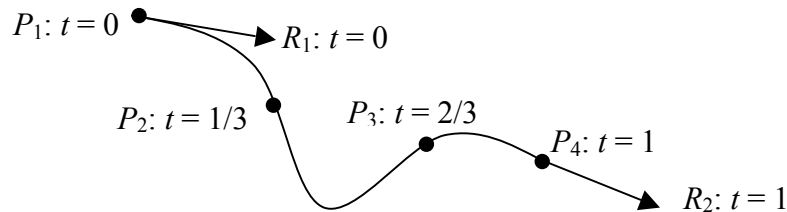


### Assignment #1 Solution

#### Problem 2 (3 marks): Parametric curve design

Determine the change of basis matrix for a quintic (that is, degree-5) parametric curve which is defined by four points and two tangent vectors, as shown below. Use the transpose of  $[P_1 R_1 P_2 P_3 P_4 R_2]$  as your vector of control points or observable quantities. Expressing your solution as the *inverse of a computed matrix* is sufficient.



[Solution] Let  $T = [1 \ t \ t^2 \ t^3 \ t^4 \ t^5]$ . Then  $x(t) = TA = TB^{-1}G$ , where

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1/3 & 1/9 & 1/27 & 1/81 & 1/243 \\ 1 & 2/3 & 4/9 & 8/27 & 16/81 & 32/243 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix} \text{ and } G = \begin{bmatrix} P_1 \\ R_1 \\ P_2 \\ P_3 \\ P_4 \\ R_2 \end{bmatrix}$$

#### Problem 3 (3 marks): Continuity of cubic B-splines

Think about how the change-of-basis matrix for cubic B-splines can be derived. Note that you *need not* submit a solution for this. You need to complete the following however. Recall that given the vector of four control points  $P = [P_0 \ P_1 \ P_2 \ P_3]^T$  and the monomial basis  $T = [1 \ t \ t^2 \ t^3]$ , the cubic B-spline curve piece defined by  $P$  is given by  $p(t) = TM_{\text{B-spline}}P$ , where  $M_{\text{B-spline}}$  is the cubic B-spline change-of-basis matrix,

$$M_{\text{B-spline}} = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

Prove that piecewise cubic B-spline curves are  $C^2$ .

[Solution] Let the first piece of curve, defined by control points  $P_0, P_1, P_2, P_3$ , be denoted by  $b_1(t)$ . Let the second piece of curve, defined by control points  $P_1, P_2, P_3, P_4$ , be denoted by  $b_2(t)$ . Do not forget that you need to show three things:  $b_1(1) = b_2(0)$ ,  $b'_1(1) = b'_2(0)$ , as

well as,  $b''_1(1) = b''_2(0)$ . With simple algebra and differentiation, you should be able to arrive at the conclusion that

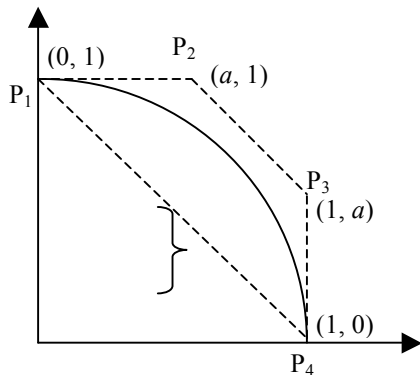
$$\begin{aligned} b_1(1) &= b_2(0) = (P_1 + 4P_2 + P_3)/6 \\ b'_1(1) &= b'_2(0) = (-3P_1 + 3P_3)/6 \\ b''_1(1) &= b''_2(0) = (6P_1 - 12P_2 + 6P_3)/6 \end{aligned}$$

To derive the change of basis matrix for cubic B-splines is a bit tedious. One can set up a system of 16 equations and solve for the entries. For example, see: <http://www2.cs.uregina.ca/~anima/408/Notes/Interpolation/UniformBSpline.htm> and search for “Deriving the weighting functions.”

#### Problem 4 (3 marks): Approximating a circular arc using a Bezier curve

Very efficient algorithms exist to draw cubic Bezier curves. In fact, these algorithms are so efficient that other types of curves are often converted to Bezier curves for display purposes. You are to approximate a  $90^\circ$  circular arc (it is part of the unit circle located in the first quadrant) with a Bezier curve. Derive the *coordinates* of the four control points of a Bezier curve that approximates the arc. This approximation should touch and be tangent to the arc at both of its endpoints as well as at its *midpoint*.

##### [Solution]



From the conditions we have  $P_1$  and  $P_4$  are  $(0, 1)$  and  $(1, 0)$ . Also, due to the tangent conditions, we have  $P_2$  and  $P_3$  are  $(a, 1)$  and  $(1, a)$  for some  $a$ .

From the equation of the unit circle, we have

$$x(t) = 1/\sqrt{2} \quad t = 1/2.$$

$$y(t) = 1/\sqrt{2}$$

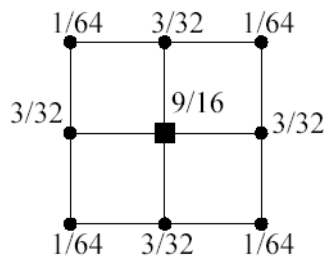
Since, the Bezier curve should also pass through this point, we have  $B_1x_1 + B_2x_2 + B_3x_3 + B_4x_4 = 1/\sqrt{2}$ , for  $t = 1/2$ , where  $B_1 = (1 - t)^3$ ,  $B_2 = 3t(1 - t)^2$ ,  $B_3 = 3t^2(1 - t)$ ,  $B_4 = t^3$  are the Bernstein polynomials of degree three (cubic Bezier basis functions). Substituting,  $B_1 = 1/8$ ,  $B_2 = 3/8$ ,  $B_3 = 3/8$ ,  $B_4 = 1/8$ ,  $x_1 = 0$ ,  $x_2 = a$ ,  $x_3 = 1$ ,  $x_4 = 1$  and solving for  $a$ , we get  $a = 0.5528$ .

It turns out that using these values for  $P_1$ ,  $P_2$ ,  $P_3$ , and  $P_4$  the Bezier approximation is also tangent to the circle at the midpoint.

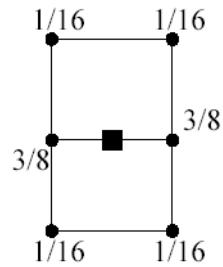
#### Problem 5 (3 marks): Analysis of midpoint subdivision

Refine an arbitrary closed manifold triangle mesh, whose connectivity is characterized by the graph  $G$ , by subdividing each triangle uniformly into four sub-triangles. Call the resulting graph  $G_1$ . Then compute the centroids of all the sub-triangles. Connect these centroids in the fashion of a dual graph  $G'_1$  of  $G_1$ . This results in a polygon mesh composed mostly of hexagons. Finally, compute the centroids of all these polygons and connect them in the fashion of a dual graph  $G''_1$  of  $G'_1$ . These constitute one step of a *midpoint subdivision scheme* for arbitrary triangle meshes.

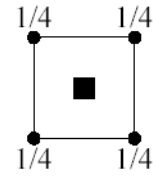
Compute the *masks* (showing topological and geometric rules of subdivision, as given in the figure below for Catmull-Clark subdivision over regular regions) for both regular and (general) extraordinary cases of this scheme.



Vertex rule

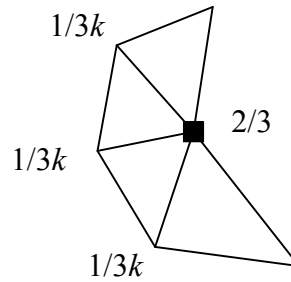
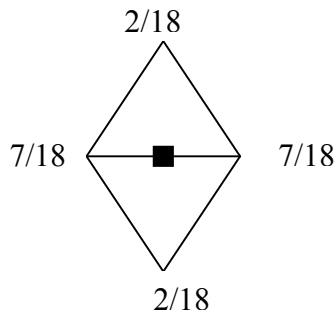


Edge rule



Face rule

**[Solution]** The subdivision masks are as shown below



On the left we have the mask for odd (new) points and on the right we have the case for the even (old) points with valence  $k$ .

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