Introductory Course on Non-smooth Optimisation

Lecture 02 - Proximal gradient method

Outline

1. Subgradient descent
2. Proximal gradient descent
3. Proximal mapping
4. Inertial proximal gradient
5. FISTA
6. Restarting FISTA
7. Numerical experiments

Outline

Problem

Problem (Unconstrained non-smooth optimisation)

Consider minising

$$\min_{x \in \mathbb{R}^n} R(x),$$

where $R : \mathbb{R}^n \to]-\infty, +\infty]$ *is proper convex and lower semi-continuous.*

Γ_0: the class of proper convex and lower semi-continuous functions on $\mathbb{R}^n$.

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- The set of minimisers, *i.e.*

  $$\mathrm{Argmin}(R) = \{x \in \mathbb{R}^n : R(x) = \min_{x \in \mathbb{R}^n} R(x)\},$$

  is non-empty
- $R(x)$ is non-differentiable...

Subdifferential

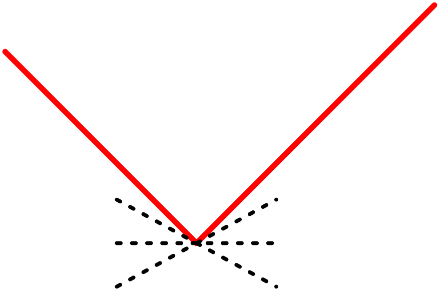
Definition

Let $R \in \Gamma_0$, the subdifferential of R at $x \in \mathrm{dom}(R)$ is defined by

$$\partial R : \mathbb{R}^n \rightrightarrows \mathbb{R}^n,\ x \mapsto \left\{ g \in \mathbb{R}^n \,|\, R(y) \geq R(x) + \langle g,\, y - x \rangle,\ \forall y \in \mathbb{R}^n \right\}.$$

Example:

$$\partial |x| = \begin{cases} +1 : x > 0 \\ [-1, 1] : x = 0 \\ -1 : x < 0 \end{cases}$$



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Lemma

Let $R \in \Gamma_0$ and $x \in \mathrm{dom}(R)$, then

- $\partial R(x) = \big\{ g \in \mathbb{R}^n : R(y) \geq R(x) + \langle g,\, y - x \rangle \big\}$;
- $\partial R(x)$ *is closed and convex;*

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Lemma (Monotonicity)

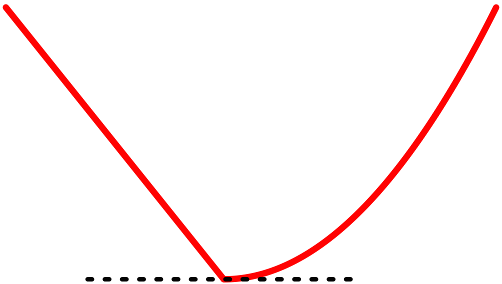
Let $R \in \Gamma_0$, then $\forall x, y \in \mathrm{dom}(R)$,

$$\langle u - v,\, x - y \rangle \geq 0,\ \forall u \in \partial R(x),\ v \in \partial R(y).$$

Optimality condition

$x^\star$ minimises $R(x)$ if and only if

$$0 \in \partial R(x^\star).$$



$$R(y) \geq R(x^\star) + \langle g,\, y - x \rangle \text{ holds for all } y \in \mathrm{dom}(R) \iff 0 \in \partial R(x^\star).$$

Subgradient descent

Subgradient descent

initial: $x_0 \in \text{dom}(R)$;

repeat:

1. Choose step-size $\gamma_k > 0$ and a subgradient $g_k \in \partial R(x_k)$
2. Update $x_{k+1} = x_k - \gamma_k g_k$

until: stopping criterion is satisfied.

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Step-size rule:

- Fixed step-size: γ_k is constant;
- Fixed length: $\gamma_k \|g_k\| = \|x_{k+1} - x_k\|$ is a constant;
- Diminishing step-size: $\gamma_k \to 0$, $\sum_i \gamma_i = +\infty$.

Assumptions

Assumptions:

- R has minimiser $x^\star$ and finite optimal value $R(x^\star)$;
- R is convex, $\mathrm{dom}(R) = \mathbb{R}^n$;
- R is Lipschitz consinuout with constant L:

$$|R(x) - R(y)| \leq L\|x - y\|, \ \forall x, y \in \mathrm{dom}(R). \tag{1.1}$$

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$$|R(x) - R(y)| \leq L\|x - y\|, \ \forall x, y \in \mathrm{dom}(R). \tag{1.1}$$

Eq. (1.1) implies $\|g\| \leq L$ for all $x \in \mathrm{dom}(R)$.

Convergence propergies

Subgradient descent is **NOT** a descent method.

Fixed step-size $\gamma_k \equiv \gamma$:

$$R_{k,best} - R(x^\star) \leq \frac{\|x_0 - x^\star\|^2}{2k\gamma} + \frac{\gamma L^2}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$
- For large k, $R_{k,best}$ is approximately $\frac{\gamma L^2}{2}$ suboptimal

Convergence propergies

Subgradient descent is **NOT** a descent method.

Fixed step-length $\gamma_k = c/\|g_k\|$:

$$R_{k,best} - R(x^\star) \leq \frac{\|x_0 - x^\star\|^2}{2kc} + \frac{cL}{2}.$$

- Does not guarantee the convergence of $R_{k,best}$
- For large k, $R_{k,best}$ is approximately $\frac{cL}{2}$ suboptimal

Convergence propergies

Subgradient descent is **NOT** a descent method.

Diminishing step-size: $\gamma_k \to 0,\ \sum_i \gamma_i = +\infty$:

$$R_{k,best} - R(x^\star) \le \frac{\|x_0 - x^\star\|^2 + L^2 \sum_{i=1}^k \gamma_i^2}{\sum_{i=1}^k \gamma_i}.$$

- If $\sum_{i=1}^k \gamma_i^2 / \sum_{i=1}^k \gamma_i \to 0$, then $R_{k,best} \to R(x^\star)$
- Choice of γ_k: $\gamma_k = c/k^q$, $q \in]1/2, 1[$

Optimal step-size

For fixed number of iterations: If $c_i = \gamma_i \|g_i\|$ and $\|x_0 - x^\star\| \leq D$,

$$R_{k,best} - R(x^\star) \leq \frac{D^2 + L^2 \sum_{i=1}^k c_i^2}{2 \sum_{i=1}^k \gamma_i / L}.$$

- For given k, rhs is minimised by $c_i = c = D/\sqrt{k}$
- Hence the rate
$$R_{k,best} - R(x^\star) \leq \frac{LD}{\sqrt{k}}.$$
- Iteration complexity: reach $R_{k,best} - R(x^\star) < \epsilon$ in $O(1/\epsilon^2)$ steps

Optimal step-size

For fixed number of iterations: If $c_i = \gamma_i \|g_i\|$ and $\|x_0 - x^\star\| \leq D$,

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When $R(x^\star)$ is available: step-size

$$\gamma_k = \frac{R(x_k) - R(x^\star)}{\|g_k\|^2}.$$

Convergence rate:

$$R_{k,best} - R(x^\star) \leq \frac{LD}{\sqrt{k}}.$$

NB: $O(1/\sqrt{k})$ is the best rate can be obtained by subgradient method.

Remarks

- Handles non-smooth problem
- Simple iterative scheme
- Slow convergence rate
- No clear stopping criterion

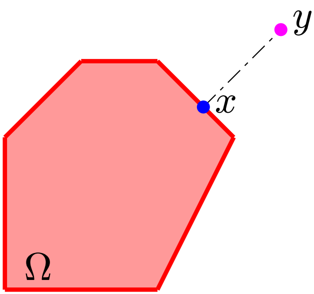
NB: need a better approach to handle non-smoothness...

Outline

Projection onto sets

Indicator function: let $\Omega \subseteq \mathbb{R}^n$

$$\iota_\Omega(x) = \begin{cases} 0 : x \in \Omega, \\ +\infty : x \notin \Omega. \end{cases}$$



Projection of y onto Ω:

$$\min_{x \in \Omega} \|x - y\|.$$

Definition (Projection)

Projection mapping onto a set is defined by

$$\mathrm{proj}_\Omega(y) \stackrel{\text{def}}{=} \mathrm{argmin}_{x \in \Omega} \|x - y\|.$$

Projected gradient descent

Problem (Constrained smooth optimisation)

Let $F \in C_L^1$ and $\Omega \subseteq \mathbb{R}^n$ be a closed and convex set

$$\min_{x \in \Omega} F(x).$$

Projected gradient descent

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Projected gradient descent

initial: $x_0 \in \Omega$;

repeat:

1. Choose step-size $\gamma_k \in]0, 2/L[$
2. Gradient descent $x_{k+1/2} = x_k - \gamma_k \nabla F(x_k)$
3. Projection $x_{k+1} = \mathrm{proj}_\Omega(x_{k+1/2})$

until: stopping criterion is satisfied.

Composite optimisation problem

As $\iota_\Omega \in \Gamma_0$, constrained optimisation problem

$$\min_{x\in\mathbb{R}^n} \iota_\Omega(x) + F(x).$$

Problem (Composite optimisation)

Consider the following optimisation problem

$$\min_{x\in\mathbb{R}^n} \left\{\Phi(x) \stackrel{\text{def}}{=} R(x) + F(x)\right\}.$$

Assumtions:

- $F \in C_L^1$
- $R \in \Gamma_0$
- $\mathrm{Argmin}(\Phi) \neq \emptyset$

Examples: regularised LSE, image processing...

Proximal gradient descent

Projection onto a set

$$\begin{aligned}\mathrm{proj}_\Omega(y) &\stackrel{\text{def}}{=} \mathrm{argmin}_{x\in\Omega} \|x-y\| \\ &= \mathrm{argmin}_{x\in\mathbb{R}^n} \iota_\Omega(x) + \frac{1}{2}\|x-y\|^2.\end{aligned}$$

Proximal mapping

$$\mathrm{proj}_{\gamma R}(y) \stackrel{\text{def}}{=} \mathrm{argmin}_{x\in\mathbb{R}^n} \gamma R(x) + \frac{1}{2}\|x-y\|^2.$$

Proximal gradient descent

Projection onto a set

$$\mathrm{proj}_{\Omega}(y) \stackrel{\text{def}}{=} \mathrm{argmin}_{x \in \Omega} \|x - y\| = \mathrm{argmin}_{x \in \mathbb{R}^n} \iota_{\Omega}(x) + \frac{1}{2}\|x - y\|^2.$$

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Interpretation

A.K.A Forward–Backward splitting:

- Forward step: gradient descent of F
- Backward step: proximity operator of R

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A.K.A Forward–Backward splitting:

- Forward step: gradient descent of F
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Interation in one line:

$$x_{k+1} = \mathrm{prox}_{\gamma_k R}(x_k - \gamma_k \nabla F(x_k)).$$

Definition of $\mathrm{prox}_{\gamma R}$,

$$\begin{aligned} x_{k+1} &= \mathrm{argmin}_x \left\{ \gamma_k R(x) + \tfrac{1}{2} \|x - (x_k - \gamma_k \nabla F(x_k))\|^2 \right\} \\ &= \mathrm{argmin}_x \left\{ \gamma_k R(x) + \gamma_k \langle \nabla F(x_k),\, x - x_k \rangle + \tfrac{1}{2} \|x - x_k\|^2 \right\} \\ &= \mathrm{argmin}_x \left\{ R(x) + \boxed{F(x_k) + \langle \nabla F(x_k),\, x - x_k \rangle + \frac{1}{2\gamma_k} \|x - x_k\|^2} \right\} \end{aligned}$$

NB: x_{k+1} minimises $R(x)$ plus the majorisation of $F(x)$ at x_k if $\gamma_k \leq \frac{1}{L}$

Special cases

Gradient descent: $R = 0$

$$x_{k+1} = x_k - \gamma_k \nabla F(x_k).$$

Proximal point algorithm: $F = 0$

$$x_{k+1} = \mathrm{prox}_{\gamma_k R}(x_k).$$

Projected gradient descent: $R = \iota_\Omega$

$$x_{k+1} = \mathrm{proj}_\Omega(x_k - \gamma_k \nabla F(x_k)).$$

ISTA: iterative shrinkage-thresholding algorithm: $R = \|x\|_1$

$$x_{k+1} = \mathcal{T}_\gamma(x_k - \gamma_k \nabla F(x_k)),$$

where

$$\big(\mathcal{T}_\gamma(y)\big)_i = \begin{cases} \mathrm{sign}(y_i) \cdot (|y_i| - \gamma) : |y_i| > \gamma \\ \hfill 0 : y_i \in [-\gamma, \gamma]. \end{cases}$$

Two basic lemmas

Define

$$E_\gamma(x, y) \stackrel{\text{def}}{=} R(x) + F(y) + \langle \nabla F(y),\, x - y \rangle + \frac{1}{2\gamma}\|x - y\|^2$$

and $y_+ \stackrel{\text{def}}{=} \mathrm{argmin}_x E_\gamma(x, y)$.

Lemma

Let $y \in \mathbb{R}^n$ and $\gamma \in]0, 2/L[$ such that

$$\Phi(y_+) \le E_\gamma(y_+, y),$$

then for any $x \in \mathbb{R}^n$,

$$\Phi(x) - \Phi(y_+) \ge \frac{1}{2\gamma}\big(\|x - y_+\|^2 - \|x - y\|^2\big).$$

Lemma

Given $y \in \mathbb{R}^n$ and $\gamma \in]0, 1/L]$, then for any $x \in \mathbb{R}^n$,

$$\Phi(y_+) + \frac{1}{2\gamma}\|y_+ - x\|^2 \le \Phi(y) + \frac{1}{2\gamma}\|y - x\|^2.$$

Convergence analysis

NB: proximal gradient is a descent method.

Consider $\gamma_k \equiv \gamma \in]0, 1/L]$

- For each step
$$\Phi(x_k) - \Phi(x_{k+1}) \geq \frac{\gamma}{2}\|x_k - x_{k+1}\|^2.$$
- Regarding $\Phi(x^\star)$
$$\Phi(x_{k+1}) - \Phi(x^\star) \leq \frac{\gamma}{2}\big(\|x_k - x^\star\|^2 - \|x_{k+1} - x^\star\|^2\big).$$
- Summing up
$$\begin{aligned} k\big(\Phi(x_k) - \Phi(x^\star)\big) &\leq \sum_{i=1}^k \big(\Phi(x_i) - \Phi(x^\star)\big) \\ &\leq \frac{\gamma}{2}\sum_{i=1}^k \big(\|x_{i-1} - x^\star\|^2 - \|x_i - x^\star\|^2\big) \leq \frac{\gamma}{2}\|x_0 - x^\star\|^2 \end{aligned}$$
- $O(1/k)$ rate
$$\Phi(x_k) - \Phi(x^\star) \leq \frac{\gamma\|x_0 - x^\star\|^2}{2k}.$$

NB: not optimal and can be accelerated

Outline

From projection to proximal mapping

Definition (Proximal mapping)

The proimal mapping (proximity operator) of a fucntion $R \in \Gamma_0$ is defined by

$$\mathrm{prox}_{\gamma R}(y) \stackrel{\mathrm{def}}{=} \mathrm{argmin}_{x \in \mathbb{R}^n} \gamma R(x) + \frac{1}{2}\|x - y\|^2.$$

Optimality condition: denote $y_+ \stackrel{\mathrm{def}}{=} \mathrm{prox}_{\gamma R}(y)$,

$$\begin{aligned} 0 \in \gamma \partial R(y_+) + y_+ - y \quad &\Longleftrightarrow \quad y \in (\mathrm{Id} + \gamma \partial R)(y_+) \\ &\Longleftrightarrow \quad y_+ = (\mathrm{Id} + \gamma \partial R)^{-1}(y). \end{aligned}$$

Examples

Projection: $R(x) = \iota_\Omega(x)$, $\partial \iota_\Omega(x) = \mathcal{N}_\Omega(x) = \{g : \langle g, v - x \rangle \le 0\}$

$$\mathrm{proj}_\Omega = (\mathrm{Id} + \mathcal{N}_\Omega)^{-1}.$$

Simple instances:

- Hyperplane: $\Omega = \{x : a^T x = b\}$, $a \neq 0$
$$\mathrm{proj}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a.$$
- Affine subspace: $\Omega = \{x : Ax = b\}$ with $A \in \mathbb{R}^{m \times n}$, $\mathrm{rank}(A) = m < n$
$$\mathrm{proj}_\Omega = x + A^T (AA^T)^{-1} (b - Ax).$$
- Half space: $\Omega = \{x : a^T x \le b\}$, $a \neq 0$
$$\mathrm{proj}_\Omega = x + \frac{b - a^T x}{\|a\|^2} a \ \text{ if } a^T x > b \quad \text{and} \quad x \ \text{ if } a^T x \le b.$$
- Nonnegative orthant: $\Omega = \mathbb{R}^n_+$
$$\mathrm{proj}_\Omega = \big(\max\{0, x_i\}\big)_i.$$

Examples

Projection: $R(x) = \iota_\Omega(x)$, $\partial \iota_\Omega(x) = \mathcal{N}_\Omega(x) = \{g : \langle g,\, v - x\rangle \leq 0\}$

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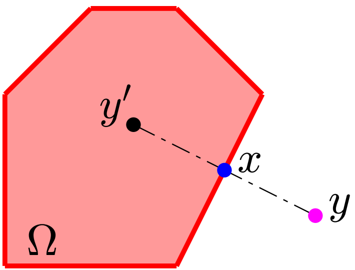
Reflection

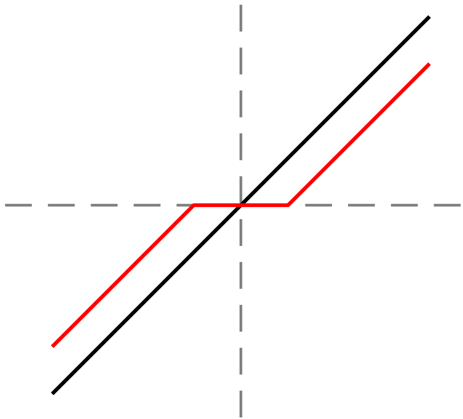
$$\mathcal{R}_{\mathcal{N}_\Omega} = 2\mathrm{proj}_\Omega - \mathrm{Id} = \mathrm{proj}_\Omega + (\mathrm{proj}_\Omega - \mathrm{Id}).$$



Examples

Soft-threshold: $R(x) = |x|$,

$$\operatorname{prox}_{\gamma R}(y) = \mathcal{T}_\gamma(y) = \begin{cases} y - \gamma : y > \gamma \\ \quad 0 : y \in [-\gamma, \gamma] \\ y + \gamma : y < -\gamma. \end{cases}$$



Examples

Quadratic function: $R(x) = \frac{1}{2}x^TAx + b^Tx + c,\ A \succeq 0$

$$\mathrm{prox}_{\gamma R}(y) = (\mathrm{Id} + \gamma A)^{-1}(y - \gamma b).$$

Euclidean norm: $R(x) = \|x\|$

$$\mathrm{prox}_{\gamma R}(y) = \begin{cases} (1 - \frac{\gamma}{\|y\|})y : \|y\| > \gamma \\ 0 : o.w. \end{cases}.$$

Logarithmic barrier: $R(x) = -\sum_i \log(x_i)$

$$\big(\mathrm{prox}_{\gamma R}(y)\big)_i = \frac{y_i + \sqrt{y_i^2 + 4\gamma}}{2},\ \ i = 1, ..., n.$$

Nuclear norm: $R(x) = \sum_i \sigma_i$

$$\mathrm{prox}_{\gamma R}(y) = U\mathcal{T}_\gamma(\Sigma)V^T.$$

Calculus rules

Quadratic perturbation: $H(x) = R(x) + \frac{\alpha}{2}\|x\|^2 + \langle x, u\rangle + \beta$, $\alpha \geq 0$

$$\mathrm{prox}_H = \mathrm{prox}_{R/(\alpha+1)}\Big(\frac{x-u}{\alpha+1}\Big).$$

Translation: $H(x) = R(x - z)$

$$\mathrm{prox}_H = z + \mathrm{prox}_R(x - z).$$

Scaling: $H(x) = R(x/\rho)$

$$\mathrm{prox}_H = \rho \mathrm{prox}_{R/\rho^2}\Big(\frac{x}{\rho}\Big).$$

Reflection: $H(x) = R(-x)$

$$\mathrm{prox}_H = -\mathrm{prox}_R(-x).$$

Composition: $H(x) = R \circ L$ with L being bijective bounded linear mapping such that $L^{-1} = L^*$,

$$\mathrm{prox}_H = L^* \circ \mathrm{prox}_R \circ L.$$

Outline

From heavy-ball to inertial proximal gradient

An inertial proximal gradient

Initial: $x_0 \in \mathbb{R}^n$ and $\gamma \in]0, 2/L[$;

$$y_k = x_k + a_k(x_k - x_{k-1}),\ a_k \in [0,1],$$
$$x_{k+1} = \mathrm{prox}_{\gamma R}(y_k - \gamma \nabla F(x_k)).$$

- Recovers inertial proximal point algorithm when $F = 0$, and heavy-ball method when $R = 0$
- Convergence via studying the Lyapunov function
$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \frac{a_k}{2\gamma}\|x_k - x_{k-1}\|^2.$$
- In general, no convergence rate.

A general inertial scheme

A general inertial proximal gradient

Initial: $x_0 \in \mathbb{R}^n, x_{-1} = x_0$ and $\gamma \in]0, 2/L[$;

$$y_k = x_k + a_k(x_k - x_{k-1}),\ a_k \in [0,1],$$

$$z_k = x_k + b_k(x_k - x_{k-1}),\ b_k \in [0,1],$$

$$x_{k+1} = \mathrm{prox}_{\gamma R}(y_k - \gamma \nabla F(z_k)).$$

- Convergence via studying the Lyapunov function

$$\mathcal{E}(x_k) \stackrel{\text{def}}{=} \Phi(x_k) + \tfrac{a_k}{2\gamma}\|x_k - x_{k-1}\|^2.$$

- In general, no convergence rate.
- Can be extend to multi-step, *e.g.*

$$y_k = x_k + a_{0,k}(x_k - x_{k-1}) + a_{1,k}(x_{k-1} - x_{k-2}) + \cdots .$$

Convergence rate

Assumption: $R = 0$, $F = \frac{1}{2}\|Ax - f\|^2$ and $(a_k, b_k) \equiv (a, b)$.

- $A^T A$ is symmetric positive definite with $A^T A \succeq \alpha \mathrm{Id}$;
- Taylor expansion
$$x_{k+1} = y_k - \gamma \nabla^2 F(x^\star)(z_k - x^\star).$$
- Let $d_k = (x_k - x^\star, x_{k-1} - x^\star)^T$ and $H = \nabla^2 F$, $G = \mathrm{Id} - \gamma H$, then
$$d_{k+1} = \underbrace{\begin{bmatrix} (a-b)\mathrm{Id} + (1+b)G, & -(a-b)\mathrm{Id} - bG \\ \mathrm{Id}, & 0 \end{bmatrix}}_{M} d_k.$$
- Spectral radius: $\eta = \rho(G) = 1 - \gamma\alpha$ and $\rho = \rho(M)$...

Spectral analysis

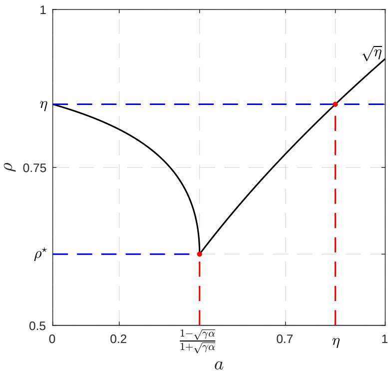
Lemma (Spectral radius ρ)

Between η and ρ,

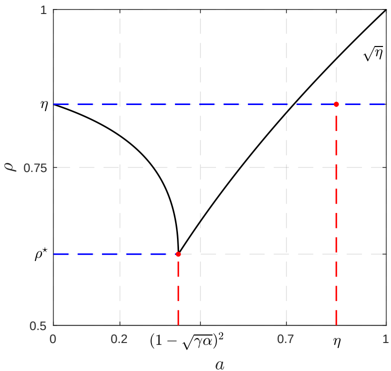
- *η and ρ satisfy the relation*

$$\rho^2 - \big((a-b) + (1+b)\eta\big)\rho + (a-b) + b\eta = 0.$$

- *Given any $(a,b) \in [0,1[^2$, then $\rho(M) < 1$ if, and only if $\frac{2(b-a)-1}{1+2b} < \eta$.*



$b = a$



$b = 0$

Remarks

- Given $b \in [0, 1]$, there exists optimal choice of $a \in [0, 1]$ such that
$$\rho = 1 - \sqrt{\gamma\alpha}$$
can be obtained.
- Take $b = a$, for
$$a \in \left] \tfrac{1-\sqrt{\gamma\alpha}}{1+\sqrt{\gamma\alpha}}, 1 \right],$$
the leading eigenvalue of M is complex.
- Continue $b = a$, for
$$a \in]\eta, 1],$$
the inertial scheme is actually slower than the original scheme.

Outline

FISTA

FISTA: fast iterative shrinkage-thresholding algorith

FISTA

Initial: $x_0 \in \mathbb{R}^n, x_{-1} = x_0$, $\gamma = 1/L$ and $t_0 = 1$;

$$t_k = \frac{1 + \sqrt{1 + 4t_{k-1}^2}}{2}, \;\; a_k = \frac{t_{k-1} - 1}{t_k},$$

$$y_k = x_k + a_k(x_k - x_{k-1}),$$

$$x_{k+1} = \mathrm{prox}_{\gamma R}(y_k - \gamma \nabla F(y_k)).$$

- $t_k \approx \frac{k+1}{2}$
- $a_k \to 1$

Relation with Nesterov's optimal scheme

Nesterov: compute $\phi_k \in]0,1[$ from equation

$$\phi_k^2 = (1-\phi_k)\phi_{k-1}^2$$

and $a_k = \frac{\phi_{k-1}(1-\phi_{k-1})}{\phi_{k-1}^2+\phi_k}$.

- ϕ_k reads

$$\phi_k = \frac{-\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}{2} = \frac{2\phi_{k-1}^2}{\phi_{k-1}^2 + \sqrt{\phi_{k-1}^4 + 4\phi_{k-1}^2}}.$$

- Let $t_k = 1/\phi_k$,

$$\frac{1}{t_k} = \frac{2}{1+\sqrt{1+4t_{k-1}^2}}.$$

- Directly

$$t_k = \frac{1+\sqrt{1+4t_{k-1}^2}}{2}.$$

- And $a_k = \dfrac{t_{k-1}-1}{t_k}$.

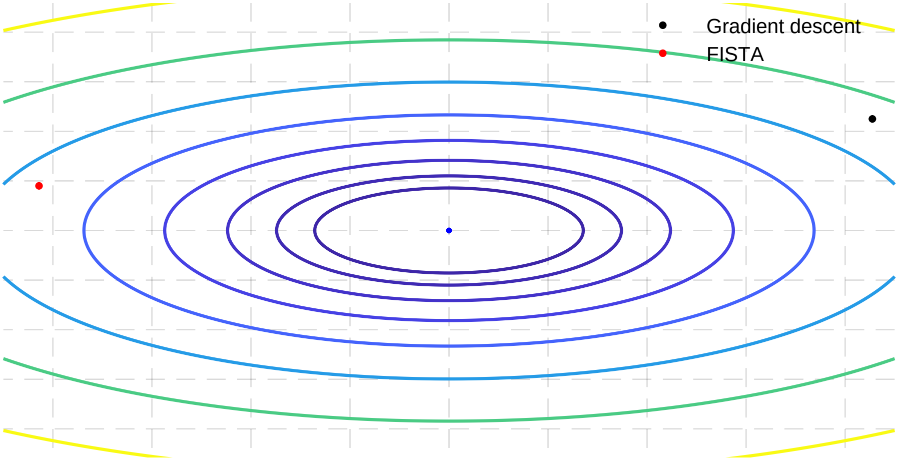
Convergence rate

NB: FISTA is not a descent method.

- Denote $f_k = \Phi(x_k) - \Phi(x^\star)$ and $u_k = t_k x_k - (t_k - 1)x_{k-1} - x^\star$, then
$$\frac{2}{L}t_k^2 f_k - \frac{2}{L}t_{k+1}^2 f_{k+1} \geq \|u_{k+1}\|^2 - \|u_k\|^2.$$
- Let c_k, d_k be positive sequences, if
$$c_k - c_{k+1} \geq d_{k+1} - d_k \forall k \geq 1, \text{ with} c_1 + d_1 < C,\ C > 0$$
then $c_k < C$ for all $k \geq 1$.
- $\frac{2}{L}t_k^2 f_k \leq \|x_0 - x^\star\|^2$
- $t_k \geq \frac{k+1}{2}$,
$$\Phi(x_k) - \Phi(x^\star) \leq \frac{L\|x_0 - x^\star\|^2}{2(k+1)^2}.$$

Outline

Oscillation of FISTA



Restarting FISTA

Why:

- for LSE, leading eigenvalue fo the system si complex.
- over extropolation, momentum beats gradient.

Restarting FISTA

Initial: $x_0 \in \mathbb{R}^n, x_{-1} = x_0$, $\gamma = 1/L$ and $t_0 = 1$;

repeat:

1. Run FISTA iteration
2. If $\langle y_k - x_{k+1},\, x_{k+1} - x_k \rangle > 0$: $t_k = 1, y_k = x_k$.

until: stopping criterion is satisfied.

Outline

Regression problems

ℓ_1-regularised least square

$$\min_{x\in\mathbb{R}^n} \mu\|x\|_1 + \frac{1}{2}\|Ax - f\|^2.$$

Sparse logistic regression

$$\min_{x\in\mathbb{R}^n} \mu\|x\|_1 + \frac{1}{m}\sum_{i=1}^{m} \log\left(1 + e^{-l_i h_i^T x}\right),$$

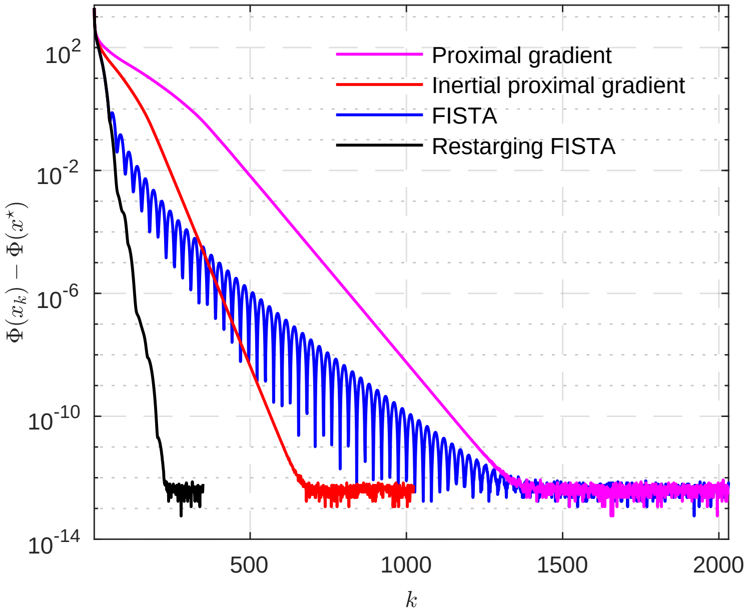
where $\mu = 10^{-2}$. The `australian` data set from LIBSVM[1] is considered.

[1] `https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/`

Compared methods

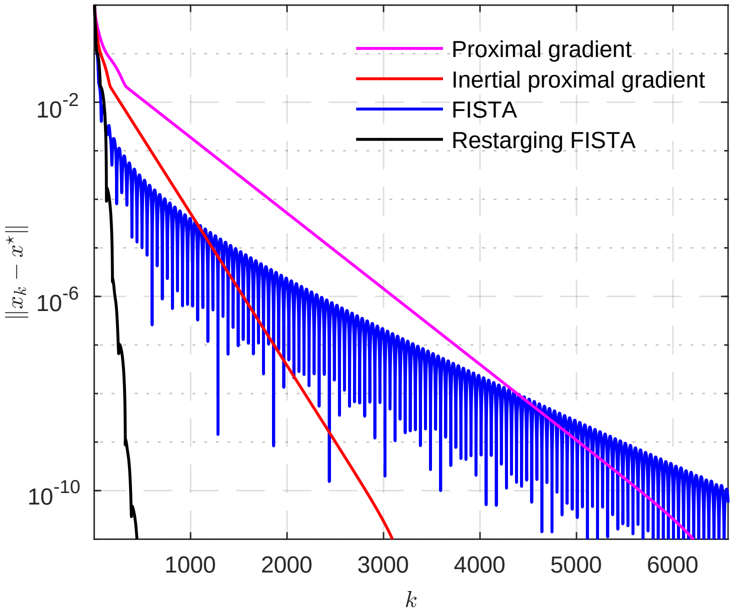
- Proximal gradient descent
- Inertial proximal gradient descent
- FISTA
- Restarting FISTA

Numerical results



ℓ_1-regularised LSE

Numerical results



Sparse logistic regression

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