Introductory Course on Non-smooth Optimisation

Lecture 06

Primal–Dual splitting

Outline

Outline

Composed monotone inclusion

Problem

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in A(x) + L^* \circ C \circ L(x).$$

Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone
- $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping
- $C : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ is maximal monotone
- $\text{zer}(A + L^* \circ C \circ L) \neq \emptyset$

Saddle-point problem

Let $x^\star \in \operatorname{zer}(A + L^* \circ C \circ L)$, then $\exists v^\star \in C \circ Lx^\star$ such that

$$0 \in A(x^\star) + L^* v^\star$$

and $Lx^\star \in C^{-1}(v^\star)$.

Saddle-point problem

$$\text{Find } \; v \in \mathbb{R}^m \; \text{ such that } \; \exists x \in \mathbb{R}^n \begin{cases} 0 \in A(x) + L^* v \\ 0 \in C^{-1}(v) - Lx \end{cases}$$

Denote $\mathcal{X}$ and $\mathcal{V}$ the set of primal and dual solutions.

Outline

Primal–Dual splitting

Primal–Dual splitting

Let $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$ and $\gamma_A, \gamma_C > 0$, $\theta \in [-1, 1]$:

$$\begin{cases} x_{k+1} = \mathcal{J}_{\gamma_A A}(x_k - \gamma_A L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C L\bar{x}_{k+1}). \end{cases}$$

- Known as Chambolle-Pock Primal–Dual method in optimisation
- Douglas–Rachford is the limiting case of Primal–Dual
- Moreau's identity

$$\mathrm{Id} = \mathcal{J}_{\gamma A}(\cdot) + \gamma \mathcal{J}_{A^{-1}/\gamma}\Big(\frac{\cdot}{\gamma}\Big).$$

PPA structure of Primal–Dual splitting

- definition of resolvent

$$\frac{1}{\gamma_A}(x_k - x_{k+1}) - L^* v_k \in A(x_{k+1})$$
$$\frac{1}{\gamma_C}(v_k - v_{k+1}) + L\big(x_{k+1} + \theta(x_{k+1} - x_k)\big) \in C^{-1}(v_{k+1})$$

- arrange terms

$$\frac{1}{\gamma_A}(x_k - x_{k+1}) - L^*(v_k - {\color{blue}v_{k+1}}) \in A(x_{k+1}) + {\color{blue}L^* v_{k+1}}$$
$$\frac{1}{\gamma_C}(v_k - v_{k+1}) + \theta L(x_{k+1} - x_k) \in C^{-1}(v_{k+1}) - Lx_{k+1}$$

- inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{bmatrix} \mathrm{Id}_n/\gamma_A & -L^* \\ -\theta L & \mathrm{Id}_m/\gamma_C \end{bmatrix} \begin{pmatrix} x_{k+1} - x_k \\ v_{k+1} - v_k \end{pmatrix}$$

PPA structure of Primal–Dual splitting

- inclusion

$$\boldsymbol{z} = \begin{pmatrix} x \\ v \end{pmatrix}, \;\; \boldsymbol{A} = \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{V} = \begin{bmatrix} \mathrm{Id}_n/\gamma_{\scriptscriptstyle A} & -L^* \\ -\theta L & \mathrm{Id}_m/\gamma_{\scriptscriptstyle C} \end{bmatrix}$$

- $\boldsymbol{A}$ is skew symmetric, hence maximal monotone

- $\boldsymbol{V}$ is symmetric if $\theta = 1$ and moreover positive definite if $\gamma_{\scriptscriptstyle A}\gamma_{\scriptscriptstyle C}\|L\|^2 < 1$

- $\boldsymbol{V}\boldsymbol{z}_k \in \boldsymbol{A}(\boldsymbol{z}_{k+1}) + \boldsymbol{V}\boldsymbol{z}_{k+1}$, hence

$$\begin{aligned} \boldsymbol{z}_{k+1} &= (\boldsymbol{V} + \boldsymbol{A})^{-1}(\boldsymbol{V}\boldsymbol{z}_k) \\ &= (\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}(\boldsymbol{z}_k) \end{aligned}$$

- which is PPA under metrix $\boldsymbol{V}$

Fixed-point equation

Fixed-point formulation

$$\mathbf{z}_{k+1} = (\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}(\boldsymbol{z}_k)$$

Property space $(\mathbb{R}^n \times \mathbb{R}^n)_{\boldsymbol{V}}$

- $\mathcal{T}_{\text{PD}} = (\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}$ is firmly non-expansive when $\theta = 1$ and $\gamma_A \gamma_C \|L\|^2 < 1$
- for $\theta \in [-1, 1[$, a correction step is needed
- Douglas–Rachford is the limiting case of Primal–Dual when

$$L = \text{Id} \quad \text{and} \quad \gamma_A \gamma_C = 1$$

Relation with Douglas–Rachford

- let $\theta = 1$ and $L = \mathrm{Id}, \gamma_A \gamma_C = 1$
- change the order of updating variables,

$$\left\lfloor \begin{aligned} v_{k+1} &= \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C \bar{x}_k) \\ x_{k+1} &= \mathcal{J}_{\gamma_A A}(x_k - \gamma_A v_{k+1}) \\ \bar{x}_{k+1} &= 2x_{k+1} - x_k. \end{aligned} \right.$$

- apply Moreau's identity to $\mathcal{J}_{\gamma_C C^{-1}}$,

$$v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C \bar{x}_k) = v_k + \gamma_C \bar{x}_k - \gamma_C \mathcal{J}_{C/\gamma_C}\left(\frac{v_k + \gamma_C \bar{x}_k}{\gamma_C}\right)$$

- let $\gamma_C = 1/\gamma_A$ and define $z_{k+1} = x_k - \gamma_A v_{k+1}$,

$$\left\lfloor \begin{aligned} u_{k+1} &= \mathcal{J}_{\gamma_A J}(2x_k - z_k) \\ z_{k+1} &= z_k + u_{k+1} - x_k \\ x_{k+1} &= \mathcal{J}_{\gamma_A R}(z_{k+1}), \end{aligned} \right.$$

Convergence rate

- Let $\mathcal{X}, \mathcal{Y}$ be two subspaces

$$\mathcal{X} = \{x : ax = 0\},\ \mathcal{Y} = \{x : bx = 0\}$$

  and assume

$$1 \le p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \le q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \le n - 1.$$

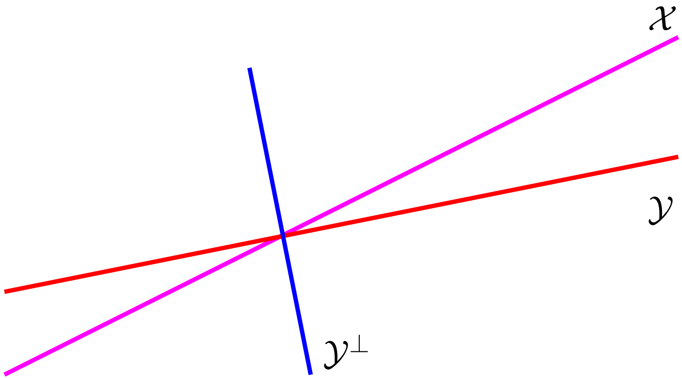
- Projection onto subspace

$$\mathrm{proj}_{\mathcal{X}}(x) = x - a^T(aa^T)^{-1}ax$$

- Moreau identity

$$x = \mathrm{proj}_{\mathcal{X}}(x) + \mathrm{proj}_{\mathcal{X}^\perp}(x)$$

Convergence rate



Convergence rate

- linearisation of PD

$$M_{\text{PD}} = \begin{bmatrix} \mathrm{Id}_n & -\gamma_A \mathrm{proj}_{\mathcal{X}} \mathrm{proj}_{\mathcal{Y}^\perp} \\ \gamma_C \mathrm{proj}_{\mathcal{Y}^\perp} \mathrm{proj}_{\mathcal{X}} & \mathrm{Id}_n - 2\gamma_C \gamma_A \mathrm{proj}_{\mathcal{Y}^\perp} \mathrm{proj}_{\mathcal{X}} \mathrm{proj}_{\mathcal{Y}^\perp} \end{bmatrix}$$

- linearisation of DR

$$M_{\text{DR}} = \begin{bmatrix} \mathrm{Id}_n & -\gamma_A \mathrm{proj}_{\mathcal{X}} \mathrm{proj}_{\mathcal{Y}^\perp} \\ \frac{1}{\gamma_A} \mathrm{proj}_{\mathcal{Y}^\perp} \mathrm{proj}_{\mathcal{X}} & \mathrm{Id}_n - 2\mathrm{proj}_{\mathcal{Y}^\perp} \mathrm{proj}_{\mathcal{X}} \mathrm{proj}_{\mathcal{Y}^\perp} \end{bmatrix}$$

- both M_{PD} and M_{DR} are convergent
- let ω be the largest principal angle (yet smaller than $\pi/2$) between $\mathcal{X}$ and $\mathcal{Y}^\perp$
- spectral radius

$$\begin{aligned} \rho(M_{\text{PD}} - M_{\text{PD}}^\infty) &= \sqrt{1 - \gamma_C \gamma_A \cos^2(\omega)} \\ &\geq \sqrt{1 - \cos^2(\omega)} = \sin(\omega) = \cos(\pi/2 - \omega) = \rho(M_{\text{DR}} - M_{\text{DR}}^\infty) \end{aligned}$$

Outline

Parallel sum

Parallel sum

Let $C, D : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be two set-valued operators, the parallel sum of C and D is defined by

$$C \Box D \stackrel{\text{def}}{=} (C^{-1} + D^{-1})^{-1}.$$

- $(C \Box D)x = \cup_{y \in \mathbb{R}^n} (A(x) \cap B(x - y))$
- if C and D are monotone, then $C \Box D$ is monotone

Monotone inclusion with parallel sum

Primal problem

$$\text{find } x \in \mathbb{R}^n \text{ such that } 0 \in (A+B)(x) + L^*\big((C \Box D)(Lx)\big)$$

Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone, $B : \mathbb{R}^n \to \mathbb{R}^n$ is β_B-cocoercive for some $\beta_B > 0$
- $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear operator
- $C, D : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ are maximal monotone, moreover D is β_D-strongly monotone for some $\beta_D > 0$
- $0 \in \operatorname{ran}(A + B + L^*(C \Box D)L)$

Saddle-point problem

$$\text{find } v \in \mathbb{R}^m \text{ such that } (\exists x \in \mathbb{R}^n) \begin{cases} 0 \in (A+B)(x) + L^* v, \\ 0 \in (C^{-1} + D^{-1})(v) - Lx. \end{cases}$$

Primal–Dual splitting

Primal–Dual splitting

Let $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$ and $\gamma_A, \gamma_C > 0$, $\theta \in [-1, 1]$:

$$\begin{cases} x_{k+1} = \mathcal{J}_{\gamma_A A}(x_k - \gamma_A B(x_k) - \gamma_A L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k - \gamma_C D^{-1}(v_k) + \gamma_C L\bar{x}_{k+1}). \end{cases}$$

- can be cast as Forward–Backward splitting

FB structure of Primal–Dual splitting

Let $\theta = 1$,

- definition of resolvent

$$\frac{1}{\gamma_A}(x_k - x_{k+1}) - B(x_k) - L^* v_k \in A(x_{k+1})$$

$$\frac{1}{\gamma_C}(v_k - v_{k+1}) - D^{-1}(v_k) + L\big(x_{k+1} + (x_{k+1} - x_k)\big) \in C^{-1}(v_{k+1})$$

- arrange terms

$$\frac{1}{\gamma_A}(x_k - x_{k+1}) - B(x_k) - L^*(v_k - v_{k+1}) \in A(x_{k+1}) + L^* v_{k+1}$$

$$\frac{1}{\gamma_C}(v_k - v_{k+1}) - D^{-1}(v_k) + L(x_{k+1} - x_k) \in C^{-1}(v_{k+1}) - L x_{k+1}$$

- inclusion

$$-\begin{bmatrix} B & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} \in \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{bmatrix} \mathrm{Id}_n/\gamma_A & -L^* \\ -L & \mathrm{Id}_m/\gamma_C \end{bmatrix} \begin{pmatrix} x_{k+1} - x_k \\ v_{k+1} - v_k \end{pmatrix}$$

FB structure of Primal–Dual splitting

- inclusion

$$\boldsymbol{z} = \begin{pmatrix} x \\ v \end{pmatrix}, \;\; \boldsymbol{A} = \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix}, \;\; \boldsymbol{B} = \begin{bmatrix} B & 0 \\ 0 & D^{-1} \end{bmatrix} \;\text{ and }\; \boldsymbol{V} = \begin{bmatrix} \mathrm{Id}_n/\gamma_A & -L^* \\ -L & \mathrm{Id}_m/\gamma_C \end{bmatrix}$$

- $\boldsymbol{A}$ is skew symmetric, hence maximal monotone
- $\boldsymbol{B}$ is $\min\{\beta_B, \beta_D\}$-cocoercive
- $\boldsymbol{V}$ is symmetric positive definite for $\gamma_A\gamma_C\|L\|^2 < 1$
- $\boldsymbol{V}\boldsymbol{z}_k - \boldsymbol{B}(z_k) \in \boldsymbol{A}(\boldsymbol{z}_{k+1}) + \boldsymbol{V}\boldsymbol{z}_{k+1}$, hence

$$\begin{aligned} \boldsymbol{z}_{k+1} &= (\boldsymbol{V} + \boldsymbol{A})^{-1}(\boldsymbol{V} - \boldsymbol{B})(\boldsymbol{z}_k) \\ &= (\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}\big(\mathbf{Id} - \boldsymbol{V}^{-1}\boldsymbol{B}\big)(\boldsymbol{z}_k) \end{aligned}$$

- which is FB under metrix $\boldsymbol{V}$

Fixed-point equation

Fixed-point formulation

$$z_{k+1} = (\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}(\mathbf{Id} - \boldsymbol{V}^{-1}\boldsymbol{B})(\boldsymbol{z}_k)$$

Property space $(\mathbb{R}^n \times \mathbb{R}^n)_{\boldsymbol{V}}$

- $(\mathbf{Id} + \boldsymbol{V}^{-1}\boldsymbol{A})^{-1}$ is firmly non-expansive when $\gamma_A \gamma_C \|L\|^2 < 1$
- $\mathbf{Id} - \boldsymbol{V}^{-1}\boldsymbol{B}$ is $\frac{1}{2\beta\nu}$-averaged non-expansive with

$$\nu = \left(1 - \sqrt{\gamma_A \gamma_C \|L\|^2}\right) \min\left\{\frac{1}{\gamma_A}, \frac{1}{\gamma_C}\right\}$$

- $\mathcal{T}_{\mathrm{PD}}$ is $\frac{2\beta\nu}{4\beta\nu - 1}$-averaged non-expansive

Outline

Infimal convolution

Infimal convolution $J, G \in \Gamma_0(\mathbb{R}^m)$

$$(J \overset{+}{\vee} G)(\cdot) \overset{\text{def}}{=} \inf_{v \in \mathbb{R}^m} J(\cdot) + G(\cdot - v)$$

- $\partial(J \overset{+}{\vee} G)(\cdot) = (\partial J \square \partial G)(\cdot)$

Example Moreau envelope

$$J \overset{+}{\vee} \frac{1}{2\gamma}\| \cdot \|^2 = \inf_{v \in \mathbb{R}^m} J(v) + \frac{1}{2\gamma}\| \cdot - v\|^2$$

Conjugate

Conjugate

Let $F : \mathbb{R}^n \to]-\infty, +\infty]$, the Fenchel conjugate of F is defined by

$$F^*(v) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^n} (\langle x, v\rangle - F(x)).$$

- F^* is closed and convex even F is not

Biconjugate $F^{**} = (F^*)^*$

Conjugate

Example Let $\mathcal{S} \subseteq \mathbb{R}^n$ be a non-empty convex set, the support function of $\mathcal{S}$ is defined by

$$\sigma_{\mathcal{S}}(v) \stackrel{\text{def}}{=} \sup_{x \in \mathcal{S}} \langle x, \; v \rangle = \iota_{\mathcal{S}}^*(y).$$

Example Let $F = \frac{1}{2}\| \cdot \|^2$,

$$F^*(y) = \frac{1}{2}\|y\|^2$$

Example Let $\| \cdot \|$ be a norm with dual norm $\| \cdot \|$. Let $F = \|x\|$, then

$$F^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ +\infty & o.w. \end{cases}$$

i.e. the indicator function of the dual norm ball.

Conjugate

Informal convolution

$$(J \overset{+}{\vee} G)^* = J^* + G^*$$

Fenchel–Moreau Let $F : \mathbb{R}^n \to]-\infty, +\infty]$ be a proper function, then F is convex and lower semi-continuous if and only if $F = F^{**}$.

Biconjugate If $F \in \Gamma_0(\mathbb{R}^n)$, then $F^* \in \Gamma_0(\mathbb{R}^n)$ and $F^{**} = F$.

Subdifferential If F is closed and convex, then

$$y \in \partial F(x) \iff x \in \partial F^*(y).$$

Moreau's identify Let function $F \in \Gamma_0(\mathbb{R}^n)$ and $\gamma > 0$, then

$$\mathrm{Id} = \mathrm{prox}_{\gamma F}(\cdot) + \gamma\, \mathrm{prox}_{F^*/\gamma}\Big(\frac{\cdot}{\gamma}\Big).$$

Strong convexity Let F be closed and α-strongly convex, then ∇F^* is $\frac{1}{\alpha}$-Lipschitz.

Problem

Primal problem

$$\min_{x\in\mathbb{R}^n} R(x) + F(x) + (J \overset{+}{\triangledown} G)(Lx)$$

Assumptions

- $R, F \in \Gamma_0(\mathbb{R}^n)$, and ∇F is $(1/\beta_F)$-Lipschitz continuous for some $\beta_F > 0$
- $J, G \in \Gamma_0(\mathbb{R}^m)$, G is β_G-strongly convex for $\beta_G > 0$
- $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping
- The inclusion $0 \in \mathrm{ran}(\partial R + \nabla F + L^*(\partial J \Box \partial G)L)$ holds

Dual problem

Saddle-point problem

$$\min_{x\in\mathbb{R}^n}\max_{v\in\mathbb{R}^m} R(x) + F(x) + \langle Lx,\, v\rangle - \big(J^*(v) + G^*(v)\big)$$

Dual problem

$$\min_{v\in\mathbb{R}^m} J^*(v) + G^*(v) + (R^* \overset{+}{\triangledown} F^*)(-L^*v)$$

Denote by $\mathcal{X}$ and $\mathcal{V}$ the sets of solutions of primal problem and dual problem, respectively.

Primal–Dual splitting

Primal–Dual splitting

Let $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$ and $\gamma_R, \gamma_J > 0$, $\theta \in [-1, 1]$:

$$\begin{cases} x_{k+1} = \mathrm{prox}_{\gamma_R R}(x_k - \gamma_R \nabla F(x_k) - \gamma_R L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \mathrm{prox}_{\gamma_J J^*}(v_k - \gamma_J \nabla G^*(v_k) + \gamma_J L \bar{x}_{k+1}), \end{cases}$$

- $A = \partial R$, $B = \nabla F$
- $C^{-1} = \partial J^*$, $D^{-1} = \nabla G^*$

Outline

Problem

Primal problem

$$\min_{x \in \mathbb{R}^n} R(x) + J(Lx)$$

Assumptions

- $R \in \Gamma_0(\mathbb{R}^n)$
- $J \in \Gamma_0(\mathbb{R}^m)$
- $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping
- The inclusion $0 \in \operatorname{ran}(\partial R + L^* \partial J L)$ holds

Dual problem

$$\min_{v \in \mathbb{R}^m} J^*(v) + R^*(-L^* v)$$

R or J is α-strongly convex

Primal–Dual splitting

Let $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$ and $\gamma_{R,0}, \gamma_{J,0} > 0$ such that $\gamma_{R,0}\gamma_{J,0}\|L\|^2 \leq 1$:

$$\begin{cases} v_{k+1} = \mathrm{prox}_{\gamma_{J,k}J^*}(v_k + \gamma_{J,k}L\bar{x}_{k+1}) \\ x_{k+1} = \mathrm{prox}_{\gamma_{R,k}R}(x_k - \gamma_{R,k}L^* v_k) \\ \quad \theta_k = \frac{1}{\sqrt{1+2\alpha\gamma_{R,k}}}, \; \gamma_{R,k+1} = \theta_k \gamma_{R,k}, \; \gamma_{J,k+1} = \gamma_{J,k}/\theta_k \\ \bar{x}_{k+1} = x_{k+1} + \theta_k(x_{k+1} - x_k) \end{cases}$$

- convergence rate

$$\|x_k - x^\star\| = O(1/k^2)$$

R and J are strongly convex

R is α-strongly covnex and J is κ-strongly convex.

Primal–Dual splitting

Let $x_0 \in \mathbb{R}^n$, $v_0 \in \mathbb{R}^n$. Choose $\mu = \frac{2\sqrt{\alpha\kappa}}{L}$, $\gamma_{_R} = \frac{\mu}{2\alpha}$, $\gamma_{_J} = \frac{\mu}{2\kappa}$ and $\theta \in [1/(\mu+1), 1]$:

$$\begin{cases} v_{k+1} = \mathrm{prox}_{\gamma_J J^*}(v_k + \gamma_J L\bar{x}_{k+1}) \\ x_{k+1} = \mathrm{prox}_{\gamma_R R}(x_k - \gamma_R L^* v_k) \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k) \end{cases}$$

- convergence rate

$$\|x_k - x^\star\| = O(\eta^k)$$

with $\eta = \frac{1+\theta}{2+\mu}$

Reference

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