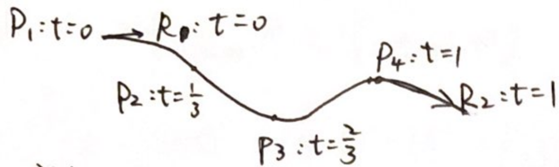
CMPT 764 Lei Pan

Problem2 : Parametric curve design



It's given that ~~we first assume that~~ $P_1 : t=0$, $P_2 : t=\frac{1}{3}$, $P_3 : t=\frac{2}{3}$, $P_4 : t=1$. $R_1 : t=0$. $R_2 : t=1$

The quintic parametric curve is defined by:

$$x(t) = a_0 + a_1(t) + a_2 \cdot t^2 + a_3 \cdot t^3 + a_4 \cdot t^4 + a_5 \cdot t^5$$

Let $T = [1\ t\ t^2\ t^3\ t^4\ t^5]$ $A = [a_0\ a_1\ a_2\ a_3\ a_4\ a_5]^{-1}$, then

$$x(t) = T \cdot A$$

$$\begin{cases} P_1 = [1\ 0\ 0\ 0\ 0\ 0]A \\ R_1 = x'(t)\big|_{t=0} = (a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4)\big|_{t=0} = a_1 \\ \quad = [0\ 1\ 0\ 0\ 0\ 0]A \\ P_2 = [1\ \frac{1}{3}\ (\frac{1}{3})^2\ (\frac{1}{3})^3\ (\frac{1}{3})^4\ (\frac{1}{3})^5] = [1\ \frac{1}{3}\ \frac{1}{9}\ \frac{1}{27}\ \frac{1}{81}\ \frac{1}{243}]A \\ P_3 = [1\ \frac{2}{3}\ (\frac{2}{3})^2\ (\frac{2}{3})^3\ (\frac{2}{3})^4\ (\frac{2}{3})^5] = [1\ \frac{2}{3}\ \frac{4}{9}\ \frac{8}{27}\ \frac{16}{81}\ \frac{32}{243}]A \\ P_4 = [1\ 1\ 1\ 1\ 1\ 1]A \\ R_2 = x'(t)\big|_{t=1} = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = [0\ 1\ 2\ 3\ 4\ 5]A \end{cases}$$

$$\begin{bmatrix} P_1 \\ R_1 \\ P_2 \\ P_3 \\ P_4 \\ R_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} \\ 1 & \frac{2}{3} & \frac{4}{9} & \frac{8}{27} & \frac{16}{81} & \frac{32}{243} \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}}_{B} \cdot A \Rightarrow A = B^{-1} \begin{bmatrix} P_1 \\ R_1 \\ P_2 \\ P_3 \\ P_4 \\ R_2 \end{bmatrix}$$

Therefore $x(t) = T \cdot A = T \cdot B^{-1} \cdot \begin{bmatrix} P_1 \\ R_1 \\ P_2 \\ P_3 \\ P_4 \\ R_2 \end{bmatrix}$, the inverse of a compute matrix is sufficient.

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Problem 3: Continuity of cubic B-splines

$$P(t) = T \cdot M_{B\text{-spline}} P = [1\ t\ t^2\ t^3] \cdot \frac{1}{6}\begin{bmatrix} 1 & 4 & 1 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \cdot P$$

$$= \frac{1}{6} \cdot [(1-t)^3 \quad (4-6t^2+3t^3) \quad (1+3t+3t^2-3t^3) \quad t^3] \cdot P$$

$$P'(t) = \frac{1}{6} \cdot [-3(1-t)^2 \quad (-12t+9t^2) \quad (3+6t-9t^2) \quad 3t^2] \cdot P$$

$$P''(t) = \frac{1}{6} [6(1-t) \quad (-12+18t) \quad (6-18t) \quad 6t] \cdot P$$

We assume that the first piece of curve $P_1(t)$ controlled by 4 control points P_0, P_1, P_2, P_3 and the second piece of curve $P_2(t)$ controlled by 4 control points P_1, P_2, P_3, P_4 as they shared 3 points.

$$P_1(1) = \frac{1}{6} \cdot [0 \quad 1 \quad 4 \quad 1] \cdot [P_0\ P_1\ P_2\ P_3]^T$$

$$= \frac{1}{6} \cdot (P_1 + 4P_2 + P_3)$$

$$P_2(0) = \frac{1}{6} \cdot [1 \quad 4 \quad 1 \quad 0] \cdot [P_1\ P_2\ P_3\ P_4]^T$$

$$= \frac{1}{6} \cdot (P_1 + 4P_2 + P_3) = P_1(1)$$

$$P_1'(1) = \frac{1}{6} [0 \quad -3 \quad 0 \quad 3] [P_0\ P_1\ P_2\ P_3]^T$$

$$= \frac{1}{6} (-3P_1 + 3P_3)$$

$$P_2'(0) = \frac{1}{6} \cdot [-3 \quad 0 \quad 3 \quad 0] [P_1\ P_2\ P_3\ P_4]^T$$

$$= \frac{1}{6} (-3P_1 + 3P_3) = P_1'(1)$$

$$P_1''(1) = \frac{1}{6} \cdot [0 \quad 6 \quad -12 \quad 6] [P_0\ P_1\ P_2\ P_3]^T$$

$$= \frac{1}{6} \cdot (6P_1 - 12P_2 + 6P_3)$$

$$P_2''(0) = \frac{1}{6} [6 \quad -12 \quad 6 \quad 0] [P_1\ P_2\ P_3\ P_4]^T$$

$$= \frac{1}{6} \cdot (6P_1 - 12P_2 + 6P_3) = P_1''(1)$$

From the calculation on the left, we have

$P_1(1) = P_2(0)$

$P_1'(1) = P_2'(0)$

$P_1''(1) = P_2''(0)$

$\Downarrow$

therefore piecewise cubic B-spline curves are C^2.

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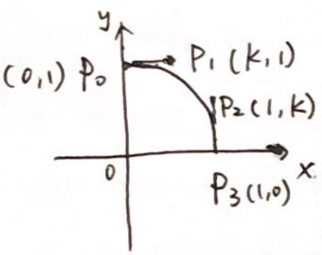
Problem 4: Approximating a circular arc using a Bezier curve.

Let P_0, P_1, P_2, P_3 be the four control points.

We assume the arc radius is 1 (unit arc)

- the start point P_0 and end point P_3 should coincide with the start and end point of circle. So $P_0 = (0,1)$ and $P_3 = (1,0)$.

- As the approximation should touch and be tangent to the arc at both endpoints, we also know that bézier curve is tangent at P_0 to $(P_1 - P_0)$ and at P_3 to $(P_3 - P_2)$. and as we are in the arc condition.

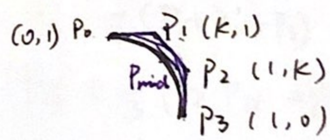


So. P_1 should have same y coordinate as P_0

P_2 should have same x coordinate as P_3

Also from the symmetric of arc. we can let $P_1(k,1)$, $P_2(1,k)$.

- the approximation should also touch the midpoint of arc, we can use midpoint subdivision rule to calculate midpoint P_{mid}. P_{mid} should equals $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ (midpoint for arc).



$P_0 \ (0,1)$

$\quad > \frac{1}{2}\cdot(k,2)$

$P_1 \ (k,1)$ $\qquad > \frac{1}{4}\cdot(2k+1, k+3)$

$\quad > \frac{1}{2}\cdot(k+1, k+1)$ $\qquad\qquad > \frac{1}{8}\cdot(3k+4, 3k+4) = P_{mid}(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

$P_2 \ (1,k)$ $\qquad > \frac{1}{4}(k+3, 2k+1)$

$\quad > \frac{1}{2}\cdot(2,k)$

$P_3 \ (1,0)$

$$\frac{1}{8}(3k+4) = \frac{\sqrt{2}}{2} \Rightarrow k = \frac{4\sqrt{2}-4}{3} = 0.5528$$

therefore. $P_0(0,1)$; $P_1(0.5528, 1)$, $P_2(1, 0.5528)$, $P_3(1,0)$ are 4 control points.

The bézier curve can be written as $P(t) = P_0 \cdot (1-t)^3 + P_1 \cdot 3t(1-t)^2 + P_2 \, 3t^2 \cdot (1-t) + P_3 t^3$

it is also tangent to the arc at the midpoint.

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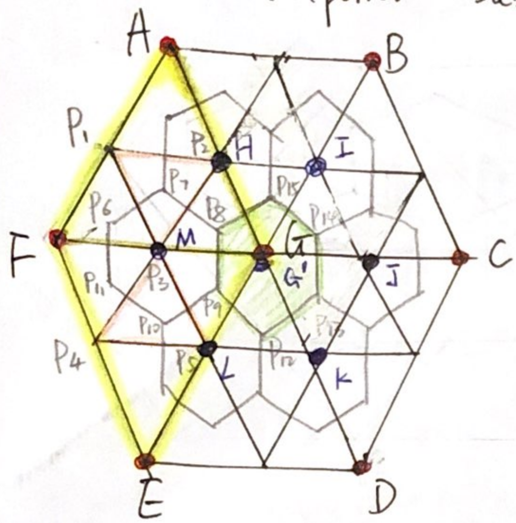
Problem 5: Midpoint Subdivision.



The red points (A.B.C.D.E.F.G) are the even points (old points)

The blue points (H.I.J.K.L.M.G') are the odd points (new points)

(1) Compute new points. (point M as an example)

we take a look at triangle $\triangle AFG$ and triangle $\triangle FGE$, the midpoints on edges are denoted as P_1, P_2, P_3, P_4, P_5. The centroids of subtriangle are denoted as $P_6, P_7, P_8, P_9, P_{10}, P_{11}$. From the characteristics of midpoints and centroids. we can write the coordinates of the points above.

$$P_1 = \frac{1}{2}(P_A + P_F)$$

$$P_2 = \frac{1}{2}(P_A + P_G)$$

$$P_3 = \frac{1}{2}(P_F + P_G)$$

$$P_4 = \frac{1}{2}(P_F + P_E)$$

$$P_5 = \frac{1}{2}(P_G + P_E)$$

$$P_6 = \frac{1}{3}(P_1 + P_F + P_3) = \frac{1}{3}\cdot\left(\frac{1}{2}(P_A+P_F) + \frac{1}{2}(P_F+P_G) + P_F\right) = \frac{1}{3} \times \left(2P_F + \frac{1}{2}P_A + \frac{1}{2}P_G\right)$$

$$P_7 = \frac{1}{3}(P_1 + P_2 + P_3) = \frac{1}{3}(P_A + P_F + P_G)$$

$$P_8 = \frac{1}{3}\left(2P_G + \frac{1}{2}P_A + \frac{1}{2}P_F\right)$$

$$P_9 = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_E + \frac{1}{2}P_F\right)$$

$$P_{10} = \frac{1}{3}(P_F + P_E + P_G)$$

$$P_{11} = \frac{1}{3}\cdot\left(2P_F + \frac{1}{2}P_E + \frac{1}{2}P_G\right)$$

the new point
$$P_M = \frac{1}{6}(P_6 + P_7 + P_8 + P_9 + P_{10} + P_{11})$$
$$= \frac{1}{6}\cdot\left(\frac{7}{3}P_F + \frac{7}{3}P_G + \frac{2}{3}P_A + \frac{2}{3}P_E\right)$$
$$= \frac{7}{18}(P_F + P_G) + \frac{2}{18}(P_A + P_E)$$

(2) compute the old points (point G update to G')

Now we take a look at the hexagon $P_8 P_9 P_{12} P_{13} P_{14} P_{15}$

$$P_8 = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_A + \frac{1}{2}P_F\right)$$

$$P_9 = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_E + \frac{1}{2}P_F\right)$$

$$P_{12} = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}\cdot P_E + \frac{1}{2}P_D\right)$$

$$P_{13} = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_D + \frac{1}{2}P_C\right)$$

$$P_{14} = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_B + \frac{1}{2}P_C\right)$$

$$P_{15} = \frac{1}{3}\cdot\left(2P_G + \frac{1}{2}P_A + \frac{1}{2}P_B\right)$$

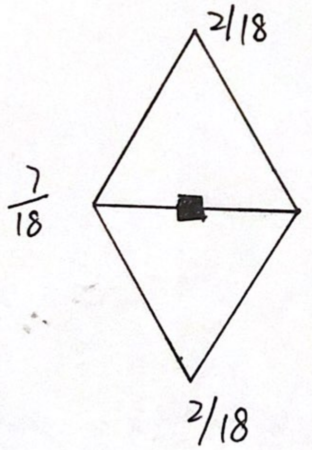
$$P_G' = \frac{1}{6}\cdot(P_8 + P_9 + P_{12} + P_{13} + P_{14} + P_{15}) = \frac{1}{6}\frac{1}{3}(12P_G + P_A + P_B + P_C + P_D + P_E + P_F)$$

$$= \frac{2}{3}P_G + \frac{1}{3\times 6}(P_A + P_B + P_C + P_D + P_E + P_F) \quad (\text{6 is the valence})$$

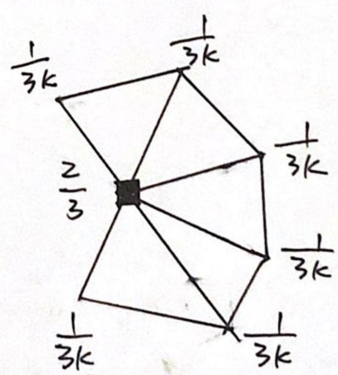
in a more general case, for old point with valence k:

$$P_G' = \frac{2}{3}P_G + \frac{1}{3k}\cdot(P_A + P_B + P_C + P_D + P_E + P_F).$$

The subdivision masks are below:



Mask for odd (new) points



Mask for even (old) points with valence k.