

# Introductory Course on Non-smooth Optimisation

## Lecture 06 - Primal-Dual splitting

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- 1 Problem
- 2 Primal-Dual splitting
- 3 Parallel sum
- 4 Composed optimisation
- 5 Acceleration

## Problem

Find  $x \in \mathbb{R}^n$  such that  $0 \in A(x) + L^* \circ C \circ L(x)$ .

## Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is maximal monotone.
- $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear mapping.
- $C : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  is maximal monotone.
- $\text{zer}(A + L^* \circ C \circ L) \neq \emptyset$ .

Let  $x^* \in \text{zer}(A + L^* \circ C \circ L)$ , then  $\exists v^* \in C \circ Lx^*$  such that

$$0 \in A(x^*) + L^*v^*$$

and  $Lx^* \in C^{-1}(v^*)$ .

## Saddle-point problem

$$\text{Find } v \in \mathbb{R}^m \text{ such that } \exists x \in \mathbb{R}^n \begin{cases} 0 \in A(x) + L^*v, \\ 0 \in C^{-1}(v) - Lx. \end{cases}$$

Denote  $\mathcal{X}$  and  $\mathcal{V}$  the set of primal and dual solutions.

1 Problem

2 Primal-Dual splitting

3 Parallel sum

4 Composed optimisation

5 Acceleration

## Primal-Dual splitting

Let  $x_0 \in \mathbb{R}^n$ ,  $v_0 \in \mathbb{R}^n$  and  $\gamma_A, \gamma_C > 0$ ,  $\theta \in [-1, 1]$ :

$$\begin{cases} x_{k+1} = \mathcal{J}_{\gamma_A A}(x_k - \gamma_A L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C L \bar{x}_{k+1}). \end{cases}$$

- Known as Chambolle-Pock Primal-Dual method in optimisation.
- Douglas-Rachford is the limiting case of Primal-Dual.
- Moreau's identity

$$\text{Id} = \mathcal{J}_{\gamma A}(\cdot) + \gamma \mathcal{J}_{A^{-1}/\gamma}\left(\frac{\cdot}{\gamma}\right).$$

- definition of resolvent

$$\begin{aligned}\frac{1}{\gamma_A}(x_k - x_{k+1}) - L^* v_k &\in A(x_{k+1}), \\ \frac{1}{\gamma_C}(v_k - v_{k+1}) + L(x_{k+1} + \theta(x_{k+1} - x_k)) &\in C^{-1}(v_{k+1}).\end{aligned}$$

- arrange terms

$$\begin{aligned}\frac{1}{\gamma_A}(x_k - x_{k+1}) - L^*(v_k - v_{k+1}) &\in A(x_{k+1}) + L^* v_{k+1}, \\ \frac{1}{\gamma_C}(v_k - v_{k+1}) + \theta L(x_{k+1} - x_k) &\in C^{-1}(v_{k+1}) - Lx_{k+1}.\end{aligned}$$

- inclusion

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{bmatrix} \text{Id}_n/\gamma_A & -L^* \\ -\theta L & \text{Id}_m/\gamma_C \end{bmatrix} \begin{pmatrix} x_{k+1} - x_k \\ v_{k+1} - v_k \end{pmatrix}.$$

- inclusion

$$\mathbf{z} = \begin{pmatrix} x \\ v \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \text{Id}_n/\gamma_A & -L^* \\ -\theta L & \text{Id}_m/\gamma_C \end{bmatrix}.$$

- $\mathbf{A}$  is skew symmetric, hence maximal monotone.
- $\mathbf{V}$  is symmetric if  $\theta = 1$  and moreover positive definite if  $\gamma_A \gamma_C \|L\|^2 < 1$ .
- $\mathbf{V}\mathbf{z}_k \in \mathbf{A}(\mathbf{z}_{k+1}) + \mathbf{V}\mathbf{z}_{k+1}$ , hence

$$\begin{aligned} \mathbf{z}_{k+1} &= (\mathbf{V} + \mathbf{A})^{-1}(\mathbf{V}\mathbf{z}_k) \\ &= (\text{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}(\mathbf{z}_k). \end{aligned}$$

- which is PPA under matrix  $\mathbf{V}$ .



## Fixed-point formulation

$$\mathbf{z}_{k+1} = (\mathbf{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}(\mathbf{z}_k).$$

**Property** space  $(\mathbb{R}^n \times \mathbb{R}^n)_V$

- $\mathcal{T}_{\text{PD}} = (\mathbf{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}$  is firmly non-expansive when  $\theta = 1$  and  $\gamma_A \gamma_C \|L\|^2 < 1$ .
- for  $\theta \in [-1, 1[$ , a correction step is needed.
- Douglas-Rachford is the limiting case of Primal-Dual when

$$L = \text{Id} \quad \text{and} \quad \gamma_A \gamma_C = 1.$$

- let  $\theta = 1$  and  $L = \text{Id}$ ,  $\gamma_A \gamma_C = 1$ .

- change the order of updating variables,

$$\begin{cases} v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C \bar{x}_k), \\ x_{k+1} = \mathcal{J}_{\gamma_A A}(x_k - \gamma_A v_{k+1}), \\ \bar{x}_{k+1} = 2x_{k+1} - x_k. \end{cases}$$

- apply Moreau's identity to  $\mathcal{J}_{\gamma_C C^{-1}}$ ,

$$v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k + \gamma_C \bar{x}_k) = v_k + \gamma_C \bar{x}_k - \gamma_C \mathcal{J}_{C/\gamma_C} \left( \frac{v_k + \gamma_C \bar{x}_k}{\gamma_C} \right).$$

- let  $\gamma_C = 1/\gamma_A$  and define  $z_{k+1} = x_k - \gamma_A v_{k+1}$ ,

$$\begin{cases} u_{k+1} = \mathcal{J}_{\gamma_A J}(2x_k - z_k), \\ z_{k+1} = z_k + u_{k+1} - x_k, \\ x_{k+1} = \mathcal{J}_{\gamma_A R}(z_{k+1}), \end{cases}$$

- Let  $\mathcal{X}, \mathcal{Y}$  be two subspaces

$$\mathcal{X} = \{x : ax = 0\}, \quad \mathcal{Y} = \{x : bx = 0\}$$

and assume

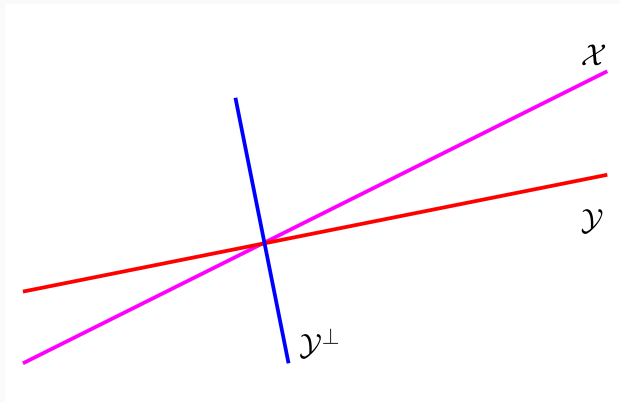
$$1 \leq p \stackrel{\text{def}}{=} \dim(\mathcal{X}) \leq q \stackrel{\text{def}}{=} \dim(\mathcal{Y}) \leq n - 1.$$

- Projection onto subspace

$$\mathcal{P}_{\mathcal{X}}(x) = x - a^T(aa^T)^{-1}ax.$$

- Moreau's identity

$$x = \mathcal{P}_{\mathcal{X}}(x) + \mathcal{P}_{\mathcal{X}^\perp}(x).$$



- linearisation of PD

$$M_{\text{PD}} = \begin{bmatrix} \text{Id}_n & -\gamma_A \mathcal{P}_{\mathcal{X}} \mathcal{P}_{\mathcal{Y}^\perp} \\ \gamma_c \mathcal{P}_{\mathcal{Y}^\perp} \mathcal{P}_{\mathcal{X}} & \text{Id}_n - 2\gamma_c \gamma_A \mathcal{P}_{\mathcal{Y}^\perp} \mathcal{P}_{\mathcal{X}} \mathcal{P}_{\mathcal{Y}^\perp} \end{bmatrix}.$$

- linearisation of DR

$$M_{\text{DR}} = \begin{bmatrix} \text{Id}_n & -\gamma_A \mathcal{P}_{\mathcal{X}} \mathcal{P}_{\mathcal{Y}^\perp} \\ \frac{1}{\gamma_A} \mathcal{P}_{\mathcal{Y}^\perp} \mathcal{P}_{\mathcal{X}} & \text{Id}_n - 2\mathcal{P}_{\mathcal{Y}^\perp} \mathcal{P}_{\mathcal{X}} \mathcal{P}_{\mathcal{Y}^\perp} \end{bmatrix}.$$

- both  $M_{\text{PD}}$  and  $M_{\text{DR}}$  are convergent.
- let  $\omega$  be the largest principal angle (yet smaller than  $\pi/2$ ) between  $\mathcal{X}$  and  $\mathcal{Y}^\perp$ .
- spectral radius

$$\begin{aligned} \rho(M_{\text{PD}} - M_{\text{PD}}^\infty) &= \sqrt{1 - \gamma_c \gamma_A \cos^2(\omega)} \\ &\geq \sqrt{1 - \cos^2(\omega)} = \sin(\omega) = \cos(\pi/2 - \omega) = \rho(M_{\text{DR}} - M_{\text{DR}}^\infty). \end{aligned}$$

1 Problem

2 Primal-Dual splitting

**3 Parallel sum**

4 Composed optimisation

5 Acceleration

## Parallel sum

Let  $C, D : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be two set-valued operators, the parallel sum of  $C$  and  $D$  is defined by

$$C \square D \stackrel{\text{def}}{=} (C^{-1} + D^{-1})^{-1}.$$

- $(C \square D)x = \bigcup_{y \in \mathbb{R}^n} (A(x) \cap B(x - y)).$
- if  $C$  and  $D$  are monotone, then  $C \square D$  is monotone.

## Primal problem

find  $x \in \mathbb{R}^n$  such that  $0 \in (A + B)(x) + L^*((C \square D)(Lx))$ .

## Assumptions

- $A : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is maximal monotone,  $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is  $\beta_B$ -cocoercive for some  $\beta_B > 0$ .
- $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear operator.
- $C, D : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  are maximal monotone,  $D$  is  $\beta_D$ -strongly monotone for some  $\beta_D > 0$ .
- $0 \in \text{ran}(A + B + L^*(C \square D)L)$ .

## Saddle-point problem

find  $v \in \mathbb{R}^m$  such that  $(\exists x \in \mathbb{R}^n) \begin{cases} 0 \in (A + B)(x) + L^*v, \\ 0 \in (C^{-1} + D^{-1})(v) - Lx. \end{cases}$



## Primal-Dual splitting

Let  $x_0 \in \mathbb{R}^n$ ,  $v_0 \in \mathbb{R}^n$  and  $\gamma_A, \gamma_C > 0$ ,  $\theta \in [-1, 1]$ :

$$\begin{cases} x_{k+1} = \mathcal{J}_{\gamma_A A}(x_k - \gamma_A B(x_k) - \gamma_A L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \mathcal{J}_{\gamma_C C^{-1}}(v_k - \gamma_C D^{-1}(v_k) + \gamma_C L \bar{x}_{k+1}). \end{cases}$$

- can be cast as Forward-Backward splitting.

Let  $\theta = 1$ ,

- definition of resolvent

$$\begin{aligned}\frac{1}{\gamma_A}(x_k - x_{k+1}) - B(x_k) - L^*v_k &\in A(x_{k+1}), \\ \frac{1}{\gamma_C}(v_k - v_{k+1}) - D^{-1}(v_k) + L(x_{k+1} + (x_{k+1} - x_k)) &\in C^{-1}(v_{k+1}).\end{aligned}$$

- arrange terms

$$\begin{aligned}\frac{1}{\gamma_A}(x_k - x_{k+1}) - B(x_k) - L^*(v_k - v_{k+1}) &\in A(x_{k+1}) + L^*v_{k+1}, \\ \frac{1}{\gamma_C}(v_k - v_{k+1}) - D^{-1}(v_k) + L(x_{k+1} - x_k) &\in C^{-1}(v_{k+1}) - Lx_{k+1}.\end{aligned}$$

- inclusion

$$-\begin{bmatrix} B & 0 \\ 0 & D^{-1} \end{bmatrix} \begin{pmatrix} x_k \\ v_k \end{pmatrix} \in \begin{bmatrix} A & L^* \\ -L & C^{-1} \end{bmatrix} \begin{pmatrix} x_{k+1} \\ v_{k+1} \end{pmatrix} + \begin{bmatrix} \text{Id}_n/\gamma_A & -L^* \\ -L & \text{Id}_m/\gamma_C \end{bmatrix} \begin{pmatrix} x_{k+1} - x_k \\ v_{k+1} - v_k \end{pmatrix}.$$

- inclusion

$$\mathbf{z} = \begin{pmatrix} \mathbf{x} \\ \mathbf{v} \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{A} & \mathbf{L}^* \\ -\mathbf{L} & \mathbf{C}^{-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \text{Id}_n / \gamma_A & -\mathbf{L}^* \\ -\mathbf{L} & \text{Id}_m / \gamma_C \end{bmatrix}.$$

- $\mathbf{A}$  is skew symmetric, hence maximal monotone.

- $\mathbf{B}$  is  $\min\{\beta_B, \beta_D\}$ -cocoercive.

- $\mathbf{V}$  is symmetric positive definite for  $\gamma_A \gamma_C \|\mathbf{L}\|^2 < 1$ .

- $\mathbf{V}\mathbf{z}_k - \mathbf{B}(\mathbf{z}_k) \in \mathbf{A}(\mathbf{z}_{k+1}) + \mathbf{V}\mathbf{z}_{k+1}$ , hence

$$\begin{aligned} \mathbf{z}_{k+1} &= (\mathbf{V} + \mathbf{A})^{-1}(\mathbf{V} - \mathbf{B})(\mathbf{z}_k) \\ &= (\text{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}(\text{Id} - \mathbf{V}^{-1}\mathbf{B})(\mathbf{z}_k). \end{aligned}$$

- which is Forward–Backward splitting under metric  $\mathbf{V}$ .

## Fixed-point formulation

$$z_{k+1} = (\text{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}(\text{Id} - \mathbf{V}^{-1}\mathbf{B})(z_k).$$

**Property** space  $(\mathbb{R}^n \times \mathbb{R}^n)_{\mathbf{V}}$

■  $(\text{Id} + \mathbf{V}^{-1}\mathbf{A})^{-1}$  is firmly non-expansive when  $\gamma_A \gamma_C \|\mathbf{L}\|^2 < 1$ .

■  $\text{Id} - \mathbf{V}^{-1}\mathbf{B}$  is  $\frac{1}{2\beta\nu}$ -averaged non-expansive with

$$\nu = \left(1 - \sqrt{\gamma_A \gamma_C \|\mathbf{L}\|^2}\right) \min \left\{ \frac{1}{\gamma_A}, \frac{1}{\gamma_C} \right\}.$$

■  $\mathcal{T}_{\text{PD}}$  is  $\frac{2\beta\nu}{4\beta\nu-1}$ -averaged non-expansive.

1 Problem

2 Primal-Dual splitting

3 Parallel sum

**4 Composed optimisation**

5 Acceleration

**Infimal convolution**  $J, G \in \Gamma_0(\mathbb{R}^m)$

$$(J \nabla G)(\cdot) \stackrel{\text{def}}{=} \inf_{v \in \mathbb{R}^m} J(\cdot) + G(\cdot - v).$$

$$\blacksquare \partial(J \nabla G)(\cdot) = (\partial J \square \partial G)(\cdot).$$

**Example** Moreau envelope

$$J \nabla \frac{1}{2\gamma} \|\cdot\|^2 = \inf_{v \in \mathbb{R}^m} J(v) + \frac{1}{2\gamma} \|\cdot - v\|^2.$$

## Conjugate

Let  $F : \mathbb{R}^n \rightarrow ]-\infty, +\infty]$ , the Fenchel conjugate of  $F$  is defined by

$$F^*(v) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^n} (\langle x, v \rangle - F(x)).$$

- $F^*$  is closed and convex even  $F$  is not.

**Biconjugate**  $F^{**} = (F^*)^*$ .

**Example** Let  $\mathcal{S} \subseteq \mathbb{R}^n$  be a non-empty convex set, the support function of  $\mathcal{S}$  is defined by

$$\sigma_{\mathcal{S}}(v) \stackrel{\text{def}}{=} \sup_{x \in \mathcal{S}} \langle x, v \rangle = \iota_{\mathcal{S}}^*(y).$$

**Example** Let  $F = \frac{1}{2} \|\cdot\|^2$ ,

$$F^*(y) = \frac{1}{2} \|y\|^2.$$

**Example** Let  $\|\cdot\|$  be a norm with dual norm  $\|\cdot\|_*$ . Let  $F = \|\cdot\|$ , then

$$F^*(y) = \begin{cases} 0 & \|y\|_* \leq 1, \\ +\infty & \text{o.w.} \end{cases}$$

i.e. the indicator function of the dual norm ball.



## Informal convolution

$$(J \uplus G)^* = J^* + G^*.$$

**Fenchel–Moreau** Let  $F : \mathbb{R}^n \rightarrow ]-\infty, +\infty]$  be a proper function, then  $F$  is convex and lower semi-continuous if and only if  $F = F^{**}$ .

**Biconjugate** If  $F \in \Gamma_0(\mathbb{R}^n)$ , then  $F^* \in \Gamma_0(\mathbb{R}^n)$  and  $F^{**} = F$ .

**Subdifferential** If  $F$  is closed and convex, then

$$y \in \partial F(x) \iff x \in \partial F^*(y).$$

**Moreau's identify** Let function  $F \in \Gamma_0(\mathbb{R}^n)$  and  $\gamma > 0$ , then

$$\text{Id} = \text{prox}_{\gamma F}(\cdot) + \gamma \text{prox}_{F^*/\gamma}\left(\frac{\cdot}{\gamma}\right).$$

**Strong convexity** Let  $F$  be closed and  $\alpha$ -strongly convex, then  $\nabla F^*$  is  $\frac{1}{\alpha}$ -Lipschitz.

## Primal problem

$$\min_{x \in \mathbb{R}^n} R(x) + F(x) + (J \nabla^\dagger G)(Lx).$$

## Assumptions

- $R, F \in \Gamma_0(\mathbb{R}^n)$ , and  $\nabla F$  is  $(1/\beta_F)$ -Lipschitz continuous for some  $\beta_F > 0$ .
- $J, G \in \Gamma_0(\mathbb{R}^m)$ ,  $G$  is  $\beta_G$ -strongly convex for  $\beta_G > 0$ .
- $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear mapping.
- The inclusion  $0 \in \text{ran}(\partial R + \nabla F + L^*(\partial J \square \partial G)L)$  holds.

## Saddle-point problem

$$\min_{x \in \mathbb{R}^n} \max_{v \in \mathbb{R}^m} R(x) + F(x) + \langle Lx, v \rangle - (J^*(v) + G^*(v)).$$

## Dual problem

$$\min_{v \in \mathbb{R}^m} J^*(v) + G^*(v) + (R^* \nabla F^*)(-L^* v).$$

Denote by  $\mathcal{X}$  and  $\mathcal{V}$  the sets of solutions of primal and dual problems, respectively.

## Primal-Dual splitting

Let  $x_0 \in \mathbb{R}^n$ ,  $v_0 \in \mathbb{R}^n$  and  $\gamma_R, \gamma_J > 0$ ,  $\theta \in [-1, 1]$ :

$$\begin{cases} x_{k+1} = \text{prox}_{\gamma_R R}(x_k - \gamma_R \nabla F(x_k) - \gamma_R L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k), \\ v_{k+1} = \text{prox}_{\gamma_J J^*}(v_k - \gamma_J \nabla G^*(v_k) + \gamma_J L \bar{x}_{k+1}). \end{cases}$$

- $A = \partial R$ ,  $B = \nabla F$ .
- $C^{-1} = \partial J^*$ ,  $D^{-1} = \nabla G^*$ .

1 Problem

2 Primal-Dual splitting

3 Parallel sum

4 Composed optimisation

**5 Acceleration**

## Primal problem

$$\min_{x \in \mathbb{R}^n} R(x) + J(Lx).$$

## Assumptions

- $R \in \Gamma_0(\mathbb{R}^n)$ .
- $J \in \Gamma_0(\mathbb{R}^m)$ .
- $L : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a linear mapping.
- The inclusion  $0 \in \text{ran}(\partial R + L^* \partial J)$  holds.

## Dual problem

$$\min_{v \in \mathbb{R}^m} J^*(v) + R^*(-L^* v)$$

## Primal-Dual splitting

Let  $x_0 \in \mathbb{R}^n$ ,  $v_0 \in \mathbb{R}^n$  and  $\gamma_{R,0}, \gamma_{J,0} > 0$  such that  $\gamma_{R,0}\gamma_{J,0}\|L\|^2 \leq 1$ :

$$\begin{cases} v_{k+1} = \text{prox}_{\gamma_{J,k}J^*}(v_k + \gamma_{J,k}L\bar{x}_{k+1}), \\ x_{k+1} = \text{prox}_{\gamma_{R,k}R}(x_k - \gamma_{R,k}L^*v_k), \\ \theta_k = \frac{1}{\sqrt{1 + 2\alpha\gamma_{R,k}}}, \gamma_{R,k+1} = \theta_k\gamma_{R,k}, \gamma_{J,k+1} = \gamma_{J,k}/\theta_k, \\ \bar{x}_{k+1} = x_{k+1} + \theta_k(x_{k+1} - x_k). \end{cases}$$

- convergence rate

$$\|x_k - x^*\| = O(1/k^2).$$

$R$  is  $\alpha$ -strongly convex and  $J$  is  $\kappa$ -strongly convex.

## Primal-Dual splitting

Let  $x_0 \in \mathbb{R}^n$ ,  $v_0 \in \mathbb{R}^n$ . Choose  $\mu = \frac{2\sqrt{\alpha\kappa}}{L}$ ,  $\gamma_R = \frac{\mu}{2\alpha}$ ,  $\gamma_J = \frac{\mu}{2\kappa}$  and  $\theta \in [1/(\mu + 1), 1]$ :

$$\begin{cases} v_{k+1} = \text{prox}_{\gamma_J J^*}(v_k + \gamma_J L \bar{x}_{k+1}), \\ x_{k+1} = \text{prox}_{\gamma_R R}(x_k - \gamma_R L^* v_k), \\ \bar{x}_{k+1} = x_{k+1} + \theta(x_{k+1} - x_k). \end{cases}$$

- convergence rate

$$\|x_k - x^*\| = O(\eta^k)$$

$$\text{with } \eta = \frac{1+\theta}{2+\mu}.$$



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