Introductory Course on Non-smooth Optimisation

Lecture 04 - Backward–Backward splitting

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Problem

Let $B : \mathbb{R}^n \to \mathbb{R}^n$ be β-cocoercive for some $\beta > 0$, $s > 1$ be a positive integer, such that for each $i \in \{1, ..., s\}$: $A_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone. Consider the problem

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in B(x) + \textstyle\sum_{i=1}^{s} A_i(x).$$

- A_i can be composed with linear mapping, *e.g.* $L^* \circ A \circ L$.
- Even if the resolvents of B and each A_i are simple, the resolvent of $B + \sum_i A_i$ in most cases is not solvable.
- Use the properties of operators and structure of problem to derive operator splitting schemes.

Outline

Monotone inclusion

$$\text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in A(x) + B(x).$$

Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is maximal monotone.
- $B : \mathbb{R}^n \to \mathbb{R}^n$ is β-cocoersive.
- $\mathrm{zer}(A + B) \neq \emptyset$.

Characterisation of minimiser: $\gamma > 0$

$$x^\star - \gamma B(x^\star) \in x^\star + \gamma A(x^\star) \quad \Leftrightarrow \quad x^\star = \mathcal{J}_{\gamma A} \circ (\mathrm{Id} - \gamma B)(x^\star).$$

Example Let $R \in \Gamma_0$ and $F \in C_L^1$,

$$\min_{x \in \mathbb{R}^n} R(x) + F(x).$$

Forward–Backward splitting

Fixed-point operator: $\gamma \in]0, 2\beta[$

$$\mathcal{T}_{\mathrm{FB}} = \mathcal{J}_{\gamma A} \circ (\mathrm{Id} - \gamma B).$$

- $\mathcal{J}_{\gamma A}$ is firmly non-expansive.
- $\mathrm{Id} - \gamma B$ is $\frac{\gamma}{2\beta}$-averaged non-expansive.
- $\mathcal{T}_{\mathrm{FB}}$ is $\frac{2\beta}{4\beta-\gamma}$-averaged non-expansive.
- $\mathrm{fix}(\mathcal{T}_{\mathrm{FB}}) = \mathrm{zer}(A+B)$.

Forward–Backward splitting

Let $\gamma \in]0, 2\beta[$, $\lambda_k \in [0, \frac{4\beta-\gamma}{2\beta}]$:

$$x_{k+1} = (1-\lambda_k)x_k + \lambda_k \mathcal{T}_{\mathrm{FB}}(x_k).$$

- Special case of Krasnosel'skiĭ-Mann iteration.
- Recovers proximal point algorithm when $B = 0$.

Outline

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ be closed convex and non-empty, such that $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$

$$\min_{x \in \mathbb{R}^n} \ \iota_{\mathcal{X}}(x) + \iota_{\mathcal{Y}}(x).$$

Method of alternating projection (MAP)

Let $x_0 \in \mathcal{X}$:

$$y_{k+1} = \mathcal{P}_{\mathcal{Y}}(x_k),$$
$$x_{k+1} = \mathcal{P}_{\mathcal{X}}(y_{k+1}).$$

Fixed-point operator: $x_{k+1} = \mathcal{T}_{\mathrm{MAP}}(x_k)$,

$$\mathcal{T}_{\mathrm{MAP}} \stackrel{\mathrm{def}}{=} \mathcal{P}_{\mathcal{X}} \circ \mathcal{P}_{\mathcal{Y}}.$$

- $\mathcal{P}_{\mathcal{X}}, \mathcal{P}_{\mathcal{Y}}$ are firmly non-expansive.
- $\mathcal{T}_{\mathrm{MAP}}$ is $\frac{2}{3}$-averaged non-expansive.
- $\mathrm{fix}(\mathcal{T}_{\mathrm{MAP}}) = \mathcal{X} \cap \mathcal{Y}$.

Derive MAP

- Feasibility problem is equivalent to

$$\min_{x,y\in\mathbb{R}^n} \ \iota_{\mathcal{X}}(x) + \frac{1}{2}\|x-y\|^2 + \iota_{\mathcal{Y}}(y).$$

- Optimality condition

$$\begin{aligned} 0 &\in \mathcal{N}_{\mathcal{Y}}(y^\star) + y^\star - x^\star, \\ 0 &\in \mathcal{N}_{\mathcal{X}}(x^\star) + x^\star - y^\star. \end{aligned}$$

- Fixed-point characterisation

$$\begin{aligned} y^\star &= \mathcal{P}_{\mathcal{Y}}(x^\star), \\ x^\star &= \mathcal{P}_{\mathcal{X}}(y^\star). \end{aligned}$$

- Fixed-point iteration

$$\begin{aligned} y_{k+1} &= \mathcal{P}_{\mathcal{Y}}(x_k), \\ x_{k+1} &= \mathcal{P}_{\mathcal{X}}(y_{k+1}). \end{aligned}$$

SDP feasibility

Find $X \in \mathcal{S}^n$ such that

$$X \succeq 0 \quad \text{and} \quad \mathrm{Tr}(A_i X) = b_i,\ i = 1, ..., m.$$

Two sets and projection:

- $\mathcal{X} = \mathcal{S}^n_+$ is the positive semidefinite cone. Let $Y_k = \sum_{i=1}^n \sigma_i u_i u_i^T$ be the eigenvalue decomposition of Y_k, then

$$\mathcal{P}_{\mathcal{X}}(Y_k) = \sum_{i=1}^n \max\{0, \sigma_i\} u_i u_i^T.$$

- $\mathcal{Y}$ is the affine set in $\mathcal{S}^n$ define by the linear inequalities,

$$\mathcal{P}_{\mathcal{Y}}(X_k) = X_k - \sum_{i=1}^m u_i A_i,$$

where u_i are found from the normal equations

$$Gu = \big(\mathrm{Tr}(A_i X_k) - b_i, \cdots, \mathrm{Tr}(A_i X_k) - b_m\big),\ \ G_{i,j} = \mathrm{Tr}(A_i A_j).$$

Let $\mathcal{X}, \mathcal{Y}$ be two subspaces, and assume

$$1 \le p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \le q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \le n-1.$$

Principal angles The principal angles $\theta_k \in [0, \frac{\pi}{2}], k = 1, \ldots, p$ between $\mathcal{X}$ and $\mathcal{Y}$ are defined by, with $u_0 = v_0 \stackrel{\text{def}}{=} 0$, and

$$\begin{aligned}\cos(\theta_k) \stackrel{\text{def}}{=} \langle u_k,\, v_k\rangle = \max\langle u,\, v\rangle \quad \text{s.t.} \quad & u \in \mathcal{X}, v \in \mathcal{Y}, \|u\| = 1, \|v\| = 1,\\ & \langle u,\, u_i\rangle = \langle v,\, v_i\rangle = 0,\ i = 0, \cdots, k-1.\end{aligned}$$

Friedrichs angle The Friedrichs angle $\theta_F \in]0, \frac{\pi}{2}]$ between $\mathcal{X}$ and $\mathcal{Y}$ is

$$\begin{aligned}\cos\big(\theta_F(\mathcal{X}, \mathcal{Y})\big) \stackrel{\text{def}}{=} \max\langle u,\, v\rangle \quad \text{s.t.} \quad & u \in \mathcal{X} \cap (\mathcal{X} \cap \mathcal{Y})^{\perp}, \|u\| = 1,\\ & v \in \mathcal{Y} \cap (\mathcal{X} \cap \mathcal{Y})^{\perp}, \|v\| = 1.\end{aligned}$$

Lemma

The Friedrichs angle is θ_{d+1} where $d \stackrel{\text{def}}{=} \dim(\mathcal{X} \cap \mathcal{Y})$. Moreover,

$$\theta_F(\mathcal{X}, \mathcal{Y}) > 0.$$

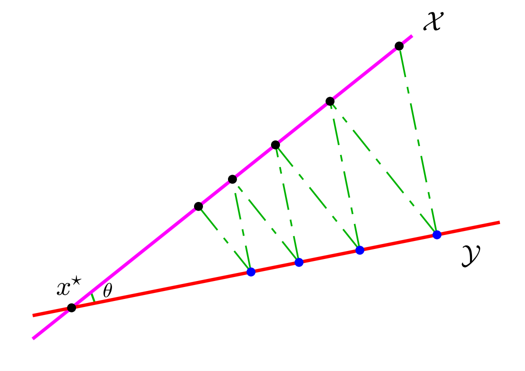
Convergence rate

Example $\mathcal{X}, \mathcal{Y}$ are defined by

$$\mathcal{X} = \{x : Ax = 0\}, \ \mathcal{Y} = \{x : Bx = 0\}.$$

Projection onto subspace

$$\mathcal{P}_{\mathcal{X}}(x) = x - A^T(AA^T)^{-1}Ax.$$



- Define diagonal matrices

$$c = \mathrm{diag}\big(\cos(\theta_1), \cdots, \cos(\theta_p)\big),$$
$$s = \mathrm{diag}\big(\sin(\theta_1), \cdots, \sin(\theta_p)\big).$$

- Suppose $p + q < n$, then there exists orthogonal matrix U such that

$$\mathcal{P}_{\mathcal{X}} = U \left[\begin{array}{cc|cc} \mathrm{Id}_p & 0 & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*,$$

$$\mathcal{P}_{\mathcal{Y}} = U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ cs & c^2 & 0 & 0 \\ \hline 0 & 0 & \mathrm{Id}_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*.$$

- Fixed-point operator

$$\begin{aligned}\mathcal{T}_{\mathrm{MAP}} &= \mathcal{P}_{\mathcal{X}} \circ \mathcal{P}_{\mathcal{Y}} \\ &= U \left[\begin{array}{cc|cc} c^2 & cs & 0 & 0 \\ 0 & 0_p & 0 & 0 \\ \hline 0 & 0 & 0_{q-p} & 0 \\ 0 & 0 & 0 & 0_{n-p-q} \end{array}\right] U^*.\end{aligned}$$

- Consider relaxation

$$\begin{aligned}\mathcal{T}_{\mathrm{MAP}}^{\lambda} &= (1-\lambda)\mathrm{Id} + \lambda \mathcal{T}_{\mathrm{MAP}} \\ &= U \left[\begin{array}{cc|c} (1-\lambda)\mathrm{Id}_p + \lambda c^2 & \lambda cs & 0 \\ 0 & (1-\lambda)\mathrm{Id}_p & 0 \\ \hline 0 & 0 & (1-\lambda)\mathrm{Id}_{n-2p} \end{array}\right] U^*.\end{aligned}$$

- Eigenvalues

$$\sigma(\mathcal{T}_{\text{MAP}}^{\lambda}) = \big\{1 - \lambda \sin^2(\theta_i) | i = 1, ..., p\big\} \cup \{1 - \lambda\}.$$

- Spectral radius

$$\rho(\mathcal{T}_{\text{MAP}}^{\lambda}) = \max\big\{1 - \lambda \sin^2(\theta_F), |1 - \lambda|\big\}.$$

- No relaxation

$$\rho(\mathcal{T}_{\text{MAP}}) = \cos^2(\theta_F).$$

- Convergence rate, $C > 0$ is some constant

$$\begin{aligned}
\|x_k - x^\star\| &= \|\mathcal{T}_{\text{MAP}} x_{k-1} - \mathcal{T}_{\text{MAP}} x^\star\| \\
&= ... \\
&= \|\mathcal{T}_{\text{MAP}}^k (x_0 - x^\star)\| \\
&\leq C \|\mathcal{T}_{\text{MAP}}\|^k \|x_0 - x^\star\|.
\end{aligned}$$

Outline

When $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, MAP returns $x_k, y_k \to x^\star \in \mathcal{X} \cap \mathcal{Y}$.

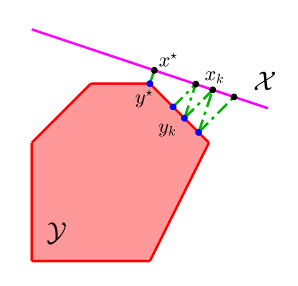
Best pair problem

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$ be closed and convex, such that

$$\mathcal{X} \cap \mathcal{Y} = \emptyset.$$

Consider finding two points in $\mathcal{X}$ and $\mathcal{Y}$ such that they are the closest, that is

$$\min_{x \in \mathcal{X}, y \in \mathcal{Y}} \|x - y\|.$$



- MAP can be applied and

$$(x_k, y_k) \to (x^\star, y^\star)$$

where $(x^\star, y^\star)$ is a best pair.

Consider

$$\text{Find } \; x, y \in \mathbb{R}^n \; \text{ such that } \; 0 \in A(x) + B(y),$$

- $A, B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are maximal monotone.
- The set of solition is non-empty.

There exists $x^\star, y^\star \in \mathbb{R}^n$ and $\gamma > 0$ such that

$$\begin{aligned} y^\star - x^\star &\in \gamma A(x^\star), \\ x^\star - y^\star &\in \gamma B(y^\star). \end{aligned}$$

Backward–Backward splitting

Let $x_0 \in \mathbb{R}^n$, $\gamma > 0$:

$$\begin{aligned} y_{k+1} &= \mathcal{J}_{\gamma B}(x_k), \\ x_{k+1} &= \mathcal{J}_{\gamma A}(y_{k+1}). \end{aligned}$$

- **Yosida approximation**

$$ {}^{\gamma}A = \frac{1}{\gamma}(\mathrm{Id} - \mathcal{J}_{\gamma A}). $$

which is γ-cocoercive.

- **Regularised monotone inclusion**

$$ \text{Find } x \in \mathbb{R}^n \text{ such that } 0 \in A(x) + {}^{\gamma}B(x). $$

- **Forward–Backward splitting** $\tau \in]0, 2\gamma]$

$$ x_{k+1} = \mathcal{J}_{\tau A} \circ (\mathrm{Id} - \tau {}^{\gamma}B)(x_k). $$

- **BB as special case of FB** let $\tau = \gamma$

$$ \begin{aligned} x_{k+1} &= \mathcal{J}_{\gamma A} \circ (\mathrm{Id} - \gamma {}^{\gamma}B)(x_k) \\ &= \mathcal{J}_{\gamma A} \circ \big(\mathrm{Id} - \gamma \tfrac{1}{\gamma}(\mathrm{Id} - \mathcal{J}_{\gamma B})\big)(x_k) \\ &= \mathcal{J}_{\gamma A} \circ \mathcal{J}_{\gamma B}(x_k). \end{aligned} $$

An inertial Backward–Backward splitting

Initial : $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0,\ \tau \in]0, 2\gamma]$;

$$\begin{aligned} y_k &= x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots\ , \\ x_{k+1} &= \mathcal{J}_{\gamma A} \circ \mathcal{J}_{\gamma B}(y_k),\ \lambda_k \in [0, 1]. \end{aligned}$$

An inertial BB splitting based on Yosida approximation

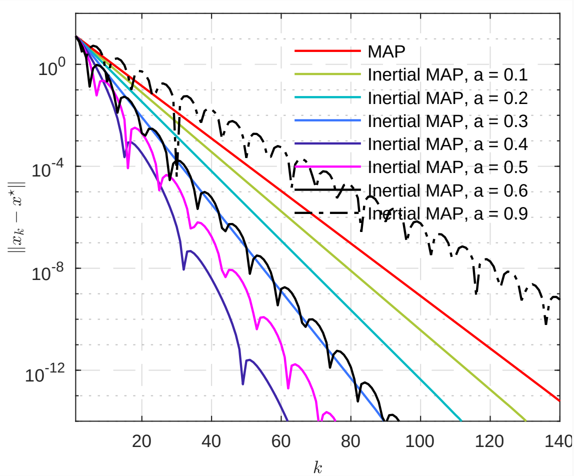
Initial : $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma > 0$;

$$\begin{aligned} y_k &= x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots\ , \\ z_k &= x_k + b_{0,k}(x_k - x_{k-1}) + b_{1,k}(x_{k-1} - x_{k-2}) + \cdots\ , \\ x_{k+1} &= \mathcal{J}_{\tau A} \circ \big(y_k - \tau^{\gamma} B(z_k)\big),\ \lambda_k \in [0, 1]. \end{aligned}$$

Numerical experiment

Feasibility problem for two subspaces:

$$a = [-4/5, 1] \quad \text{and} \quad b = [-1/5, 1]$$



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