Notes on invariant forms on generalized Bott–Samelson bimodules

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In these notes we introduce invariant forms on generalized Bott-Samelson bimodules, generalizing the definitions from [EW14, §3.1]. We then show that flipping a singular diagram is the same as taking the adjoint with respect to the invariant forms.

Let (W, S) be a Coxeter system and $I \subset S$ be finitary. Consider the category $\mathbb{S}Bim^I$ of I-singular Soergel bimodules. Let $\mathfrak{h}^\vee$ be a representation of W as in [Soe07, Proposition 2.1] with simple roots $\alpha_s \in \mathfrak{h}^\vee$ and simple coroots $\alpha_s \in \mathfrak{h}^\vee$.

For $s \in S$, let $\partial_s : R \to R(-2)$ denote the Demazure operator

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}.$$

Note that if f is of degree 2 then $\partial_j(f) = \alpha_j^\vee(f)$. The Demazure operators satisfy the braid relations, hence for any $x \in W$ with reduced expression $\underline{x} = s_1 s_2 \dots s_k$ we can define $\partial_x := \partial_1 \partial_2 \dots \partial_k : R \to R(-2k)$.

1 Invariant forms

Let $B^I \in \mathbb{S}Bim^I$ be a I-singular Soergel bimodule. A left invariant bilinear form on B is a graded bilinear map

$$\langle -, - \rangle : B^I \times B^I \to R$$

such that for any $b, b' \in B^I$, $f \in R$ and $g \in R^I$ we have

$$\langle fb, b' \rangle = \langle b, fb' \rangle = f \langle b, b' \rangle$$

$$\langle bg, b' \rangle = \langle b, b'g \rangle.$$

Let $(\vec{I}, \vec{J})$ be a translation pair with $\vec{J} = (J_1, J_2, \dots, , J_k)$ and $\vec{I} = (I_0, I_1, I_2 \dots, I_k)$ and assume that $I_0 = \emptyset$ and $I_k = I$. Consider the translation pair $(\vec{I'}, \vec{J'})$ with

$$\vec{J'} = (J_2, J_3, \dots, J_k) \qquad \text{and} \qquad \vec{I'} = (\emptyset, I_2, I_3, \dots, I_k). \tag{1}$$

Recall the notation of [Pat19, Table 1]. The morphism $f_{\smile} : R^{I_1} \to R^{I_1} \otimes_{R^{J_1}} R^{I_1}(\ell(J_1) - \ell(I_1))$ induces a morphism

$$f_{\smile} \otimes \mathrm{Id} : BS(\vec{I'}, \vec{J'}) \to BS(\vec{I}, \vec{J}).$$

Definition 1.1. We define, by induction on k, an element $c_{top}(\vec{I}, \vec{J})$ as $(f_{\smile} \otimes \mathrm{Id})(c_{top}(\vec{I'}, \vec{J'}))$, where $c_{top}((\emptyset), \emptyset) = 1 \in R$.

If $\underline{w} = s_1 s_2 \dots s_k$ is a word with $s_i \in S$, then the ordinary Bott–Samelson bimodule $BS(\underline{w})$ can be realized as a generalized Bott–Samelson bimodule $BS(\vec{I}, \vec{J})$, with $\vec{J} = (\{s_1\}, \{s_2\}, \dots \{s_k\})$ and $\vec{I} = (\emptyset, \emptyset, \dots, \emptyset)$. In this case, the corresponding element $c_{top}(\underline{w}) := c_{top}(\vec{I}, \vec{J}) \in BS(\underline{w})$ coincides with the element c_{top} defined in [EW14, §3.4].

Let $t \in W$ be a reflection. Then we can write $t = wsw^{-1}$ with $ws > w$ and $s \in S$. The root $\alpha_t := w(\alpha_s) \in \mathfrak{h}^*$ is well defined and it is positive, i.e. it can be written as a positive linear combination of simple roots. We can consider the operator

$$\partial_t^T : R \to R(-2)$$

defined by $\partial_t^T(f) = \frac{f - t(f)}{\alpha_t}$. (We use the non-standard notation ∂_t^T to distinguish it from ∂_t, as they only coincide when $t \in S$.) If $t = wsw^{-1}$ with $ws > w$ and $s \in S$, then $\partial_t^T(f) = w\left(\partial_s(w^{-1}(f))\right)$ for all $f \in R$. In particular, if $f \in R$ is of degree 2, then $\partial_t^T(f) = \partial_s(w^{-1}(f)) = \alpha_s^\vee(w^{-1}(f))$. We write $y \xrightarrow[R]{t} x$ if $t \in T$, $yt = x$ and $\ell(y) + 1 = \ell(x)$.

Lemma 1.2. *Let $x \in W$. Let $f, g \in R$ be such that f is of degree 2 and g of degree $2\ell(x) - 2$. Then*

$$\partial_x(fg) = \sum_{y \xrightarrow[R]{t} x} \partial_t^T(f) \partial_y(g).$$

Proof. The proof is by induction on $\ell(x)$. The case $\ell(x) \leq 1$ is trivial. Let $x = x's$ with $s \in S$ and $x > x'$. By induction on the length we obtain

$$\partial_x(fg) = \partial_{x'}\partial_s(fg) = \partial_{x'}\left(g\partial_s(f) + s(f)\partial_s(g)\right) = \partial_s(f)\partial_{x'}(g) + \sum_{y' \xrightarrow[R]{u} x'} \partial_u^T(s(f))\partial_{y'}(\partial_s(g)).$$

If $y's < y'$ the term $\partial_{y'}(\partial_s(g))$ vanishes, while if $y's > y'$ and $y' \xrightarrow[R]{u} x'$, then also $y's \xrightarrow[R]{sus} x$. In particular we have $s \neq u$, so $s(\alpha_u)$ is a positive root and $\partial_u^T(s(f)) = \partial_{sus}^T(f)$. The claim now follows. □

Consider the right R^I-module $\overline{BS(\vec{I}, \vec{J})} := \mathbb{R} \otimes_R BS(\vec{I}, \vec{J})$. Its degree $\ell(\vec{I}, \vec{J})$ component is of dimension one. Obvioulsy, in the bimodule $BS(\vec{I}, \vec{J})$ the dimension of the degree $\ell(\vec{I}, \vec{J})$ component is much bigger. However, among these, the element $c_{top}(\vec{I}, \vec{J})$ is basically unique up to lower degree terms. We make this precise with the following definition.

Definition 1.3. We define $LT(BS(\vec{I}, \vec{J}))$ to be the left R-submodule of $BS(\vec{I}, \vec{J})$ generated by elements of degree less than $\ell(\vec{I}, \vec{J})$. We refer to the elements of $LT(BS(\vec{I}, \vec{J}))$ as *lower degree terms.*

With this definition, every two elements of $BS(\vec{I}, \vec{J})$ of degree $\ell(\vec{I}, \vec{J})$ are multiple of each other up to lower degree terms. Let $(\mathfrak{h}^*)^I$ be the subspace of W_I-invariant elements in $\mathfrak{h}^*$ (or, equivalently, $(\mathfrak{h}^*)^I$ is the subspace of elements of degree 2 in R^I). We say that $\rho \in (\mathfrak{h}^*)^I$ is *ample* if $\partial_s(\rho) > 0$ for every $s \in S \setminus I$. Since the simple coroots $\alpha_s^\vee$ are linearly independent there always exists an ample element $\rho \in (\mathfrak{h}^*)^I$.

Lemma 1.4. *Let $\rho \in (\mathfrak{h}^*)^I$ be ample. Let $w \in W^I$ and $w = s_1 s_2 \dots s_l$ be a reduced expression for w. Then for any $i \leq l$ we have*

$$\partial_{s_i}(s_{i+1}s_{i+2}\dots s_l(\rho)) > 0$$

Proof. See [Pat18, Lemma 4.2.1]. □

We say that a word $\underline{w}$ is I-reduced if $\underline{w}$ is a reduced word for $w \in W^I$. If $B \in \mathbb{S}Bim^J$ and $J \subset I$ we denote by $B_I = B \otimes_{R^I} R^I \in \mathbb{S}Bim^I$ its restriction to a (R, R^I)-bimodule.

Lemma 1.5. *Let $\underline{w}$ be a word of length k, $\rho \in (\mathfrak{h}^*)^I$ and consider $1^\otimes := 1 \otimes 1 \otimes \ldots \otimes 1 \in BS(\underline{w})_I$. Then $1^\otimes \cdot \rho^{\ell(\underline{w})} = Nc_{top}(\underline{w})$ up to lower degree terms. If $\underline{w}$ is not I-reduced then $N = 0$ while if $\underline{w}$ is a I-reduced word for w then $N = \partial_w(\rho^k)$.*

If, moreover, ρ is ample then $N > 0$.

Proof. If $\underline{w}$ is not I-reduced then

$$BS(\underline{w})_I = \bigoplus_z B_z^I(m) \tag{2}$$

and all the elements z occurring in the RHS in (2) satisfy $\ell(z) < k$. Then, by degree reasons $1^\otimes \cdot \rho^k$ has to be an element of $LT(BS(\underline{w})_I)$, hence $N = 0$.

Assume now that $\underline{w}$ is a I-reduced word for $w \in W^I$ with $\underline{w} = s_1 s_2 \ldots s_k$. For any i, let $\underline{w}_i = s_1 s_2 \ldots s_{i-1} s_{i+1} \ldots s_k$. Let $v_i = s_i s_{i+1} \ldots s_k$. As in the proof of [EW14, Lemma 3.10] one can see that

$$1^\otimes \cdot \rho^k = Nc_{top}(\underline{w}) + \text{ lower degree terms}$$

with $N = \sum_i \partial_{s_i}(v_{i+1}(\rho))N_i$, where N_i is the coefficient of $c_{top}(\underline{w}_i)$ in $1^\otimes \cdot \rho^{k-1} \in BS(\underline{w}_i)$. By induction we have

$$N = \sum_{i\,:\,\underline{w}_i \text{ } I\text{-reduced}} \partial_{s_i}(v_{i+1}(\rho))\partial_{w_i}(\rho^{k-1}) = \sum_{y \xrightarrow[R]{t} w} \partial_t^T(\rho)\partial_y(\rho^{k-1})$$

Finally, applying Lemma 1.2 we get $N = \partial_w(\rho^k)$.

Assume now that ρ is ample. Then, by induction, for any i such that $\underline{w}_i$ is I-reduced we have $N_i > 0$ and also $\partial_{s_i}(v_{i+1}(\rho)) > 0$ by Lemma 1.4. Hence $N > 0$. □

Let $I \subset J$ be finitary subsets and fix a reduced expression $\underline{w_J w_I}$ for $w_J w_I$. Then $BS((\emptyset, I), (J)) = R \otimes_{R^J} R^I(\ell(J) - \ell(I))$ is a direct summand of $BS(\underline{w_J w_I})_I$ and we have a natural embedding of (R, R^I)-bimodules

$$\vartheta : R \otimes_{R^J} R^I(\ell(J) - \ell(I)) \to BS(\underline{w_J w_I})_I$$

induced by sending $1^\otimes \in R \otimes_{R^J} R^I$ to $1^\otimes \in BS(\underline{w_J w_I})_I$. Recall the element $c_{top}((\emptyset, I), (J)) \in R \otimes_{R^J} R^I(\ell(J) - \ell(I))$. By definition, we have $c_{top}((\emptyset, I), (J)) = \sum_i x_i \otimes y_i$ where $\{x_i\}$ and $\{y_i\}$ are bases of R^I over R^J with $\partial_{w_I w_J}(x_i y_j) = \delta_{ij}$.

Lemma 1.6. *The embedding $\vartheta : R \otimes_{R^J} R^I(\ell(J) - \ell(I)) \to BS(\underline{w_J w_I})_I$ sends $c_{top}((\emptyset, I), (J))$ to $c_{top}(\underline{w_J w_I})$, up to lower degree terms.*

Proof. Since ϑ sends $1^\otimes$ to $1^\otimes$, it also sends $1^\otimes \cdot \rho^{\ell(w_J w_I)}$ to $1^\otimes \cdot \rho^{\ell(w_J w_I)}$ for every $\rho \in (\mathfrak{h}^*)^I$. Fix ρ ample so that $\partial_{w_J w_I}(\rho^{\ell(w_J w_I)}) \neq 0$. Then, by Lemma 1.5, it suffices to show that

$$1 \otimes \rho^{\ell(w_J w_I)} = \partial_{w_J w_I}(\rho^{\ell(w_J w_I)}) c_{top}((\emptyset, I), (J)) + \text{lower degree terms}$$

If $c_{top}((\emptyset, I), (J)) = \sum_i x_i \otimes y_i$, there exists a unique index i_0 such that $x_{i_0} \in R^I$ is of degree 0 and $y_{i_0} \in R^I$ is of degree $2\ell(w_J w_I)$. Then $c_{top}((\emptyset, I), (J))$ is equal to

$x_{i_0} \otimes y_{i_0} = 1 \otimes x_{i_0} y_{i_0}$ up to lower degree terms. On the other hand, when expressing $\rho^{\ell(w_J w_I)}$ in the $\{y_i\}$ basis, the coefficient of y_{i_0} is

$$\frac{\partial_{w_J w_I}(\rho^{\ell(w_J w_I)})}{\partial_{w_J w_I}(y_{i_0})} = x_{i_0} \partial_{w_J w_I}(\rho^{\ell(w_J w_I)}),$$

so we have $1 \otimes \rho^{\ell(w_J w_I)} = \partial_{w_J w_I}(\rho^{\ell(w_J w_I)}) \cdot 1 \otimes x_{i_0} y_{i_0}$ up to lower degree terms and the claim follows. ∎

Reiterating Lemma 1.6, we get a chain of embeddings

$$\begin{aligned} BS(\vec{I},\vec{J}) = R \otimes_{R^{J_1}} R^{I_1} \otimes_{R^{J_2}} \dots \otimes_{R_{J_k}} R^{I_k}(\ell(\vec{I},\vec{J})) \hookrightarrow \\ \hookrightarrow BS(\underline{w_J w_I}) \otimes_{R^{J_2}} R^{I_2} \otimes_{R_{J_k}} R^{I_k}(\ell(\vec{I},\vec{J}) - \ell(w_J w_I)) \hookrightarrow \dots \hookrightarrow \\ \hookrightarrow BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})_I. \quad (3) \end{aligned}$$

Lemma 1.7. *The embedding in* (3) *sends* $c_{top}(\vec{I},\vec{J})$ *to* $c_{top} \in BS(\underline{w_{J_1} w_{I_1}} \dots \underline{w_{J_k} w_{I_k}})_I$ *up to lower degree terms.*

Proof. Consider the following diagram, where $\vec{I}'$ and $\vec{J}'$ are defined as in (1) and α and β are embeddings as in (3).

$$\begin{tikzcd} & BS(\vec{I},\vec{J}) \arrow[dr, hook, "\alpha"] & \\ BS(\vec{I}',\vec{J}') \arrow[ur, "f_{\curvearrowleft} \otimes \mathrm{Id}"] \arrow[rr, "c_{top}(\underline{w_{J_1} w_{I_1}}) \otimes \mathrm{Id}"] \arrow[d, hook, "\beta"] & & BS(\underline{w_{J_1} w_{I_1}}) \otimes_R BS(\vec{I}',\vec{J}') \arrow[d, hook, "\mathrm{Id} \otimes \beta"] \\ BS(\underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})_I \arrow[rr, "c_{top}(\underline{w_{J_1} w_{I_1}}) \otimes \mathrm{Id}"] & & BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})_I \end{tikzcd}$$

Notice that the lower square is commutative. By Lemma 1.6 the top triangle is "commutative up to lower degree terms," meaning that we have

$$\alpha \circ (f_{\curvearrowleft} \otimes \mathrm{Id}) c_{top}(\vec{I}',\vec{J}') = c_{top}(\underline{w_J w_I}) \otimes c_{top}(\vec{I}',\vec{J}')$$

up to lower degree terms. Therefore we have

$$\begin{aligned} (c_{top}(\underline{w_J w_I}) \otimes \mathrm{Id}) \circ \beta(c_{top}(\vec{I}',\vec{J}')) = (\mathrm{Id} \otimes \beta) \circ \alpha \circ (f_{\curvearrowleft} \otimes \mathrm{Id})(c_{top}(\vec{I}',\vec{J}')) = \\ = (\mathrm{Id} \otimes \beta) \circ \alpha(c_{top}(\vec{I},\vec{J})) \end{aligned}$$

up to lower degree terms. We conclude since by induction we have $\beta(c_{top}(\vec{I}',\vec{J}')) = c_{top}(\underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})$ up to lower degree terms. ∎

Lemma 1.8. *The element* $c_{top}(\vec{I},\vec{J}) \in BS(\vec{I},\vec{J})$ *together with* $LT(BS(\vec{I},\vec{J}))$ *generates* $BS(\vec{I},\vec{J})$ *as a left* R*-module.*

Proof. By looking at $\mathrm{grrk}(BS(\vec{I},\vec{J}))$ it is enough to show that $c_{top}(\vec{I},\vec{J})$ does not belong to $LT(BS(\vec{I},\vec{J}))$. This is clear by looking at the embedding in (3), since the image of $c_{top}(\vec{I},\vec{J})$ contains $c_{top}(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})$ with non-trivial coefficient, hence it is not contained in the submodule $LT(BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \dots \underline{w_{J_k} w_{I_k}})_I)$. ∎

In other words, we have the following decomposition of left R-bimodules.

$$BS(\vec{I},\vec{J}) = R \cdot c_{top}(\vec{I},\vec{J}) \oplus LT(BS(\vec{I},\vec{J})). \tag{4}$$

We can finally define the intersection form of a generalized Bott–Samelson bimodule.

Definition 1.9. Let $\mathrm{Tr} : BS(\vec{I},\vec{J}) \to R$ be the R-linear map which returns the coefficient of $c_{top}(\vec{I},\vec{J})$ with respect to the decomposition (4). We denote by $(-)\cdot(-)$ the term-wise multiplication on $BS(\vec{I},\vec{J})$. We call *intersection form* the invariant form $\langle -,-\rangle_{(\vec{I},\vec{J})}$ on $BS(\vec{I},\vec{J})$ defined by

$$\langle x,y\rangle_{(\vec{I},\vec{J})} = \mathrm{Tr}(x\cdot y).$$

1.1 Adjoint and flipped maps

Our next goal is to show that taking the adjoint with respect of the intersection forms is the same as flipping the corresponding diagrams. It is enough to show this for the "building boxes" of Recall the notation of [Pat19, Table 1].

We consider first the clockwise cup $f_{\circlearrowright\smile}$ and its flip, the clockwise cap $f_{\frown\!\!\circlearrowright}$. Let $\vec{I}$ and $\vec{J}$ be as in the previous section and let

$$\vec{I'} = (\emptyset, I_1, \ldots, I_{h-1}, I_h, I_h, I_{h+1}, \ldots, I_k),$$

$$\vec{J'} = (J_1, \ldots, J_{h-1}, K, J_h, J_{h+1}, \ldots, J_k)$$

with $I_h \subset K$. Let

$$F := \mathrm{Id} \otimes f_{\circlearrowright\smile} \otimes \mathrm{Id} : BS(\vec{I},\vec{J}) \to BS(\vec{I'},\vec{J'})$$

be the morphism induced by $f_{\circlearrowright\smile} : R^{I_h} \to R^{I_h} \otimes_{R^K} R^{I_h}(\ell(K) - \ell(I_h))$ and let

$$\overline{F} := \mathrm{Id} \otimes f_{\frown\!\!\circlearrowright} \otimes \mathrm{Id} : BS(\vec{I'},\vec{J'}) \to BS(\vec{I},\vec{J})$$

be the morphism induced by $f_{\frown\!\!\circlearrowright} : R^{I_h} \otimes_{R^K} R^{I_h}(\ell(K) - \ell(I_h)) \to R^{I_h}$.

Lemma 1.10. *The maps F and $\overline{F}$ above are adjoint to each other with respect to the corresponding intersection forms, i.e. we have*

$$\langle F(x), y\rangle_{(\vec{I'},\vec{J'})} = \langle x, \overline{F}(y)\rangle_{(\vec{I},\vec{J})} \text{ for all } x \in BS(\vec{I},\vec{J}),\ y \in BS(\vec{I'},\vec{J'}).$$

Proof. In $R^{I_h} \otimes_{R^K} R^{I_h}$ we have

$$f_{\circlearrowright\smile}(z)\cdot(y_1\otimes y_2) = f_{\circlearrowright\smile}(1) z y_1 y_2 = f_{\circlearrowright\smile}(z \cdot f_{\frown\!\!\circlearrowright}(y_1\otimes y_2)) \text{ for all } z, y_1, y_2 \in R^{I_h}$$

where the first equality follows from the fact that $g f_{\circlearrowright\smile}(1) = f_{\circlearrowright\smile}(g) = f_{\circlearrowright\smile}(1) g$ for any $g \in R^{I_h}$. It follows that for every $x \in BS(\vec{I},\vec{J})$ and $y \in BS(\vec{I'}, vecJ')$ we have $F(x)\cdot y = F(x\cdot\overline{F}(y))$ and $\langle x,y\rangle_{(\vec{I},\vec{J})} = \mathrm{Tr}(F(x\cdot\overline{F}(y))$. Thus, to conclude that F and $\overline{F}$ are adjoint to each other it is enough to show that, for all $b \in BS(\vec{I},\vec{J})$ we have $\mathrm{Tr}(F(b)) = \mathrm{Tr}(b)$.

Assume that $b = g c_{top}(\vec{I},\vec{J}) + b'$ with $g \in R$ and $b' \in LT(BS(\vec{I},\vec{J}))$. By degree reasons, also $F(b') \in LT(BS(\vec{I'},\vec{J'}))$. We conclude that $\mathrm{Tr}(F(b)) = \mathrm{Tr}(b)$ by showing $F(c_{top}(\vec{I},\vec{J})) = c_{top}(\vec{I'},\vec{J'})$. This follows since the following diagram is commutative:

$$\begin{tikzcd}
R \otimes_{R^{I_h}} R^{I_h} \arrow[r, "\mathrm{Id} \otimes f_{\circlearrowright}"] \arrow[d, "f_{\circlearrowright}"'] & R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^K} R^{I_h} \arrow[d, "\mathrm{Id} \otimes f_{\circlearrowright} \otimes \mathrm{Id}"] \\
R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^{J_h}} R^{I_h} \arrow[d, phantom, "\cong" rotate=90] & R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^{J_h}} R^{I_h} \otimes_{R^K} R^{I} \arrow[d, phantom, "\cong" rotate=90] \\
R \otimes_{R^{J_h}} R^{I_h} \arrow[r, "\mathrm{Id} \otimes f_{\circlearrowright}"] & R \otimes_{R^{J_h}} R^{I_h} \otimes_{R^K} R^{I_h}
\end{tikzcd}$$

as it can be easily checked, for example, by looking at the corresponding diagrams. $\square$

We consider next the counterclockwise cup $f_{\circlearrowleft}$ and its flip, the counterclockwise cap $f_{\frown}$. Let now

$$\vec{I''} = (\emptyset, I_1, \ldots, I_{h-1}, K, I_{h+1}, \ldots, I_k) \qquad \text{and } \vec{J''} = (J_1, \ldots, J_{h-1}, J_h, J_{h+1}, \ldots, J_k)$$

with $K \subset I_h$. Let

$$G = \mathrm{Id} \otimes f_{\circlearrowleft} \otimes \mathrm{Id} : BS(\vec{I}, \vec{J}) \to BS(\vec{I''}, \vec{J''})$$

be the morphism induced by $f_{\circlearrowleft} : R^{I_h} \to {}_{I_h}(R^K)_{I_h}(\ell(I_k) - \ell(K))$ and let

$$\overline{G} = \mathrm{Id} \otimes f_{\frown} \otimes \mathrm{Id} : BS(\vec{I''}, \vec{J''}) \to BS(\vec{I}, \vec{J})$$

be the morphism induced by $f_{\frown} : {}_{R^{I_h}}(R^K)_{R^{I_h}}(\ell(I_k) - \ell(K)) \to R^{I_h}$.

Lemma 1.11. *The maps G and $\overline{G}$ above are adjoint with respect to the corresponding intersection forms.*

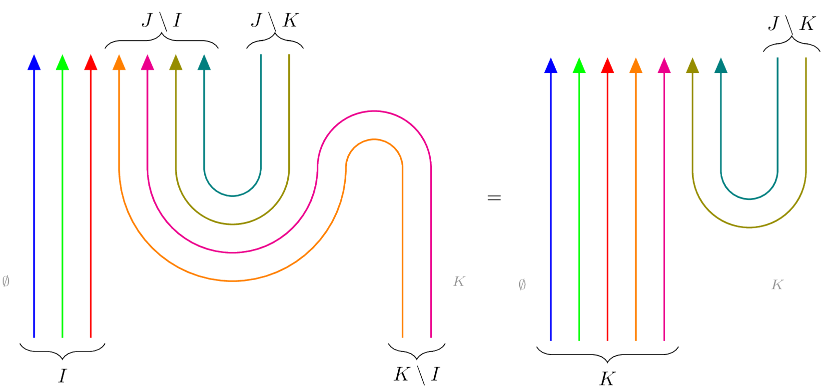
Proof. We have

$$z_1 \cdot f_{\frown}(z_2) = f_{\frown}(z_1 z_2) = f_{\frown}(f_{\circlearrowleft}(z_1) z_2) \text{ for all } z_1 \in R^{I_h} \text{ and } z_2 \in R^K.$$

and so $x \cdot \overline{G}(y) = \overline{G}(G(x) \cdot y)$ for any $x \in BS(\vec{I}, \vec{J})$ and $y \in BS(\vec{I''}, \vec{J''})$. Therefore, it is enough to show that for every $b \in BS(\vec{I''}, \vec{J''})$ we have $Tr(b) = Tr(\overline{G}(b))$.

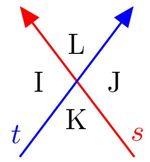
Let $b \in BS(\vec{I''}, \vec{J''})$. We can write $b = g c_{top}(\vec{I''}, \vec{J''}) + b'$, with $g \in R$ and $b' \in LT(BS(\vec{I''}, \vec{J''}))$, so that $\mathrm{Tr}(b) = g$.

By degree reasons, we have $\overline{G}(b') \in LT(BS(\vec{I}, \vec{J}))$. To conclude we just need to show that $\overline{G}(c_{top}(\vec{I''}, \vec{J''})) = c_{top}(\vec{I}, \vec{J})$. This follows from the commutativity of the following diagram.



$\square$

Finally, it remains to consider the last building box:

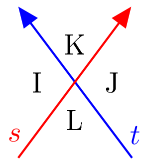


(5)

On bimodules $f_{\times}$ induces the canonical identification

$$R^I \otimes_{R^K} R^K \otimes_{R^J} R^J(\ell(J) - \ell(I)) \cong R^I \otimes_{R^L} R^L \otimes_{R^J} R^J(\ell(J) - \ell(I)).$$

By construction, $\mathrm{Id} \otimes f_{\times} \otimes \mathrm{Id}$ identifies the corresponding c_{top} elements. In particular, $\mathrm{Id} \otimes f_{\times} \otimes \mathrm{Id}$ is an isometry with respect to the corresponding intersection forms and if $f_{\underline{\times}}$ denotes the inverse morphism of $f_{\times}$ given by the following diagram



(6)

then $\mathrm{Id} \otimes f_{\underline{\times}} \otimes \mathrm{Id}$ is also the adjoint $\mathrm{Id} \otimes f_{\times} \otimes \mathrm{Id}$. Notice that the diagram in (6) is the flip of (5).

We can condense the results of this section in the following proposition.

Proposition 1.12. *Let $f : BS(\vec{I}, \vec{J}) \to BS(\vec{I'}, \vec{J'})$ be a morphism. Then the flipped morphism $\overline{f} : BS(\vec{I'}, \vec{J'}) \to BS(\vec{I}, \vec{J})$ is the adjoint of f with respect to the corresponding intersection forms.*

1.2 Reduced translation pairs

Reduced translating sequences [Wil08, Definition 1.3.1] are the analogue of reduced expression for double cosets of Coxeter groups. Since we are only interested in one-sided singular Soergel bimodules, we give a simpler definition which is valid in our setting.

Definition 1.13. Let $(\vec{I}, \vec{J})$ be a translation pair with $\vec{I} = (\emptyset, I_1, I_2, \ldots, I_k)$. Let $v_1 = w_{J_1} w_{I_1}$ and $v_h = v_{h-1} w_{J_h} w_{I_h}$ for every $h \leq k$. We call v_k the *end-point* of $(\vec{I}, \vec{J})$.

We say that $(\vec{I}, \vec{J})$ is *reduced* if for every $h < k$ we have

$$\ell(v_h w_{J_{h+1}}) = \ell(v_h) + \ell(w_{J_{h+1}}).$$

If $(\vec{I}, \vec{J})$ is a reduced translation pair with end-point $w \in W^I$, then B^I_w is a direct summand of $BS(\vec{I}, \vec{J})$ with multiplicity 1.

Recall the embedding of $BS(\vec{I}, \vec{J})$ in $BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}})_I$ from (3). If $(\vec{I}, \vec{J})$ is a reduced translation pair, then $\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}}$ is a reduced word. Combining Lemma 1.7 and Lemma 1.5 we have for every $\rho \in (h^*)^I$

$$1 \otimes \rho^{\ell(\vec{I}, \vec{J})} = \partial_w(\rho^{\ell(\vec{I}, \vec{J})}) c_{top}(\vec{I}, \vec{J}) + \text{lower degree terms}$$

and

$$\langle 1^{\otimes}, 1^{\otimes} \cdot \rho^{\ell(\vec{I}, \vec{J})} \rangle = \partial_w(\rho^{\ell(\vec{I}, \vec{J})}).$$

Lemma 1.14. *Let $(\vec{I},\vec{J})$ and $(\vec{I'},\vec{J'})$ be two reduced translation pairs both having the same end-point $w \in W^I$. Let $\varphi : BS(\vec{I},\vec{J}) \to BS(\vec{I'},\vec{J'})$ be a morphism such that $\varphi(1^{\otimes}) = 1^{\otimes}$. Then also $\overline{\varphi}(1^{\otimes}) = 1^{\otimes}$.*

Proof. Since $\varphi(1^{\otimes}) = 1^{\otimes}$, the morphism φ must be of degree 0, and so does $\overline{\varphi}$. Then $\overline{\varphi}(1^{\otimes}) = c1^{\otimes}$ for some scalar c. Let $\rho \in (\mathfrak{h}^*)^I$. Then we have

$$\partial_w(\rho^k) = \langle \varphi(1^{\otimes}), 1^{\otimes}\rho^k\rangle_{(\vec{I'},\vec{J'})} = \langle 1^{\otimes}\rho^k, \overline{\varphi}(1^{\otimes})\rangle_{(\vec{I},\vec{J})} = c\partial_w(\rho^k).$$

We can choose ρ to be ample. Since $\partial_w(\rho^k) > 0$, we obtain $c = 1$. □

Corollary 1.15. *Let $(\vec{I},\vec{J})$, $(\vec{I'},\vec{J'})$ and $\varphi : BS(\vec{I},\vec{J}) \to BS(\vec{I'},\vec{J'})$ be as above. Then $\varphi(c_{top}(\vec{I},\vec{J})) = c_{top}(\vec{I'},\vec{J'})$ up to lower degree terms.*

Proof. We have $c_{top}(\vec{I},\vec{J}) = \psi(1)$ where $\psi := (f_{\smile} \otimes \mathrm{Id}) \circ (f_{\smile} \otimes \mathrm{Id}) \circ \ldots : R_I \to BS(\vec{I},\vec{J})$, and similarly $c_{top}(\vec{I'},\vec{J'}) = \psi'(1)$ with $\psi' : R_I \to BS(\vec{I'},\vec{J'})$. The morphisms $\varphi \circ \psi$ and ψ' are both of degree $\ell(\vec{I},\vec{J})$, so they coincide up to scalar and up to lower degree terms.

However, after taking the flip, we have

$$\overline{\psi} \circ \overline{\varphi}(1^{\otimes}) = 1 = \overline{\psi'}(1^{\otimes}),$$

hence $\varphi \circ \psi = \psi'$ up to lower degree terms. □

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