Introductory Course on Non-smooth Optimisation

Lecture 05 - Peaceman–Rachford, Douglas–Rachford splitting

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Sum of two operators

Problem

$$\text{Find } \ x \in \mathbb{R}^n \ \text{ such that } \ 0 \in A(x) + B(x).$$

Assumptions

- $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are maximal monotone.
- the resolvents of A, B are simple, *i.e.* easy to compute.
- $\mathrm{zer}(A + B) \neq \emptyset$.

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Peaceman–Rachford splitting

Let $z_0 \in \mathbb{R}^n$, $\gamma > 0$:

$$\begin{aligned} x_k &= \mathcal{J}_{\gamma B}(z_k), \\ y_k &= \mathcal{J}_{\gamma A}(2x_k - z_k), \\ z_{k+1} &= z_k + 2(y_k - x_k). \end{aligned}$$

- dates back to 1950s for solving numerical PDEs.
- the resolvents of A, B are evaluated separately.

How to derive

- given $x^\star \in \mathrm{zer}(A+B)$, there exists $z^\star \in \mathbb{R}^n$ such that

$$\begin{cases} z^\star - x^\star \in \gamma A(x^\star) \\ x^\star - z^\star \in \gamma B(x^\star) \end{cases} \Longrightarrow \begin{cases} z^\star \in x^\star + \gamma A(x^\star), \\ 2x^\star - z^\star \in x^\star + \gamma B(x^\star). \end{cases}$$

- apply the resolvent

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star), \\ x^\star = \mathcal{J}_{\gamma B}(2x^\star - z^\star). \end{cases}$$

- equivalent formulation

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star), \\ z^\star = z^\star + 2\big(\mathcal{J}_{\gamma B}(2x^\star - z^\star) - x^\star\big). \end{cases}$$

- fixed-point iteration

$$\begin{cases} x_k = \mathcal{J}_{\gamma A}(z_k), \\ z_{k+1} = z_k + 2\big(\mathcal{J}_{\gamma B}(2x_k - z_k) - x_k\big). \end{cases}$$

Fixed-point characterisartion

Fixed-point formulation Recall reflection operator $\mathcal{R}_{\gamma A} = 2\mathcal{J}_{\gamma A} - \mathrm{Id}$.

- $y_k = \mathcal{J}_{\gamma A}(2x_k - z_k) = \mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k)$.
- For z_k,

$$\begin{aligned} z_{k+1} &= z_k + 2(y_k - x_k) \\ &= z_k + 2\big(\mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - \mathcal{J}_{\gamma B}(z_k)\big) \\ &= 2\mathcal{J}_{\gamma A} \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) \\ &= (2\mathcal{J}_{\gamma A} - \mathrm{Id}) \circ (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k). \end{aligned}$$

Property

- $\mathcal{R}_{\gamma A} = 2\mathcal{J}_{\gamma A} - \mathrm{Id}, \mathcal{R}_{\gamma B} = 2\mathcal{J}_{\gamma B} - \mathrm{Id}$ are non-expansive.
- $\mathcal{T}_{\mathrm{PR}} = \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B}$ is non-expansive.

NB: Cannot guarantee convergence in general.

Convergence

- Uniform monotonicity: $\phi : \mathbb{R}_+ \to [0, +\infty]$ is increasing and vanishes only at 0

$$\langle u - v,\, x - y\rangle \geq \phi(\|x - y\|),\ \ (x, u), (y, v) \in \mathrm{gra}(B).$$

- If B is uniformly monotone, then $\mathrm{zer}(A + B) = \{x^\star\}$ and $\mathrm{fix}(\mathcal{T}_{\mathrm{PR}}) \neq \emptyset$. Moreover

$$\langle x - y,\, \mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y)\rangle \geq \|\mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y)\|^2 + \gamma\phi(\|\mathcal{J}_{\gamma B}(x) - \mathcal{J}_{\gamma B}(y)\|).$$

- Let $z^\star \in \mathrm{fix}(\mathcal{T}_{\mathrm{PR}})$, then $x^\star = \mathcal{J}_{\gamma A}(z^\star)$, and

$$\begin{aligned}
\|z_{k+1} - z^\star\|^2 &= \|\mathcal{R}_{\gamma A}\mathcal{R}_{\gamma B}(z_k) - \mathcal{R}_{\gamma A}\mathcal{R}_{\gamma B}(z^\star)\|^2 \\
&\leq \|(2\mathcal{J}_{\gamma B} - \mathrm{Id})(z_k) - (2\mathcal{J}_{\gamma B} - \mathrm{Id})(z^\star)\|^2 \\
&= \|z_k - z^\star\|^2 - 4\langle z_k - z^\star,\, \mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star)\rangle + 4\|\mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star)\|^2 \\
&\leq \|z_k - z^\star\|^2 - 4\gamma\phi(\|\mathcal{J}_{\gamma B}(z_k) - \mathcal{J}_{\gamma B}(z^\star)\|).
\end{aligned}$$

- $\phi(\|z_k - z^\star\|) \to 0$ and $\|z_k - z^\star\| \to 0$.

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To overcome the drawback of Peaceman–Rachford splitting.

Douglas–Rachford splitting

Let $z_0 \in \mathbb{R}^n,\ \gamma > 0,\ \lambda \in]0, 2[$:

$$\begin{aligned} x_k &= \mathcal{J}_{\gamma B}(z_k), \\ y_k &= \mathcal{J}_{\gamma A}(2x_k - z_k), \\ z_{k+1} &= z_k + \lambda(y_k - x_k). \end{aligned}$$

How to derive

- given $x^\star \in \mathrm{zer}(A+B)$, there exists $z^\star \in \mathbb{R}^n$ such that

$$\begin{cases} z^\star - x^\star \in \gamma A(x^\star) \\ x^\star - z^\star \in \gamma B(x^\star) \end{cases} \Longrightarrow \begin{cases} z^\star \in x^\star + \gamma A(x^\star), \\ 2x^\star - z^\star \in x^\star + \gamma B(x^\star). \end{cases}$$

- apply the resolvent

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star), \\ x^\star = \mathcal{J}_{\gamma B}(2x^\star - z^\star). \end{cases}$$

- equivalent formulation

$$\begin{cases} x^\star = \mathcal{J}_{\gamma A}(z^\star), \\ z^\star = z^\star + \big(\mathcal{J}_{\gamma B}(2x^\star - z^\star) - x^\star\big). \end{cases}$$

- fixed-point iteration

$$\begin{cases} x_k = \mathcal{J}_{\gamma A}(z_k), \\ z_{k+1} = z_k + \big(\mathcal{J}_{\gamma B}(2x_k - z_k) - x_k\big). \end{cases}$$

Fixed-point characterisartion

Fixed-point formulation Same as PR, $y_k = \mathcal{J}_{\gamma A} \circ \mathcal{R}_{\gamma B}(z_k)$

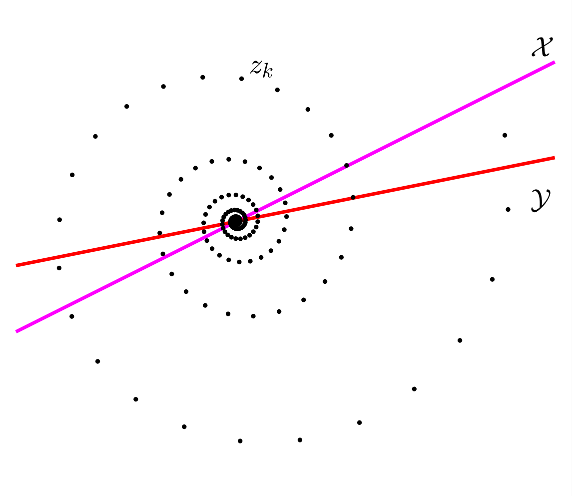
$$\begin{aligned} z_{k+1} &= (1-\lambda)z_k + \lambda\big(z_k + (y_k - x_k)\big) \\ &= (1-\lambda)z_k + \lambda\big(\tfrac{1}{2}z_k + \tfrac{1}{2}(z_k + 2(y_k - x_k))\big) \\ &= (1-\lambda)z_k + \lambda\tfrac{1}{2}(\mathrm{Id} + \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B})(z_k). \end{aligned}$$

Property

- $\mathcal{T}_{\mathrm{DR}} = \frac{1}{2}(\mathrm{Id} + \mathcal{R}_{\gamma A} \circ \mathcal{R}_{\gamma B})$ is firmly non-expansive.
- $\mathcal{T}_{\mathrm{DR}}^{\lambda} = (1-\lambda)\mathrm{Id} + \lambda\mathcal{T}_{\mathrm{DR}}$ is $\frac{\lambda}{2}$-averaged non-expansive.
- Peaceman–Rachford is the limiting case of Douglas–Rachford, $\lambda = 2$.

NB: guaranteed convergence if $\lambda(2-\lambda) > 0$.

Convergence rate



Convergence rate

- Let $\mathcal{X}, \mathcal{Y}$ be two subspaces

$$\mathcal{X} = \{x : ax = 0\},\ \mathcal{Y} = \{x : bx = 0\}$$

  and assume

$$1 \leq p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \leq q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \leq n - 1.$$

- Projection onto subspace

$$\mathcal{P}_{\mathcal{X}}(x) = x - a^T (aa^T)^{-1} ax.$$

- Define diagonal matrices

$$c = \mathrm{diag}\big(\cos(\theta_1), \cdots, \cos(\theta_p)\big),$$
$$s = \mathrm{diag}\big(\sin(\theta_1), \cdots, \sin(\theta_p)\big).$$

- Suppose $p+q<n$, then there exists orthogonal matrix U such that

$$\mathcal{P}_{\mathcal{X}} = U \left[\begin{array}{cc|cc} \mathrm{Id}_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

and

$$\mathcal{P}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ cs & c^2 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*.$$

Convergence rate

- For the composition

$$\mathcal{P}_{\mathcal{X}} \circ \mathcal{P}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*$$

  and

$$\mathcal{P}_{\mathcal{X}^\perp} \circ \mathcal{P}_{\mathcal{Y}^\perp} = U \left[\begin{array}{cc|cc} 0_p & 0 & 0 & 0 \\ -cs & c^2 & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-p-q} \end{array}\right] U^*.$$

- Fixed-point operator

$$
\begin{aligned}
\mathcal{T}_{\mathrm{DR}} &= \mathcal{P}_{\mathcal{X}} \circ \mathcal{P}_{\mathcal{Y}} + \mathcal{P}_{\mathcal{X}^\perp} \circ \mathcal{P}_{\mathcal{Y}^\perp} \\
&= U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ -cs & c^2 & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-p-q} \end{array}\right] U^*.
\end{aligned}
$$

- Consider relaxation

$$
\begin{aligned}
\mathcal{T}_{\mathrm{DR}}^{\lambda} &= (1-\lambda)\mathrm{Id} + \lambda \mathcal{T}_{\mathrm{DR}} \\
&= U \left[\begin{array}{cc|cc} \mathrm{Id}_p - \lambda s^2 & \lambda cs & 0 & 0 \\ -\lambda cs & \mathrm{Id}_p - \lambda s^2 & 0 & 0 \\ \hline 0 & 0 & (1-\lambda)\mathrm{Id}_{q-p} & 0 \\ 0 & 0 & 0 & \mathrm{Id}_{n-p-q} \end{array}\right] U^*.
\end{aligned}
$$

- Eigenvalues

$$\sigma(\mathcal{T}_{\mathrm{DR}}^{\lambda}) = \begin{cases} \big\{1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i) | i=1,...,p\big\} \cup \{1\} : q=p, \\ \big\{1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i) | i=1,...,p\big\} \cup \{1\} \cup \{1-\lambda\} : q>p. \end{cases}$$

- Complex eigenvalues

$$|1-\lambda\sin^2(\theta_i) \pm \mathrm{i}\lambda\cos(\theta_i)\sin(\theta_i)| = \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2}$$

  and

$$1 \geq \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2} \geq |1-\lambda|.$$

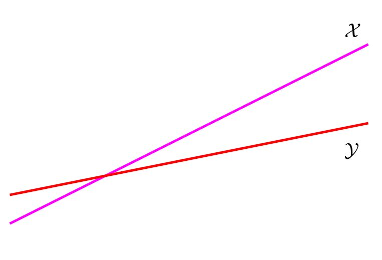
- $\lim_{k\to+\infty} \mathcal{T}_{\mathrm{DR}}^{k} = \mathcal{T}_{\mathrm{DR}}^{\infty}$ and $z_k - z^\star = (\mathcal{T}_{\mathrm{DR}} - \mathcal{T}_{\mathrm{DR}}^{\infty})(z_{k-1} - z^\star)$.

- Spectral radius, minimises at $\lambda = 1$

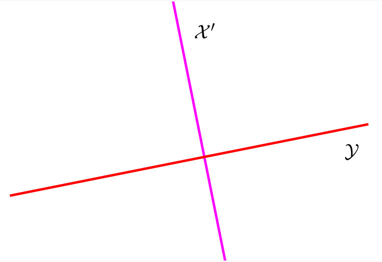
$$\rho(\mathcal{T}_{\mathrm{DR}} - \mathcal{T}_{\mathrm{DR}}^{\infty}) = \sqrt{\lambda(2-\lambda)\cos^2(\theta_i) + (1-\lambda)^2}.$$

- $\widetilde{\mathcal{T}_{\mathrm{DR}}} = \mathcal{T}_{\mathrm{DR}} - \mathcal{T}_{\mathrm{DR}}^{\infty}$

$$\begin{aligned} \|z_k - z^\star\| &= \|\widetilde{\mathcal{T}_{\mathrm{DR}}} z_{k-1} - \widetilde{\mathcal{T}_{\mathrm{DR}}} z^\star\| = \ldots = \|\widetilde{\mathcal{T}_{\mathrm{DR}}}^{k}(z_0 - z^\star)\| \\ &\leq C\big(\rho(\widetilde{\mathcal{T}_{\mathrm{DR}}})\big)^{k}\|z_0 - z^\star\|. \end{aligned}$$



$\mathcal{X}$ and $\mathcal{Y}$



$\mathcal{X}'$ and $\mathcal{Y}$

Optimal metric A invertable operation which makes the Friedrichs angle between $\mathcal{X}'$ and $\mathcal{Y}$ the largest, *e.g.* $\frac{\pi}{2}$...

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More than two operators

Problem $s \in \mathbb{N}_+$ and $s \geq 2$

$$\text{Find } \; x \in \mathbb{R}^n \; \text{ such that } \; 0 \in \textstyle\sum_i A_i(x).$$

Assumptions

- for each $i = 1, ..., s$, $A_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone.
- $\mathrm{zer}(\sum_i A_i) \neq \emptyset$.

Product space

- Let $\mathcal{H} = \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{s \text{ times}}$ endowed with the scalar inner-product and norm

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{H},\ \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \textstyle\sum_{i=1}^{s} \langle x_i,\ y_i \rangle,\ \|\boldsymbol{x}\| = \sqrt{\textstyle\sum_{i=1}^{s} \|x_i\|^2}.$$

- Let

$$\mathcal{S} = \{\boldsymbol{x} = (x_i)_i \in \mathcal{H} : x_1 = \cdots = x_s\}$$

  and its orthogonal complement

$$\mathcal{S}^{\perp} = \{\boldsymbol{x} = (x_i)_i \in \mathcal{H} : \textstyle\sum_{i=1}^{s} x_i = 0\}.$$

Define $\boldsymbol{A}$ by

$$\boldsymbol{A}(\boldsymbol{x}) : \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \to A_1(x_1) \times \cdots \times A_s(x_s).$$

Lifted problem

$$\text{Find } \ \boldsymbol{x} \in \boldsymbol{\mathcal{H}} \ \text{ such that } \ 0 \in \boldsymbol{A}(\boldsymbol{x}) + \mathcal{N}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}).$$

- the resolvent of $\boldsymbol{A}$ is seperable, *i.e.* $\mathcal{J}_{\gamma \boldsymbol{A}} = (\mathcal{J}_{\gamma A_i})_i$.
- define the canonical isometry,

$$\boldsymbol{C} : \mathbb{R}^n \to \boldsymbol{\mathcal{S}},\ x \mapsto (x, \cdots, x),$$

then $\mathcal{P}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{z}) = \boldsymbol{C}(\frac{1}{s}\sum_{i=1}^{s} z_i)$.

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DR in product space for $\boldsymbol{x}^\star \in \boldsymbol{\mathcal{S}}, \exists -\boldsymbol{v} \in \boldsymbol{\mathcal{S}}$ such that

$$-\boldsymbol{v} \in \boldsymbol{\mathcal{S}}^\perp = \mathcal{N}_{\boldsymbol{\mathcal{S}}}(\boldsymbol{x}^\star) \quad \text{and} \quad \boldsymbol{v} \in \boldsymbol{A}(\boldsymbol{x}^\star).$$

Problem V is a close subspace

$$\text{Find } \; x \in V \text{ and } v \in V^\perp \; \text{ such that } \; v \in A(x).$$

Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone.
- admits at least one solution.

Partial inverse

Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be set-valued and $V \subseteq \mathbb{R}^n$ be a closed subspace. The partial inverse of A respect to V is the operator $A_V : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ define by

$$\mathrm{gra}(A_V) = \big\{ \big(\mathcal{P}_V(x) + \mathcal{P}_{V^\perp}(u), \mathcal{P}_{V^\perp}(x) + \mathcal{P}_V(u) \big) : (x, u) \in \mathrm{gra}(A) \big\}.$$

Example Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, then $A_{\mathbb{R}^n} = A$ and $A_{\{0\}} = A^{-1}$.

An application of Proximal Point Algorithm.

Spingarn

Let $x_0 \in V,\ u_0 \in V^{\perp}$:

$$\begin{aligned} y_k &= \mathcal{J}_A(x_k + u_k), \\ v_k &= x_k + u_k - y_k, \\ (x_{k+1}, u_{k+1}) &= \big(\mathcal{P}_V(y_k), \mathcal{P}_{V^{\perp}}(v_k)\big). \end{aligned}$$

Fixed-point characterisation

- define mapping

$$L : \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{R}^n \oplus \mathbb{R}^n : (x, u) \to \big(\mathcal{P}_V(x) + \mathcal{P}_{V^\perp}(u), \mathcal{P}_{V^\perp}(x) + \mathcal{P}_V(u)\big).$$

- $$\begin{aligned} p = \mathcal{J}_{A_V}(x) &\iff (p, x-p) \in \mathrm{gra}(A_V) \\ &\iff L(p, x-p) \in L\big(\mathrm{gra}(A_V)\big) = \mathrm{gra}(A) \\ &\iff \big(\mathcal{P}_V(p) + \mathcal{P}_{V^\perp}(x-p), \mathcal{P}_V(x-p) + \mathcal{P}_{V^\perp}(p)\big) \in \mathrm{gra}(A). \end{aligned}$$

- let $q = \mathcal{P}_V(p) + \mathcal{P}_{V^\perp}(x-p)$

$$\begin{aligned} p = \mathcal{J}_{A_V}(x) &\iff x - q = \mathcal{P}_V(x-p) + \mathcal{P}_{V^\perp} p \in A(q) \\ &\iff q = \mathcal{J}_A(x). \end{aligned}$$

- let $z_k = x_k + u_k$, since $x_k \in V$ and $u_k \in V^\perp$

$$\begin{aligned} \mathcal{P}_V(z_{k+1}) + \mathcal{P}_{V^\perp}(z_k - z_{k+1}) &= x_{k+1} + \mathcal{P}_{V^\perp}(u_k) - u_{k+1} \\ &= \mathcal{P}_V(y_k) + \mathcal{P}_{V^\perp}(v_k - x_k + y_k) - \mathcal{P}_{V^\perp}(v_k) \\ &= \mathcal{P}_V(y_k) + \mathcal{P}_{V^\perp}(v_k) + \mathcal{P}_{V^\perp}(y_k) - \mathcal{P}_{V^\perp}(v_k). \end{aligned}$$

- $z_{k+1} = \mathcal{J}_A(z_k)$.

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An inertial DR splitting

Initial : $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0$;

$$\begin{aligned} y_k &= z_k + a_{0,k}(z_k - z_{k-1}) + a_{1,k}(z_{k-1} - z_{k-2}) + \cdots , \\ z_{k+1} &= \mathcal{T}_{\mathrm{DR}}(y_k) \end{aligned}$$

- relaxation can be applied.

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Basis pursuit

$$\min_{x \in \mathbb{R}^n} \quad \|x\|_1 \qquad \text{such that} \quad Ax = b,$$

- $A : \mathbb{R}^n \to \mathbb{R}^m$ with $m << n$.
- $b \in \mathrm{Img}(A)$.

Image inpainting

$$\min_{X \in \mathbb{R}^{n \times n}} \quad \|WX\|_1$$
$$\text{such that} \quad \mathcal{P}_\Omega(X) = \bar{X},$$

- W: total variation, orthonomal basis, redundant wavelet frame.
- Observation constraint
$$\big(\mathcal{P}_\Omega(X)\big)_{i,j} = \begin{cases} \bar{X}_{i,j} : (i,j) \in \Omega, \\ 0 : (i,j) \notin \Omega. \end{cases}$$
- Painting reconstruction in museum.

Example: matrix completion

Matrix completion

$$\min_{X \in \mathbb{R}^{n \times n}} \quad \|X\|_* \\ \text{such that} \quad \mathcal{P}_\Omega(X) = \bar{X},$$

- Observation constraint

$$\big(\mathcal{P}_\Omega(X)\big)_{i,j} = \begin{cases} \bar{X}_{i,j} : (i,j) \in \Omega, \\ 0 : (i,j) \notin \Omega. \end{cases}$$

- Netflix prize, recommendation system.

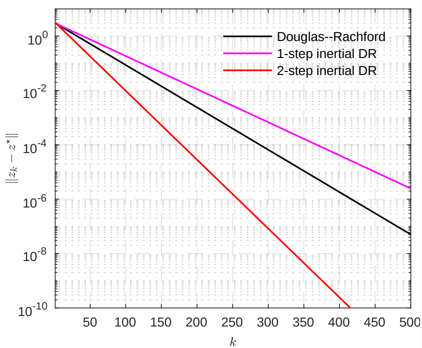
Variation ineuality

$$\text{Find } x \in \mathbb{R}^n \text{ such that } \quad \exists u \in A(x), \forall y \in \mathbb{R}^n : \ \langle x - y, u \rangle + R(x) \le R(y).$$

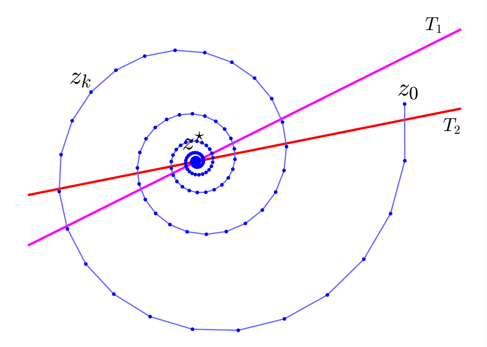
- $R \in \Gamma_0$.
- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone.

Example Let $R, J \in \Gamma_0$, and $x^\star \in \mathrm{Argmin}(R + J)$, then $\exists u \in \partial J(x^\star)$ s.t. $-u \in \partial R(x^\star)$ and

$$\begin{aligned} & \langle y - x^\star, -u \rangle + R(x^\star) \le R(y) \\ \iff \ & \langle x^\star - y, u \rangle + R(x^\star) \le R(y). \end{aligned}$$



Comparison



Tracjectory

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