

Introductory Course on Non-smooth Optimisation

Lecture 02 - Proximal gradient descent

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- 1 Subgradient descent
- 2 Proximal gradient descent
- 3 Proximal mapping
- 4 Inertial proximal gradient
- 5 Fast iterative shrinkage-thresholding algorithm (FISTA)
- 6 Restarting FISTA
- 7 Numerical experiments

Unconstrained non-smooth optimisation

Consider minimising

$$\min_{x \in \mathbb{R}^n} R(x),$$

where $R : \mathbb{R}^n \rightarrow]-\infty, +\infty]$ is proper convex and **lower semi-continuous**.

- Γ_0 : the class of proper convex and lower semi-continuous functions on \mathbb{R}^n .
- The set of minimisers, *i.e.*

$$\text{Argmin}(R) = \{x \in \mathbb{R}^n : R(x) = \min_{x \in \mathbb{R}^n} R(x)\}$$

is non-empty.

- $R(x)$ is non-differentiable...

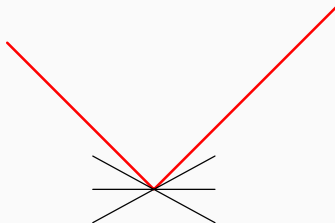
Subdifferential

Let $R \in \Gamma_0$, the subdifferential of R at $x \in \text{dom}(R)$ is defined by

$$\partial R : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, x \mapsto \{g \in \mathbb{R}^n \mid R(y) \geq R(x) + \langle g, y - x \rangle, \forall y \in \mathbb{R}^n\}.$$

Example absolute value function

$$\partial|x| = \begin{cases} +1 : x > 0 \\ [-1, 1] : x = 0 \\ -1 : x < 0 \end{cases}$$



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Convexity

Let $R \in \Gamma_0$ and $x \in \text{dom}(R)$, then

- $\partial R(x) = \{g \in \mathbb{R}^n : R(y) \geq R(x) + \langle g, y - x \rangle\}.$
- $\partial R(x)$ is closed and convex.

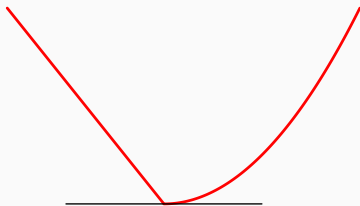
Monotonicity

Let $R \in \Gamma_0$, then $\forall x, y \in \text{dom}(R)$,

$$\langle u - v, x - y \rangle \geq 0, \forall u \in \partial R(x), v \in \partial R(y).$$

Given $x^* \in \mathbb{R}^n$, it minimises $R(x)$ if and only if

$$0 \in \partial R(x^*).$$



$$R(y) \geq R(x^*) + \langle g, y - x^* \rangle \text{ holds for all } y \in \text{dom}(R) \iff 0 \in \partial R(x^*).$$

Subgradient descent

initial : $x_0 \in \text{dom}(R)$;

repeat :

1. Choose step-size $\gamma_k > 0$ and a subgradient $g_k \in \partial R(x_k)$
2. Update $x_{k+1} = x_k - \gamma_k g_k$

until : stopping criterion is satisfied.

Subgradient descent

initial : $x_0 \in \text{dom}(R)$;

repeat :

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until : stopping criterion is satisfied.

Step-size rule:

- Fixed step-size: γ_k is constant.
- Fixed length: $\gamma_k \|g_k\| = \|x_{k+1} - x_k\|$ is a constant.
- Diminishing step-size: $\gamma_k \rightarrow 0$, $\sum_i \gamma_i = +\infty$.

Assumptions:

- R has minimiser x^* and finite optimal value $R(x^*)$.
- R is convex, $\text{dom}(R) = \mathbb{R}^n$.
- R is Lipschitz continuous with constant L :

$$|R(x) - R(y)| \leq L\|x - y\|, \quad \forall x, y \in \text{dom}(R).$$

NB: the Lipschitz continuity implies $\|g\| \leq L$ for all $x \in \text{dom}(R)$.

Subgradient descent is **NOT** a descent method.

Fixed step-size $\gamma_k \equiv \gamma$

$$R_{k,best} - R(x^*) \leq \frac{\|x_0 - x^*\|^2}{2k\gamma} + \frac{\gamma L^2}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$.
- For large k , $R_{k,best}$ is approximately $\frac{\gamma L^2}{2}$ suboptimal.

Subgradient descent is **NOT** a descent method.

Fixed step-length $\gamma_k = c/\|g_k\|$

$$R_{k,best} - R(x^*) \leq \frac{\|x_0 - x^*\|^2}{2kc} + \frac{cL}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$.
- For large k , $R_{k,best}$ is approximately $\frac{cL}{2}$ suboptimal.

Subgradient descent is **NOT** a descent method.

Diminishing step-size $\gamma_k \rightarrow 0$, $\sum_i \gamma_i = +\infty$:

$$R_{k,best} - R(x^*) \leq \frac{\|x_0 - x^*\|^2 + L^2 \sum_{i=1}^k \gamma_i^2}{\sum_{i=1}^k \gamma_i}.$$

- If $\sum_{i=1}^k \gamma_i^2 / \sum_{i=1}^k \gamma_i \rightarrow 0$, then $R_{k,best} \rightarrow R(x^*)$.
- Choice of γ_k : $\gamma_k = c/k^q$, $q \in]1/2, 1[$.

For fixed number of iterations if $c_i = \gamma_i \|g_i\|$ and $\|x_0 - x^*\| \leq D$,

$$R_{k,best} - R(x^*) \leq \frac{D^2 + L^2 \sum_{i=1}^k c_i^2}{2 \sum_{i=1}^k \gamma_i / L}.$$

- For given k , rhs is minimised by $c_i = c = D/\sqrt{k}$.
- Hence the rate

$$R_{k,best} - R(x^*) \leq \frac{LD}{\sqrt{k}}.$$

- Iteration complexity: reach $R_{k,best} - R(x^*) < \epsilon$ in $O(1/\epsilon^2)$ steps.

When $R(x^*)$ is available step-size

$$\gamma_k = \frac{R(x_k) - R(x^*)}{\|g_k\|^2}.$$

Convergence rate

$$R_{k,best} - R(x^*) \leq \frac{LD}{\sqrt{k}}.$$

NB: $O(1/\sqrt{k})$ is the best rate can be obtained by subgradient method.

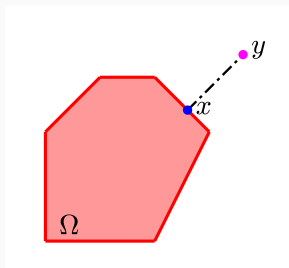
- Handles non-smooth problem
- Simple iterative scheme
- Slow convergence rate
- No clear stopping criterion

NB: need a better approach to handle non-smoothness...

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Indicator function : let $\Omega \subseteq \mathbb{R}^n$

$$\iota_{\Omega}(x) = \begin{cases} 0 & : x \in \Omega, \\ +\infty & : x \notin \Omega. \end{cases}$$



Projection of y onto Ω :

$$\min_{x \in \Omega} \|x - y\|.$$

Projection

Projection mapping onto a set is defined by

$$\mathcal{P}_{\Omega}(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in \Omega} \|x - y\|.$$

Constrained smooth optimisation

Let $F \in C_L^1$ and $\Omega \subseteq \mathbb{R}^n$ be a closed and convex set

$$\min_{x \in \Omega} F(x).$$

Projected gradient descent

initial : $x_0 \in \Omega$;

repeat :

1. Choose step-size $\gamma_k \in]0, 2/L[$
2. Gradient descent $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$
3. Projection $x_{k+1} = \mathcal{P}_\Omega(x_{k+1/2})$

until : stopping criterion is satisfied.

As $\iota_\Omega \in \Gamma_0$, the constrained optimisation problem is equivalent to

$$\min_{x \in \mathbb{R}^n} \iota_\Omega(x) + F(x).$$

Composite optimisation

Consider the following optimisation problem

$$\min_{x \in \mathbb{R}^n} \{ \Phi(x) \stackrel{\text{def}}{=} R(x) + F(x) \}.$$

Assumptions

- $F \in C_L^1$
- $R \in \Gamma_0$
- $\text{Argmin}(\Phi) \neq \emptyset$

Examples regularised LSE, image processing...

Projection

$$\begin{aligned}\mathcal{P}_\Omega(y) &\stackrel{\text{def}}{=} \operatorname{argmin}_{x \in \Omega} \|x - y\| \\ &= \operatorname{argmin}_{x \in \mathbb{R}^n} \iota_\Omega(x) + \frac{1}{2} \|x - y\|^2.\end{aligned}$$

Proximal mapping

$$\operatorname{prox}_R(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in \mathbb{R}^n} R(x) + \frac{1}{2} \|x - y\|^2.$$

Projected gradient descent

initial : $x_0 \in \Omega$;

repeat :

1. Choose step-size $\gamma_k \in]0, 2/L[$
2. Gradient descent $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$
3. Projection $x_{k+1} = \operatorname{prox}_{\gamma_k R}(x_{k+1/2})$

until : stopping criterion is satisfied.

- A.K.A Forward-Backward splitting
 - Forward step: gradient descent of F .
 - Backward step: proximal mapping of R .

- Iteration in one line

$$x_{k+1} = \text{prox}_{\gamma_k R}(x_k - \gamma_k \nabla F(x_k)).$$

- Definition of $\text{prox}_{\gamma R}$,

$$\begin{aligned} x_{k+1} &= \operatorname{argmin}_x \left\{ \gamma_k R(x) + \frac{1}{2} \|x - (x_k - \gamma_k \nabla F(x_k))\|^2 \right\} \\ &= \operatorname{argmin}_x \left\{ \gamma_k R(x) + \gamma_k \langle \nabla F(x_k), x - x_k \rangle + \frac{1}{2} \|x - x_k\|^2 \right\} \\ &= \operatorname{argmin}_x \left\{ R(x) + \boxed{F(x_k) + \langle \nabla F(x_k), x - x_k \rangle + \frac{1}{2\gamma_k} \|x - x_k\|^2} \right\}. \end{aligned}$$

NB: x_{k+1} minimises $R(x)$ plus the majorisation of $F(x)$ at x_k if $\gamma_k \leq \frac{1}{L}$.

- **Gradient descent** $R = 0$

$$x_{k+1} = x_k - \gamma_k \nabla F(x_k).$$

- **Proximal point algorithm** $F = 0$

$$x_{k+1} = \text{prox}_{\gamma_k R}(x_k).$$

- **Projected gradient descent** $R = \iota_\Omega$

$$x_{k+1} = \mathcal{P}_\Omega(x_k - \gamma_k \nabla F(x_k)).$$

- **ISTA: iterative shrinkage-thresholding algorithm** $R = \|x\|_1$

$$x_{k+1} = \mathcal{T}_\gamma(x_k - \gamma_k \nabla F(x_k)),$$

where

$$(\mathcal{T}_\gamma(y))_i = \begin{cases} \text{sign}(y_i) \cdot (|y_i| - \gamma) & : |y_i| > \gamma, \\ 0 & : |y_i| \leq \gamma. \end{cases}$$

Define

$$E_\gamma(x, y) \stackrel{\text{def}}{=} R(x) + F(y) + \langle \nabla F(y), x - y \rangle + \frac{1}{2\gamma} \|x - y\|^2$$

and $y_+ \stackrel{\text{def}}{=} \operatorname{argmin}_x E_\gamma(x, y)$.

Lemma

Let $y \in \mathbb{R}^n$ and $\gamma \in]0, 2/L[$ such that

$$\Phi(y_+) \leq E_\gamma(y_+, y),$$

then for any $x \in \mathbb{R}^n$,

$$\Phi(x) - \Phi(y_+) \geq \frac{1}{2\gamma} (\|x - y_+\|^2 - \|x - y\|^2).$$

Lemma

Given $y \in \mathbb{R}^n$ and $\gamma \in]0, 1/L]$, then for any $x \in \mathbb{R}^n$,

$$\Phi(y_+) + \frac{1}{2\gamma} \|y_+ - x\|^2 \leq \Phi(y) + \frac{1}{2\gamma} \|y - x\|^2.$$

NB: proximal gradient is a descent method.

Consider $\gamma_k \equiv \gamma \in]0, 1/L]$

- For each step

$$\Phi(x_k) - \Phi(x_{k+1}) \geq \frac{\gamma}{2} \|x_k - x_{k+1}\|^2.$$

- Regarding $\Phi(x^*)$

$$\Phi(x_{k+1}) - \Phi(x^*) \leq \frac{\gamma}{2} (\|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2).$$

- Summing up

$$\begin{aligned} k(\Phi(x_k) - \Phi(x^*)) &\leq \sum_{i=1}^k (\Phi(x_i) - \Phi(x^*)) \\ &\leq \frac{\gamma}{2} \sum_{i=1}^k (\|x_{i-1} - x^*\|^2 - \|x_i - x^*\|^2) \leq \frac{\gamma}{2} \|x_0 - x^*\|^2 \end{aligned}$$

- $O(1/k)$ rate

$$\Phi(x_k) - \Phi(x^*) \leq \frac{\gamma \|x_0 - x^*\|^2}{2k}.$$

NB: not optimal and can be accelerated.

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Proximal mapping

The proximal mapping (proximity operator) of a function $R \in \Gamma_0$ is defined by

$$\text{prox}_{\gamma R}(y) \stackrel{\text{def}}{=} \operatorname{argmin}_{x \in \mathbb{R}^n} \gamma R(x) + \frac{1}{2} \|x - y\|^2.$$

Optimality condition denote $y_+ \stackrel{\text{def}}{=} \text{prox}_{\gamma R}(y)$,

$$\begin{aligned} 0 \in \gamma \partial R(y_+) + y_+ - y &\iff y \in (\text{Id} + \gamma \partial R)(y_+) \\ &\iff y_+ = (\text{Id} + \gamma \partial R)^{-1}(y). \end{aligned}$$

Projection $R(x) = \iota_{\Omega}(x)$, $\partial \iota_{\Omega}(x) = \mathcal{N}_{\Omega}(x) = \{g : \langle g, v - x \rangle \leq 0\}$

$$\mathcal{P}_{\Omega} = (\text{Id} + \mathcal{N}_{\Omega})^{-1}.$$

Examples

- Hyperplane: $\Omega = \{x : a^T x = b\}$, $a \neq 0$

$$\mathcal{P}_{\Omega} = x + \frac{b - a^T x}{\|a\|^2} a.$$

- Affine subspace: $\Omega = \{x : Ax = b\}$ with $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = m < n$

$$\mathcal{P}_{\Omega} = x + A^T (AA^T)^{-1} (b - Ax).$$

- Half space: $\Omega = \{x : a^T x \leq b\}$, $a \neq 0$

$$\mathcal{P}_{\Omega} = x + \frac{b - a^T x}{\|a\|^2} a \text{ if } a^T x > b \quad \text{and} \quad x \text{ if } a^T x \leq b.$$

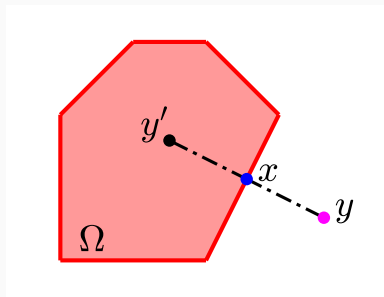
- Nonnegative orthant: $\Omega = \mathbb{R}_+^n$

$$\mathcal{P}_{\Omega} = (\max\{0, x_i\})_i.$$

Projection $R(x) = \iota_{\Omega}(x)$, $\partial \iota_{\Omega}(x) = \mathcal{N}_{\Omega}(x) = \{g : \langle g, v - x \rangle \leq 0\}$
 $\mathcal{P}_{\Omega} = (\text{Id} + \mathcal{N}_{\Omega})^{-1}.$

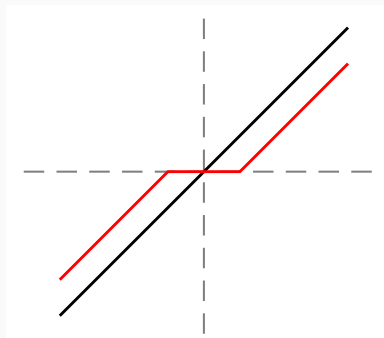
Reflection

$$\mathcal{R}_{\mathcal{N}_{\Omega}} = 2\mathcal{P}_{\Omega} - \text{Id} = \mathcal{P}_{\Omega} + (\mathcal{P}_{\Omega} - \text{Id}).$$



Soft-threshold : $R(x) = |x|$,

$$\text{prox}_{\gamma R}(y) = \mathcal{T}_{\gamma}(y) = \begin{cases} y - \gamma : y > \gamma, \\ 0 : y \in [-\gamma, \gamma], \\ y + \gamma : y < -\gamma. \end{cases}$$



Quadratic function $R(x) = \frac{1}{2}x^T Ax + b^T x + c, A \succeq 0$

$$\text{prox}_{\gamma R}(y) = (\text{Id} + \gamma A)^{-1}(y - \gamma b).$$

Euclidean norm $R(x) = \|x\|$

$$\text{prox}_{\gamma R}(y) = \begin{cases} (1 - \frac{\gamma}{\|y\|})y : \|y\| > \gamma, \\ 0 : \text{o.w.} \end{cases}$$

Logarithmic barrier $R(x) = -\sum_i \log(x_i)$

$$(\text{prox}_{\gamma R}(y))_i = \frac{y_i + \sqrt{y_i^2 + 4\gamma}}{2}, \quad i = 1, \dots, n.$$

Nuclear norm $R(x) = \sum_i \sigma_i$

$$\text{prox}_{\gamma R}(y) = U\mathcal{T}_{\gamma}(\Sigma)V^T.$$

Quadratic perturbation $H(x) = R(x) + \frac{\alpha}{2} \|x\|^2 + \langle x, u \rangle + \beta, \alpha \geq 0$

$$\text{prox}_H = \text{prox}_{R/(\alpha+1)}\left(\frac{x-u}{\alpha+1}\right).$$

Translation $H(x) = R(x - z)$

$$\text{prox}_H = z + \text{prox}_R(x - z).$$

Scaling $H(x) = R(x/\rho)$

$$\text{prox}_H = \rho \text{prox}_{R/\rho^2}\left(\frac{x}{\rho}\right).$$

Reflection $H(x) = R(-x)$

$$\text{prox}_H = -\text{prox}_R(-x).$$

Composition $H = R \circ L$ with L being bijective bounded linear mapping such that $L^{-1} = L^*$,

$$\text{prox}_H = L^* \circ \text{prox}_R \circ L.$$

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An inertial proximal gradient

Initial : $x_0 \in \mathbb{R}^n$ and $\gamma \in]0, 2/L[$;

$$\begin{aligned}y_k &= x_k + a_k(x_k - x_{k-1}), \quad a_k \in [0, 1], \\x_{k+1} &= \text{prox}_{\gamma R}(y_k - \gamma \nabla F(x_k)).\end{aligned}$$

- Recovers inertial PPA when $F = 0$, and heavy-ball method when $R = 0$.
- Convergence via studying the Lyapunov function

$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \frac{a_k}{2\gamma} \|x_k - x_{k-1}\|^2.$$

- In general, no convergence rate.

A general inertial proximal gradient

Initial : $x_0 \in \mathbb{R}^n$, $x_{-1} = x_0$ and $\gamma \in]0, 2/L[$;

$$y_k = x_k + a_k(x_k - x_{k-1}), \quad a_k \in [0, 1],$$

$$z_k = x_k + b_k(x_k - x_{k-1}), \quad b_k \in [0, 1],$$

$$x_{k+1} = \text{prox}_{\gamma R}(y_k - \gamma \nabla F(z_k)).$$

- Convergence via studying the Lyapunov function

$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \frac{a_k}{2\gamma} \|x_k - x_{k-1}\|^2.$$

- In general, no convergence rate.
- Can be extend to multiple steps, e.g.

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots.$$

Assumption $R = 0$, $F = \frac{1}{2}\|Ax - f\|^2$ and $(a_k, b_k) \equiv (a, b)$.

- $A^T A$ is symmetric positive definite with $A^T A \succeq \alpha \text{Id}$.

- Taylor expansion

$$x_{k+1} = y_k - \gamma \nabla^2 F(x^*)(z_k - x^*).$$

- Let $d_k = (x_k - x^*, x_{k-1} - x^*)^T$ and $H = \nabla^2 F$, $G = \text{Id} - \gamma H$, then

$$d_{k+1} = \underbrace{\begin{bmatrix} (a-b)\text{Id} + (1+b)G, & -(a-b)\text{Id} - bG \\ \text{Id}, & 0 \end{bmatrix}}_M d_k.$$

- Spectral radius: $\eta = \rho(G) = 1 - \gamma\alpha$ and $\rho = \rho(M)$...

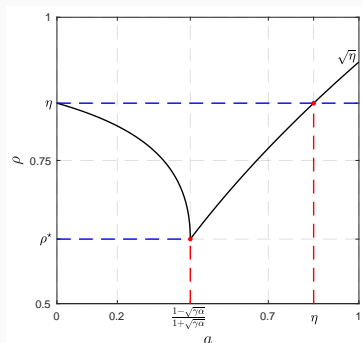
Spectral radius ρ

Between η and ρ ,

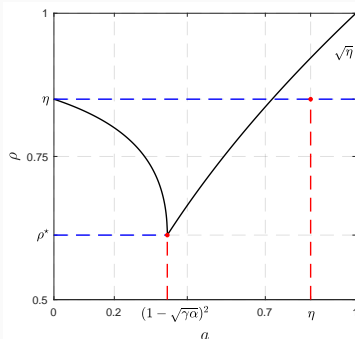
- η and ρ satisfy the relation

$$\rho^2 - ((a - b) + (1 + b)\eta)\rho + (a - b) + b\eta = 0.$$

- Given any $(a, b) \in [0, 1]^2$, then $\rho(M) < 1$ if, and only if $\frac{2(b-a)-1}{1+2b} < \eta$.



$b = a$



$b = 0$

- Given $b \in [0, 1]$, there exists optimal choice of $a \in [0, 1]$ such that

$$\rho = 1 - \sqrt{\gamma\alpha}$$

can be obtained.

- Take $b = a$, for

$$a \in \left] \frac{1 - \sqrt{\gamma\alpha}}{1 + \sqrt{\gamma\alpha}}, 1 \right],$$

the leading eigenvalue of M is complex.

- Continue $b = a$, for

$$a \in]\eta, 1],$$

the inertial scheme is actually **slower** than the original scheme.

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FISTA

Initial : $x_0 \in \mathbb{R}^n$, $x_{-1} = x_0$, $\gamma = 1/L$ and $t_0 = 1$;

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \quad a_k = \frac{t_{k-1} - 1}{t_k},$$

$$y_k = x_k + a_k(x_k - x_{k-1}),$$

$$x_{k+1} = \text{prox}_{\gamma R}(y_k - \gamma \nabla F(y_k)).$$

- A special case of inertial proximal gradient descent.
- Inertial parameters

$$t_k \approx \frac{k+1}{2} \quad \text{and} \quad a_k \rightarrow 1.$$

Nesterov compute $\phi_k \in]0, 1[$ from equation

$$\phi_k^2 = (1 - \phi_k)\phi_{k-1}^2$$

and $a_k = \frac{\phi_{k-1}(1 - \phi_{k-1})}{\phi_{k-1}^2 + \phi_k}.$

- ϕ_k reads

$$\phi_k = \frac{-\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}{2} = \frac{2\phi_{k-1}^2}{\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}.$$

- Let $t_k = 1/\phi_k$,

$$\frac{1}{t_k} = \frac{2}{1 + \sqrt{1 + 4t_{k-1}^2}}.$$

- Which leads to

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}.$$

- Moreover, $a_k = \frac{t_{k-1} - 1}{t_k}.$

NB: FISTA is not a descent method.

- Denote $f_k = \Phi(x_k) - \Phi(x^*)$ and $u_k = t_k x_k - (t_k - 1)x_{k-1} - x^*$, then

$$\frac{2}{L} t_k^2 f_k - \frac{2}{L} t_{k+1}^2 f_{k+1} \geq \|u_{k+1}\|^2 - \|u_k\|^2.$$

- Let c_k, d_k be positive sequences, if

$$c_k - c_{k+1} \geq d_{k+1} - d_k \forall k \geq 1, \text{ with } c_1 + d_1 < C, C > 0$$

then $c_k < C$ for all $k \geq 1$.

- $\frac{2}{L} t_k^2 f_k \leq \|x_0 - x^*\|^2$

- $t_k \geq \frac{k+1}{2},$

$$\Phi(x_k) - \Phi(x^*) \leq \frac{L \|x_0 - x^*\|^2}{2(k+1)^2}.$$

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Why FISTA oscillates

- for LSE, leading eigenvalue for the system is complex.
- over extrapolation, momentum beats gradient.

Restarting FISTA

Initial : $x_0 \in \mathbb{R}^n$, $x_{-1} = x_0$, $\gamma = 1/L$ and $t_0 = 1$;

repeat :

1. Run FISTA iteration
2. If $\langle y_k - x_{k+1}, x_{k+1} - x_k \rangle > 0$: $t_k = 1$, $y_k = x_k$.

until : stopping criterion is satisfied.

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ℓ_1 -regularised least square (LASSO)

$$\min_{x \in \mathbb{R}^n} \mu \|x\|_1 + \frac{1}{2} \|Ax - f\|^2.$$

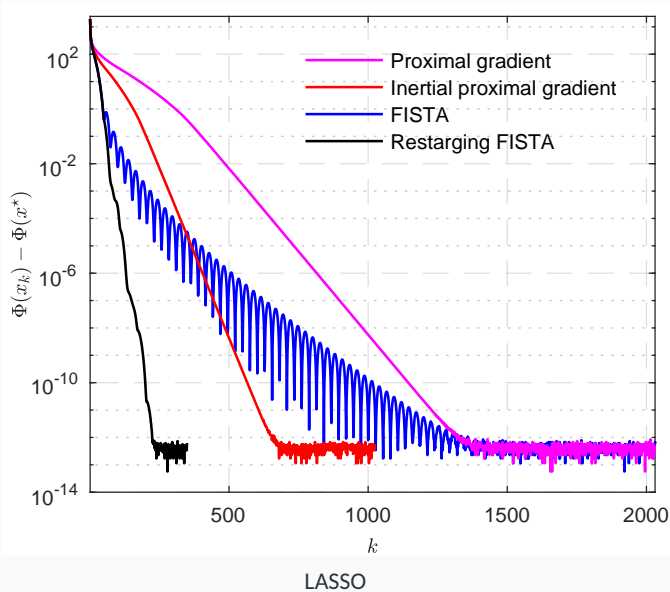
Sparse logistic regression

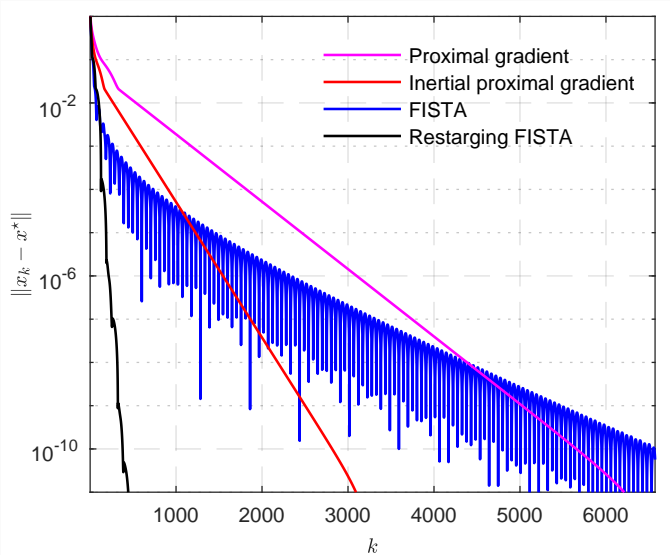
$$\min_{x \in \mathbb{R}^n} \mu \|x\|_1 + \frac{1}{m} \sum_{i=1}^m \log(1 + e^{-h_i^T x}),$$

where $\mu = 10^{-2}$. The australian data set from LIBSVM¹ is considered.

¹<https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/>

- Proximal gradient descent
- Inertial proximal gradient descent
- FISTA
- Restarting FISTA





Sparse logistic regression

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