

Introductory Course on Non-smooth Optimisation

Lecture 03

Krasnosel'skiĭ-Mann iteration

Outline

- 1 Method of alternating projection
- 2 Monotone and non-expansive mappings
- 3 Krasnosel'skiĭ-Mann iteration
- 4 “Accelerated” Krasnosel'skiĭ-Mann iteration

Recap

Recap of descent methods:

- include gradient descent, proximal gradient descent
- convergence (rate) properties
 - objective function value
 - $O(1/k)$ convergence rate
 - optimal $O(1/k^2)$ convergence rate
 - sequence
 - $O(1/\sqrt{k})$ convergence rate
 - optimal $O(1/k)$ convergence rate
 - linear convergence under e.g. strong convexity

Operator splitting

Consider the problem

$$\min_{x \in \mathbb{R}^n} \mu_1 \|x\|_1 + \mu_2 \|\nabla x\|_1 + \frac{1}{2} \|Ax - f\|^2.$$

In 1-D, both

$$\text{prox}_{\gamma \|\cdot\|_1}(\cdot) \quad \text{and} \quad \text{prox}_{\gamma \|\nabla \cdot\|_1}(\cdot)$$

have close form solution. However, not for

$$\text{prox}_{\gamma(\|\cdot\|_1 + \|\nabla \cdot\|_1)}(\cdot).$$

Operator splitting design properly structured scheme such that

- the proximal mapping of non-smooth functions are evaluated separated
- gradient descent is applied to the smooth part

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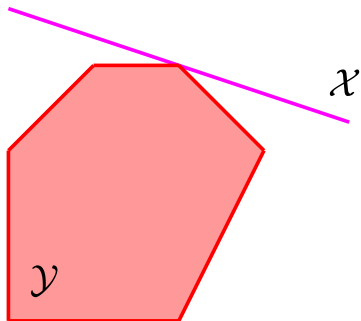
Feasibility problem

Problem (Feasibility problem)

Consider finding a common point

$$\text{find } x \in \mathcal{X} \cap \mathcal{Y},$$

where $\mathcal{X}, \mathcal{Y} \in \mathbb{R}^n$ are two closed and convex sets.



Method of alternating projection

Equivalent formulation

$$\min_{x \in \mathbb{R}^n} \iota_{\Omega_1}(x) + \iota_{\Omega_2}(x).$$

Method of alternating projection (MAP)

initial: $x_0 \in \mathcal{X}$;

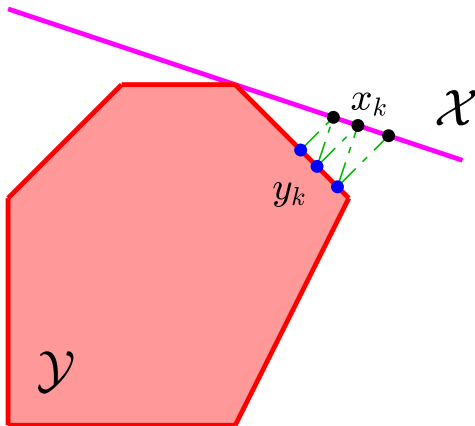
repeat:

1. Projection onto Ω_2 : $y_k = \text{proj}_{\mathcal{Y}}(x_k)$
2. Projection onto Ω_1 : $x_{k+1} = \text{proj}_{\mathcal{X}}(y_k)$

until: stopping criterion is satisfied.

- The projection onto two sets are computed separately
- Stopping criterion: $\|x_k - x_{k-1}\| \leq \epsilon$

Method of alternating projection



Convergence analysis

MAP:

$$x_{k+1} = \text{proj}_{\mathcal{X}} \circ \text{proj}_{\mathcal{Y}}(x_k).$$

How to:

- analyse the convergence properties
- not convergence result for the objective function value
- how about the sequence $\{x_k\}_{k \in \mathbb{N}}$

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Notations

Given two non-empty sets $\mathcal{X}, \mathcal{U} \subseteq \mathbb{R}^n$, $A : \mathcal{X} \rightrightarrows \mathcal{U}$ is called set-valued operator if A maps every point in \mathcal{X} to a subset of \mathcal{U} , *i.e.*

$$A : \mathcal{X} \rightrightarrows \mathcal{U}, x \in \mathcal{X} \mapsto A(x) \subseteq \mathcal{U}.$$

- The graph of A is defined by

$$\text{gra}(A) \stackrel{\text{def}}{=} \{(x, u) \in \mathcal{X} \times \mathcal{U} : u \in A(x)\}.$$

- The domain and range of A are

$$\text{dom}(A) \stackrel{\text{def}}{=} \{x \in \mathcal{X} : A(x) \neq \emptyset\}, \text{ran}(A) \stackrel{\text{def}}{=} A(\mathcal{X}).$$

- The inverse of A defined through its graph

$$\text{gra}(A^{-1}) \stackrel{\text{def}}{=} \{(u, x) \in \mathcal{U} \times \mathcal{X} : u \in A(x)\}.$$

- The set of zeros of A are the points such that

$$\text{zer}(A) \stackrel{\text{def}}{=} A^{-1}(0) = \{x \in \mathcal{X} : 0 \in A(x)\}.$$

Monotone operator

Definition (Monotone operator)

Let $\mathcal{X}, \mathcal{U} \subseteq \mathbb{R}^n$ be two non-empty convex sets, $A : \mathcal{X} \rightrightarrows \mathcal{U}$ is monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in \text{gra}(A).$$

It is moreover maximal monotone if $\text{gra}(A)$ is not strictly contained in the graph of any other monotone operators.

A is called α -strongly monotone for some $\kappa > 0$ if

$$\langle x - y, u - v \rangle \geq \kappa \|x - y\|^2.$$

Lemma

Let $R \in \Gamma_0$, then ∂R is maximal monotone.

Definition (Resolvent and reflection)

The resolvent and reflection of $A : \mathcal{X} \rightrightarrows \mathcal{U}$ are defined by

$$\mathcal{J}_A \stackrel{\text{def}}{=} (\text{Id} + A)^{-1} \quad \text{and} \quad \mathcal{R}_A \stackrel{\text{def}}{=} 2\mathcal{J}_A - \text{Id}.$$

Cocoercive operator

Definition (Cocoercive operator)

An operator $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called β -cocoercive if there exists $\beta > 0$ such that

$$\beta \|B(x) - B(y)\|^2 \leq \langle B(x) - B(y), x - y \rangle, \quad \forall x, y \in \mathbb{R}^n.$$

The above equation implies that B is $(1/\beta)$ -Lipschitz continuous.

Theorem (Baillon-Haddad)

Let $F \in C_L^1$, then ∇F is β -cocoercive.

Lemma

Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be β -strongly monotone, then its inverse C^{-1} is β -cocoercive.

Resolvent of monotone operator

Definition (Resolvent)

Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone operator and $\gamma > 0$, the resolvent of A is defined by

$$\mathcal{J}_{\gamma A} \stackrel{\text{def}}{=} (\text{Id} + \gamma A)^{-1}.$$

The reflection of $\mathcal{J}_{\gamma A}$ is defined by

$$\mathcal{R}_{\gamma A} \stackrel{\text{def}}{=} 2\mathcal{J}_{\gamma A} - \text{Id}.$$

Given a function $R \in \Gamma_0$ and its sub-differential ∂R ,

$$\text{prox}_R = \mathcal{J}_{\partial R}.$$

Set of fixed points,

$$\text{fix}(\text{prox}_R) = \text{fix}(\mathcal{J}_{\partial R}) = \text{zer}(\partial R).$$

Yosida approximation

Theorem (Yosida approximation)

Let $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a maximal monotone operator and $\gamma > 0$, the Yosida approximation of A with γ is

$$\gamma A \stackrel{\text{def}}{=} \frac{1}{\gamma}(\text{Id} - \mathcal{J}_{\gamma A}) = (\gamma \text{Id} + A^{-1})^{-1} = \mathcal{J}_{A^{-1}/\gamma}(\cdot/\gamma).$$

Moreover,

$$\text{Id} = \mathcal{J}_{\gamma A}(\cdot) + \gamma \mathcal{J}_{A^{-1}/\gamma}\left(\frac{\cdot}{\gamma}\right).$$

- γA is γ -cocoercive

Non-expansive operator

Definition (Non-expansive operator)

An operator $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called non-expansive if it is 1-Lipschitz continuous, *i.e.*

$$\|\mathcal{T}(x) - \mathcal{T}(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

For any $\alpha \in]0, 1[$, \mathcal{T} is α -averaged if there exists a non-expansive operator \mathcal{T}' such that

$$\mathcal{T} = \alpha\mathcal{T}' + (1 - \alpha)\text{Id}.$$

- $\mathcal{A}(\alpha)$ denotes the class of α -averaged operators on \mathbb{R}^n
- $\mathcal{A}(\frac{1}{2})$ is the class of firmly non-expansive operators

Properties: α -averaged operators

Lemma

Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be non-expansive and $\alpha \in]0, 1[$. The following statements are equivalent:

- \mathcal{T} is α -averaged non-expansive.
- The operator

$$\left(1 - \frac{1}{\alpha}\right)\text{Id} + \frac{1}{\alpha}\mathcal{T}$$

is non-expansive.

- For any $x, y \in \mathbb{R}^n$,

$$\|\mathcal{T}(x) - \mathcal{T}(y)\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha} \|(\text{Id} - \mathcal{T})(x) - (\text{Id} - \mathcal{T})(y)\|^2.$$

Properties: α -averaged operators

$\mathcal{A}(\alpha)$ is closed under relaxations, convex combinations and compositions.

Lemma

Let $m \in \mathbb{N}_+$, $\{\mathcal{T}_i\}_{i \in \{1, \dots, m\}}$ be non-expansive operators on \mathbb{R}^n , $(\omega_i)_i \in]0, 1]^m$ and $\sum_i \omega_i = 1$, and $(\alpha_i)_i \in]0, 1]^m$ such that $\mathcal{T}_i \in \mathcal{A}(\alpha_i)$, $i \in \{1, \dots, m\}$. Then,

- $\text{Id} + \lambda_i(\mathcal{T}_i - \text{Id}) \in \mathcal{A}(\lambda_i \alpha_i)$, $\lambda_i \in]0, \frac{1}{\alpha_i}[$ and $i \in \{1, \dots, m\}$.
- $\sum_i \omega_i \mathcal{T}_i \in \mathcal{A}(\alpha)$ with $\alpha = \max_i \alpha_i$.
- $\mathcal{T}_1 \cdots \mathcal{T}_m \in \mathcal{A}(\alpha)$ with $\alpha = \frac{m}{m-1+1/\max_{i \in \{1, \dots, m\}} \alpha_i}$.

Remark

When having the composition of two averaged operators, a sharper bound of α can be obtained for $m = 2$,

$$\alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{1 - \alpha_1\alpha_2} \in]0, 1[.$$

Properties: firmly non-expansive operators

Lemma

Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be non-expansive. The following statements are equivalent:

- *\mathcal{T} is firmly non-expansive.*
- *$\text{Id} - \mathcal{T}$ is firmly non-expansive.*
- *$2\mathcal{T} - \text{Id}$ is non-expansive.*
- *$\|\mathcal{T}(x) - \mathcal{T}(y)\|^2 \leq \langle \mathcal{T}(x) - \mathcal{T}(y), x - y \rangle, \forall x, y \in \mathbb{R}^n$.*
- *\mathcal{T} is the resolvent of a maximal monotone operator A , i.e. $\mathcal{T} = \mathcal{J}_A$.*

Lemma

Let operator $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be β -cocoercive for some $\beta > 0$. Then

- *$\beta B \in \mathcal{A}(\frac{1}{2})$, i.e. is firmly non-expansive.*
- *$\text{Id} - \gamma B \in \mathcal{A}(\frac{\gamma}{2\beta})$ for $\gamma \in]0, 2\beta[$.*

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Fixed point

Definition (Fixed point)

Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-expansive operator, $x \in \mathbb{R}^n$ is called the fixed point of \mathcal{T} if

$$x = \mathcal{T}(x).$$

The set of fixed points of \mathcal{T} is denoted as $\text{fix}(\mathcal{T})$.

$\text{fix}(\mathcal{T})$ may be empty, e.g. translation by a non-zero vector.

Theorem

Let \mathcal{X} be a non-empty bounded closed convex subset of \mathbb{R}^n and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a non-expansive operator, then $\text{fix}(\mathcal{T}) \neq \emptyset$.

Lemma

Let \mathcal{X} be a non-empty closed convex subset of \mathbb{R}^n and $\mathcal{T} : \mathcal{X} \rightarrow \mathbb{R}^n$ be a non-expansive operator, then $\text{fix}(\mathcal{T})$ is closed and convex.

Krasnosel'skiĭ-Mann iteration

Fixed-point iteration

$$x_{k+1} = \mathcal{T}(x_k).$$

However, it may not converge, e.g. $\mathcal{T} = -\text{Id}$...

Definition (Krasnosel'skiĭ-Mann iteration)

Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-expansive operator such that $\text{fix}(\mathcal{T}) \neq \emptyset$. Let $\lambda_k \in [0, 1]$ and choose x_0 arbitrarily from \mathbb{R}^n , then the Krasnosel'skiĭ-Mann iteration of \mathcal{T} reads

$$x_{k+1} = x_k + \lambda_k(\mathcal{T}(x_k) - x_k). \quad (3.1)$$

- If $\mathcal{T} \in \mathcal{A}(\alpha)$, then $\lambda_k \in [0, 1/\alpha]$

Fejér monotonicity

Definition (Fejér monotonicity)

Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a non-empty set and $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in \mathbb{R}^n . Then

- $\{x_k\}_{k \in \mathbb{N}}$ is Fejér monotone with respect to \mathcal{S} if

$$\|x_{k+1} - x\| \leq \|x_k - x\|, \quad \forall x \in \mathcal{S}, \forall k \in \mathbb{N}.$$

- $\{x_k\}_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to \mathcal{S} , if there exists a summable sequence $\{\epsilon_k\}_{k \in \mathbb{N}} \in \ell_+^1$ such that

$$\forall k \in \mathbb{N}, \quad \|x_{k+1} - x\| \leq \|x_k - x\| + \epsilon_k, \quad \forall x \in \mathcal{S}.$$

Example

Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a non-empty convex set, and $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ be a non-expansive operator such that $\text{fix}(\mathcal{T}) \neq \emptyset$. The sequence $\{x_k\}_{k \in \mathbb{N}}$ generated by $x_{k+1} = \mathcal{T}(x_k)$ is Fejér monotone with respect to $\text{fix}(\mathcal{T})$.

Convergence

Lemma

Let $S \subseteq \mathbb{R}^n$ be a non-empty set and $\{x_k\}_{k \in \mathbb{N}}$ be a sequence in \mathbb{R}^n . Assume the $\{x_k\}_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to S , then the following holds

- *$\{x_k\}_{k \in \mathbb{N}}$ is bounded.*
- *$\|x_k - x\|$ is bounded for any $x \in S$.*
- *$\{\text{dist}(x_k, S)\}_{k \in \mathbb{N}}$ is decreasing and convergent.*

If every sequential cluster point of $\{x_k\}_{k \in \mathbb{N}}$ belongs to S , then $\{x_k\}_{k \in \mathbb{N}}$ converges to a point in S .

- Weak convergence in general real Hilbert space

Convergence

Theorem

Let $\mathcal{T} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a non-expansive operator such that $\text{fix}(\mathcal{T}) \neq \emptyset$. Consider the Krasnosel'skiĭ-Mann iteration of \mathcal{T} , and choose $\lambda_k \in [0, 1]$ such that

$$\sum_{k \in \mathbb{N}} \lambda_k (1 - \lambda_k) = +\infty,$$

then the following holds

- $\{x_k\}_{k \in \mathbb{N}}$ *is Fejér monotone with respect to $\text{fix}(\mathcal{T})$.*
- $\{x_k - \mathcal{T}(x_k)\}_{k \in \mathbb{N}}$ *converges strongly to 0.*
- $\{x_k\}_{k \in \mathbb{N}}$ *converges to a point in $\text{fix}(\mathcal{T})$.*

When \mathcal{T} is α -averaged, then

$$\lambda_k \in [0, 1/\alpha] \text{ such that } \sum_{k \in \mathbb{N}} \lambda_k (1/\alpha - \lambda_k) = +\infty.$$

Preliminary

Krasnosel'skiĭ-Mann iteration with constant relaxation

$$\begin{aligned}x_{k+1} &= x_k + \lambda(\mathcal{T}(x_k) - x_k) \\ &= ((1 - \lambda)\text{Id} + \lambda\mathcal{T})(x_k).\end{aligned}$$

Denote $\mathcal{T}_\lambda = (1 - \lambda)\text{Id} + \lambda\mathcal{T}$, and define residual

$$e_k = (\text{Id} - \mathcal{T})(x_k) = \frac{1}{\lambda}(x_k - x_{k+1}).$$

Preliminary

Krasnosel'skiĭ-Mann iteration with constant relaxation

$$\begin{aligned}x_{k+1} &= x_k + \lambda(\mathcal{T}(x_k) - x_k) \\ &= ((1 - \lambda)\text{Id} + \lambda\mathcal{T})(x_k).\end{aligned}$$

Denote $\mathcal{T}_\lambda = (1 - \lambda)\text{Id} + \lambda\mathcal{T}$, and define residual

$$e_k = (\text{Id} - \mathcal{T})(x_k) = \frac{1}{\lambda}(x_k - x_{k+1}).$$

- $\mathcal{T}_\lambda \in \mathcal{A}(\lambda)$ if $\lambda \in]0, 1[$. If $\mathcal{T} \in \mathcal{A}(\alpha)$, then $\mathcal{T}_\lambda \in \mathcal{A}(\lambda\alpha)$
- For any $x^* \in \text{fix}(\mathcal{T})$,

$$x^* \in \text{fix}(\mathcal{T}) \Leftrightarrow x^* \in \text{fix}(\mathcal{T}_\lambda) \Leftrightarrow x^* \in \text{zer}(\text{Id} - \mathcal{T})$$

- If $\lambda \in [\epsilon, 1 - \epsilon]$, $\epsilon \in]0, 1/2]$,
 - e_k converges to 0.
 - $\{x_k\}_{k \in \mathbb{N}}$ is quasi-Fejér monotone with respect to $\text{fix}(\mathcal{T})$, and converges to a $x^* \in \text{fix}(\mathcal{T})$.

Pointwise convergence rate

Rate of $\|e_k\|^2$:

- For residual

$$\|e_{k+1}\|^2 \leq \|e_k\|^2 - \frac{1-\lambda}{\lambda} \|e_k - e_{k+1}\|^2.$$

- $\mathcal{T}_\lambda \in \mathcal{A}(\lambda)$, $\tau = \lambda(1 - \lambda)$

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \tau \|e_k\|^2.$$

- Summation

$$(k+1)\|e_k\|^2 \leq \tau \sum_{i=0}^k \|e_i\|^2 \leq \|x_0 - x^*\|^2 - \|x_{k+1} - x^*\|^2.$$

- Rate

$$\|e_k\|^2 \leq \frac{\|x_0 - x^*\|^2}{k+1}.$$

If $T \in \mathcal{A}(\alpha)$, then the above holds for $\lambda \in [\epsilon, 1/\alpha - \epsilon]$.

Ergodic convergence rate

Define $\bar{e}_k = \frac{1}{k+1} \sum_{i=0}^k e_i$.

- Boundedness

$$\begin{aligned}\|x_{k+1} - x^*\| &= \|\mathcal{T}_\lambda(x_k) - \mathcal{T}_\lambda(x^*)\| \leq \|x_k - x^*\| \\ &\leq \|x_0 - x^*\|\end{aligned}$$

- $\lambda e_k = x_k - x_{k+1}$

$$\begin{aligned}\|\bar{e}_k\| &= \frac{1}{k+1} \left\| \sum_{i=0}^k e_i \right\| = \frac{1}{\lambda(k+1)} \left\| \sum_{i=0}^k (x_i - x_{i+1}) \right\| \\ &= \frac{1}{\lambda(k+1)} \|x_0 - x_{k+1}\| \\ &\leq \frac{1}{\lambda(k+1)} (\|x_0 - x^*\| + \|x_{k+1} - x^*\|) \\ &\leq \frac{2\|x_0 - x^*\|}{\lambda(k+1)}\end{aligned}$$

Both rates (pointwise and ergodic) can be extended to the inexact case...

Metric sub-regularity

Definition (Metric sub-regularity)

A set-valued mapping $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is called metrically sub-regular at \bar{x} for $\bar{u} \in A(\bar{x})$ if there exists $\kappa \geq 0$ along with neighbourhood \mathcal{X} of \bar{x} such that

$$\text{dist}(x, A^{-1}(\bar{u})) \leq \kappa \text{dist}(\bar{u}, A(x)), \quad \forall x \in \mathcal{X}.$$

The infimum of all κ such that above holds is called the modulus of metric sub-regularity, and denoted by $\text{subreg}(A; \bar{x}|\bar{u})$.

Example Let $F \in S_{\alpha, L}^1$ and $A = \gamma \nabla F$ with $\gamma \leq 1/L$: $\bar{x} = \text{argmin}_{\mathbb{R}^n} F$ and $\bar{u} = 0$,

$$\begin{aligned} \text{dist}(\bar{u}, A(x)) &= \|\gamma \nabla F(x) - \gamma \nabla F(\bar{x})\| \\ &\geq \gamma \alpha \|x - \bar{x}\| \end{aligned}$$

(Local) linear convergence

Let $x^* \in \text{fix}(\mathcal{T})$, suppose $\mathcal{T}' \stackrel{\text{def}}{=} \text{Id} - \mathcal{T}$ is metrically sub-regular at x^* with neighbourhood \mathcal{X} of x^* , let $\kappa > \text{subreg}(\mathcal{T}'; x^*|0)$:

- $0 = \mathcal{T}'(x^*)$, $\mathcal{T}'^{-1}(0) = \text{fix}(\mathcal{T})$

$$\text{dist}(x, \text{fix}(\mathcal{T})) \leq \kappa \text{dist}(0, \mathcal{T}'(x)) = \kappa \|x - \mathcal{T}(x)\|.$$

- Denote $d_k = \text{dist}(x_k, \text{fix}(\mathcal{T}))$, $\bar{x} \in \text{fix}(\mathcal{T})$

$$\begin{aligned} d_{k+1}^2 &\leq \|x_{k+1} - \bar{x}\|^2 \leq \|x_k - \bar{x}\|^2 - \tau \|\mathcal{T}'(x_k) - \mathcal{T}'(\bar{x})\|^2 \\ &\leq d_k^2 - \frac{\tau_k}{\kappa^2} d_k^2 \\ &= \left(1 - \frac{\tau_k}{\kappa^2}\right) d_k^2. \end{aligned}$$

NB: As metric sub-regularity is a local property, the linear convergence will happen only when x_k is close enough to $\text{fix}(\mathcal{T})$

Optimal relaxation parameter?

Consider $\lambda_k \in [0, 1]$ and $x_{k+1} = (1 - \lambda_k)x_k + \lambda_k\mathcal{T}(x_k)$. Then

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|(1 - \lambda_k)(x_k - x^*) + \lambda_k(\mathcal{T}(x_k) - x^*)\|^2 \\&= (1 - \lambda_k)\|x_k - x^*\|^2 + \lambda_k\|\mathcal{T}(x_k) - x^*\|^2 \\&\quad - \lambda_k(1 - \lambda_k)\|x_k - \mathcal{T}(x_k)\|^2 \\&= \lambda_k^2\|x_k - \mathcal{T}(x_k)\|^2 \\&\quad - \lambda_k(\|x_k - x^*\|^2 - \|\mathcal{T}(x_k) - x^*\|^2 + \|x_k - \mathcal{T}(x_k)\|^2) \\&\quad + \|x_k - x^*\|^2,\end{aligned}$$

which is a quadratic function of λ_k , and minimises at

$$\lambda = \frac{1}{2} + \frac{\|x_k - x^*\|^2 - \|\mathcal{T}(x_k) - x^*\|^2}{2\|x_k - \mathcal{T}(x_k)\|^2}.$$

Approximation:

$$\lambda = \frac{1}{2} + \frac{\|x_k - \mathcal{T}(x_k)\|^2 - \|\mathcal{T}(x_k) - \mathcal{T}^2(x_k)\|^2}{2\|(x_k - \mathcal{T}(x_k)) - (\mathcal{T}(x_k) - \mathcal{T}^2(x_k))\|^2}.$$

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Inertial Krasnosel'skiĭ-Mann iteration

An inertial Krasnosel'skiĭ-Mann iteration

Initial: $x_0 \in \mathbb{R}^n$, $x_{-1} = x_0$;

$$y_k = x_k + a_k(x_k - x_{k-1}), \quad a_k \in [0, 1],$$

$$z_k = x_k + b_k(x_k - x_{k-1}), \quad b_k \in [0, 1],$$

$$x_{k+1} = (1 - \lambda_k)y_k + \lambda_k \mathcal{T}(z_k), \quad \lambda_k \in [0, 1].$$

- Covers heavy-ball method, Nesterov's scheme, inertial FB and FISTA
- Convergence analysis is much harder than the inertial version of descent methods
- No convergence rate
- May perform very poorly in practice, slower than the original scheme

A multi-step inertial scheme

A multi-step inertial Krasnosel'skiĭ-Mann iteration

Initial: $x_0 \in \mathbb{R}^n$, $x_{-1} = x_0$ and $\gamma \in]0, 2/L[$;

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots ,$$

$$z_k = x_k + b_{0,k}(x_k - x_{k-1}) + b_{1,k}(x_{k-1} - x_{k-2}) + \cdots ,$$

$$x_{k+1} = (1 - \lambda_k)y_k + \lambda_k \mathcal{T}(z_k), \lambda_k \in [0, 1].$$

- Even harder to analyse convergence
- No rate
- However, can outperform the original scheme...

Convergence

- Conditional convergence, $i = 0, 1, \dots$

$$\sum_{k \in \mathbb{N}} \max \left\{ \max_i |a_{i,k}|, \max_i |b_{i,k}| \right\} \sum_i \|x_{k-i} - x_{k-i-1}\| < +\infty.$$

- Online updating rule

$$a_{i,k} = \min \{a_i, c_{i,k}\}$$

where

$$c_{i,k} = \frac{c_i}{k^{1+\delta} \sum_i \|x_{k-i} - x_{k-i-1}\|}, \quad \delta > 0.$$

Reference

- H. Bauschke and P. L. Combettes. Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer, 2011.
- J. Liang, J. Fadili, and G. Peyré. Convergence rates with inexact non-expansive operators. Mathematical Programming 159.1-2 (2016): 403-434.