geq \Phi(\bar{x}).$$

- KL inequality

$$\varphi'\big(v - R(\bar{x})\big)\|g_k\| \geq 1$$

indicates $v = \Phi(\bar{x})$. Lower semi-continuous yields $\Phi(x^\star) \leq \Phi(\bar{x})$.

Consider minimising

$$\min_{x\in\mathbb{R}^n} \left\{\Phi(x) \stackrel{\text{def}}{=} R(x) + F(x)\right\},$$

- $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is proper l.s.c. and bounded from below.
- $F : \mathbb{R}^n \to \mathbb{R}$ is finite-valued, differentiable and ∇F is L-Lipschitz.

Forward–Backward splitting

Let $\gamma \in]0, 1/L[$:

$$x_{k+1} \in \mathrm{prox}_{\gamma R}\big(x_k - \gamma \nabla F(x_k)\big).$$

- **Sufficient decrease**

$$\Phi(x_{k+1}) + \frac{1-\gamma L}{2\gamma}\|x_k - x_{k+1}\|^2 \leq \Phi(x_j).$$

- **Relative error** $g_{k+1} \stackrel{\text{def}}{=} \frac{1}{\gamma}(x_k - x_{k+1}) - \nabla F(x_k) + \nabla F(x_{k+1}) \in \partial\Phi(x_{k+1})$

$$\|g_{k+1}\| \leq \left(\frac{1}{\gamma} + L\right)\|x_k - x_{k+1}\|.$$

- **Continuity** sequence $\{x_k\}_{k\in\mathbb{N}}$ is bounded.

A coupled problem

Consider minimising

$$\min_{x\in\mathbb{R}^n, y\in\mathbb{R}^m} \left\{ E(x,y) \stackrel{\text{def}}{=} R(x) + F(x,y) + J(y) \right\},$$

- $R : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, J : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ are proper l.s.c. and bounded from below.
- $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is finite-valued, differentiable and ∇F is L-Lipschitz.

Subdifferential

$$\partial E(x,y) = \left\{ \partial R(x) + \nabla_x F(x,y) \right\} \times \left\{ \partial J(x) + \nabla_y F(x,y) \right\} = \partial_x E(x,y) \times \partial_y E(x,y).$$

Separate Lipschitz continuity for F: $\nabla_x F$ is L_x-Lip. and $\nabla_y F$ is L_y-Lip.

Proximal alternating minimisation (PAM)

PAM is an alternating minimisation algorithm.

PAM

Let $\gamma_x, \gamma_y \in]0, 1/L[$:

$$x_{k+1} \in \mathrm{argmin}_{x\in\mathbb{R}^n} E(x, y_k) + \frac{1}{2\gamma_x}\|x - x_k\|^2,$$

$$y_{k+1} \in \mathrm{argmin}_{y\in\mathbb{R}^m} E(x_{k+1}, y) + \frac{1}{2\gamma_y}\|y - y_k\|^2.$$

- PAM is an instance of PPA.
- Convergence, let $\Phi(x, y) = E(x, y)$.
- No closed form solution,

$$\begin{aligned} x_{k+1} &\in \mathrm{argmin}_{x\in\mathbb{R}^n} E(x, y_k) + \frac{1}{2\gamma_x}\|x - x_k\|^2 \\ &= \mathrm{argmin}_{x\in\mathbb{R}^n} R(x) + F(x, y_k) + \frac{1}{2\gamma_x}\|x - x_k\|^2. \end{aligned}$$

PALM is linearised PAM.

$$F(x, y_k) \le F(x_k, y_k) + \langle \nabla_x F(x_k, y_k),\ x - x_k \rangle + \frac{1}{2\gamma_x}\|x - x_k\|^2.$$

PALM

Let $\gamma_x, \gamma_y \in]0, 1/L[$:

$$x_{k+1} \in \mathrm{prox}_{\gamma_x R}\big(x_k - \gamma_x \nabla_x F(x_k, y_k)\big),$$
$$y_{k+1} \in \mathrm{prox}_{\gamma_y J}\big(y_k - \gamma_y \nabla_y F(x_{k+1}, y_k)\big).$$

- PAM is an instance of Forward–Backward.
- Convergence, let $\Phi(x, y) = E(x, y)$.

Remarks

- Converges to global minimiser if starts close enough.
- Inertial acceleration can be applied to all of them.
- Step-size v.s. inertial parameter.
- Step-size and critical points.
- Stochastic optimisation methods can escape saddle-point or find global minimiser...

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