Bases of the Intersection Cohomology of Grassmannian Schubert Varieties

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Abstract

In [SZJ12] the parabolic Kazhdan-Lusztig polynomials for Grassmannians are computed by counting certain Dyck partitions. We "lift" this combinatorial formula to the intersection cohomology of Schubert varieties in Grassmannians and we obtain many bases of the intersection cohomology which extend (after dualizing) the classical Schubert basis of the ordinary cohomology.

The cohomology of the Grassmannians has been studied since the late 19th century. The original motivation was given by Schubert calculus: it turns out that many basis questions in enumerative geometry (for example: how many lines intersect four given lines in a 3 dimensional space?) can be approached via some computations in the cohomology ring.

We denote by $\mathrm{Gr}(i,n)$ the Grassmannians of vector spaces of dimension i inside $\mathbb{C}^n$. The Schubert varieties $X_\lambda \subset \mathrm{Gr}(i,n)$ are parameterized by piece-wise linear paths λ in $\mathbb{R}^2$ from $(0,i)$ to $(n,n-i)$ which can be obtained joining segments of the form $(x,y)-(x+1,y+1)$ and $(x,y)-(x+1,y-1)$. From the Schubert varieties we get a cell decomposition of $\mathrm{Gr}(i,n)$ and, by taking their characteristic classes, we obtain a distinguished basis of the cohomology ring $H^\bullet(\mathrm{Gr}(i,n),\mathbb{Q})$, called the Schubert basis. Classical results such as Pieri's formula, which describes how to multiply the class of a Schubert variety with the Chern class of a line bundle, and the more general Littlewood-Richardson rule, which describes how to multiply two arbitrary Schubert classes, allow us to fully understand the ring structure of $H^\bullet(\mathrm{Gr}(i,n),\mathbb{Q})$ and $H^\bullet(X_\lambda,\mathbb{Q})$.

However, for applications in representation theory, when working with Schubert varieties it is often more natural to study instead their intersection cohomology $IH^\bullet(X_\lambda,\mathbb{Q})$. The Schubert varieties are in general singular and the ordinary singular cohomology embeds as a submodule in the intersection cohomology. It is then natural to ask whether (and how) one can extend the Schubert basis in a natural way to $IH^\bullet(X_\lambda,\mathbb{Q})$, that is if one could find a distinguished basis for $IH^\bullet(X_\lambda,\mathbb{Q})$, for a Schubert variety X_λ. The study of bases of the intersection cohomology of Schubert varieties of a Grassmannian is the main subject of the present paper.

The dimension of the intersection cohomology of Schubert varieties can be computed in terms of the Kazhdan-Lusztig polynomials. As usually, Kazhdan-Lusztig polynomials can be computed via a recursive formula. However, Schubert varieties in Grassmannians are very special among Schubert varieties in that they all admit small resolution of singularities [Zel83]. At the level of Kazhdan-Lusztig polynomials this is reflected in the existence of combinatorial non-recursive formulas to compute them.

Originally, Lascoux and Schützenberger describe a combinatorics for these polynomials involving "binary trees" [LS81]. More recently, Shigechi and Zinn-Justin gave an equivalent formulation of this combinatorics, this time involving Dyck partitions [SZJ12]. A major advantage of using Dyck partitions is that in this setting is also possible to describe formulas for the inverse Kazhdan-Lusztig polynomials (this was originally shown by Brenti [Bre02]).

If λ and μ are paths with $\lambda \leq \mu$ (i.e. λ lies completely below μ), a Dyck partition between λ and μ is a partition of the region between λ and μ into Dyck strips as in the following figure.

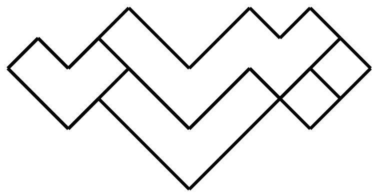

Figure 1: An example of a Dyck partition

By counting the number of Dyck partition satisfying some specific condition (see Definition 2.7) we can recover the Kazhdan-Lusztig polynomial $h_{\lambda,\mu}(v)$. More precisely, the coefficient of v^k in $h_{\lambda,\mu}(v)$ is equal to the number of Dyck partitions with k strips.

The main goal of the present paper is to "lift" this combinatorics to the equivariant intersection cohomology of Schubert varieties, obtaining a basis of the intersection cohomology parameterized by Dyck partitions.

It is very convenient to reinterpret equivariant intersection cohomology of Schubert varieties as indecomposable singular Soergel bimodules as this allows us to have the rich technology of Soergel bimodules at disposal. For example, morphisms between singular Soergel bimodules can be depicted using diagrams and the dimension of the spaces morphisms can be computed using the corresponding Hecke algebra. Also the inverse Kazhdan-Lusztig polynomial naturally occur in the setting of singular Soergel bimodules: they can be extrapolated by looking at the direct summands occurring in singular Rouquier complexes [Pat19].

The basic idea is to reinterpret any Dyck strip D as morphisms of degree one (denoted by f_D) between the corresponding singular Soergel bimodules. Because the space of degree one morphisms between these indecomposable singular Soergel bimodules is one-dimensional the morphism f_D is actually uniquely determined up to a scalar. (In §2.3 we will fix a precise choice for the morphisms f_D.)

For an arbitrary Dyck partition $\mathbf{P} = \{D_1, D_2, \dots, D_k\}$ we can consider the morphism $f_{\mathbf{P}} = f_{D_1} \circ f_{D_2} \circ \dots \circ f_{D_k}$. This is however not well defined: in general, different orders of the elements in $\mathbf{P}$ (i.e. different orders for the composition of the morphisms f_{D_i}) may lead to different morphisms.

A crucial technical point is to define a partial order $\succ$ on the set of Dyck partitions. We show that with respect to this partial order the morphism $f_{\mathbf{P}}$ is well defined, up to a scalar and up to smaller morphisms in the partial order. As a corollary, we conclude that after fixing arbitrarily for any Dyck partition an order of its strips (the order must be admissible, see Definition 2.25) the set $\{f_{\mathbf{P}}\}$ gives us a basis of the morphisms between singular Soergel bimodules. Hence, by evaluating this morphisms on the unity of the cohomology ring, we also get bases of the (equivariant) intersection cohomology of Schubert variety.

Unfortunately, the basis we obtain is highly not canonical as there does not seem to exist any distinguished way of choosing the order of the strips in the Dyck partitions. If the Schubert variety is very simple, e.g. the corresponding partition has only two rows, we do get a canonical bases.

Acknowledgments

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1 Singular Soergel Bimodules

We first recall some notation about Coxeter groups and their Hecke algebra from [EW14] and [Wil11]. Let (W, S) be a Coxeter group with Bruhat order $\geq$ and length function ℓ.

Let $\mathcal{H}$ be the Hecke algebra of W. The Hecke algebra has a basis $\{\mathbf{H}_x\}_{x\in W}$ as a $\mathbb{Z}[v, v^{-1}]$-module, called the standard basis of $\mathcal{H}$. We denote by $\{\underline{\mathbf{H}}_x\}_{x\in W}$ the Kazhdan-Lusztig basis.

If $I \subset S$, we denote by W_I the subgroup of W generated by I. We say that I is finitary if W_I is finite. In this case we denote by w_I its longest element and by $\ell(I)$ the length of w_I. For I finitary let $\underline{\mathbf{H}}_I := \underline{\mathbf{H}}_{w_I}$. We denote by W^I the set of minimal coset representatives in W/W_I, i.e. $W^I = \{x \in W \mid xs > x \text{ for all } s \in I\}$.

We define $\mathcal{H}^I = \mathcal{H}\underline{\mathbf{H}}_I$. This is a left ideal of $\mathcal{H}$. For $x \in W^I$ we define $\mathbf{H}^I_x = \mathbf{H}_x\underline{\mathbf{H}}_I$ and $\underline{\mathbf{H}}^I_x = \underline{\mathbf{H}}_{xw_I}$. Both $\{\mathbf{H}^I_x\}_{x\in W^I}$ and $\{\underline{\mathbf{H}}^I_x\}_{x\in W^I}$ are $\mathbb{Z}[v, v^{-1}]$-basis of $\mathcal{H}^I$, called respectively the *I-standard* and the *I-parabolic Kazhdan-Lusztig basis*. The *I-parabolic Kazhdan-Lusztig polynomials* $h^I_{x,y}(v) \in v\mathbb{N}[v]$ are the coefficient of the change-of-basis matrix between these two bases, namely

$$\underline{\mathbf{H}}^I_x = \mathbf{H}^I_x + \sum_{W^I \ni y < x} h^I_{x,y}(v)\mathbf{H}^I_y.$$

The inverse *I-parabolic Kazhdan-Lusztig polynomials* are the coefficients of the inverse matrix, i.e. we have

$$\sum_{y\in W^I} (-1)^{\ell(y)-\ell(x)} g^I_{x,y}(v) h^I_{y,z}(v) = \delta_{x,z}.$$

In Section §2 we describe some morphisms between singular Soergel bimodules using the diagrammatics of [ESW17]. For this reason, even if our focus is mainly on *one-sided* singular Soergel bimodules, we need to recall some notation and a few facts about *two-sided* singular Soergel bimodules.

For simplicity, we work in the same setting as in [EW14]. Let $\mathfrak{h}^*$ be the reflection faithful representation of W over $\mathbb{R}$ defined in [Soe07, Proposition 2.1]. There are linearly independent sets $\{\alpha_s\}_{s\in S} \subseteq \mathfrak{h}^*$ and $\{\alpha_s^\vee\}_{s\in S} \subseteq \mathfrak{h}$ such that for any $s \in S$ we have

$$s(v) = v - \alpha_s^\vee(v)\alpha_s \text{ for all } v \in \mathfrak{h}^*.$$

Remark 1.1. If W is a crystallographic Coxeter group there are other very natural choice for $\mathfrak{h}^*$ coming from the theory of Kac-Moody groups (see [Ric17, Proposition 1.1]) for which the results in [EW14] (and in this section) remain valid.

Moreover, if $\mathfrak{h}^*$ is defined to a subfield $\mathbb{K} \subset \mathbb{R}$, we can work over $\mathbb{K}$ instead of $\mathbb{R}$.

Let $R = \mathrm{Sym}^\bullet(\mathfrak{h}^*)$ be the symmetric algebra of $\mathfrak{h}^*$. We regard it as a graded algebra with $\deg(\mathfrak{h}^*) = 2$. If I is finitary we define R^I to be the subalgebra of W_I-invariants in R. We denote by (1) the grading shift.

For a simple reflection $s \in S$ the Demazure operator $\partial_s : R \to R(-2)$ is defined as

$$\partial_s(f) = \frac{f - s(f)}{\alpha_s}.$$

Note that if f is of degree 2 then $\partial_s(f) = \alpha_s^\vee(f)$. The Demazure operators satisfy the braid relation, hence for any $x \in W$ with reduced expression $\underline{x} = s_1 s_2 \dots s_k$ we can define $\partial_x := \partial_{s_1}\partial_{s_2}\dots\partial_{s_k} : R \to R(-2\ell(x))$.

If $J \subseteq I$ then the ring R^J is a Frobenius extension of R^I, i.e. R^J is free and finitely generated as a R^I-module there is a non-degenerate morphism of R^I-modules, namely $\partial_{w_J w_I} : R^J \to R^I$. Here non-degenerate means that there exists bases $\{x_i\}$ and $\{y_i\}$ of R^J over R^I such that $\partial_{w_J w_I}(x_i y_j) = \delta_{ij}$. The comultiplication $R^J \to R^J \otimes_{R^I} R^J$ is the map of R^J-bimodules which sends 1 to $\Delta_I^J := \sum x_i \otimes y_i$. This does not depend on the choice of dual bases.

Definition 1.2. Let $\vec{I} = (I_0, \dots, I_k)$ and $\vec{J} = (J_1, \dots, J_k)$ be sequences of finitary subsets of S. We say that $(\vec{I}, \vec{J})$ is a *translation pair* if for any h we have $I_{h-1} \subseteq J_h \supseteq I_h$.

If $(\vec{I}, \vec{J})$ is a translation pair let

$$\ell(\vec{I}, \vec{J}) := \sum_{i=1}^{k} \ell(J_i) - \ell(I_i).$$

The *generalized Bott-Samelson bimodule* $BS(\vec{I}, \vec{J})$ is the graded (R^{I_0}, R^{I_k})-bimodule

$$BS(\vec{I}, \vec{J}) = R^{I_0} \otimes_{R_{J_0}} R^{I_1} \otimes_{R^{J_1}} \dots \otimes_{R^{J_k}} R^{I_k}(\ell(\vec{I}, \vec{J})).$$

Definition 1.3. Let J, I be finitary subsets of S. We defined the category ${}^J\mathbb{S}Bim^I$ of (J, I)-singular Soergel bimodules as the full subcategory of (R^J, R^I)-bimodules whose objects are direct sums of direct summands of shifts of generalized Bott-Samelson bimodules of the form $BS(\vec{I}, \vec{J})$, where $(\vec{I}, \vec{J})$ is a translation pair with $I_0 = J$ and $I_k = I$.

If $J = \emptyset$ we call ${}^\emptyset\mathbb{S}Bim^I$ the category of I-singular Soergel bimodules, and we denote it by $\mathbb{S}Bim^I$

There is a duality functor $\mathbb{D}$ on ${}^J\mathbb{S}Bim^I$ defined by $B \mapsto \mathrm{Hom}^\bullet_{R^J\_}(B, R^J)$, where here $\mathrm{Hom}^\bullet_{R^J\_}(-,-)$ denotes the space of morphisms of left R^J-modules. The indecomposable self-dual elements in ${}^J\mathbb{S}Bim^I$ are in bijection with the double cosets in $W_J\backslash W/W_I$, and we denote by ${}^JB_x^I$ the indecomposable self-dual bimodule corresponding to $x \in W_J\backslash W/W_I$. There exists an isomorphism of $\mathbb{Z}[v, v^{-1}]$-modules

$$\mathrm{ch} : [{}^J\mathbb{S}Bim^I] \to {}^J\mathcal{H}^I := \underline{\mathbf{H}}_J\mathcal{H} \cap \mathcal{H}\underline{\mathbf{H}}_I.$$

If $x \in W_J\backslash W/W_I$ is a double coset, we denote by x_+ and x_- resp. the longest and the shortest elements in x.

Theorem 1.4 (Soergel's conjecture for singular Soergel bimodules [EW14, Wil11])**.** *For any $x \in W_J\backslash W/W_I$ we have $\mathrm{ch}({}^JB_x^I) = {}^J\underline{\mathbf{H}}_x^I = \underline{\mathbf{H}}_{x_+}$*

Proposition 1.5 ([Wil11, Prop. 7.4.3])**.** *Let $x \in W_J\backslash W/W_I$ and let $w \in W^I$ such that the coset wW_I is the unique maximal W_I-coset contained in x. Then*

$$R \otimes_{R^J} {}^JB_x^I \cong B_w^I(\ell(x_-) - \ell(w)).$$

When w is as in Proposition 1.5 then ww_I is the maximal element in the coset W_JwW_I, hence

$$\underline{\mathbf{H}}_w^I = \underline{\mathbf{H}}_{ww_I} \in {}^J\mathcal{H}^I.$$

Going in the opposite direction, for $w \in W^I$ we define $S_w = \{s \in S \mid sw \leq w \text{ in } W/W_I\}$. (When W is a Weyl group. then the parabolic subgroup W_{S_w} is the stabilizer of the Schubert variety corresponding to w.) We have

$$B_w^I = R \otimes_{R^J} {}^JB_w^I \qquad \text{for any } J \subseteq S_w. \tag{1}$$

Theorem 1.6 (Soergel's Hom Formula for Singular Soergel Bimodules [Wil11, Theorem 7.4.1])**.** *Let $B_1, B_2 \in {}^J\mathbb{S}Bim^I$. Then $\mathrm{Hom}^\bullet(B_1, B_2)$ is a free graded left R^J-module and we have an isomorphism*

$$\mathrm{Hom}^\bullet(B_1, B_2) = \langle \overline{\mathrm{ch}(B_1)}, \mathrm{ch}(B_2)\rangle \cdot R^J,$$

of graded R^J-modules. where $\langle -, - \rangle$ is the pairing in the Hecke algebra defined in [Wil11, §2.3].

We can bundle together all the categories of (J, I)-singular Soergel bimodules into a 2-category.

Definition 1.7. The 2*-category of singular Soergel bimodules* $\mathcal{S}\mathbb{S}Bim$ is the 2-category whose objects are the finitary subsets of S and such that, given two finitary subsets $J, I \subseteq S$, we have

$$\mathrm{Hom}_{\mathcal{S}\mathbb{S}Bim}(J, I) = {}^J\mathbb{S}Bim^I.$$

1.1 Diagrammatic singular Soergel bimodules

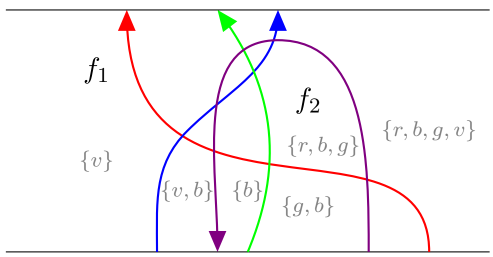
Definition 1.8. A *singular S-diagrams* is a finite collection of oriented 1-manifolds with boundary, colored by elements of S, embedded in the strip $\mathbb{R} \times [0, 1]$ and whose boundary is embedded in $\mathbb{R} \times \{0, 1\}$, the boundary of the strip.

Two 1-manifolds associated to colors $s \neq t$ can intersect only transversely and cannot intersect on the boundary of the strip. Moreover, two 1-manifolds associated to the same color cannot intersect at all.

We refer to connected components of the complement of the 1-manifolds in the strip as *regions*. Each region is labeled by a finitary subset $I \subseteq S$ compatibly with the following rule: if two finitary subsets are on two regions bordering the same 1-manifold labeled by s, then they differ by the single element $\{s\}$ with the one on the right side of the 1-manifold (while looking in the direction of its orientation) being bigger, i.e. if I is on the right side we have $I = J \sqcup \{s\}$.

Moreover, each region labeled by I can be decorated by a polynomial $f \in R^I$.

Example 1.9. Let $S = \{r, b, g, v\}$. The following is an example of a singular S-diagram with some of the regions labeled (in gray). In this example $f_1 \in R^v$ and $f_2 \in R^{\{r,g,b\}}$



To a singular S-diagram we associate a bottom sequence and a top sequence of finitary sets, respectively given by the sequence of finitary sets appearing on the lower and on the upper boundary of the strip.

Notice that the only sequences of finitary sets which can appear as bottom (or top) sequence of some S-diagram are those in which two consecutive sets in the sequence differ exactly by a single element of S. We call these sequences *strict translation sequence*.

Singular S-diagrams will be always considered up to isotopy. If the top sequence of a diagram $\mathcal{D}_1$ coincides with the bottom sequence of a diagram $\mathcal{D}_2$ then, up to an isotopy, we can stack $\mathcal{D}_2$ on top of $\mathcal{D}_1$ to obtain a new diagram which we denote by $\mathcal{D}_2 \circ \mathcal{D}_1$.

Similarly, if the last element of the bottom (and top) sequence of $\mathcal{D}_1$ coincides with the first element of the bottom sequence of $\mathcal{D}_2$ we can stack horizontally two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ to obtain a new diagram which we denote by $\mathcal{D}_1 \otimes \mathcal{D}_2$.

Each S-diagram can be obtained, up to isotopy, by stacking together the following "building boxes":

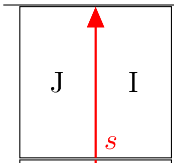
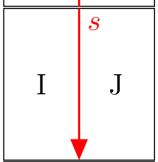
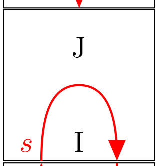
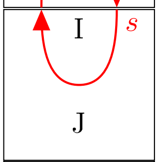
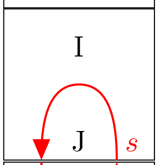
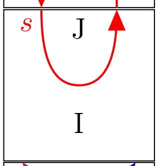
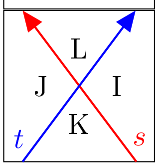
$\mathfrak{D}$	$\mathcal{S}\mathbb{S}Bim$	degree
	$R^J \otimes_{R^I} R^I(\eta) \xrightarrow{id} R^J \otimes_{R^I} R^I(\eta)$	0
	${}_I R^J \xrightarrow{id} {}_I R^J$	0
	$f_{\frown} : R^J \otimes_{R^I} R^J(\eta) \to R^J$ $f \otimes g \mapsto fg$	$\ell(I) - \ell(J)$
	$f_{\smile} : R^J \to R^J \otimes_{R^I} R^J(\eta)$ $1 \mapsto \Delta_I^J$	$\ell(I) - \ell(J)$
	$f_{\frown} : {}_I R_I^J(\eta) \to R^I$ $f \mapsto \partial_I^J(f)$	$\ell(J) - \ell(I)$
	$f_{\smile} : R^I \to {}_I R_I^J(\eta)$ $f \mapsto f$	$\ell(J) - \ell(I)$
	$f_{\times} : R^J \otimes_{R^K} R^K \otimes_{R^I} R^I(\eta) \to R^J \otimes_{R^L} R^L \otimes_{R^I} R^I(\eta)$ $f \otimes 1 \otimes 1 \mapsto f \otimes 1 \otimes 1$	0

Table 1: The building boxes

In every box I denotes the largest subset and J the smallest. The degree of the building boxes is defined as in the third column of Table 1. We define the degree of a diagram to be the sum of the degree of its building boxes.

Definition 1.10. We define the 2-category $\mathfrak{D}$ as follows: the objects are finitary subset, 1-morphisms between I and J are the strict translation sequences starting in I and ending

J, and 2-morphisms between two sequences K_1 and K_2 are isotopy classes of singular S-diagrams with bottom sequence K_1 and top sequence K_2 up to isotopy.

Now we define a 2-functor $\Xi : \mathfrak{D} \to \mathcal{S}\mathbb{S}Bim$. The functor Ξ is the identity on objects while on 1-morphisms sends a strict translation sequence to the corresponding induction or restriction bimodules, i.e. $\Xi(I) = R^I$ and

$$\Xi(I_1, \ldots, I_k, I_{k+1}) = \begin{cases} \Xi(I_1, \ldots, I_k) \otimes_{R^{I_k}} R^{I_{k+1}}(\ell(I_k) - \ell(I_{k+1})) & \text{if } I_{k+1} \subset I_k \\ \Xi(I_1, \ldots, I_k) \otimes_{R^{I_{k+1}}} R^{I_{k+1}} & \text{if } I_k \subset R^{I_{k+1}}. \end{cases}$$

To define Ξ on 2-morphisms it is enough to specifies the image of the generating 2-morphisms as in Table 1.

Theorem 1.11. *[ESW17] The 2-functor $\Xi : \mathfrak{D} \to \mathcal{S}\mathbb{S}Bim$ is well-defined and it is full on 2-morphisms.*

The functor Ξ is not an equivalence of 2-category, even after taking the Karoubi envelope of $\mathfrak{D}$. To achieve an equivalence, one would need to impose several relations to the category $\mathfrak{D}$. Some of these relations are general for any *hypercube* of Frobenius extensions and are discussed in [ESW17]. In (finite or affine) type A the full list of relations that we need to add to $\mathfrak{D}$ to obtain an equivalence with $\mathcal{S}\mathbb{S}Bim$ is described in [EL17] (a proof of this fact is therein announced, but has not yet been published). In this paper we will not actually need any of this as we will only make use of the diagrammatics as they provide an insightful and convenient way to draw and describe morphisms between singular Soergel bimodules.

The following definition is the diagrammatic counterpart of the duality functor $\mathbb{D}$.

Definition 1.12. Given a S-diagram $\mathcal{D}$ we define $\overline{\mathcal{D}}$ the S-diagram obtaining by reflecting $\mathcal{D}$ over the line $\{y = \frac{1}{2}\}$ and then changing the orientation of all the 1-manifolds.

We say that $\overline{\mathcal{D}}$ is the *flip (upside down)* of $\mathcal{D}$.

If f is a 2-morphism in $\mathcal{S}\mathbb{S}Bim$, there exists a S-diagram $\mathcal{D}$ such that $\Xi(\mathcal{D}) = f$. Then we can define the flip $\overline{f}$ of f as $\Xi(\overline{\mathcal{D}})$. This is well defined: if $\mathcal{D}'$ is another S-diagram such that $\Xi(\mathcal{D}') = f$, then we also have $\Xi(\overline{\mathcal{D}}) = \Xi(\overline{\mathcal{D}'})$.

1.2 Invariant forms on generalized Bott-Samelson bimodules

To better understand how the flip $\overline{f}$ of a morphism f behaves, we need to identify it with the adjoint of f with respect to certain invariant forms. In this perspective, we need to generalize the definition of intersection form from [EW14, §3.1] to generalized Bott-Samelson bimodules.

Let $B^I \in \mathbb{S}Bim^I$ be a I-singular Soergel bimodules. A left invariant bilinear form on B is a graded bilinear map

$$\langle -, - \rangle : B^I \times B^I \to R$$

such that for any $b, b' \in B^I$, $f \in R$ and $g \in R^I$ we have

$$\langle fb, b' \rangle = \langle b, fb' \rangle = f \langle b, b' \rangle$$

$$\langle b, b'g \rangle = \langle b, b'g \rangle.$$

Let $(\vec{I}, \vec{J})$ be a translation pair with $\vec{J} = (J_1, \ldots, J_{k-1}, J_k)$ and $\vec{I} = (I_0 = \emptyset, I_1, \ldots, I_{k-1}, I_k = I)$, i.e. we assume that $I_0 = \emptyset$ and $I_k = I$. Consider the translation pair $(\vec{I'}, \vec{J'})$ with

$\vec{J'} = (J_2, J_3, \dots, J_k)$ and $\vec{I'} = (\emptyset, I_2, I_3, \dots, I_k)$. The morphism $f_{\curvearrowleft} : R^{I_k} \to R^{I_k} \otimes_{R^{J_k}} R^{I_k}$ induces a morphism

$$f_{\curvearrowleft} \otimes \mathrm{Id} : BS(\vec{I'}, \vec{J'}) \to BS(\vec{I}, \vec{J}).$$

By induction on k we can define an element $\Delta(\vec{I}, \vec{J})$ as $(f_{\curvearrowleft} \otimes \mathrm{Id})(\Delta(\vec{I'}, \vec{J'}))$.

If $\underline{w} = s_1 s_2 \dots s_k$ is a word with $s_i \in S$, then the ordinary Bott-Samelson bimodule $BS(\underline{w})$ can be realized as a generalized Bott-Samelson bimodule $BS(\vec{I}, \vec{J})$, with $\vec{J} = (\{s_1\}, \{s_2\}, \dots \{s_k\})$ and $\vec{I} = (\emptyset, \emptyset, \dots, \emptyset)$. In this case, the corresponding element $\Delta(\underline{w}) := \Delta(\vec{I}, \vec{J}) \in BS(\underline{w})$ coincides with the element c_{top} defined in [EW14, §3.4].

Let $t \in W$ be a reflection. Then we can write $t = wsw^{-1}$ with $ws > w$ and $s \in S$. The root $\alpha_t := w(\alpha_s) \in \mathfrak{h}^*$ is well defined and it is positive. We can consider the operator

$$\partial_t^T : R \to R(-2)$$

defined by $\partial_t^T(f) = \frac{f - t(f)}{\alpha_t}$. (We use the non-standard notation ∂_t^T to distinguish it from ∂_t, as they only coincide for $t \in S$.) If $t = wsw^{-1}$ with $ws > w$ and $s \in S$, then $\partial_t^T(f) = w\left(\partial_s(w^{-1}(f))\right)$ for all $f \in R$. In particular, if $f \in R$ is of degree 2, then $\partial_t^T(f) = \partial_s(w^{-1}(f)) = \alpha_s^\vee(w^{-1}(f))$. We write $y \xrightarrow[R]{t} x$ if $t \in T$, $yt = x$ and $\ell(y) + 1 = \ell(x)$.

Lemma 1.13. *Let $x \in W$. Let $f, g \in R$ with f of degree 2 and g of degree $2\ell(x) - 2$. Then*

$$\partial_x(fg) = \sum_{y \xrightarrow[R]{t} x} \partial_t^T(f) \partial_y(g).$$

Proof. The proof is by induction on $\ell(x)$. The case $\ell(x) \leq 1$ is trivial. Let $x = ys$ with $s \in S$ and $x > y$. By induction we obtain

$$\partial_x(fg) = \partial_y \partial_s(fg) = \partial_y\left(g\partial_s(f) + s(f)\partial_s(g)\right) = \partial_s(f)\partial_y(g) + \sum_{z \xrightarrow[R]{t} y} \partial_t^T(s(f))\partial_z(\partial_s(g)).$$

If $zs < z$ the term $\partial_z(\partial_s(g))$ vanishes, while if $zs > z$ and $z \xrightarrow[R]{t} y$, then also $zs \xrightarrow[R]{sts} x$. In particular we have $s \neq t$, so $s(\alpha_t)$ is a positive root and $\partial_t^T(s(f)) = \partial_{sts}^T(f)$. The claim now follows. ☐

The right R^I-module $\overline{BS(\vec{I}, \vec{J})} := \mathbb{R} \otimes_R BS(\vec{I}, \vec{J})$ is one-dimensional in degree

$$\ell(\vec{I}, \vec{J}) := \sum_{i=1}^k \ell(J_i) - \ell(I_i).$$

Obvioulsy, in the bimodule $BS(\vec{I}, \vec{J})$ the dimension of the vector space of elements of degree $\ell(\vec{I}, \vec{J})$ is much bigger. For this reason, we need to give the following definition.

Definition 1.14. We define $LT(BS(\vec{I}, \vec{J}))$ to be the left R-module generated by elements of degree less than $\ell(\vec{I}, \vec{J})$. In this section, we call *lower terms* the elements of $LT(BS(\vec{I}, \vec{J}))$.

Let $(\mathfrak{h}^*)^I$ be the subspace of W_I-invariant elements in $\mathfrak{h}^*$ (or, equivalently, $(\mathfrak{h}^*)^I$ is the subspace of elements of degree 2 in R^I). We say that $\rho \in \mathfrak{h}^*$ is ample if $\partial_s(\rho) > 0$ for every $s \in S \setminus I$. Since the simple coroots $\alpha_s^\vee$ are linearly independent there always exist a ample ρ.

Lemma 1.15. *Let $\rho \in (\mathfrak{h}^*)^I$ be ample. Let $w \in W^I$ and $w = s_1 s_2 \dots s_l$ be a reduced expression for w. Then for any $i \leq l$ we have*

$$\partial_{s_i}(s_{i+1}s_{i+2}\dots s_l(\rho)) > 0$$

Proof. See [Pat18, Lemma 4.2.1] □

With this definition, every two elements of $BS(\vec{I},\vec{J})$ of degree $\ell(\vec{I},\vec{J})$ are multiple of each other up to lower terms.

We say that a word $\underline{w}$ is I-reduced if $\underline{w}$ is a reduced word for $w \in W^I$. If $B \in \mathbb{S}Bim^J$ and $J \subset I$ we denote by $B_I = B \otimes_{R^I} R^I \in \mathbb{S}Bim^I$ its restriction to a (R, R^I)-bimodule.

Lemma 1.16. *Let $\underline{w}$ be a word, $\rho \in R^I$ of degree 2 and consider $1^\otimes := 1 \otimes 1 \otimes \dots \otimes 1 \in BS(\underline{w})_I$. Then $1^\otimes \cdot \rho^{\ell(\underline{w})} = Nc_{top}$ up to lower terms. If $\underline{w}$ is not I-reduced then $N = 0$ while if $\underline{w}$ is a I-reduced word for w then $N = \partial_w(\rho^k)$.*

If, moreover, ρ is ample then $N > 0$.

Proof. If $\underline{w}$ is not I-reduced then

$$BS(\underline{w})_I = \bigoplus_z B_z(m) \tag{2}$$

and all the elements z occurring in the RHS in (2) satisfy $\ell(z) < k$. Then, by degree reasons the coefficient of c_{top} in $1^\otimes \cdot \rho^k$ has to be 0.

Assume now that $\underline{w}$ is I-reduced. Let $\underline{w} = s_1 s_2 \dots s_k$ and let $\underline{w_i} = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_k$. Let $v_i = s_i s_{i+1} \dots s_k$. As in the proof of [EW14, Lemma 3.10] one can see that

$$1^\otimes \cdot \rho^k = Nc_{top} + \text{ lower terms}$$

with $N = \sum_i \partial_{s_i}(v_{i+1}(\rho))N_i$, where N_i is the coefficient of c_{top} in $1^\otimes \cdot \rho^{k-1} \in BS(\underline{w_i})$. By induction we have

$$N = \sum_{i\,:\,\underline{w_i}\ I\text{-reduced}} \partial_{s_i}(v_{i+1}(\rho))\partial_{w_i}(\rho^{k-1}) = \sum_{y \xrightarrow[R]{t} w} \partial_t(\rho)\partial_y(\rho^{k-1})$$

Finally, applying Lemma 1.13 we get $N = \partial_x(\rho^k)$.

If ρ is ample, then by induction for any i such that $\underline{w_i}$ is I-reduced we have $N_i > 0$ and also $\partial_{s_i}(v_{i+1}(\rho)) > 0$ by Lemma 1.15. Hence $N > 0$. □

Let $I \subset J$ be finitary subsets and fix a reduced expression $\underline{w_J w_I}$ for $w_J w_I$. Then $R \otimes_{R^J} R^I$ is a direct summand of $BS(\underline{w_J w_I})_I$ and we have a natural embedding of (R, R^I)-bimodules

$$\vartheta : R \otimes_{R^J} R^I(\ell(J) - \ell(I)) \to BS(\underline{w_J w_I})_I$$

induced by sending $1^\otimes \in R \otimes_{R^J} R^I$ to $1^\otimes \in BS(\underline{w_J w_I})_I$.

Lemma 1.17. *The embedding $\vartheta : R \otimes_{R^J} R^I \to BS(\underline{w_J w_I})_I$ sends $\Delta((I),(J))$ to c_{top}, up to lower terms.*

Proof. Since ϑ sends $1^\otimes$ to $1^\otimes$, it also sends $1^\otimes \cdot \rho^{\ell(w_J w_I)}$ to $1^\otimes \cdot \rho^{\ell(w_J w_I)}$ for every $\rho \in (\mathfrak{h}^*)^I$. Fix ρ ample so that $\partial_{w_J w_I}(\rho^{\ell(w_J w_I)}) \neq 0$. Then it suffices to show that

$$1 \otimes \rho^{\ell(w_J w_I)} = \partial_{w_J w_I}(\rho^{\ell(w_J w_I)})\Delta((I),(J)) + \text{lower terms}$$

By definition $\Delta((I),(J)) = \sum_i x_i \otimes y_i$ where $\{x_i\}$ and $\{y_i\}$ are bases of R^I over R^J with $\partial_{w_I w_J}(x_i \cdot y_i) = 1$. There exists a unique index i_0 such that x_{i_0} is of degree 0 and y_{i_0} is of degree $2\ell(w_J w_I)$. Then $\Delta((I),(J))$ is equal to $x_{i_0} \otimes y_{i_0} = 1 \otimes x_{i_0} y_{i_0}$ up to lower terms. Furthermore, we also have $1 \otimes \rho^{\ell(w_J w_I)} = \partial_{w_J w_I}(\rho^{\ell(w_J w_I)}) \cdot 1 \otimes x_{i_0} y_{i_0}$ up to lower terms and the claim follows. $\square$

Reiterating, we get a chain of embeddings

$$\begin{aligned} BS(\vec{I},\vec{J}) = R \otimes_{R^{J_1}} R^{I_1} \otimes_{R^{J_2}} \ldots \otimes_{R_{J_k}} R^{I_k}(\ell(\vec{I},\vec{J})) \hookrightarrow \\ \hookrightarrow BS(\underline{w_J w_I}) \otimes_{R^{J_2}} R^{I_2} \otimes_{R_{J_k}} R^{I_k}(\ell(\vec{I},\vec{J}) - \ell(w_J w_I)) \hookrightarrow \ldots \hookrightarrow \\ \hookrightarrow BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}}) \end{aligned} \tag{3}$$

Lemma 1.18. *The embedding in* (3) *sends* $\Delta(\vec{I},\vec{J})$ *to* $c_{top} \in BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}})$ *up to lower terms.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & BS(\vec{I},\vec{J}) \arrow[dr, hook, "\alpha"] & \\
BS(\vec{I'},\vec{J'}) \arrow[ur, "f_{\circlearrowleft} \otimes \mathrm{Id}"] \arrow[rr, "c_{top} \otimes \mathrm{Id}"] \arrow[d, hook, "\beta"] & & BS(\underline{w_{J_1} w_{I_1}}) \otimes_R BS(\vec{I'},\vec{J'}) \arrow[d, hook, "\mathrm{Id} \otimes \beta"] \\
BS(\underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}}) \arrow[rr, "c_{top} \otimes \mathrm{Id}"] & & BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}})
\end{tikzcd}$$

Notice that the lower square is commutative. The top triangle is "commutative up to lower terms," meaning that we have

$$\alpha(f_{\circlearrowleft} \otimes \mathrm{Id})\Delta(\vec{I'},\vec{J'}) = c_{top} \otimes \Delta(\vec{I'},\vec{J'})$$

up to lower terms (here we used the fact that $f c_{top} = c_{top} f$ for every $f \in R$). Therefore

$$(c_{top} \otimes \mathrm{Id}) \circ \beta(\Delta(\vec{I'},\vec{J'})) = (Id \otimes \beta) \circ \alpha \circ (f_{\circlearrowleft} \otimes \mathrm{Id})(\Delta(\vec{I'},\vec{J'})) = (Id \otimes \beta) \circ \alpha(\Delta(\vec{I},\vec{J}))$$

up to lower terms. We can conclude since by induction $\beta(\Delta(\vec{I'},\vec{J'})) = c_{top}$ up to lower terms. $\square$

Lemma 1.19. *The element* $\Delta(\vec{I},\vec{J}) \in BS(\vec{I},\vec{J})$ *together with* $LT(BS(\vec{I},\vec{J}))$ *generates* $BS(\vec{I},\vec{J})$ *as a left* R*-module.*

Proof. By looking at $\mathrm{grrk}(BS(\vec{I},\vec{J}))$ is it enough to show that $\Delta(\vec{I},\vec{J})$ does not belong to $LT(BS(\vec{I},\vec{J}))$. This is clear by looking at the embedding in (3), since the image of $\Delta(\vec{I},\vec{J})$ contains c_{top} with non-trivial coefficient, hence it is not contained in the submodule $LT(BS(\underline{w_{J_1} w_{I_1}}\ \underline{w_{J_2} w_{I_2}} \ldots \underline{w_{J_k} w_{I_k}}))$. $\square$

Therefore, we have a decomposition of left R-bimodules.

$$BS(\vec{I},\vec{J}) = R \cdot \Delta(\vec{I},\vec{J}) \oplus LT(BS(\vec{I},\vec{J})). \tag{4}$$

Definition 1.20. Let $\mathrm{Tr} : BS(\vec{I},\vec{J}) \to R$ be R-linear map which returns the coefficient of $\Delta(\vec{I},\vec{J})$ with respect to the decomposition (4). We denote by $\cdot$ the term-wise multiplication on $BS(\vec{I},\vec{J})$. We define an invariant form $\langle -,- \rangle_{(\vec{I},\vec{J})}$ on $BS(\vec{I},\vec{J})$, called the *intersection form*, by

$$\langle x,y \rangle_{(\vec{I},\vec{J})} = Tr(x \cdot y)$$

Remark 1.21. Let $(\vec{I},\vec{J})$ and $(\vec{I}',\vec{J}')$ be translation pairs with end-point $w \in W^I$. Assume that there exists an isomorphism $\varphi : BS(\vec{I},\vec{J}) \xrightarrow{\sim} BS(\vec{I}',\vec{J}')$ with $\varphi(1^{\otimes}) = 1^{\otimes}$. Then by Lemma 1.18 we also obtain that $\varphi(\Delta(\vec{I},\vec{J})) = \Delta(\vec{I}',\vec{J}')$ up to lower terms. In particular, the intersection form on $BS(\vec{I},\vec{J})$ does not depend on the choice of the reduced translation pair.

Moreover, if $(\vec{I},\vec{J})$ is I-reduced and $\rho \in (h^*)^I$ is ample we have

$$\langle 1^{\otimes}, 1^{\otimes} \cdot \rho^{\ell(\vec{I},\vec{J})} \rangle = \partial_w(\rho^{\ell(\vec{I},\vec{J})})$$

1.2.1 Adjoint and flipped maps

Our next goal is to show that taking the adjoint with respect of the intersection forms is the same as flipping the corresponding diagram. It is enough to show this for the "building boxes" in Figure 1.

We consider first the clockwise cup and its flip, the clockwise cap. Let

$$\vec{I}' = (\emptyset, I_1, \dots, I_{h-1}, I_h, I_h, I_{h+1}, \dots, I_k),$$

$$\vec{J}' = (J_1, \dots, J_{h-1}, K, J_h, J_{h+1}, \dots, J_k)$$

with $I_h \subset K$. Let

$$F := \mathrm{Id} \otimes f_{\circlearrowright\!\smile} \otimes \mathrm{Id} : BS(\vec{I},\vec{J}) \to BS(\vec{I}',\vec{J}')$$

be the morphism induced by $f_{\circlearrowright\!\smile} : R^{I_h} \to R^{I_h} \otimes_{R^K} R^{I_h}(\ell(K) - \ell(I_h))$ and let

$$\overline{F} := \mathrm{Id} \otimes f_{\frown} \otimes \mathrm{Id} : BS(\vec{I}',\vec{J}') \to BS(\vec{I},\vec{J})$$

be the morphism induced by $f_{\frown} : R^{I_h} \otimes_{R^K} R^{I_h}(\ell(K) - \ell(I_h)) \to R^{I_h}$. We

Lemma 1.22. *The maps F and $\overline{F}$ above are adjoint to each other with respect to the corresponding intersection forms, i.e.*

$$\langle F(x), y \rangle_{(\vec{I}',\vec{J}')} = \langle x, \overline{F}(y) \rangle_{(\vec{I},\vec{J})} \ \text{ for all } x \in BS(\vec{I},\vec{J}),\ y \in BS(\vec{I}',\vec{J}').$$

Proof. We have

$$f_{\circlearrowright\!\smile}(z) \cdot (y_1 \otimes y_2) = f_{\circlearrowright\!\smile}(1) z y_1 y_2 = f_{\circlearrowright\!\smile}(x \cdot f_{\frown}(y_1 \otimes y_2) \ \text{ for all } z, y_1, y_2 \in R^{I_h}$$

hence $F(x) \cdot y = F(x \cdot \overline{F}(y))$. Thus, to conclude that F and G are adjoint it is enough to show that, for all $x \in BS(\vec{I},\vec{J})$ we have $\mathrm{Tr}(F(x)) = \mathrm{Tr}(x)$.

Assume that $x = g\Delta(\vec{I},\vec{J}) + b$ with $g \in R$ and $b \in LT(BS(\vec{I},\vec{J}))$. By degree reasons, also $F(b) \in LT(BS(\vec{I}',\vec{J}'))$. We conclude that $\mathrm{Tr}(F(x)) = \mathrm{Tr}(x)$ by showing $F(\Delta(\vec{I},\vec{J})) = \Delta(\vec{I}',\vec{J}')$. This follows since the following diagram is commutative

$$
\begin{tikzcd}
R \otimes_{R^{J_h}} R^{I_h} \arrow[r, "\mathrm{Id} \otimes f_{\circlearrowleft}"] & R \otimes_{R^{J_h}} R^{I_h} \otimes_{R^K} R^{I_h} \\
R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^{J_h}} R^{I_h} \arrow[u, phantom, sloped, "\cong"] & R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^{J_h}} R^{I_h} \otimes_{R^K} R^{I} \arrow[u, phantom, sloped, "\cong"] \\
R \otimes_{R^{I_h}} R^{I_h} \arrow[u, "f_{\circlearrowleft}"] \arrow[r, "\mathrm{Id} \otimes f_{\circlearrowleft}"] & R \otimes_{R^{I_h}} R^{I_h} \otimes_{R^K} R^{I_h} \arrow[u, "\mathrm{Id} \otimes f_{\circlearrowleft} \otimes \mathrm{Id}"']
\end{tikzcd}
$$

as it can be easily checked, for example, by looking at the corresponding diagrams. $\square$

We consider next the counterclockwise cap and its flip, the counterclockwise cup. Let now $\vec{I''} = (\emptyset, I_1, \dots, I_{h-1}, K, I_{h+1}, \dots, I_k)$ and $\vec{J''} = (J_1, \dots, J_{h-1}, J_h, J_{h+1}, \dots, J_k)$ with $K \subset I_h$. Let

$$G = \mathrm{Id} \otimes f_{\circlearrowright} \otimes \mathrm{Id} : BS(\vec{I}, \vec{J}) \to BS(\vec{I''}, \vec{J''})$$

be the morphism induced by $f_{\circlearrowright} : R^{I_h} \to {}_{I_h}(R^K)_{I_h}(\ell(I_k) - \ell(K))$ and let

$$\overline{G} = \mathrm{Id} \otimes f_{\curvearrowleft} \otimes \mathrm{Id} : BS(\vec{I''}, \vec{J''}) \to BS(\vec{I}, \vec{J})$$

be the morphism induced by $f_{\curvearrowleft} : {}_{R^{I_h}}(R^K)_{R^{I_h}}(\ell(I_k) - \ell(K)) \to R^{I_h}$.

Lemma 1.23. *The maps G and $\overline{G}$ above are adjoint with respect to the corresponding intersection forms.*

Proof. We have

$$z_1 \cdot f_{\circlearrowleft}(z_2) = f_{\circlearrowleft}(z_1 z_2) = f_{\curvearrowleft}(f_{\circlearrowright}(z_1) z_2) \text{ for all } z_1 \in R^K \text{ and } z_2 \in R^{I_h}.$$

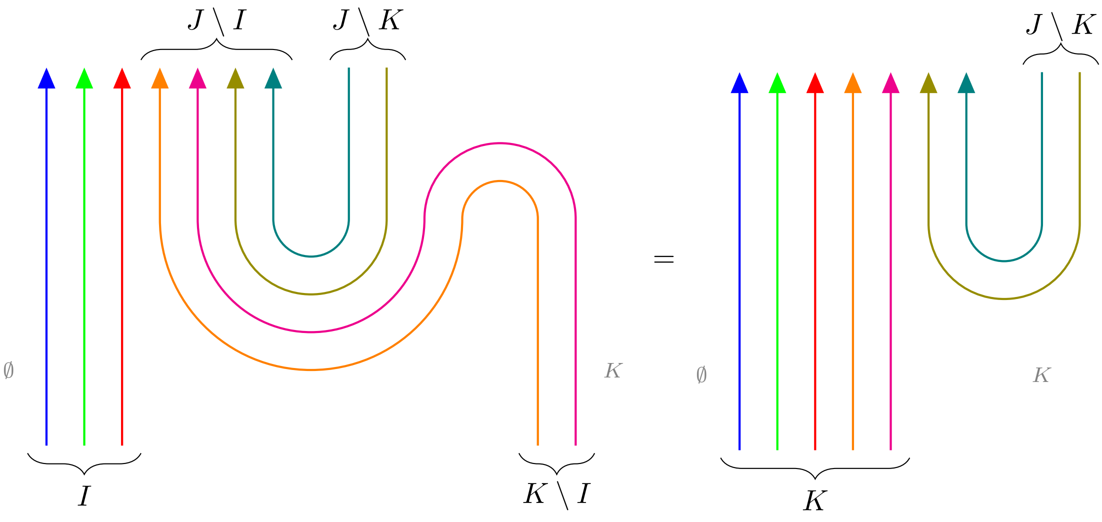
so $x \cdot \overline{G}(y) = \overline{G}(G(x) \cdot y)$. Therefore, it is enough to show that $Tr(x) = Tr(\overline{G}(x))$ for all $x \in BS(\vec{I''}, \vec{J''})$.

Let $x \in BS(\vec{I''}, \vec{J''})$. We can write $x = g\Delta(\vec{I''}, \vec{J''}) + b$, with $g \in R$ and $b \in LT(BS(\vec{I''}, \vec{J''}))$, so that $\mathrm{Tr}(x) = g$.

By degree reasons, we have $\overline{G}(b) \in LT(BS(\vec{I}, \vec{J}))$. To conclude we just need to show that $\overline{G}(\Delta(\vec{I''}, \vec{J''})) = \Delta(\vec{I}, \vec{J})$. This follows from the fact that the following diagram is commutative.

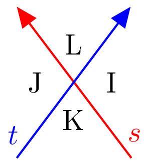
$$
\begin{tikzcd}
R \otimes_{R^J} R^I \arrow[r, "\mathrm{Id} \otimes f_{\curvearrowleft}"] & R \otimes_{R^J} R^K \\
R \otimes_{R^I} R^I \otimes_{R^J} R^I \arrow[u, phantom, sloped, "\cong"] & R \otimes_{R^K} R^K \otimes_{R^J} R^K \arrow[u, phantom, sloped, "\cong"] \\
R \otimes_{R^I} R^I \arrow[u, "\mathrm{Id} \otimes f_{\circlearrowleft}"] \arrow[r, "\sim"] & R \otimes_{R^K} R^K \arrow[u, "\mathrm{Id} \otimes f_{\circlearrowleft}"']
\end{tikzcd}
$$

which can be illustrated by the following simple diagram equality.



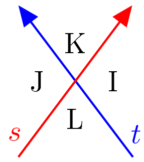
∎

Finally, we consider the last building box:



(5)

On bimodules $f_{\times}$ induces the canonical identification $R^J \otimes_{R^K} R^K \otimes_{R^I} R^I(\ell(I) - \ell(J)) \cong R^J \otimes_{R^L} R^L \otimes_{R^I} R^I(\ell(I) - \ell(J))$ and, by construction, $f_{\times}$ clearly identifies the corresponding Δ elements. In particular, $f_{\times}$ is an isometry with respect to the corresponding intersection forms and the inverse morphism of $f_{\times}$, which is given by the following diagram



(6)

is also its adjoint. Notice that the diagram in (6) is the flip of (5).

We just proved the following fact.

Proposition 1.24. *Let $f : BS(\vec{I}, \vec{J}) \to BS(\vec{I}', \vec{J}')$ be a morphism. Then the flipped morphism $\overline{f} : BS(\vec{I}', \vec{J}') \to BS(\vec{I}, \vec{J})$ is the adjoint of f with respect to the corresponding intersection forms.*

1.2.2 Reduced translation pairs

Reduced translating sequences [Wil08, Definition 1.3.1] are the analogue of reduced expression for double cosets of Coxeter groups. Since we are only interested in one-sided singular Soergel bimodules, we give an equivalent definition which is valid in our setting.

Definition 1.25. Let $(\vec{I}, \vec{J})$ be a translation pair with $\vec{I} = (\emptyset, I_1, I_2, \ldots, I_k)$. Let $v_1 = w_{J_1} w_{I_1}$ and $v_h = v_{h-1} w_{J_h} w_{I_h}$ for every $h \leq k$. We call v_k the *end-point* of $(\vec{I}, \vec{J})$.

We say that $(\vec{I}, \vec{J})$ is *reduced* if for every $h < k$ we have

$$\ell(v_h w_{J_{h+1}}) = \ell(v_h) + \ell(w_{J_{h+1}}).$$

If $(\vec{I},\vec{J})$ is a reduced translation pair with end-point $w \in W^I$, then B^I_w is a direct summand of $BS(\vec{I},\vec{J})$ with multiplicity 1. Note that $(B^I_w)^{-\ell(w)} = (BS(\vec{I},\vec{J}))^{-\ell(w)}$ and they are both spanned by $1^\otimes$.

Lemma 1.26. *Let $(\vec{I},\vec{J})$ and $(\vec{I}',\vec{J}')$ be two reduced translation pairs both having the same end-point $w \in W^I$. Let $\varphi : BS(\vec{I},\vec{J}) \to BS(\vec{I}',\vec{J}')$ be a morphism such that $\varphi(1^\otimes) = 1^\otimes$. Then also $\overline{\varphi}(1^\otimes) = 1^\otimes$.*

Proof. Since $\varphi(1^\otimes) = 1^\otimes$, then φ must be of degree 0, and so does $\overline{\varphi}$. Then $\overline{\varphi}(1^\otimes) = c1^\otimes$ for some scalar c. Let $\rho \in (\mathfrak{h}^*)^I$. Then we have (cf. Remark 1.21

$$\partial_w(\rho^k) = \langle \varphi(1^\otimes), 1^\otimes \rho^k \rangle_{(\vec{I}',\vec{J}')} = \langle 1^\otimes \rho^k, \overline{\varphi}(1^\otimes) \rangle_{(\vec{I},\vec{J})} = c\partial_w(\rho^k).$$

If ρ is ample, then $\partial_w(\rho^k) > 0$ and $c = 1$. □

2 Singular Soergel calculus for Grassmannians

In this chapter we specialize to the case of the complex Grassmannians.

From now on W will denote the symmetric group S_n and $\mathfrak{h}^*$ the $(n-1)$-dimensional geometric representation over $\mathbb{Q}$ from [Hum90], that is, $\mathfrak{h}^*$ is the $\mathbb{Q}$-vector space

$$\mathfrak{h}^* := \left\{ (x_i)_{i=1}^n \in \mathbb{Q}^n \mid \sum_{i=1}^n x_i = 0 \right\}$$

on which W acts by permuting the coordinates. The representation $\mathfrak{h}^*$ is reflection faithful and Soergel's conjecture holds for $\mathfrak{h}^*$.

We denote by $s_k \in W$ the simple transposition $(k\ \ k{+}1)$, so that $S = \{s_1, s_2, \dots, s_{n-1}\}$. We fix now once for all i with $1 \leq i \leq n-1$ and $I = S \setminus \{s_i\} = \{s_1, s_2, \dots s_{i-1}, s_{i+1}, \dots, s_{n-1}\}$ so that $W_I \cong S_i \times S_{n-i} \subseteq S_n$.

The Grassmannian $\mathrm{Gr}(i,n)$ is a smooth complex projective variety, isomorphic to $SL_n(\mathbb{C})/P$, where P is a maximal parabolic subgroup of $SL_n(\mathbb{C})$ containing the upper triangular matrices. Let $T \subseteq SL_n(\mathbb{C})$ be the maximal torus given by diagonal matrices. The T-equivariant cohomology of $Gr(i,n)$ (cf. [Bri98]) is

$$H^\bullet_T(\mathrm{Gr}(i,n), \mathbb{Q}) \cong R \otimes_{R^W} R^I.$$

Let $\mathcal{D}^b_T(\mathrm{Gr}(i,n), \mathbb{Q})$ denote the T-equivariant derived category of constructible sheaves of $\mathbb{Q}$-vector spaces on $\mathrm{Gr}(i,n)$. If $\mathcal{F} \in \mathcal{D}^b_T(\mathrm{Gr}(i,n))$, the hypercohomology $\mathbb{H}(\mathcal{F})$ is in a natural way a module over $H^\bullet_T(\mathrm{Gr}(i,n), \mathbb{Q})$, hence in particular a (R, R^I)-bimodule.

Let $\mathcal{K}$ be the full subcategory of $\mathcal{D}^b_T(\mathrm{Gr}(i,n))$ whose objects are direct sums of shifts of simple T-equivariant $\mathbb{Q}$-perverse sheaves on $\mathrm{Gr}(i,n)$. Then the hypercohomology induces an equivalence $\mathbb{H} : \mathcal{K} \to \mathbb{S}Bim^I$ [Soe90, Gin91]. As a corollary, if $X_w \subseteq \mathrm{Gr}(i,n)$ is the Schubert Schubert variety associated to $w \in W^I$ we have $IH^\bullet_T(X_w, \mathbb{Q}) \cong B^I_w$.

Consider the following set of n-tuples:

$$\Lambda_{n,i} := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \{\nearrow, \searrow\}^n \mid \#\{k \mid \lambda_k = \searrow\} = i\}$$

The natural action of S_n induces a bijection

$$W^I \longleftrightarrow \Lambda_{n,i}$$

$$w \mapsto \lambda^w = (\lambda^w_1, \lambda^w_2, \dots, \lambda^w_n) := w(\searrow, \searrow, \dots, \searrow, \nearrow, \nearrow, \dots, \nearrow)$$

Definition 2.1. By *path* we mean a piecewise linear path in $\mathbb{R}^2$ which is union of segments of the form $(x,y) \to (x+1,y+1)$ or $(x,y) \to (x+1,y-1)$, for $x,y \in \mathbb{Z}$. Every element $\lambda \in \Lambda_{n,i}$ can be thought as a path from $(0,i)$ to $(n,n-i)$, where to every $\nearrow$ in λ it corresponds a segment $(x,y) \to (x+1,y+1)$ and to every $\searrow$ corresponds a segment $(x,y) \to (x+1,y-1)$. Hence, we can identify $\Lambda_{n,i}$ with the set of paths from $(0,i)$ to $(n,n-i)$. By a slight abuse of terminology we also call the elements in $\Lambda_{n,i}$ paths.

In the following we will often identify an element $w \in W^I$ with its corresponding path $\lambda^w \in \Lambda_{n,i}$. For instance, we will denote indecomposable singular Soergel bimodules by B_w^I or by $B_{\lambda^w}^I$ indistinctly.

We can deduce many properties of an element $w \in W^I$ directly in terms of the corresponding path λ^w. For example, the length $\ell(w)$ of an element $w \in W^I$ is half the area of the region between λ^w and λ^{id}.

Definition 2.2. Let j be an integer with $0 \leq j \leq n$ and λ be a path. The *height* of λ at j is the integer $\mathrm{ht}_j(\lambda)$ such that the point $(j, \mathrm{ht}_j(\lambda))$ belongs to the path λ.

We have $v \leq w$ if and only if the path λ^v lies completely below λ^w, i.e. if $\mathrm{ht}_j(\lambda^v) \leq \mathrm{ht}_j(\lambda^w)$ for every j. We will simply write $\lambda^v \leq \lambda^w$ in this case. If λ and μ are paths with $\lambda \leq \mu$ we denote by $\mathcal{A}(\lambda,\mu)$ the region of the plane delimited by λ and μ. If $\lambda = \lambda^{id}$, we denote $\mathcal{A}(\lambda^{id},\mu)$ simply by $\mathcal{A}(\mu)$.

From the path representation λ^w of an element w it is easy to recover its left descent set. We have:

- $s_j w \in W^I$ and $\ell(s_j w) = \ell(w) - 1$ if j is a *peak* for λ^w, i.e. if $(\lambda^w_j, \lambda^w_{j+1}) = (\nearrow, \searrow)$,
- $s_j w \in W^I$ and $\ell(s_j w) = \ell(w) + 1$ if j is a *valley* for λ^w, i.e. if $(\lambda^w_j, \lambda^w_{j+1}) = (\searrow, \nearrow)$,
- $s_j w \notin W^I$ and $s_j w W_I = w W_I$ if j is on a *slope* of λ^w, i.e. if $(\lambda^w_j, \lambda^w_{j+1}) = (\nearrow, \nearrow)$ or $(\lambda^w_j, \lambda^w_{j+1}) = (\searrow, \searrow)$.

Definition 2.3. By *box* we mean a square rotated by 45° with side length $\sqrt{2}$ and whose center is a point (x,y) with integral coordinates such that $x+y+i$ is odd.

We fill the region $\mathcal{A}(\lambda^w)$ with boxes and we label each boxes by the horizontal coordinate of its highest point as in Figure 2.

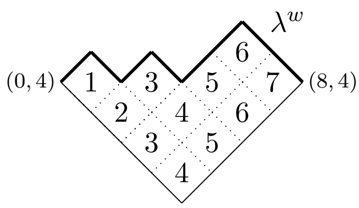


Figure 2: The path λ^w for $w = s_6 s_1 s_3 s_5 s_7 s_2 s_4 s_6 s_3 s_5 s_4 \in S_8$.

Any order in which we can remove a box from $\mathcal{A}(\lambda^w)$ so that at any step the upper boundary of the remaining region is still a path in $\Lambda_{n,i}$ gives rise to a reduced expression. Moreover, all the reduced expressions of w arise in this way.

From the description above of reduced expression it is easy to see that no reduced expression can contain a subword of the form $s_j s_{j+1} s_j$.

In particular, the function $\mathrm{rex}_w : \{1, 2, \ldots, \} \to \mathbb{N}$ given by

$$\begin{aligned}\mathrm{rex}_w(j) = \#\{k \mid i_k = j \text{ for a reduced expression } s_{i_1}s_{i_2}\ldots s_{i_k} \text{ of } w\} = \\ = \{\text{boxes labeled by } j \text{ between } \lambda^w \text{ and } \lambda^{id}\}.\end{aligned}$$

is well-defined. We have

$$\mathrm{ht}_{\lambda^w}(j) = 2\,\mathrm{rex}_j(w) + |i - j|$$

Notice that for $v, w \in W^I$ we have $v \geq w$ if and only if $\mathrm{rex}_v(j) \geq \mathrm{rex}_w(j)$ for all j.

Moreover, if $v \geq w$ then there exists $x \in W$ such that $v = xw$ and $\ell(v) = \ell(x) + \ell(w)$, therefore the Bruhat order on W^I is generated by $sx > x$ with $s \in S$ and $\ell(sx) > \ell(x)$.

To simplify the notation, we will often use $J \subseteq \{1, 2, \ldots, n\}$ to denote the set $\{s_j \mid j \in J\}$, e.g. we use W^J and W_J to denote $W^{\{s_j \mid j \in J\}}$ and $W_{\{s_j \mid j \in J\}}$ respectively.

We have now all the tools to prove the following simple lemma.

Lemma 2.4. *Assume that $w \in W^I$ has a valley in j and that $\lambda^w_a = \ldots = \lambda^w_j = \searrow$ and $\lambda^w_{j+1} = \ldots = \lambda^w_b = \nearrow$. Assume that $v < w$.*

i) If $x \in W_{[a,b-1]}$, then $xv \not\geq w$.

ii) Let $\hat{J} = [a, b-1] \setminus \{j\}$. Then, if $x \in W^{\hat{J}}_{[a,b-1]}$ we have $xw \in W^I$ and $\ell(xw) = \ell(x) + \ell(w)$.

Roughly, the second point is stating that we can stack a "small tableau" on top of λ^w to obtain the tableau of λ^{xw}.

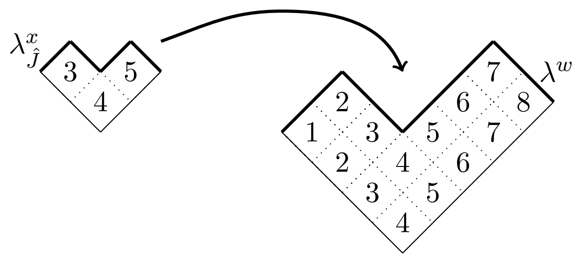


Figure 3: In this example we have $a = 3$, $b = 7$ and $j = 4$.

Proof. Let $\lambda = \lambda^w$ and $\mu = \lambda^v$. Since $\mu < \lambda$ there exists k with $\mathrm{ht}_k(\mu) < \mathrm{ht}_k(\lambda)$. Actually, there exists always such a k with $k < a$ or $k \geq b$. Otherwise, if $\mathrm{ht}_{a-1}(\lambda) = \mathrm{ht}_{a-1}(\mu)$ and $\mathrm{ht}_b(\lambda) = \mathrm{ht}_b(\mu)$ then, since $\lambda^w_a = \ldots = \lambda^w_j = \searrow$ and $\lambda^w_{j+1} = \ldots = \lambda^w_b = \nearrow$ we also have $\mathrm{ht}_k(\mu) = \mathrm{ht}_k(\lambda)$ for all $k \in [a, b-1]$.

Let now k with $k < a$ or $k \geq b$ be such that $\mathrm{ht}_k(\mu) > \mathrm{ht}_k(\lambda)$. Since $\mathrm{rex}_k(w) > \mathrm{rex}_k(v) = \mathrm{rex}_k(xv)$ we have $xv \not\geq w$.

ii) To any element $x \in W^{\hat{J}}_{[a,b-1]}$ we can associate a path $\lambda^x_{\hat{J}} \in \Lambda_{b-a,j-a}$. We claim that the path $\lambda^{xw} \in \Lambda_{n,i}$ corresponding to xw can be obtained by "stacking" the path $\widetilde{\lambda^x_{\hat{J}}}$ on top of λ^w. In formulas, we have

$$\lambda^{xw}_k = \begin{cases} \lambda^w_k & \text{if } k < a \text{ or } k \geq b-1 \\ (\lambda^w_{\hat{J}})_{k-a} & \text{if } a \leq k \leq b. \end{cases}.$$

If x is a simple reflection (i.e. if $x = s_j$) the claim is clear. If $\ell(x) > 1$ then we can write $x = s_k x'$ with $x' \in W^{\hat{J}}$ and $\ell(x') = \ell(x) - 1$. By induction on $\ell(x)$ we have $\ell(x'w) = \ell(x') + \ell(w)$, and k is a valley for $\lambda^{x'w}$ since it is a valley for $\lambda^{x'}_{\hat{J}}$. We conclude that $\ell(s_k x'w) = \ell(x') + \ell(w) + 1$. $\square$

Lemma 2.5. *Let $y \in W^I$ and let $a \leq j < b$ such that $\lambda^y_a = \ldots = \lambda^y_j = \searrow$ and $\lambda^y_{j+1} = \ldots = \lambda^y_b = \nearrow$. Let $\hat{J} = [a, b-1] \setminus \{j\}$ and let $x \in W^{\hat{J}}_{[a,b-1]}$.*

Consider the (R, R^I)-bimodule $B := B^{\hat{J}}_x \otimes_{R^{\hat{J}}} {}^{\hat{J}}B^I_y$ and assume $B^I_z(m_z)$ is a direct summand of B. If $z \geq y$, then $z = xy$ and $m_z = 0$.

Proof. By Soergel's conjecture (Theorem 1.4), it is sufficient to prove the correspondent statement in the Hecke algebra. We can write

$$\mathrm{ch}(B) = \underline{\mathbf{H}}^{\hat{J}}_x *_{\hat{J}} {}^{\hat{J}}\underline{\mathbf{H}}^I_y = \sum_z p_z(v) \underline{\mathbf{H}}^I_z \tag{7}$$

for some $p_z(v) \in \mathbb{Z}[v, v^{-1}]$. Assume there exists $z \geq y$ such that $p_z(v) \neq 0$. Since B is self-dual, we have $p_z(v) = \overline{p_z(v)} = p_z(v^{-1})$. So, if we write $\mathrm{ch}(B)$ in the I-standard basis of $\mathcal{H}^I$, the term

$$\tau := v^{-\deg p_z(v)} \mathbf{H}^I_z$$

must occur with a non-trivial coefficient in $\mathbb{N}$.

Note that k is on a slope of λ^y for all $k \in \hat{J}$. It follows that for any $u \in W_{\hat{J}}$ we have $uyW_I = yW_I$, and so $W_{\hat{J}} y W_I = yW_I$. This implies $\underline{\mathbf{H}}^I_y = {}^{\hat{J}}\underline{\mathbf{H}}^I_y$ and, in particular, $\underline{\mathbf{H}}^I_y \in \underline{\mathbf{H}}_{\hat{J}}\mathcal{H}$. We can write

$$\underline{\mathbf{H}}^{\hat{J}}_x = \mathbf{H}^{\hat{J}}_x + \sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}^{\hat{J}}_r = \mathbf{H}_x \underline{\mathbf{H}}_{\hat{J}} + \sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}_r \underline{\mathbf{H}}_{\hat{J}}.$$

Since $\underline{\mathbf{H}}^I_y \in \underline{\mathbf{H}}_J \mathcal{H}$ and $\underline{\mathbf{H}}_{\hat{J}} *_{\hat{J}} \underline{\mathbf{H}}_{\hat{J}} = \underline{\mathbf{H}}_{\hat{J}}$, we have

$$\begin{aligned} \underline{\mathbf{H}}^{\hat{J}}_x *_{\hat{J}} \underline{\mathbf{H}}^I_y &= \mathbf{H}_x \underline{\mathbf{H}}^I_y + \sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}_r \underline{\mathbf{H}}^I_y = \\ &= \mathbf{H}_x \mathbf{H}^I_y + \sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}_r \mathbf{H}^I_y + \sum_{\substack{t \in W^I \\ t < y}} h^I_{t,y}(v) \mathbf{H}_x \mathbf{H}^I_t + \sum_{\substack{r \in W^{\hat{J}},\, t \in W^I \\ r < x,\, t < y}} h^{\hat{J}}_{r,x}(v) h^I_{t,y}(v) \mathbf{H}_r \mathbf{H}^I_t. \end{aligned} \tag{8}$$

Recall that for any $r < x$ we have $h^{\hat{J}}_{r,x}(v) \in v\mathbb{N}[v]$. Using Lemma 2.4(ii), we can rewrite the first sum in (8) as

$$\sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}_r \mathbf{H}^I_y = \sum_{\substack{r \in W^{\hat{J}} \\ r < x}} h^{\hat{J}}_{r,x}(v) \mathbf{H}^I_{ry},$$

so it cannot contain the term τ.

By Lemma 2.4(i) if $t < y$ we have $rt \not\geq y$ for any $r \in W^{\hat{J}}_{[a,b-1]}$. It follows that the term τ cannot occur neither in the second or in the third sum of (8). The only remaining possibility is $\tau = \mathbf{H}^I_z = \mathbf{H}_x \mathbf{H}^I_y$, hence $z = xy$ and $\deg p_z(v) = 0$.

□

2.1 Dyck strips

Dyck paths are a very classical combinatorial object studied, for example for their connection to Catalan numbers. In this section we recall the main result from [SZJ12] which allows us to express Kazhdan-Lusztig polynomials in terms Dyck paths.

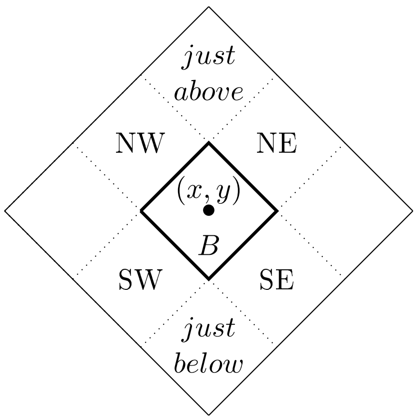
Definition 2.6. A *Dyck path* is a path (in the sense of Definition 2.1) from (x_0, y_0) to $(x_0 + 2l, y_0)$, with $x_0, y_0 \in \mathbb{Z}$ and $l \in \mathbb{Z}_{\geq 0}$, and such that it is contained in the half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \leq y_0\}$.

To any Dyck path we associate a *Dyck strip*: this is the set of boxes given by squares (rotated by $45°$ with side length $\sqrt{2}$) with center in the integral coordinates of the Dyck path. If D is a Dyck strip corresponding to a Dyck path going from (x_0, y_0) to $(x_0 + 2l, y_0)$ we define its *height* to be $\mathrm{ht}(D) := y_0$ and its length to be $\ell(D) := 2l + 1$, i.e. the length of a Dyck strip is the number of its boxes.

Let λ, μ be paths with $\lambda \leq \mu$. We call a *Dyck partition* a partition of $\mathcal{A}(\lambda, \mu)$ into Dyck strips. Given a Dyck partition $\mathbf{P}$, we denote by $|\mathbf{P}|$ the number of Dyck strips in $\mathbf{P}$. We denote by $\mathrm{ht}(\lambda, \mu)$ the maximal height of a Dyck strip in $\mathbf{P}$, for any Dyck partition of $\mathcal{A}(\lambda, \mu)$.

Our convention is to denote Dyck partitions by bold capital letters and Dyck strips by capital letter, so a typical Dyck partition would be $\mathbf{P} = \{D_1, \ldots, D_k\}$.

Given a box B centered in (x, y) we call the box centered in $(x, y+2)$ (resp. $(x, y-2)$, $(x+1, y+1)$, $(x+1, y-1)$, $(x-1, y-1)$, $(x-1, y+1)$) the box just above (resp. just below, NE, SE, SW, NW) of B.



There are two special type of Dyck partitions of $\mathcal{A}(\lambda, \mu)$ describing the I-parabolic (or Grassmannians) Kazhdan-Lusztig polynomials.

Definition 2.7. We say that a Dyck partition $\mathbf{P}$ is of type 1 (resp. of type 2) if it satisfies the following rules:

Type 1: For any Dyck strip $D \in \mathbf{P}$ if there exists a a Dyck strip $D' \in \mathbf{P}$ containing a box just above a box of D, then every box just above a box of D is in D'.

Type 2: For any Dyck strip $D \in \mathbf{P}$, if there exists a Dyck strip $D' \in \mathbf{P}$ containing a box just below, SW or SE a box of D then every box just below, SW or SE a box of D belongs either to D or D'.

Definition 2.8. Let $\lambda, \mu \in \Lambda_{n,i}$ with $\lambda \leq \mu$. For $? \in \{1, 2\}$ we denote by $\mathrm{Conf}^?(\lambda, \mu)$ be the set of Dyck partitions of Type ? in $\mathcal{A}(\lambda, \mu)$.

We define the following polynomials

$$Q^{(?)}_{\lambda,\mu}(v) = \sum_{\mathbf{P} \in \mathrm{Conf}^?(\lambda,\mu)} v^{|\mathbf{P}|}.$$

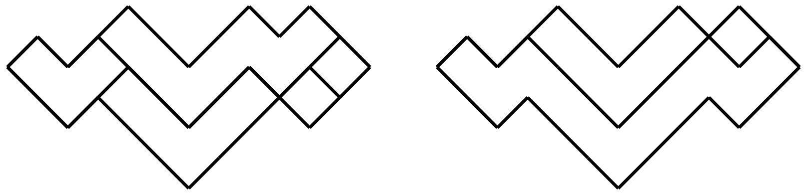


Figure 4: Two partitions resp. of Type 1 and 2 of the same region.

Theorem 2.9 ([SZJ12]). *For any $x, y \in W$ with $x \leq y$ we have*

$$Q^{(1)}_{\lambda^x,\lambda^y}(v) = h^I_{x,y}(v) \quad \text{and} \quad Q^{(2)}_{\lambda^x,\lambda^y}(v) = g^I_{x,y}(v)$$

where $h^I_{x,y}(v)$ and $g^I_{x,y}(v)$ denote respectively the Kazhdan-Lusztig polynomials and the inverse Kazhdan-Lusztig polynomials.

Moreover, for $\lambda, \mu \in \Lambda_{n,i}$ with $\lambda \leq \mu$, the set $\mathrm{Conf}^2(\lambda, \mu)$ has at most one element, thus $g^I_{\lambda,\mu}(v)$ is a monomial.

We introduce some additional notation. Let $\lambda, \mu \in \Lambda_{n,i}$ with $\lambda \leq \mu$ be such that the region $\mathcal{A}(\lambda, \mu)$ consists of a single Dyck strip D. Then we say that D *can be added to* λ and we write $\lambda + D := \mu$. Vice versa, we say that D *can be removed from* μ and we write $\mu - D := \lambda$.

2.2 Small Resolutions of Schubert varieties

A small resolution of singularities of a complex algebraic variety X is a morphism $p : \widetilde{X} \to X$ where $\widetilde{X}$ is smooth variety such that for every $r > 0$ we have

$$\operatorname{codim}\{x \in X \mid \dim(p^{-1}(x)) \geq r\} > 2r.$$

For every small resolution we have that $H^\bullet(\widetilde{X}, \mathbb{Q}) \cong IH^{\bullet - \dim X}(X, \mathbb{Q})$.

Zelevinsky [Zel83] showed that all Schubert varieties in a Grassmannian admit a small resolution. In our setting this means that, for any indecomposable Soergel bimodule B^I_x is isomorphic to a generalized Bott-Samelson bimodule, i.e. that there exists a sequence of finitary sets $J_1 \supseteq J_2 \subseteq \ldots \subseteq J_k \supseteq I$ such that

$$B^I_x := R \otimes_{R^{J_1}} R^{J_2} \otimes_{R^{J_3}} \ldots \otimes_{R^{J_k}} R^I(\ell(x)). \tag{9}$$

Following Zelevinsky, we describe how to construct such a sequence of finitary sets.

Let $x \in W^I$ and $\lambda = \lambda^x$. We denote by $\mathrm{Peaks}(\lambda) = \{p \in [1, n] \mid (\lambda_p, \lambda_{p+1}) = (\nearrow, \searrow)\}$ the set of peaks of λ.

Choose a peak $p \in \mathrm{Peaks}(\lambda)$. Let a the maximum index with the property that $a < p$ and $\lambda_a = \searrow$ and let b the minimum index with the property that $b > p$ and $\lambda_b = \nearrow$. We obtain a new path $\lambda^p \in \Lambda_{n,i}$ by setting

$$\lambda^p_i = \begin{cases} \lambda_{a+b-i} & \text{for all } i \in [a+1, b-1], \\ \lambda_i & \text{otherwise.} \end{cases}$$

We denote by x^p the element $x^p \in W^I$ such that $\lambda^{x^p} = \lambda^p$. Clearly $\lambda^p < \lambda$ and $\mathrm{Peaks}(\lambda^p) = \mathrm{Peaks}(\lambda) \setminus \{p\}$.

Definition 2.10. An ordering $(p_1, p_2, \ldots, p_k)$ of the elements of $\mathrm{Peaks}(\lambda)$ is called *neat* if

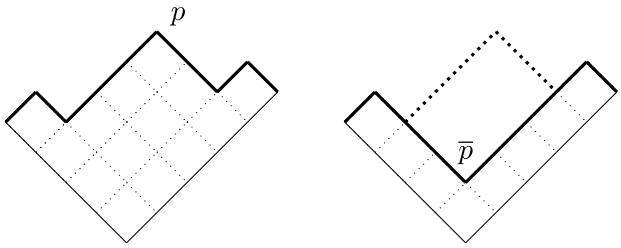


Figure 5: From λ to λ^p.

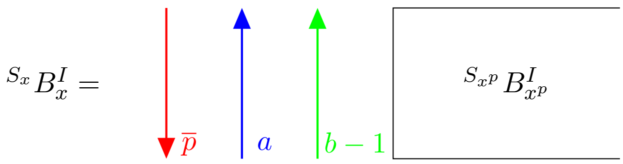
- the height of p_1 is less or equal than the height of the peaks adjacent to p_1 in λ,
- $(p_2, p_3, \dots, p_k)$ is a neat ordering of $\mathrm{Peaks}(\lambda^{p_1})$.

For $x \in W^I$ let $S_x = \{s \in S \mid sx \leq x \text{ in } W/W_I\}$. Recall from (1) that we have $B^I_x = R \otimes_{R^{S_x}} {}^{S_x}B^I_x$.

Theorem 2.11 ([Zel83])**.** *Let $x \in W^I$, $\lambda = \lambda^x$ and let $(p_1, \dots, p_k)$ be a neat ordering of* $\mathrm{Peaks}(\lambda)$*. Fix $p = p_1$ and let a and b be defined as above. Let $\widetilde{S} := S_{x^p} \setminus \{a, b-1\}$ and let $\overline{p} = a + b - p$, so that $S_x = \widetilde{S} \sqcup \{\overline{p}\}$. Then we have:*

$$ {}^{S_x}B^I_x \cong {}_{S_x}\left(R^{\widetilde{S}} \otimes_{R^{S_{x^p}}} {}^{S_{x^p}}B^I_{x^p}\right)\left(\ell(S_x)) - \ell(\widetilde{S})\right) $$

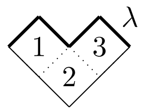
In terms of S-diagrams we can obtain ${}^{S_x}B^I_x$ from ${}^{S_{x^p}}B^I_{x^p}$ as follows:



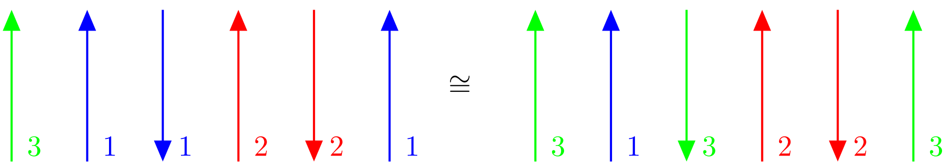
where the arrow labeled by a (resp. by $b-1$) is neglected if $a \leq 0$ (resp. $b \geq n+1$).

In general, there are several possible neat orders for a path λ, thus the algorithm of Theorem 2.11 gives rise to several different descriptions of the bimodule ${}^{S_x}B^I_x$. Since $\mathrm{End}^0({}^{S_x}B^I_x) \cong \mathbb{Q}$, we can canonically identify these bimodules via the unique isomorphism sending $1^{\otimes} := 1 \otimes 1 \otimes \dots 1$ to $1^{\otimes}$.

Example 2.12. Consider the path $\lambda = (\nearrow, \searrow, \nearrow, \searrow) \in \Lambda_{4,2}$.



Then $(1,3)$ and $(3,1)$ are both neat ordering of $\mathrm{Peaks}(\lambda)$. They give rise to the two following different diagrammatic description of the bimodule B^I_λ:



while in terms of bimodules we have

$$ R \otimes_{R^{1,3}} R^1 \otimes_{R^{1,2}} R^1 \otimes_{R^{1,3}} R^{1,3}(3) \cong R \otimes_{R^{1,3}} R^3 \otimes_{R^{2,3}} R^3 \otimes_{R^{1,3}} R^{1,3}(3). $$

2.3 Morphisms of degree one

From Theorem 2.9 and Soergel's Hom formula [Wil11, Theorem 7.4.1] we obtain that

$$\mathrm{Hom}^1(B^I_\lambda, B^I_\mu) = \begin{cases} \mathbb{Q} & \text{if } \lambda = \mu \pm D \text{ for some Dyck strip } D, \\ 0 & \text{otherwise.} \end{cases}$$

This means that to any Dyck strip we can associate uniquely up to a scalar a degree 1 morphisms of I-singular Soergel bimodules.

Assume that λ, μ are two paths such that $\lambda = \mu + D$ for some Dyck strip D. Most of the peaks of λ are also peaks for μ. There are exactly two integers $a, b+1$ with $a < b+1$ such that $\lambda_a \neq \mu_a$ and $\lambda_{b+1} \neq \mu_{b+1}$. We have $a, b \in \mathrm{Peaks}(\lambda) \setminus \mathrm{Peaks}(\mu)$. On the other hand we have $(a-1) \notin \mathrm{Peaks}(\lambda)$ and

$$(a-1) \in \mathrm{Peaks}(\mu) \iff \lambda_{a-1} = \mu_{a-1} = \nearrow.$$

Similarly, $(b+1) \notin \mathrm{Peaks}(\lambda)$ and $(b+1) \in \mathrm{Peaks}(\mu) \iff \lambda_{b+2} = \mu_{b+2} = \searrow$.

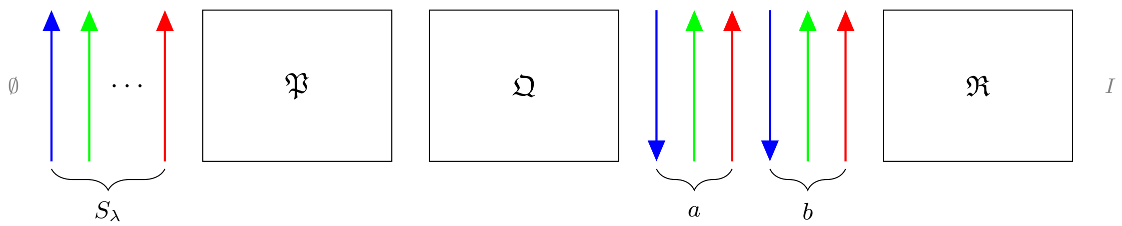
Since D is a Dyck strip, we have $\mathrm{ht}_a(\lambda) = \mathrm{ht}_b(\lambda)$ and all the peaks of λ between a and b are of height smaller or equal than $\mathrm{ht}_a(\lambda)$. This means that we can find a neat ordering of $\mathrm{Peaks}(\lambda)$ of the form

$$(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots q_{m'}, a, b, r_1, r_2, \ldots, r_{m''}) \tag{10}$$

satisfying the following rules:

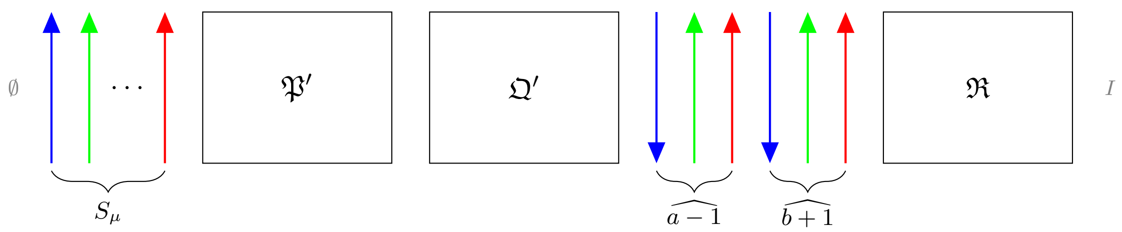
- for every peak $p \in \mathrm{Peaks}(\lambda)$ such that $p \notin [a, b]$ and $\mathrm{ht}_p(\lambda) < \mathrm{ht}(D)$ we have $p = p_i$ for some $i \in [1, m]$,
- for every peak $p \in \mathrm{Peaks}(\lambda)$ such that $p \in [a, b]$ we have $p = q_i$ for some $i \in [1, m']$,
- for every peak $p \in \mathrm{Peaks}(\lambda)$ such that $p \notin [a, b]$ and $\mathrm{ht}_p(\lambda) \geq \mathrm{ht}(D)$ we have $p = r_i$ for some $i \in [1, m'']$,

For $\lambda = \lambda^x$, let $S_\lambda := S_x$. In other words, S_λ is the set of peaks and slopes of λ. Using Theorem 2.11, from the neat order (10) we can construct a diagram representing the object B^I_λ of the form



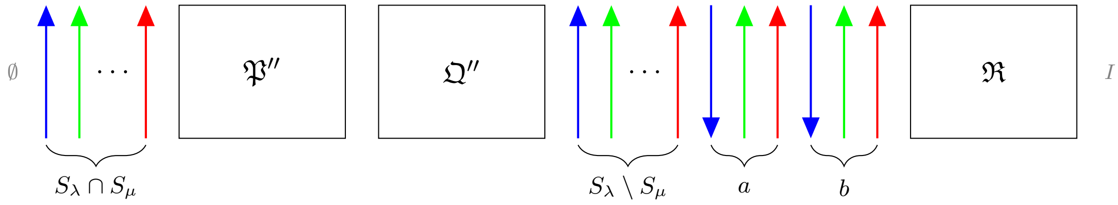
where the upward arrows on the left are labeled by the elements of S_λ, the box labeled by $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{R}$ correspond respectively to the peaks p_i's, q_i's and r_i's.

Similarly, we obtain a neat ordering $(p_1, \ldots, p_m, q_1, \ldots q_{m'}, \widehat{a-1}, \widehat{b+1}, r_1, \ldots, r_{m''})$ of $\mathrm{Peaks}(\mu)$ (here the notation $\widehat{j}$ means that j is neglected if $j \notin \mathrm{Peaks}(\mu)$) and we obtain a diagrammatic presentation of B^I_μ as follows.



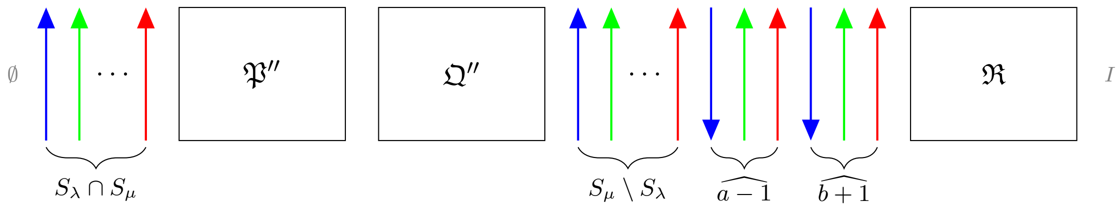
Apart from having differently labeled regions, the diagrams in the boxes $\mathfrak{P}$ and $\mathfrak{P}'$ actually coincide. The same holds for $\mathfrak{Q}$ and $\mathfrak{Q}'$.

Let $\Delta = (S_\lambda \cup S_\mu) \setminus (S_\lambda \cap S_\mu)$ be the symmetric difference of S_λ and S_μ. Then $\Delta \subseteq \{a-1, a, b, b+1\}$. Any $j \in \Delta$ is distinct from the label of any arrow appearing in $\mathfrak{P}$ and $\mathfrak{Q}$, and it is distant from any label of a downward arrow in $\mathfrak{P}$ and $\mathfrak{Q}$. Hence, up to a canonical isomorphism we can draw B^I_λ as follows.

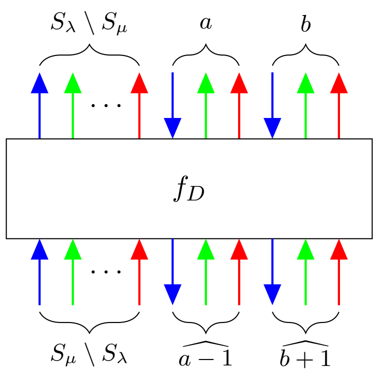


The canonical isomorphism between these two diagrammatic presentations of B^I_λ is simply given by crossing all the arrows in $S_\lambda \setminus S_\mu$ over $\mathfrak{P}$ and $\mathfrak{Q}$. The box $\mathfrak{P}''$ is equal to $\mathfrak{P}$ up to a relabeling of the regions.

We can similarly do the same for μ and obtain the following diagram representing B^I_μ:



Now, to specify a morphism between B^I_λ and B^I_μ it is enough to do it "locally" via some diagram of the form:

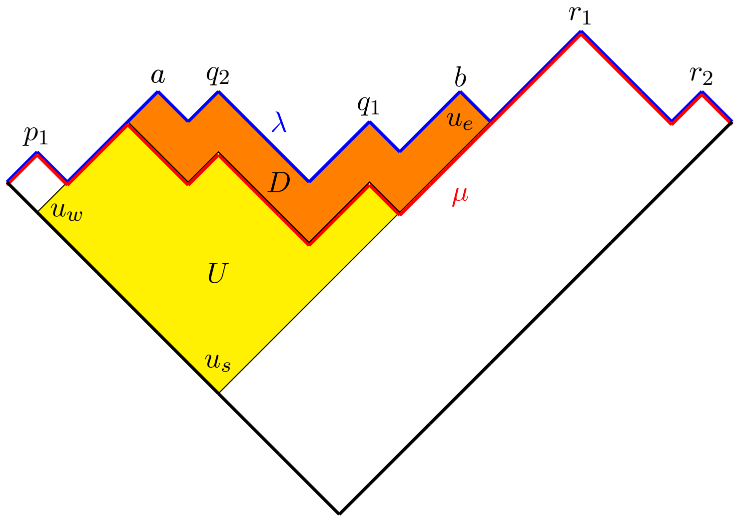


(11)

Following [Per07], we define a partial order $\lhd$ on the set of boxes in $\mathcal{A}(\lambda)$. The order $\lhd$ is generated by the relation $\theta \lhd \theta'$ if θ is SW or SE of θ. If $p \in \mathrm{Peaks}(\lambda)$ we denote by $\theta(p)$ the unique box containing p. Clearly, this is a maximal element with respect to $\lhd$. Define:

$$U := \left\{ \theta \text{ box in } \mathcal{A}(\lambda) \;\middle|\; \begin{array}{c} \exists q \in \{q_1, \dots, q_{m'}, a, b-1\} \text{ such that } \theta \lhd \theta(q) \\ \text{and } \theta \not\lhd \theta(r_i) \text{ for any } i \in [1, m''] \end{array} \right\}.$$

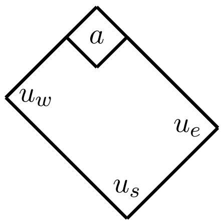
Let us call u_s, u_e and u_w respectively the labels of the southernmost, easternmost and westernmost boxes of U.

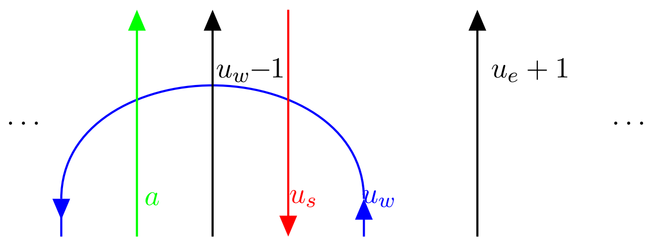


We proceed to draw the morphisms f_D as in (11) explicitly. To do this, we need to divide into 8 cases, which depend on the precise form of D and U. All the crossing involving a black arrow are isomorphism of degree 0. The black arrows are meant to be neglected whenever their label is not an element of $\{1, \dots, n\}$.

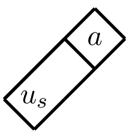
Case 1: We have $a = b$, i.e. D consists of a single box.

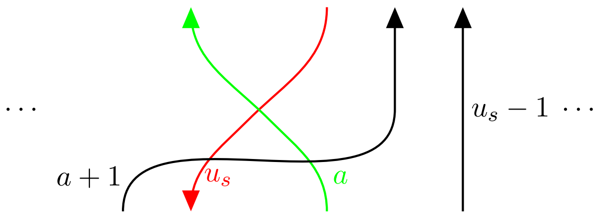
Case 1a) $a - 1$ and $b + 1$ are both peaks.





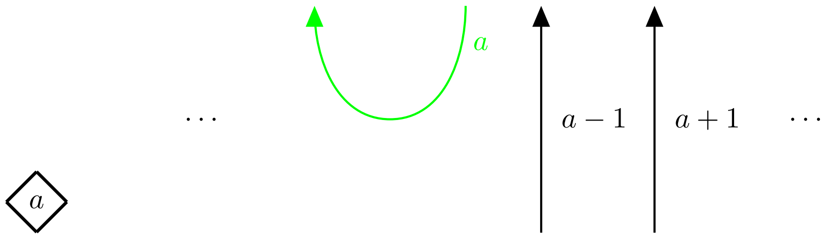
Case 1b) $a - 1 \in \mathrm{Peaks}(\mu)$ and $b + 1 \notin \mathrm{Peaks}(\mu)$. In this case $u_w = u_s$ and $u_e = a$





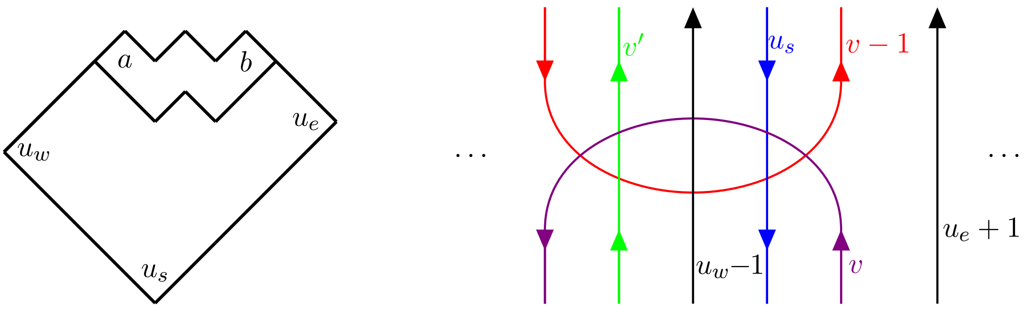
Case 1c) $a - 1 \notin \mathrm{Peaks}(\mu)$ and $b + 1 \in \mathrm{Peaks}(\mu)$. This is completely analogous to case 1b).

Case 1d) $a - 1, b + 1 \notin \mathrm{Peaks}(\mu)$. In this case U is a single box labeled by a.



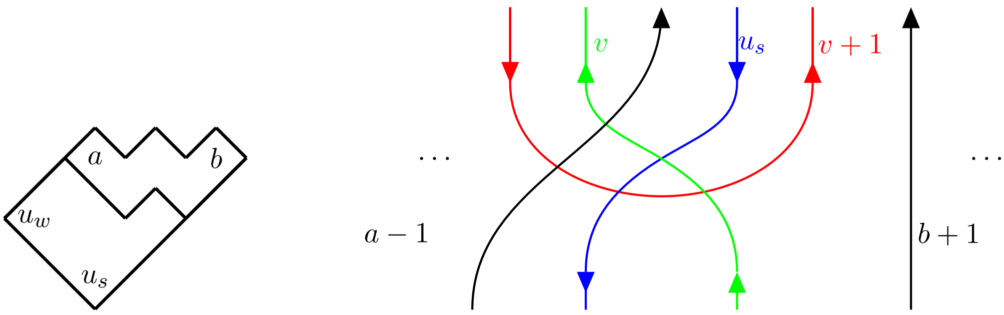
Case 2: We have $a < b$, i.e. the Dyck strip D has length > 1.

Case 2a) $a - 1$ and $b + 1$ are both peaks



where $v = u_s - u_e + b$ and $v' = v - u_e + b + a - u_w$.

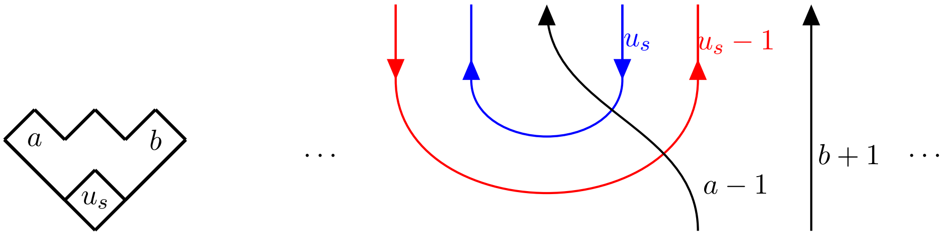
Case 2b) $a - 1 \in \mathrm{Peaks}(\mu)$ and $b + 1 \notin \mathrm{Peaks}(\mu)$. In this case $u_e = b$.



where $v = u_s + a - u_e$

Case 2c) $a - 1 \notin \mathrm{Peaks}(\mu)$ and $b + 1 \in \mathrm{Peaks}(\mu)$. This is completely analogous to case 2b).

Case 2d) $a - 1, b + 1 \notin \mathrm{Peaks}(\mu)$.

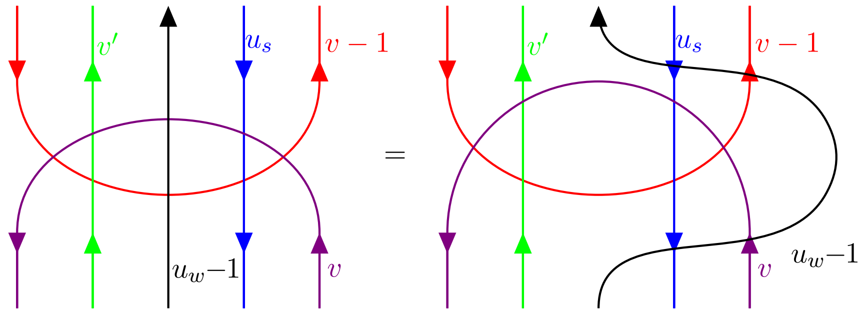


Proposition 2.13. *i) The diagrams shown in cases* 1a)-2d) *are non-trivial morphisms of degree one, hence they provide a basis of the one-dimensional space* $\mathrm{Hom}^1(B^I_\lambda, B^I_\mu)$.

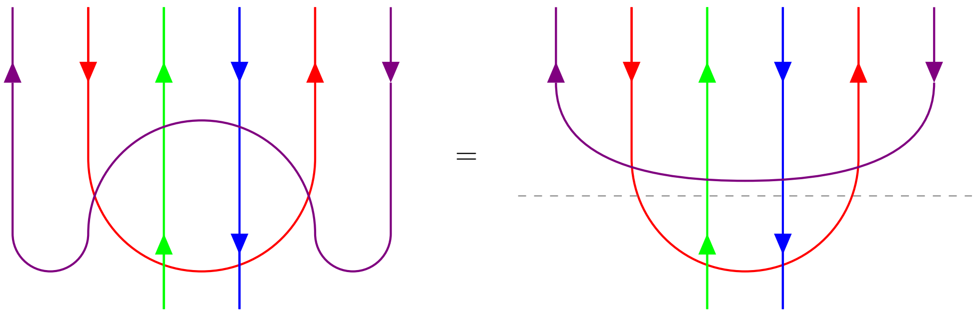
ii) Similarly, by taking the flips of the diagram shown above upside down we obtain again non trivial morphisms of degree one, hence they provide a basis of $\mathrm{Hom}^1(B^I_\mu, B^I_\lambda)$.

iii) These morphisms do not depend on the different possible choice of a neat order of the form as in (10) *if we identify all the different bimodules so obtained via the canonical isomorphisms.*

Proof. It is a straightforward computation to check that all the listed morphisms are of degree 1. To show that they are non-trivial, it is enough to check that the image of $1^\otimes$ is not trivial. For example, we consider the case 2a, which is the least trivial. We have



Thus, we can forget about about the arrow labeled by $u_w - 1$. By isotopy, it is enough to check that morphism corresponding to the following diagram is not trivial.



We compute now the image of $1^\otimes$. Notice that the morphism corresponding to the bottom half of the diagram, up to the dashed line, sends $1^\otimes$ to $1^\otimes$. The top half is given by the composition of a clockwise cup and four left pointing crossings. The image of $1^\otimes$ under a clockwise cup is non trivial. It follows from [ESW17, Relation 1.12] that a left pointing crossing represents an injective morphisms of bimodules. This concludes our proof.

Hence, they are the generators of the one-dimensional space of morphisms of degree 1. The second statement immediately follows from the first one.

For the third statement, notice that if we choose a different order of the peaks p_i's, then we will obtain a new box $\hat{\mathfrak{P}}$ instead of $\mathfrak{P}$. However, by Theorem 2.11, the bimodules corresponding to the diagrams in the boxes $\mathfrak{P}$ and $\hat{\mathfrak{P}}$ are canonically isomorphic. Similar statements hold when we change the orders of the peaks q_i's or r_i's. $\square$

Definition 2.14. If $\lambda = \mu + D$ for a Dyck strip D, we denote by $f_D^{\mu,\lambda}$ (or simply by f_D) the map $f_D^{\mu,\lambda} : B^I_\mu \to B^I_\lambda$ defined above. We denote by $g_D^{\mu,\lambda}$ (or simply by g_D) the map $g_D^{\mu,\lambda} : B^I_\lambda \to B^I_\mu$ obtained by taking the flip of the diagram of $f_D^{\mu,\lambda}$.

Remark 2.15. Also the morphisms g_D do not depend on the different form of the neat order, as long as it is of the form as in (10). For this we need to use Lemma 1.26 which implies that the flip of a canonical isomorphism is also a canonical isomorphism.

2.4 Singular Rouquier Complexes and Morphisms of degree two

Let $\mathcal{C}^b(\mathbb{S}Bim^I)$ be the bounded category of complexes of I-singular Soergel bimodules and $\mathcal{K}^b(\mathbb{S}Bim^I)$ be the corresponding homotopy category. For $\mu \in \Lambda_{n,i}$ we recall the definition of the singular Rouquier complex

$$E^I_\mu = [\dots {}^{-2}E^I_\mu \to {}^{-1}E^I_\mu \to {}^{0}E^I_\mu (= B^I_\mu) \to 0] \in \mathcal{C}^b(\mathbb{S}Bim^I)$$

defined in [Pat19]. If $x \in W^I$ is such that $\mu = \lambda^x$ then E^I_μ can be obtained by taking the minimal complex in $\mathcal{C}^b(\mathbb{S}Bim^I)$ of the restriction of the (ordinary) Rouquier complex E_x to a complex of (R, R^I)-bimodules. The singular Rouquier complex is perverse [Pat19, Theorem 4.13], i.e. for any i the bimodule ${}^{-i}E^I_\mu(i)$ is a direct sum of indecomposable self-dual singular Soergel bimodules. Moreover, by Theorem 2.9 and [Pat19, Remark 4.14] we have

$${}^{-i}E^I_\mu \cong \bigoplus_{\substack{\lambda \in \Lambda_{n,i} \\ g^I_{\lambda,\mu}(v) = v^i}} B^I_\lambda(-i)$$

Let λ and μ be two paths with $\lambda < \mu$. Assume that there exists a Dyck partition $\{C, D\} \in \mathrm{Conf}^2(\lambda, \mu)$ with two elements. Notice that this implies that $B^I_\lambda(-2)$ is a direct summand of ${}^{-2}E^I_\mu$.

From Soergel's Hom formula we have

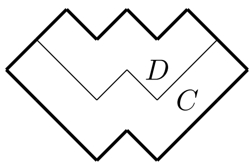
$$\dim \mathrm{Hom}^2(B^I_\lambda, B^I_\mu) = \begin{pmatrix} \text{coeff. of } v^2 \\ \text{in } h^I_{\lambda,\mu}(v) \end{pmatrix} + \# \left\{ \nu \in \Lambda_{n,i} \mid \begin{array}{c} \nu \leq \lambda \text{ and there exist} \\ \text{Dyck strips } T_1 \text{ and } T_2 \text{ with} \\ \nu + T_1 = \lambda \text{ and } \nu + T_2 = \mu \end{array} \right\}. \tag{12}$$

To compute this dimension we thus need to classify all Dyck partitions in $\mathrm{Conf}^1(\lambda, \mu)$ with two elements and all all the way in which one can obtain $\mathcal{A}(\lambda, \mu)$ as a difference of two Dyck paths. We divide this task into few cases.

2.4.1 The case of two overlying strips

Let λ, μ, C and D be as above, that is $\{C, D\} \in \mathrm{Conf}^2(\lambda, \mu)$. Assume that there exists a box of C just below a box of D, hence any box just below a box of D is in C. Let $\{C', D'\}$ be any other Dyck partition of $\mathcal{A}(\lambda, \mu)$. Two boxes lying one just above the other cannot belong to the same Dyck strip, so we can assume $D \subseteq D'$ and $C' \subseteq C$.

Assume first $\mathrm{ht}(C) < \mathrm{ht}(D)$. Then, since $\mathrm{ht}(D) \leq \mathrm{ht}(D')$, we must have $D' = D$ and $C' = C$. In particular there are no Dyck partitions in $\mathrm{Conf}^1(\lambda, \mu)$ with exactly two elements.



Lemma 2.16. *Let $\{C, D\} \in \mathrm{Conf}^2(\lambda, \mu)$ with $\mathrm{ht}(C) < \mathrm{ht}(D)$. If there exists a box in C just below a box of D then $f_D \circ f_C = 0$*

Proof. There exist no strips T_1 and T_2 such that $T_1 \sqcup D \sqcup C = T_2$. From (12) it follows that $\mathrm{Hom}^2(B^I_\lambda, B^I_\mu) = 0$, so $f_D \circ f_C = 0$. $\square$

Assume now $\mathrm{ht}(C) \geq \mathrm{ht}(D)$. Then there exists a unique Dyck partition of type 1 obtained by taking as C' the set of all boxes just below a box of D and as D' its complement.

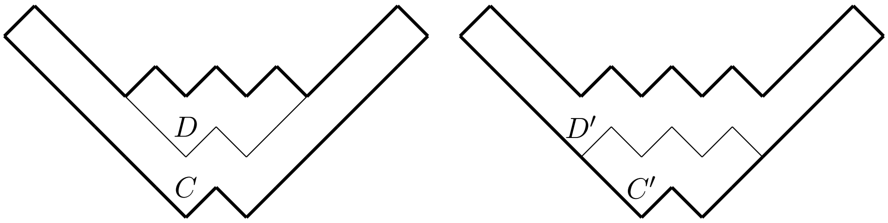


Figure 6: If $\mathrm{ht}(C) \geq \mathrm{ht}(D)$ there exists a unique type 1 Dyck partition $\{D', C'\}$ of $\mathcal{A}(\lambda, \mu)$.

Lemma 2.17. *Let D, C, D', C' be as above. Then the morphism $f_C \circ f_D$ is not trivial and we have $f_C \circ f_D = f_{D'} \circ f_{C'}$.*

Proof. From (12) we have $\dim \mathrm{Hom}^2(B^I_\mu, B^I_\lambda) = 1$, so it is clear that $f_D \circ f_C$ is equal to $f_{D'} \circ f_{C'}$ up to a scalar. To see that they actually coincide notice that we can choose a neat ordering of $\mathrm{Peaks}(\lambda)$ of the form

$$(p_1, p_2, \ldots, p_m, q_1, q_2, \ldots q_{m'}, a_D, b_D, r_1, r_2, \ldots, r_{m''}, a_C, b_C, \widetilde{r}_1, \widetilde{r}_2, \ldots, \widetilde{r}_{m'''})$$

where a_D, b_D are the extreme boxes of D and a_C and b_C are the extreme boxes of C and $\mathrm{ht}(p_i) < \mathrm{ht}(a_C) \leq \mathrm{ht}(r_j)$ for any i and j. Now it follows from the explicit description of the morphisms above that starting from $f_D \circ f_C$, we can apply an isotopy that "slides" f_D down and f_C up to obtain $f_{D'} \circ f_{C'}$.

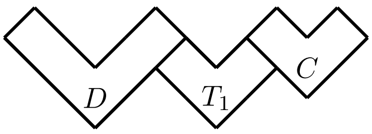
Finally, since $\mathrm{Hom}^2(B^I_\mu, B^I_\lambda)$ is generated by $f_C \circ f_D (= f_{D'} \circ f_{C'})$, the morphism $f_C \circ f_D$ must be non-trivial. $\square$

2.4.2 The case of two distant strips

Assume now that no box in D is just above or just below a box of C. In this case we say that the two strips are *distant*. Notice that in this case we can apply f_D and f_C in any order. However, as we will see, in general we have $f_D \circ f_C \neq f_C \circ f_D$.

It is evident that the only Dyck partition of $\mathcal{A}(\lambda, \mu)$ with two elements into two strips is $\{C, D\}$.

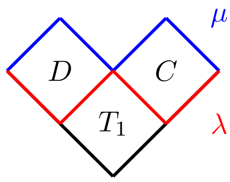
There exists a Dyck strip T_1 that can be removed from λ such that $T_2 = D \sqcup T_1 \sqcup C$ is also a Dyck strip if and only if $\mathrm{ht}(D) = \mathrm{ht}(C)$ and there are no peaks in λ between D and C of height at least $\mathrm{ht}(D)$.



Assume that there does not exist such a strip T_1. Then, by (12), $\dim \mathrm{Hom}^2(B^I_\mu, B^I_\lambda) = 1$ and there exists $c \in \mathbb{Q}$ so that $f_D \circ f_C \cong c f_C \circ f_D$. In order to show that $c \in \mathbb{Q}^*$ it is enough to show that $f_D \circ f_C$ and $f_C \circ f_D$ are non-zero. We will show this in §2.5.

Assume now that there exists such a T_1. Then $\dim \mathrm{Hom}^2(B^I_\lambda, B^I_\mu) = 2$. In general, as the next example shows, the two morphisms $f_D \circ f_C$ and $f_C \circ f_D$ are not even multiple of each others.

Example 2.18. Let $n = 4$ and $i = 2$. Let $\lambda = (\searrow, \nearrow, \searrow, \nearrow)$ and $\mu = (\nearrow, \searrow, \nearrow, \searrow)$ and let D and C be the single-box Dyck strips as in the following picture.



Let $\lambda + D := (\nearrow, \searrow, \searrow, \nearrow)$ and $\lambda + C := (\searrow, \nearrow, \nearrow, \searrow)$. Let T_1 be the Dyck strip consisting of the only box in $\mathcal{A}(\mu)$ and let $T_2 = C \sqcup D \sqcup T_1$. From Theorem [Pat19, Lemma 4.15] we can compute the Rouquier complex E^I_μ.

$$\begin{tikzcd}
{}^{-3}E^I_\mu & {}^{-2}E^I_\mu & {}^{-1}E^I_\mu & {}^{0}E^I_\mu & \\
& & B^I_{\lambda+D}(-1) \arrow[dr, "f_C"] & & \\
& & \oplus & & \\
0 \arrow[r] & B^I_\lambda(-2) \arrow[uur, "c_1 \cdot f_D"] \arrow[r, "c_2 \cdot f_C"] \arrow[ddr, "c_3 \cdot g_{T_1}"'] & B^I_{\lambda+C}(-1) \arrow[r, "f_D"] & B^I_\mu \arrow[r] & 0 \\
& & \oplus & & \\
& & B^I_{id}(-1) \arrow[uur, "f_{T_2}"'] & &
\end{tikzcd}$$

with $c_1, c_2, c_3 \in \mathbb{Q}$. Since $H^{-2}(E^I_\mu) = 0$ we immediately see that at least one between c_1 and c_2 must be non-zero. We can assume $c_1 \neq 0$. Now a simple computation shows that $f_C \circ f_D(1^\otimes) \neq 0$. Since $g_{T_1}(1^\otimes) = 0$ and $d^2 \neq 0$, we see that also $c_2 \neq 0$.

Since $d^2 = 0$ we see that $c_1 f_D \circ f_C - c_2 f_C \circ f_D = c_3 f_{T'} \circ g_T$ with $c_1, c_2 \neq 0$.

One can actually explicitly compute the maps above and check that $f_D \circ f_C$ is not a scalar multiple of $f_C \circ f_D$, i.e. that also $c_3 \neq 0$.

We record the following lemma that will be proved in §2.5.

Lemma 2.19. *Let D and C be two distant Dyck strips that can be removed from a strip λ.*

- *If $C \sqcup D$ cannot be obtained as the difference of two Dyck strips then there exists $c \in \mathbb{Q}^*$ such that $f_D \circ f_C = c \cdot f_C \circ f_D$.*
- *If $\mathrm{ht}(D) = \mathrm{ht}(C)$ then there exists $c \in \mathbb{Q}^*$ such that $f_D \circ f_C = c \cdot f_C \circ f_D \in \mathrm{Hom}^\bullet_{\not< \lambda}(B^I_\lambda, B^I_\mu)$.*

It is also important to study the composition between a morphism f_C and the flipped morphism g_D, when C and D are distant.

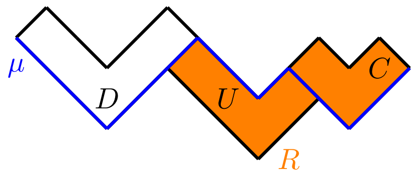
Lemma 2.20. *Assume that D and C are distant strips. Then*

$$f_D \circ g_C = c \cdot g_C \circ f_D : B^I_{\lambda+C} \to B^I_{\lambda+D}$$

for some $c \in \mathbb{Q}^*$.

Proof. From [Pat19, Lemma 4.15], every map of degree 2 between $B^I_{\lambda+C}$ and $B^I_{\lambda+D}$ can be realized as a sum of maps factoring through $B^I_{\mu+C} \xrightarrow{g_R} B^I_{\mu+C-R} \to B^I_{\mu+D}$ for some Dyck strip R. This can happen only if there exists a Dyck strip T such that $\mu+D = \mu+C-R+T$.

Assume such a strip T exists. We claim that this implies $T = D$ and $R = C$. In fact, we have $T \sqcup C = D \sqcup R$, so $D \subseteq T$ and $C \subseteq R$. Let $U := T \setminus D = R \setminus C$. We can assume that C lies on the right of D, so U lies on the left side of C and on the right side of D. In particular the set of boxes U is either empty or connected. Assume that U is not empty.



Then the left and the right extreme boxes of R are resp. in C and in U, so in particular $\mathrm{ht}(R) = \mathrm{ht}(C)$. Similarly, the right extreme box of U must be an extreme box of T. But this box as height $\mathrm{ht}(C) - 1$, hence T cannot be a Dyck strip as it should contain also a box of height $\mathrm{ht}(C)$. $\square$

2.5 Description of Rouquier complexes

The Example 2.18 above exhibits how the knowledge of singular Rouquier complexes is a useful tool for computations involving the maps f_D's. The goal of this section is to generalize this Example 2.18 to further describe the singular Rouquier complexes E^I_λ. This will allow us to prove Lemma 2.19 in general.

Recall that the terms ${}^{-i}E^I_\lambda$ of the singular Rouquier complex E^I_λ. are determined by the inverse I-parabolic Kazhdan-Lusztig polynomials $g^I_{\mu,\lambda}(v)$.

In Lemma [Pat19, Lemma 4.15] we described the first differential $d^\mu_{-1} : {}^{-1}E^I_\mu \to B^I_\mu$. We need now to understand the second differential $d^\mu_{-2} : {}^{-2}E^I_\mu \to {}^{-1}E^I_\mu$.

Proposition 2.21. *Let $\nu < \lambda < \mu \in \Lambda_{n,i}$ be such that $B^I_\nu(-2) \overset{\oplus}{\subseteq} {}^{-2}E^I_\mu$ and $B^I_\lambda(-1) \overset{\oplus}{\subseteq} {}^{-1}E^I_\mu$. Assume there exists a Dyck strip C such that $\nu+C = \lambda$. Then the second differential d^μ_{-2} of E_μ induces a non-trivial map between $B^I_\nu(-2)$ and $B^I_\lambda(-1)$.*

Proof. It is a consequence of the proof of [Pat19, Lemma 4.12] that for any summand $B^I_\nu(-2)$, the restriction of d^μ_{-2} is not trivial. Let D be the Dyck strip such that $\lambda + D = \mu$. We divide the proof into four cases.

Case 1: D contains a box just above a box of C and $\mathrm{ht}(C) \geq \mathrm{ht}(D)$.

Notice that in this case $\{C, D\}$ is of type 2. The only path κ, with $\kappa \neq \lambda$, such that $B^I_\kappa(-1) \overset{\oplus}{\subseteq} {}^{-1}E_\mu$ and $\mathrm{Hom}^1(B^I_\nu, B^I_\kappa) \neq 0$ is $\kappa = \mu - D' = \nu + C'$, where C' and D' are obtained as in Figure 6. At least one of the two maps $B^I_\nu(-2) \to B^I_\kappa(-1)$ and $B^I_\nu(-2) \to B^I_\lambda(-1)$ is not trivial. Since $(d^\mu)^2 = 0$, we see from Lemma 2.17 that they must be both not trivial. The claim follows in this case.

Case 2: D contains a box just above a box of C and $\mathrm{ht}(C) = \mathrm{ht}(D) - 1$.

Also in this case $\{C, D\}$ is of type 2. Then λ is the only path such that $B^I_\lambda(-1) \overset{\oplus}{\subseteq} {}^{-1}E_\mu$ and $\mathrm{Hom}^1(B^I_\nu, B^I_\lambda) \neq 0$. Since the restriction of d^μ_{-2} to $B^I_\nu(-2)$ is not trivial, the induced map $B^I_\nu(-2) \to B^I_\lambda(-1)$ must be also non-trivial.

Case 3: D contains a box just above a box of C, $\mathrm{ht}(D) > \mathrm{ht}(C)$

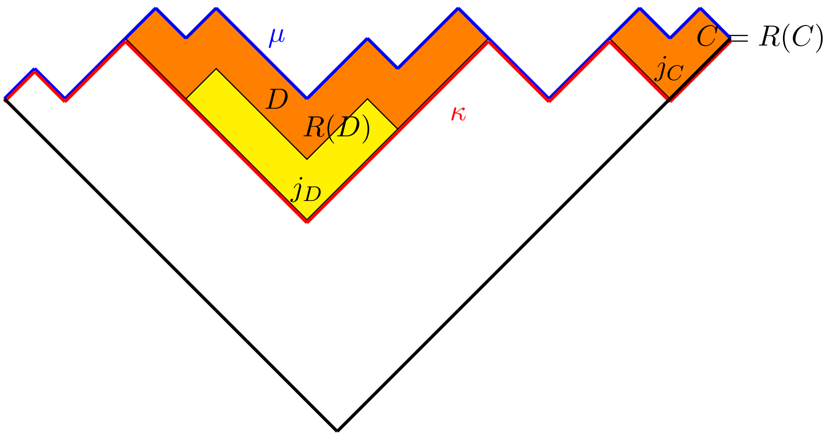
In this case $\{C, D\}$ is of type 1 and proof is the same as in Case 1.

Case 4: C and D are distant Dyck strips.

Since C and D are distant Dyck strips, they can be removed from μ in any order and we have $\nu = \mu - D - C$ and $\lambda = \mu - C$. The remaining part of this section is devoted to this case. Our strategy is to construct a new complex $\widetilde{E} \in \mathcal{C}^b(\mathbb{S}Bim^I)$ whose minimal complex is E^I_μ and easier to study than E^I_μ itself.

Definition 2.22. We call $R(D)$ the intersection of $\mathcal{A}(\mu)$ and the smallest square (with sides tilted by 45° from the axes) containing the Dyck strip D.

Let $\widetilde{J_D}$ be the set of simple reflections occurring as labels of boxes of D (or, equivalently, of $R(D)$). Let j_D be the label of southernmost box of $R(D)$ and let $J_D = \widetilde{J_D} \setminus \{j_D\}$.



Since D and C are distant, the regions $R(D)$ and $R(C)$ are disjoint. Let κ be the path obtained by removing from μ all the boxes in $R(D)$ and $R(C)$.

Let $J = J_D \cup J_C$. Notice that every $s \in J$ is on a slope of κ. It follows that $B^I_\kappa = R \otimes_{R^J} {}^J B^I_\kappa$.

We can think of $R(D)$ and $R(C)$ as sub-tableaux, and we define $x_{R(D)}$ (resp. $x_{R(C)}$) to be the corresponding element in $W^{J_D}_{\widetilde{J_D}} \subseteq W$ (resp. $W^{J_C}_{\widetilde{J_C}} \subseteq W$).

Consider the complex

$$E := E_{x_{R(D)}} \otimes_R E_{x_{R(C)}} \otimes_R E^I_\kappa \in \mathcal{C}^b(\mathbb{S}Bim^I).$$

In general the complex E is not minimal, and the minimal complex of E is E^I_μ. For any $i \geq 0$, the bimodules ${}^{-i}E$ are a direct sum of bimodules of the form

$$B_{y_{R(D)}} B_{y_{R(C)}} B^I_{\kappa'}(-i), \qquad \text{with} \quad y_{R(D)} \leq x_{R(D)},\ y_{R(C)} \leq x_{R(C)} \text{ and } \kappa' \leq \kappa.$$

Consider the subcomplex

$$E_1 := E_{x_{R(D)}} \otimes_R E_{x_{R(C)}} \otimes_R B^I_\kappa \in \mathcal{C}^b(\mathbb{S}Bim^I)$$

of E.

Since J_C and J_D are distant, we have $E^J_{x_{R(C)}} \cong (E^{J_C}_{x_{R(C)}})_J \in \mathcal{C}^b(\mathbb{S}Bim^J)$. Moreover, there exists a complex of (R^{J_D}, R^J)-bimodules ${}^{J_D}E^J_{x_{R(C)}}$ such that

$$E^J_{x_{R(C)}} \cong R \otimes_{R^{J_D}} {}^{J_D}E^J_{x_{R(C)}} \in \mathcal{C}^b(\mathbb{S}Bim^J)$$

In fact, for any i we have
$$ {}^{-i}E^J_{x_{R(C)}} \cong \bigoplus_k B^J_{z_k}(-i), $$
with $z_k \in W^J_{\widetilde{J}_C} = W^{J_C}_{\widetilde{J}_C}$. Since $z_k W_J = W_{J_D} z_k W_J$ by Proposition 1.5 there exists an indecomposable bimodule ${}^{J_D}B^J_{z_k}$ such that $B^J_{z_k} \cong R \otimes_{J_D} {}^{J_D}B^J_{z_k}$. Moreover, since by [Wil11, Theorem 7.4.1].
$$ \mathrm{Hom}({}^{J_D}B^J_{z_k}, {}^{J_D}B^J_{z_k}) \cong \mathrm{Hom}(B^J_{z_k}, B^J_{z_k}) $$
we can construct the complex ${}^{J_D}E^J_{x_{R(C)}}$ with
$$ {}^{-i}\left({}^{J_D}E^J_{x_{R(C)}}\right) \cong \bigoplus_k {}^{J_D}B^J_{z_k}(-i), $$

Consider now the complex
$$ E_2 := E^{J_D}_{x_{R(D)}} \otimes_{R^{J_D}} {}^{J_D}E^J_{x_{R(C)}} \otimes_{R^J} {}^J B^I_\kappa \in \mathcal{C}^b(\mathbb{S}Bim^I) $$
We have a decomposition $E_1 \cong E_2 \oplus T$, with T a contractible summand in $\mathcal{C}^b(\mathbb{S}Bim^I)$.

As explained in [EW14, Footnote 3], since T is a subcomplex of E, we can always find a decomposition of complexes $E = \widetilde{E} \oplus T$ such that E_2 is a subcomplex of $\widetilde{E}$. Clearly, $E \cong \widetilde{E} \in \mathcal{K}^b(\mathbb{S}Bim^I)$, so the minimal complex of $\widetilde{E}$ is again E^I_μ and moreover, E_2 is a subcomplex of $\widetilde{E}$.

We can now go back to the original question. consider the indecomposable bimodule $B^I_\nu(-2)$: it is a direct summand of ${}^{-2}E^I_\mu$ and therefore of ${}^{-2}\widetilde{E}$.

Let $x_{R(C)-C}$ be the element in $W^{J_C}_{\widetilde{J}_C}$ corresponding to the tableau $R(C) - C$. Define $x_{R(D)-D}$ similarly.

Claim 2.23. The indecomposable bimodule $B^I_\nu(-2)$ is a direct summand of
$$ B^{J_D}_{x_{R(D)-D}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa(-2) \overset{\oplus}{\subseteq} {}^{-2}E_2. $$

Proof of the claim. From Lemma 2.4(i) we see that the bimodule $B^I_\nu(-2)$ cannot be a summand of $B_{y_{R(D)}} B_{y_{R(C)}} B^I_{\kappa'}(-2)$ if $\kappa' < \kappa$. Hence, it must be a summand of ${}^{-2}E_2$. We see from Lemma 2.5 that the only summand of ${}^{-2}E_2$ containing $B^I_\nu(-2)$ is $B^{J_D}_{x_{R(D)-D}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa(-2)$. ☐

Similarly, if $\kappa_D = \mu - D$ and $\kappa_C = \mu - C$, we have
$$ B^I_{\kappa_D}(-1) \overset{\oplus}{\subseteq} B^{J_D}_{x_{R(D)-D}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)}} \otimes_{R^J} {}^J B^I_\kappa(-1) \overset{\oplus}{\subseteq} {}^{-1}E_2, $$
$$ B^I_{\kappa_C}(-1) \overset{\oplus}{\subseteq} B^{J_D}_{x_{R(D)}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa(-1) \overset{\oplus}{\subseteq} {}^{-1}E_2. $$

In $E^{J_D}_{x_{R(D)}}$ the differential $d^{x_{R(D)}}_{-1}$ induces a non-trivial map between the summands $B^{J_D}_{x_{R(D)-D}}(-1) \overset{\oplus}{\subseteq} {}^{-1}E^{J_D}_{x_{R(D)}}$ and $B^{J_D}_{x_{R(D}} = {}^0E^{J_D}_{x_{R(D)}}$ (cf. [Pat19, Lemma 4.15]). We call this map φ. It follows that in the complex $\widetilde{E}$ there is a non-trivial map
$$ \varphi \otimes id \otimes id : B^{J_D}_{x_{R(D)-D}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa(-1) \to B^{J_D}_{x_{R(D)}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)}} \otimes_{R^J} {}^J B^I_\kappa. $$

Claim 2.24. For any choice of the summands

$$B^I_\nu \overset{\oplus}{\subseteq} B^{J_D}_{x_{R(D)-D}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa$$

$$B^I_{\kappa_D} \overset{\oplus}{\subseteq} B^{J_D}_{x_{R(D)}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa$$

the map $\varphi \otimes id \otimes id$ induces a non-trivial map between $B^I_\nu(-2)$ and $B^I_{\kappa_D}(-1)$.

Proof of the claim. Assume that there exists a decomposition such that the induced map $B^I_\nu(-2) \to B^I_{\kappa_D}(-1)$ vanishes. Because of Lemma 2.5 this means that $\varphi \otimes id \otimes id$ would vanish modulo terms smaller that κ. Recall the support functor $\Gamma^I_{<\kappa}$ from [Pat19, §4.1]. Then we have

$$\varphi(1^\otimes)\otimes 1^\otimes \otimes 1^\otimes = (\varphi \otimes id \otimes id)(1^\otimes \otimes 1^\otimes \otimes 1^\otimes) \in \Gamma^I_{<\kappa}\left(B^{J_D}_{x_{R(D)}} \otimes_{R^{J_D}} {}^{J_D}B^J_{x_{R(C)-C}} \otimes_{R^J} {}^J B^I_\kappa\right),$$

which is impossible since $\varphi(1^\otimes) \neq 0$. □

It follows from the claim that for any choice of the minimal complex $E^I_\mu \overset{\oplus}{\subseteq} \widetilde{E}$, the map induced by the differentials between the summands $B^I_\nu(-2) \subseteq E^{-2}_\mu$ and $B^I_{\kappa_D}(-1) \subseteq E^{-1}_\mu$ must be non-trivial. □

Proposition 2.21 allows us to finally give a proof of Lemma 2.19.

Proof of Lemma 2.19. Consider the Rouquier complex E^I_λ. The summand $B^I_{\lambda-D-C}(-2)$ occurs in ${}^{-2}E^I_\lambda(-1)$ while $B^I_{\lambda-D}$ and $B^I_{\lambda-C}(-1)$ occur in ${}^{-1}E^I_\lambda$. As shown above the maps between these summands are non-trivial, i.e. there exists $c_1, c_2 \in \mathbb{Q}^*$ such that the first two terms of Rouquier complex E^I_λ look like

$$\begin{tikzcd}
{}^{-3}E^I_\lambda & {}^{-2}E^I_\lambda & {}^{-1}E^I_\lambda & {}^{0}E^I_\lambda & \\
 & & B^I_{\lambda-D} \arrow[dr, "f_C"] & & \\
\cdots \arrow[r] & B^I_{\lambda-D-C} \arrow[ur, "c_1 \cdot f_D"] \arrow[dr, "c_2 \cdot f_C"'] \arrow[dddr, dashed] & \oplus & B^I_\lambda \arrow[r] & 0 \\
 & \oplus & B^I_{\lambda-C} \arrow[ur, "f_D"'] & & \\
 & \vdots & \oplus & & \\
 & & \vdots \arrow[uuur, dashed] & &
\end{tikzcd}$$

Since $d^\lambda_{-2} \circ d^\lambda_{-1} = 0$ and since $\{D, C\}$ is the only Dyck partition of $\mathcal{A}(\lambda - C - D, \lambda)$ with two elements, it follows that

$$f_D \circ f_C = \frac{c_2}{c_1} f_C \circ f_D \in \mathrm{Hom}^2_{\not<\lambda-C-D}(B^I_{\lambda-C-D}, B_\lambda). \qquad \square$$

2.6 The construction of the basis

Let $\lambda < \mu$ be two paths and let $\mathbf{P} = \{D_1, D_2, \ldots, D_k\} \in \mathrm{Conf}^1(\lambda, \mu)$.

Definition 2.25. We say that an ordering $(D_1, D_2, \ldots, D_k)$ of $\mathbf{P}$ is *admissible* if for any j the Dyck strip D_j can be added to $\lambda + D_1 + \ldots + D_{j-1}$. We denote by $\mathrm{Adm}(\mathbf{P})$ the set of admissible orderings of $\mathbf{P}$.

To any admissible ordering $o = (D_{\sigma(1)}, \ldots, D_{\sigma(k)}) \in \mathrm{Adm}(\mathbf{P})$ we can associate a map $f_{\mathbf{P},o} = f_{D_{\sigma(k)}} \circ f_{D_{\sigma(k-1)}} \circ \ldots \circ f_{D_{\sigma(1)}} : B^I_\lambda \to B^I_\mu$. In general, the map $f_{\mathbf{P},o}$ will depend on the choice of the order o. The goal of this section is to show that, fixing for any Dyck partition $\mathbf{P}$ an order $o \in \mathrm{Adm}(\mathbf{P})$, the set $\{f_{\mathbf{P},o}\}_{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu)}$ gives a basis of $\mathrm{Hom}_{\not<\mu}(B^I_\lambda, B^I_\mu)$.

Given a Dyck partition $\mathbf{P} = \{D_1, D_2, \ldots, D_k\} \in \mathrm{Conf}^1(\mu, \lambda)$ we call $\mathbf{P}(h)$ the subset of strips in $\mathbf{P}$ of height h. Notice that all the Dyck strips in $\mathbf{P}(h)$ are pairwise distant.

Since $\mathbf{P}$ is of type 1, if $\mathrm{ht}(D_i) > \mathrm{ht}(D_j)$ then there is no box in D_j which is just above, NW or NE a box of D_i. It follows that any order of $(D_{\sigma(1)}, D_{\sigma(2)}, \ldots, D_{\sigma(k)})$ for which $\mathrm{ht}(D_{\sigma(i)}) \leq \mathrm{ht}(D_{\sigma(i+1)})$ for all i is admissible.

Definition 2.26. Let $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$ and $o \in \mathrm{Adm}(\mathbf{P})$. We define o_{ht} to be the order given by taking first all the strips of height 1 (in the same order as they occur in o), then all the strips of height 2 and so on. We have $o_{\mathrm{ht}} \in \mathrm{Adm}(\mathbf{P})$.

Lemma 2.27. *Let $\mathbf{P} \in \mathrm{Conf}^1(\mu, \lambda)$ and let $o \in \mathrm{Adm}(\mathbf{P})$. We denote by $o(h)$ the restriction of o to $\mathbf{P}(h)$. Then there exists $c \in \mathbb{Q}^*$ such that*

$$f_{\mathbf{P},o} = c \cdot f_{\mathbf{P}(m),o(m)} \circ f_{\mathbf{P}(m-1),o(m-1)} \circ \ldots \circ f_{\mathbf{P}(1),o(1)} = c \cdot f_{\mathbf{P},o_{\mathrm{ht}}}.$$

Proof. Let $o_{\mathrm{ht}} = (D_1, \ldots, D_k)$ and $o = (D_{\sigma(1)}, D_{\sigma(2)}, \ldots, D_{\sigma(k)})$. Assume that $o \neq o_{\mathrm{ht}}$, i.e. that σ is not the trivial permutation. Then, there exists an index j such that $\mathrm{ht}(D_{\sigma(j)}) > \mathrm{ht}(D_{\sigma(j+1)})$. Since $o \in \mathrm{Adm}(\mathbf{P})$, this implies that the strips $D_{\sigma(j)}$ and $D_{\sigma(j+1)}$ can be added in any order to $\mu + D_{\sigma(1)} + \ldots + D_{\sigma(j-1)}$, so in particular they are distant. Hence the ordering

$$o' := (D_{\sigma(1)}, \ldots D_{\sigma(j-1)}, D_{\sigma(j+1)}, D_{\sigma(j)}, D_{\sigma(j+2)}, \ldots D_{\sigma(k)})$$

is also admissible.

By induction on the length of the permutation σ we can assume $f_{\mathbf{P},o'} = c' f_{\mathbf{P},o_{\mathrm{ht}}}$ for some $c' \in \mathbb{Q}^*$. Moreover, since $\mathrm{ht}(D_{\sigma(j)}) \neq \mathrm{ht}(D_\sigma(j+1))$, by Lemma 2.19, we have $f_{\mathbf{P},o} = c'' f_{\mathbf{P},o'}$ for some $c'' \in \mathbb{Q}^*$. □

2.6.1 A partial order on Dyck partitions

Definition 2.28. Given two paths λ, μ with $\lambda \leq \mu$ and $\mathbf{P}, \mathbf{Q} \in \mathrm{Conf}^1(\lambda, \mu)$, we write $\mathbf{P} \succ \mathbf{Q}$ if there exists an integer $h > 0$ such that $\mathbf{P}(j) = \mathbf{Q}(j)$ for any $j > h$ and $\mathbf{P}(h)$ is *finer* than $\mathbf{Q}(h)$. By finer we mean that $\mathbf{P}(h) \neq \mathbf{Q}(h)$ and for any strip in $\mathbf{P}(h)$ there exists a strip in $\mathbf{Q}(h)$ containing it.

Lemma 2.29. *The relation $\succ$ defines a partial order on Dyck partitions.*

Proof. Assume $\mathbf{P} \succ \mathbf{Q}$ and $\mathbf{Q} \succ \mathbf{P}$. Then for the largest index h for which $\mathbf{P}(h)$ and $\mathbf{Q}(h)$ differ we would have that $\mathbf{P}(h)$ is finer than $\mathbf{Q}(h)$ and $\mathbf{Q}(h)$ is finer that $\mathbf{P}(h)$. This is a contradiction.

Assume $\mathbf{P} \succ \mathbf{Q}$ and $\mathbf{Q} \succ \mathbf{R}$. Let h be such that $\mathbf{P}(j) = \mathbf{Q}(j)$ for all $j > h$ and $\mathbf{P}(h)$ is finer than $\mathbf{Q}(h)$. Let k be such that $\mathbf{Q}(j) = \mathbf{R}(j)$ for all $j > k$ and $\mathbf{Q}(k)$ is finer than $\mathbf{R}(k)$.

We have $\mathbf{P}(j) = \mathbf{R}(j)$ for any $j > \max\{h, k\}$. If $k = h$ then $\mathbf{P}(h)$ is also finer than $\mathbf{R}(h)$. If $k > h$ then $\mathbf{P}(k) = \mathbf{Q}(k)$ is finer than $\mathbf{R}(k)$. If $k < h$ then $\mathbf{P}(h)$ is finer than $\mathbf{Q}(h) = \mathbf{R}(h)$. In any case we have $\mathbf{P} \succ \mathbf{R}$. □

For any two paths λ, μ with $\lambda \leq \mu$ and for any Dyck partition $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$ we arbitrarily fix an admissible order $o_{\mathbf{P}} \in \mathrm{Adm}(\mathbf{P})$.

Notation. To keep the notation under control we write $f_{\mathbf{P}}$ instead of $f_{\mathbf{P},o_{\mathbf{P}}}$ where $o_{\mathbf{P}} \in \mathrm{Adm}(\mathbf{P})$ is the order that we have just fixed.

Our next goal is to show that the set $\{f_{\mathbf{P}}\}_{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu)}$ is a basis of $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$. This will be achieved via a "double induction" on the following statements.

Let λ and μ denote paths with $\lambda \leq \mu$ and $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$.

$A(\lambda, \mu, \mathbf{P}) :=$ For any $o' \in \mathrm{Adm}(\mathbf{P})$ we have

$$f_{\mathbf{P},o'} \in c f_{\mathbf{P}} + \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \preceq \mathbf{P}\rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$$

for some $c \in \mathbb{Q}^*$.

$A(\lambda, \mu) :=$ $A(\lambda, \mu, \mathbf{P})$ holds for any $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$,

Let now D be a Dyck strip that can be removed from μ and $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \mu - D)$.

$B(\lambda, \mu, D, \mathbf{R}) :=$ For any $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$ such that $D \in \mathbf{P}$ and $\mathbf{P} \setminus \{D\} \succeq \mathbf{R}$ and for any $o' \in \mathrm{Adm}(\mathbf{R})$ we have

$$f_D \circ f_{\mathbf{R},o'} \in \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \preceq \mathbf{P}\rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu).$$

If, moreover, $\mathbf{P} \setminus \{D\} \succ \mathbf{R}$ then

$$f_D \circ f_{\mathbf{R},o'} \in \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P}\rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu).$$

$B(\lambda, \mu) :=$ $B(\lambda, \mu, D, \mathbf{R})$ holds for any D that can be removed from μ and any $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \mu - D)$.

Notice that $B(\lambda, \mu, D, \mathbf{R})$ implies that if $\mathbf{R} \preceq \mathbf{S} \in \mathrm{Conf}^1(\lambda, \mu - D)$ and $\{D\} \cup \mathbf{S}$ is of type 1, then

$$f_D \circ f_{\mathbf{R},o'} \in \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \preceq \mathbf{S} \cup \{D\}\rangle.$$

Moreover, if $\mathbf{R} \prec \mathbf{S}$ we have

$$f_D \circ f_{\mathbf{R},o'} \in \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{S} \cup \{D\}\rangle.$$

Lemma 2.30. *Let λ, μ be paths with $\lambda \leq \mu$ and assume $A(\lambda, \mu')$ for all $\mu' \leq \mu$. Then $\{f_{\mathbf{P}}\}_{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu)}$ is a left R-basis of* $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$.

Proof. The case $\mu = \lambda$ is clear. By induction we can assume the statement for any μ' with $\lambda \leq \mu' < \mu$. First notice that $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu) = 0$ if $\mu \not\geq \lambda$.

Consider the Rouquier complex E^I_μ. Let

$$M^1(\mu) = \{\nu \in \Lambda_{n,i} \mid \exists \text{ Dyck strip } D_\nu \text{ with } \nu + D_\nu = \mu\},$$

so that we have

$$^{-1}E^I_\mu \cong \bigoplus_{\nu\in M^1(\mu)} B^I_\nu(-1)$$

From [Pat19, Lemma 4.15] and the induction hypothesis we see that

$$\begin{aligned}\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu) = \mathrm{span}\left\langle f_{D_\nu}\circ g \mid \nu\in M^1(\mu),\ g\in \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\nu)\right\rangle = \\ = \mathrm{span}\left\langle f_{D_\nu}\circ f_{\mathbf{P}} \mid \nu\in M^1(\mu),\ \mathbf{P}\in \mathrm{Conf}^1(\lambda,\nu)\right\rangle.\end{aligned}$$

Let $V := \mathrm{span}\left\langle f_{\mathbf{Q}} \mid \mathbf{Q}\in\mathrm{Conf}^1(\lambda,\mu)\right\rangle$. We want to show $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)\subseteq V$. This will follow from the next claim.

Claim 2.31. Let D be a Dyck strip that can be removed from μ and let $\mathbf{P}\in\mathrm{Conf}^1(\lambda,\mu-D)$. Then $f_D\circ f_{\mathbf{P}}\in V$.

Proof of the claim. By induction, we can assume that the claim holds for all the pairs $(D',\mathbf{Q})$ satisfying one of the following conditions:

- D' is any Dyck strip that can be removed from μ with $\mathrm{ht}(D') > \mathrm{ht}(D)$ or $\mathrm{ht}(D') = \mathrm{ht}(D)$ and $\ell(D') > \ell(D)$, and $\mathbf{Q}\in\mathrm{Conf}^1(\lambda,\mu-D')$
- $D' = D$ and $\mathbf{Q}\in\mathrm{Conf}^1(\lambda,\mu-D)$ with $\mathbf{Q}\prec\mathbf{P}$.

If $\{D\}\cup\mathbf{P}$ is of type 1, then by $A(\lambda,\mu,\mathbf{P})$ we have $f_D\circ f_{\mathbf{P}}\in\mathrm{span}\langle f_{\mathbf{S}}\mid \mathbf{S}\preceq\{D\}\cup\mathbf{P}\rangle$. Hence, we can assume $\{D\}\cup\mathbf{P}$ not of type 1. This means there exists a strip $C\in\mathbf{P}$ which contains a box just below a box of D and such that $\mathrm{ht}(C)\geq\mathrm{ht}(D)-1$.

Let $h:=\mathrm{ht}(D)$. Assume there exists a strip $D'\in\mathbf{P}$ with $D'\neq C$ that can be removed from μ such that $\mathrm{ht}(D')>h$. Then there exists an admissible order $\widetilde{o}$ ending in a strip D' as above. Notice that D and D' are distant, hence by Lemma 2.19 we have

$$f_{\mathbf{P},o} - cf_{\mathbf{P},\widetilde{o}}\in\mathrm{span}\langle f_{\mathbf{Q}}\mid\mathbf{Q}\prec\mathbf{P}\rangle$$

for some scalar $c\in\mathbb{Q}$. By induction, we can assume $o=\widetilde{o}$. Since $f_D\circ f_{D'} = cf_{D'}\circ f_D$ for some scalar $c\in\mathbb{Q}$ and $\mathrm{ht}(D')>\mathrm{ht}(D)$ we conclude by induction. Hence, we can assume that there are not such strips D'.

Assume now $\mathrm{ht}(C)\geq h$. Then, there exists an admissible order $o'\in\mathrm{Adm}(\mathbf{P})$ ending in C and we can assume, as before, that $o=o'$. By Lemma 2.17 there exist Dyck strips C',D' with $C'\sqcup D' = C\sqcup D$, such that $\{C',D'\}$ is a partition of type 1 and that $f_D\circ f_C = f_{D'}\circ f_{C'}$.

We have $f_{C'}\circ f_{\mathbf{P}\setminus\{C\},o'}\in\mathrm{span}\langle f_{\mathbf{Q}}\mid\mathbf{Q}\in\mathrm{Conf}^1(\lambda,\mu-D')\rangle$. Since $\mathrm{ht}(D')\geq\mathrm{ht}(D)$ and $\ell(D')>\ell(D)$ the claim now follows by our induction hypothesis.

It remains to consider the case $\mathrm{ht}(C)=h-1$. By the assumption above there are no strips in $\mathbf{P}$ of height $>h$. There can be at most two strips $R_1,R_2\in\mathbf{P}$ with $\mathrm{ht}(R_1)=\mathrm{ht}(R_2)=\mathrm{ht}(D)$ touching C, as in the picture below.

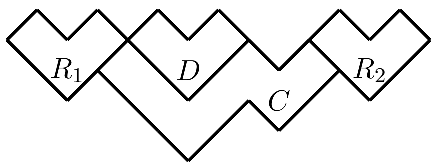


Figure 7: The strips R_1 and R_2 are of maximal height in $\mathbf{P}$

There exists an admissible order $o'' \in \mathrm{Adm}(\mathbf{P})$ ending in C, R_2, R_1. As before, it is enough to consider the case $o = o''$. Let T_1, T_2 and S_1, S_2 be Dyck strips such that $T_1 = R_1 \sqcup D \sqcup T_2$ and $S_1 = R_2 \sqcup D \sqcup S_2$.

By Lemma 2.19 and Lemma 2.20 there exists $c_1, c_2, c_3 \in \mathbb{Q}$ such that

$$f_D \circ f_{R_1} \circ f_{R_2} = c_1 f_{R_1} \circ f_{R_2} \circ f_D + c_2 f_{T_1} \circ f_{R_2} \circ g_{T_2} + c_3 f_{R_1} \circ f_{S_1} \circ g_{S_2}. \tag{13}$$

Let $\mathbf{R} = \mathbf{P} \setminus \{R_1, R_2\}$ and $o''_{\mathbf{R}}$ be the restriction of o'' to $\mathbf{R}$. We have $f_D \circ f_C = 0$ and $\ell(R_1) > \ell(D)$, so when precomposing with $f_{\mathbf{R},o''}$ the first term in the RHS of (13) vanishes and the second term is in V by induction since $\ell(T_1) > \ell(D)$.

It remains to consider the third term in (13). Notice that either f_{R_1} and f_{S_1} commute (up to a scalar), or there exists Dyck strips V_1 and V_2 with $V_1 = R_1 \sqcup S_1 \sqcup V_2$ and

$$f_{R_1} \circ f_{S_1} = d_1 f_{S_1} \circ f_{R_1} + d_2 f_{V_1} \circ g_{V_2}$$

for some $d_1, d_2 \in \mathbb{Q}$. Since $\ell(V_1) > \ell(S_1) > \ell(D)$ and $\mathrm{ht}(V_1) = \mathrm{ht}(S_1) = \mathrm{ht}(D)$, by induction we obtain also $f_{R_1} \circ f_{S_1} \circ g_{S_2} \circ f_{\mathbf{R},o''_{\mathbf{R}}} \in V$. □

We go back to the proof of the Lemma. It follows that $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$ is generated by the set $\{f_{\mathbf{P}}\}_{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu)}$. We conclude since, if M is a finitely generated graded free R-module, every set $S \subset M$ of homogeneous elements which generates M over R and such that

$$\mathrm{grrk}\, M = \sum_{s \in S} v^{\deg(s)}$$

is a R-basis of M.

Then since $\mathrm{grrk}\,\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu) = h^I_{\mu,\lambda}(v)$, from Theorem 2.9 we see that the generating set $\{f_{\mathbf{P}}\}_{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu)}$ has the right graded size, and it must be a basis. □

Let λ, μ be paths and D be a Dyck strip removable from μ and such that $\lambda \leq \mu - D$. For $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \mu - D)$ we define the sets

$$X(\mathbf{R}, D) := \{\mathbf{P} \in \mathrm{Conf}^1(\lambda,\mu) \mid D \in \mathbf{P} \text{ and } \mathbf{P} \setminus \{D\} \succ \mathbf{R}\}$$

$$Y(\mathbf{R}, D) := \{\mathbf{Q} \in \mathrm{Conf}^1(\lambda,\mu) \mid \mathbf{Q} \prec \mathbf{P} \text{ for all } \mathbf{P} \in X(\mathbf{R}, D)\}.$$

Notice that if $\mathbf{R} \cup \{D\}$ is not of type 1 then $B(\lambda, \mu, D, \mathbf{R})$ is equivalent to

$$f_D \circ f_{\mathbf{R},o'} \in \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \in Y(\mathbf{R}, D)\rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu).$$

We write $\mathbf{P} \succ Y(\mathbf{R}, D)$ if $\mathbf{P} \succ \mathbf{Q}$ for any $\mathbf{Q} \in Y(\mathbf{R}, D)$. Notice that if $\mathbf{P} \in X(\mathbf{R}, D)$, then $\mathbf{P} \succ Y(\mathbf{R}, D)$. Moreover, if $\mathbf{R} \prec \mathbf{S}$ then $Y(\mathbf{R}, D) \subseteq Y(\mathbf{S}, D)$.

Lemma 2.32. *Let λ, μ be paths with $\lambda \leq \mu$ and let $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$. Assume*

i) $A(\lambda, \mu')$ *for all* $\mu' < \mu$*;*

ii) $B(\lambda, \mu')$ *for all* $\mu' < \mu$*;*

iii) $A(\lambda, \mu, \mathbf{Q})$ *for all* $\mathbf{Q} \in \mathrm{Conf}^1(\lambda, \mu)$ *such that* $\mathbf{Q} \prec \mathbf{P}$*;*

iv) $B(\lambda, \mu, D, \mathbf{R})$ *for all Dyck strips D with* $\mathrm{ht}(D) = \mathrm{ht}(\lambda, \mu)$ *that can be removed from* μ *and for all* $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \mu - D)$ *such that* $\mathbf{P} \succ Y(\mathbf{R}, D)$.

Then $A(\lambda, \mu, \mathbf{P})$ *holds.*

Proof. Let $o_{\mathbf{P}} = (D_1, \dots, D_k)$ and $o' = (D_{\sigma(1)}, \dots, D_{\sigma(k)}) \in \mathrm{Adm}(\mathbf{P})$. Assume that σ is not the trivial permutation. Then there exists j such that $\sigma(j) > \sigma(j+1)$. This implies that the strips $D_{\sigma(j)}$ and $D_{\sigma(j+1)}$ can be added in any order, in particular they are distant. Hence $o'' := (D_{\sigma(1)}, \dots D_{\sigma(j-1)}, D_{\sigma(j+1)}, D_{\sigma(j)}, D_{\sigma(j+2)}, \dots D_{\sigma(k)}) \in \mathrm{Adm}(\mathbf{P})$.

By induction on the length of the permutation σ we can assume that

$$f_{\mathbf{P},o''} = d \cdot f_{\mathbf{P}} + \mathrm{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P} \rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu),$$

for some $d \in \mathbb{Q}^*$. Now, up to replacing $o_{\mathbf{Q}}$ with o'', it is enough to prove the statement when σ is the simple transposition $(i\ i+1)$, i.e we can assume

$$o' = (D_1, \dots, D_{i-1}, D_{i+1}, D_i, D_{i+2}, \dots, D_k).$$

If $D_i \sqcup D_{i+1}$ cannot be obtained as the difference of two Dyck strips then, from Lemma 2.19, we have $f_{\mathbf{P},o'} = d' f_{\mathbf{P}}$ for some $d' \in \mathbb{Q}^*$. So we can assume in the following that $h := \mathrm{ht}(D_i) = \mathrm{ht}(D_{i+1})$ and that there exists Dyck strips T and T' such that $D_i \sqcup D_{i+1} \sqcup T' = T$. Moreover, due to Lemma 2.27, we can also replace o by o_{ht} and o' by o'_{ht}. Notice that o_{ht} and o'_{ht} also differ by a single transposition.

Let $\mathbf{P}^{<i} = \{D_1, \dots, D_{i-1}\}$, $o^{<i}_{\mathbf{P}} = (D_1, \dots, D_{i-1})$. Define similarly $\mathbf{P}^{>i+1}$ and $o^{>i+1}_{\mathbf{P}}$. Since $o = o_{\mathrm{ht}}$, all the strips in $\mathbf{P}^{<i}$ have height $\leq h$. Let $\mathbf{P}^{<i}(h)$ be the subset of $\mathbf{P}^{<i}$ of strips of height h and $\mathbf{P}^{<i}(<h)$ the subset of strips of height $< h$. By Lemma 2.19, there exist $c \in \mathbb{Q}^*$ and $c' \in \mathbb{Q}$ such that

$$\begin{aligned} f_{\mathbf{P},o'} - c \cdot f_{\mathbf{P}} = c' \cdot f_{\mathbf{P}^{>i+1}, o^{>i+1}_{\mathbf{P}}} \circ f_T \circ g_{T'} \circ f_{\mathbf{P}^{<i}, o^{<i}_{\mathbf{P}}} = \\ = c' \cdot f_{\mathbf{P}^{>i+1}, o^{>i+1}_{\mathbf{P}}} \circ f_T \circ g_{T'} \circ f_{\mathbf{P}^{<i}(h), o^{<i}_{\mathbf{P}}(h)} \circ f_{\mathbf{P}^{<i}(<h), o^{<i}_{\mathbf{P}}(<h)}. \end{aligned}$$

Notice that T' is distant from all the strips in $\mathbf{P}^{<i}(h)$, hence by Lemma 2.20 there exists $c'' \in \mathbb{Q}^*$ such that

$$g_{T'} \circ f_{\mathbf{P}^{<i}(h), o^{<i}_{\mathbf{P}}(h)} = c'' \cdot f_{\mathbf{P}^{<i}(h), o^{<i}_{\mathbf{P}}(h)} \circ g_{T'}.$$

Let $\nu = \lambda + \sum_{C \in \mathbf{P}^{<i}(<h)} C - T'$. Since $\nu < \mu$, we can use the hypothesis $A(\lambda, \kappa)$ for all $\kappa \leq \nu$ and Lemma 2.30 to write

$$g_{T'} \circ f_{\mathbf{P}^{<i}(<h), o^{<i}(<h)} \in \mathrm{span}\langle f_{\mathbf{R}} \mid \mathbf{R} \in \mathrm{Conf}^1(\lambda, \nu) \rangle \subset \mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\nu)$$

and

$$f_{\mathbf{P},o'} - c \cdot f_{\mathbf{P}} \in \mathrm{span}\langle f_{\mathbf{P}^{>i+1}, o^{>i+1}_{\mathbf{P}}} \circ f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} \mid \mathbf{R} \in \mathrm{Conf}^1(\lambda, \nu) \rangle.$$

We need to "bound" each term of the form $f_{\mathbf{P}^{>i+1}, o^{>i+1}_{\mathbf{P}}} \circ f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}}$ for $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \nu)$, i.e. we need to show that

$$f_{\mathbf{P}^{>i+1}, o^{>i+1}_{\mathbf{P}}} \circ f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} \in \mathrm{span}\langle f_{\mathbf{S}} \mid \mathbf{S} \prec \mathbf{P} \rangle$$

Let $\mathbf{U}$ be the maximal Dyck partition in $\mathrm{Conf}^1(\lambda, \nu)$, i.e. the partition consisting only of single boxes. Let $\nu' = \nu + \sum_{C \in P^{<i}(h)} C + T$. Clearly $\{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U} \in \mathrm{Conf}^1(\lambda, \nu')$ and

$$\{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U} \prec \{D_i, D_{i+1}\} \cup \mathbf{P}^{<i}.$$

Assume first that $\nu' = \mu$, i.e. $\mathbf{P}^{>i} = \emptyset$ and $\mathbf{P} = \{D_i, D_{i+1}\} \cup \mathbf{P}^{<i}$. Furthermore, in this case we have $h = \mathrm{ht}(T) = \mathrm{ht}(\lambda, \mu)$. Then $\mathbf{P} \succ \{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U}$, so we can apply the hypothesis $A(\lambda, \mu, \{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U})$ to deduce

$$f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{U}} \in \mathrm{span}\langle f_{\mathbf{S}} \mid \mathbf{S} \prec \mathbf{P} \rangle.$$

If $\mathbf{R} \prec \mathbf{U}$, we have by $B(\lambda, \mu')$ for $\mu' \leq \mu - T$ that

$$f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} \in \operatorname{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P}^{<i}(h) \cup \mathbf{U} \rangle.$$

If $\mathbf{Q} \prec \mathbf{P}^{<i}(h) \cup \mathbf{U}$, since $\{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U} \in X(\mathbf{Q}, T)$, we also have $\mathbf{P} \succ Y(\mathbf{Q}, T)$. We can now apply the hypothesis iv) and obtain as desired

$$f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} \in \operatorname{span}\langle f_{\mathbf{S}} \mid \mathbf{S} \prec \{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U} \rangle \subseteq \operatorname{span}\langle f_{\mathbf{S}} \mid \mathbf{S} \prec \mathbf{P} \rangle.$$

We can now assume $\nu' < \mu$. Applying repeatedly the hypothesis $B(\lambda, \mu')$ for some $\mu' < \mu$, we get

$$\begin{aligned} f_{\mathbf{P}^{>i+1}, o_{\mathbf{P}}^{>i+1}} \circ f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} &\in \operatorname{span}\langle f_{\mathbf{P}^{>i+1}, o_{\mathbf{P}}^{>i+1}} \circ f_{\mathbf{V}} \mid \mathbf{V} \preceq \{T\} \cup \mathbf{P}^{<i}(h) \cup \mathbf{U} \rangle \subseteq \\ &\subseteq \operatorname{span}\langle f_{\mathbf{P}^{>i+1}, o_{\mathbf{P}}^{>i+1}} \circ f_{\mathbf{V}} \mid \mathbf{V} \prec \{D_i, D_{i+1}\} \cup \mathbf{P}^{<i} \rangle. \end{aligned}$$

Let D_k be the last strip in $\mathbf{P}$. Since $o = o_{\mathrm{ht}}$ we have $\operatorname{ht}(D_k) = \operatorname{ht}(\lambda, \mu)$. Applying again repeatedly the hypothesis $B(\lambda, \mu')$ for $\mu' < \mu$ we get

$$f_{\mathbf{P}^{>i+1}, o_{\mathbf{P}}^{>i+1}} \circ f_T \circ f_{\mathbf{P}^{<i}(h), o^{<i}(h)} \circ f_{\mathbf{R}} \in \operatorname{span}\langle f_{D_k} \circ f_{\mathbf{V}} \mid \mathbf{V} \prec \mathbf{P} \setminus \{D_k\} \rangle.$$

Since $\mathbf{V} \prec \mathbf{P} \setminus \{D_k\}$, we have $\mathbf{P} \in X(\mathbf{V}, D_k)$ and $Y(\mathbf{V}, D_k) \prec \mathbf{P}$. The claim finally follows from $B(\lambda, \mu, D_k, \mathbf{V})$. □

Lemma 2.33. *Let λ, μ be paths with $\lambda \leq \mu$. Let D a Dyck strip that can be removed from μ and let $\nu = \mu - D$. Let $\mathbf{R} \in \operatorname{Conf}^1(\lambda, \nu)$. Assume:*

i) $A(\lambda, \mu')$ for all $\mu' < \mu$;

ii) $B(\lambda, \mu')$ for all $\mu' < \mu$;

iii) $A(\lambda, \mu, \mathbf{Q})$ if $\mathbf{Q} \in Y(\mathbf{R}, D)$;

iv) $B(\lambda, \mu, D, \mathbf{Q})$ for all $\mathbf{Q} \in \operatorname{Conf}^1(\lambda, \nu)$ with $\mathbf{Q} \prec \mathbf{R}$;

v) $B(\lambda, \mu, D', \mathbf{Q})$ for all Dyck strips D' with $\operatorname{ht}(D') > \operatorname{ht}(D)$ and for all $\mathbf{Q} \in \operatorname{Conf}^1(\lambda, \mu - D')$.

vi) $B(\lambda, \mu, D', \mathbf{Q})$ for all Dyck strips D' with $\operatorname{ht}(D') = \operatorname{ht}(D)$ and $\ell(D') > \ell(D)$ and for all $\mathbf{Q} \in \operatorname{Conf}^1(\lambda, \mu - D')$ such that $Y(\mathbf{Q}, D') \subseteq Y(\mathbf{R}, D)$.

Then $B(\lambda, \mu, D, \mathbf{R})$ holds.

Proof. If $\mathbf{R} \cup \{D\}$ is of type 1, then $\mathbf{R} \cup \{D\} \in Y(\mathbf{R}, D)$. Hence the claim follows from $A(\lambda, \mu, \mathbf{R} \cup \{D\})$. We assume now that $\mathbf{R} \cup \{D\}$ is not of Type 1. Let

$$V := \operatorname{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \in Y(\mathbf{R}, D) \rangle \subset \operatorname{Hom}_{\not\leq \lambda}(B^I_\lambda, B^I_\mu).$$

We need to show that $f_D \circ f_{\mathbf{R}, o'} \in V$ for all $o' \in \operatorname{Adm}(\mathbf{R})$. However, if $o', o'' \in \operatorname{Adm}(\mathbf{R})$, by $A(\lambda, \nu)$ we have $f_{\mathbf{R}, o'} - c f_{\mathbf{R}, o''} \in \operatorname{span}\langle f_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{R} \rangle$ for some scalar $c \in \mathbb{Q}^*$. By iv) is it thus enough to show that $f_D \circ f_{\mathbf{R}, o'} \in V$ for some $o' \in \operatorname{Adm}(\mathbf{R})$.

Let $h := \operatorname{ht}(D)$. Since $\mathbf{R} \cup \{D\}$ is not of type 1, there exists a strip $C \in \mathbf{R}$ which contains a box just below a box of D and such that $\operatorname{ht}(C) \geq h - 1$.

The strategy is similar to the proof of Claim 2.31. Assume there exists a strip $D' \in \mathbf{R}$ with $D' \neq C$ that can be removed from ν such that $\operatorname{ht}(D') > h$, then D' and D are distant.

Then there exists $o' \in \mathrm{Adm}(\mathbf{R})$ ending in a strip D' with this property. Since for some scalar $c \in \mathbb{Q}$ we have $f_D \circ f_{D'} = c f_{D'} \circ f_D$ and we can conclude by hypothesis v). We can assume from now on that there are not such strips D'.

Assume first that $\mathrm{ht}(C) = h - 1$. In this case there are no strips of height $> h$ in $\mathbf{R}$ and there are at most two strips $R_1, R_2 \in \mathbf{R}$ with $\mathrm{ht}(R_1) = \mathrm{ht}(R_2) = h$ touching C (see Figure 7). Since there are no removable strips of height $> h$ we can find an admissible order $o' \in \mathrm{Adm}(\mathbf{R})$ whose last three elements are C, R_2, R_1.

As in the proof of Claim 2.31 we have

$$f_D \circ f_{R_1} \circ f_{R_2} = c_1 f_{R_1} \circ f_{R_2} \circ f_D + c_2 f_{T_1} \circ f_{R_2} \circ g_{T_2} + d_1 f_{S_1} \circ f_{R_1} \circ g_{S_2} + d_2 f_{V_1} \circ g_{V_2} \circ g_{S_2}. \quad (14)$$

for some scalars $c_1, c_2, d_1, d_2 \in \mathbb{Q}$ (the Dyck strips S_1, S_2,etc. have the same meaning as in Claim 2.31).

Let $o'_{\mathbf{R}\setminus\{R_1,R_2\}}$ be the restriction of o' to $\mathbf{R} \setminus \{R_1, R_2\}$. We need to compute

$$f_D \circ f_{\mathbf{R},o'} = f_D \circ f_{R_1} \circ f_{R_2} \circ f_{\mathbf{R}\setminus\{R_1,R_2\},o'_{\mathbf{R}\setminus\{R_1,R_2\}}}.$$

Since $f_D \circ f_C = 0$ after precomposing with $f_{\mathbf{R}\setminus\{R_1,R_2\},o'_{\mathbf{R}\setminus\{R_1,R_2\}}}$ the first term in the RHS of (14) vanishes.

Consider the term $f_{T_1} \circ f_{R_2} \circ g_{T_2} \circ f_{\mathbf{R}\setminus\{R_1,R_2\},o'_{\mathbf{R}\setminus\{R_1,R_2\}}}$. Let $\mathbf{U}$ be the Dyck partition in $\mathrm{Conf}^1(\lambda, \nu - R_1 - T_2)$ which contains of all the Dyck strips in $\mathbf{R} \setminus \{R_1\}$ of height h and such that all the other Dyck strips in $\mathbf{U}$ are single boxes. Since g_{T_2} commutes with all the strips of height $\geq h$ we have, by $B(\lambda, \mu - T_1)$,

$$f_{R_2} \circ g_{T_2} \circ f_{\mathbf{R}\setminus\{R_1,R_2\},\widetilde{o}_{\mathbf{R}\setminus\{R_1,R_2\}}} \in \mathrm{span}\langle f_{\mathbf{S}} \mid \mathbf{S} \preceq \mathbf{U}\rangle.$$

Claim 2.34. We have $\mathbf{U} \cup \{T_1\} \in Y(\mathbf{R}, D)$.

Proof of the claim. Let $\mathbf{P} \in X(\mathbf{R}, D)$. We have $\mathbf{P}(j) = \emptyset$ if $j > h$. Moreover, $\mathbf{P}(h)$ is finer or equal than $\mathbf{R}(h) \cup \{D\}$ which in turn is finer than $\mathbf{U}(h) \cup \{T_1\}$. □

Notice that $\mathbf{U} \cup \{T_1\}$ is of type 1. By iii) we have $f_{T_1} \circ f_{\mathbf{U}} \in V$. If $\mathbf{S} \prec \mathbf{U}$, then $Y(\mathbf{S}, T_1) \prec \mathbf{U} \cup \{T_1\}$, hence $Y(\mathbf{S}, T_1) \subseteq Y(\mathbf{R}, D)$ and by vi) we conclude that

$$f_{T_1} \circ f_{R_2} \circ g_{T_2} \circ f_{\mathbf{R}\setminus\{R_1,R_2\},o'_{\mathbf{R}\setminus\{R_1,R_2\}}} \in V.$$

By a similar proof for the remaining terms in the RHS of (14) we finally obtain

$$f_D \circ f_{R_1} \circ f_{R_2} \circ f_{\mathbf{R}\setminus\{R_1,R_2\},o'_{\mathbf{R}\setminus\{R_1,R_2\}}} \in V$$

as desired.

It remains to consider the case $\mathrm{ht}(C) \geq h$. Since we assumed there are no strips D' which are removable from ν, distant from D and of height $> h$, there exists an admissible order $o' \in \mathrm{Adm}(\mathbf{R})$ ending in C.

Let $o'_{\mathbf{R}\setminus\{C\}}$ be the restriction of o' to $\mathbf{R} \setminus \{C\}$. As in Figure 6, there exists $\{C', D'\}$ of Type 1 such that $D' \sqcup C' = D \sqcup C$. By Lemma 2.17 we have

$$f_D \circ f_C \circ f_{\mathbf{R}\setminus\{C\},o'_{\mathbf{R}\setminus\{C\}}} = f_{D'} \circ f_{C'} \circ f_{\mathbf{R}\setminus\{C\},o'_{\mathbf{R}\setminus\{C\}}}$$

Consider now the Dyck partition $\widetilde{\mathbf{U}} \in \mathrm{Conf}^1(\lambda, \nu - C)$ obtained by taking all the Dyck strips in $\mathbf{R} \setminus \{C\}$ of height $\geq \mathrm{ht}(C)$, and by replacing every strip in $\mathbf{R} \setminus \{C\}$ of height $< \mathrm{ht}(C)$ with the single boxes of which it consists. We have $\widetilde{\mathbf{U}} \succeq \mathbf{R} \setminus \{C\}$ and, moreover, $\widetilde{\mathbf{U}} \cup \{D', C'\} \in \mathrm{Conf}^1(\lambda, \mu)$. By applying $B(\lambda, \mu - D')$ we obtain

$$f_D \circ f_{\mathbf{R},o'} = f_{D'} \circ f_{C'} \circ f_{\mathbf{R}\setminus\{C\},o'_{\mathbf{R}\setminus\{C\}}} \in \mathrm{span}\langle f_{D'} \circ f_{\mathbf{S}} \mid \mathbf{S} \preceq \widetilde{\mathbf{U}} \cup \{C'\}\rangle.$$

Claim 2.35. We have $\widetilde{\mathbf{U}} \cup \{C', D'\} \in Y(\mathbf{R}, D)$.

Proof of the claim. Let $\mathbf{P} \in X(\mathbf{R}, D)$ and let $\mathbf{Q} = \mathbf{P} \setminus \{D\}$. We need to show that $\widetilde{\mathbf{U}} \cup \{D', C'\} \prec \mathbf{P}$. Notice that there cannot be any strip in $\mathbf{Q}$ containing C, as this would contradict the hypotheses that $D \in \mathbf{P}$ and that $\mathbf{P}$ is of Type 1. In particular this means that $\mathbf{Q}(\mathrm{ht}(C)) \neq \mathbf{R}(\mathrm{ht}(C))$. Since $\mathbf{Q} \succ \mathbf{R}$, the maximal index j such that $\mathbf{Q}(j) \neq \mathbf{R}(j)$ must be $j \geq \mathrm{ht}(C)$.

If $j > \mathrm{ht}(C)$ it is easy to see that $(\widetilde{\mathbf{U}} \cup \{D', C'\})(j) = \widetilde{\mathbf{U}}(j) = \mathbf{R}(j)$ and also $\mathbf{Q}(j) = \mathbf{P}(j)$. Since $\mathbf{Q}(j)$ is finer than $\mathbf{R}(j)$ the claim follows.

If $j = \mathrm{ht}(C)$ then $(\widetilde{\mathbf{U}} \cup \{D', C'\})(\mathrm{ht}(C)) = (\mathbf{R}(\mathrm{ht}(C)) \setminus \{C\}) \cup \{D'\}$ and

$$\mathbf{P}(\mathrm{ht}(C)) = \begin{cases} \mathbf{Q}(\mathrm{ht}(C)) & \text{if } \mathrm{ht}(D) < \mathrm{ht}(C) \\ \mathbf{Q}(\mathrm{ht}(C)) \cup \{D\} & \text{if } \mathrm{ht}(D) = \mathrm{ht}(C). \end{cases}$$

We have $D \subseteq D'$. We need to show that any other strip in $\mathbf{Q}$ of height $\mathrm{ht}(C)$ that is contained in C is also contained in D'.

Let $D_{\mathbf{Q}}$ be such a strip. Since $\mathbf{P}$ is of type 1, $D_{\mathbf{Q}}$ cannot contain any box just below a box in D, so it cannot contain any box in C'. Since $C \subseteq D' \sqcup C'$ it follows that $D_{\mathbf{Q}}$ must be contained in D'. Hence $\mathbf{P}(\mathrm{ht}(C))$ is finer than $(\widetilde{\mathbf{U}} \cup \{C', D'\})(\mathrm{ht}(C))$, thus $\mathbf{P} \succ \widetilde{\mathbf{U}} \cup \{C', D'\}$. $\square$

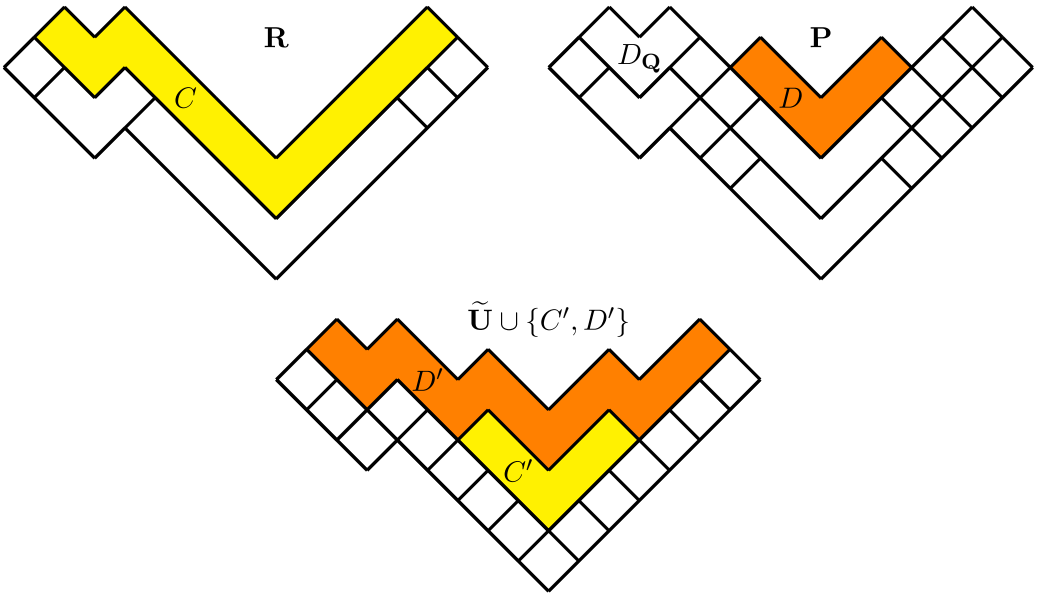


Figure 8: An illustration of Claim 2.35.

Since $\widetilde{\mathbf{U}} \cup \{C', D'\}$ is of type 1, we have, by $A(\lambda, \mu, \widetilde{\mathbf{U}} \cup \{C', D'\})$

$$f_{D'} \circ f_{\widetilde{\mathbf{U}} \cup \{C'\}} \in \mathrm{span}\langle f_{\mathbf{V}} \mid \mathbf{V} \prec \mathbf{P} \rangle.$$

Notice that $\ell(D') > \ell(D)$. From Claim 2.35 it follows that for any $\mathbf{S} \prec \widetilde{\mathbf{U}} \cup \{C'\}$ we have $Y(\mathbf{S}, D') \prec \widetilde{\mathbf{U}} \cup \{C', D'\} \prec X(\mathbf{R}, D)$, hence $Y(\mathbf{S}, D') \subseteq Y(\mathbf{R}, D)$. We can now apply $B(\lambda, \mu, D', \mathbf{S})$ to obtain

$$f_D \circ f_{\mathbf{R}, \widetilde{\sigma}} \in \mathrm{span}\langle f_{\mathbf{V}} \mid \mathbf{V} \preceq \widetilde{\mathbf{U}} \cup \{C', D'\} \rangle \subseteq \mathrm{span}\langle f_{\mathbf{V}} \mid \mathbf{V} \prec \mathbf{P} \rangle. \qquad \square$$

We can now put together the results of this section and obtain the main result of this paper. We denote by $g_{\mathbf{P}} \in \mathrm{Hom}(B^I_\mu, B^I_\lambda)$ the map obtained by taking the flip of $f_{\mathbf{P}}$. Recall the definition of a cellular category from [Wes09].

Theorem 2.36. *For any two paths λ, μ with $\lambda \leq \mu$ the set $\{f_{\mathbf{P}}\}_{\mathbf{P}\in \mathrm{Conf}^1(\lambda,\mu)}$ is a basis of $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$.*

For any two paths λ, μ the set $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$ where $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \nu)$, $\mathbf{Q} \in \mathrm{Conf}^1(\nu, \mu)$ and $\lambda \leq \nu \leq \mu$ is a basis of $\mathrm{Hom}_{\not<\lambda}(B^I_\lambda, B^I_\mu)$.

The category $\mathbb{S}Bim^I$ is cellular with cellular basis $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$.

Proof. It is easy to see from Lemma 2.32 that $B(\lambda, \mu')$ for all $\mu' \leq \mu$ implies $A(\lambda, \mu')$ for all $\mu' \leq \mu$.

Fix λ and assume $B(\lambda, \mu)$ does not hold for some μ.

- Choose μ_0 minimal such that $B(\lambda, \mu_0)$ does not hold.
- Choose a pair $(\mathbf{R}_0, D_0)$ such that $B(\lambda, \mu_0, D_0, \mathbf{R}_0)$ does not hold for which $\mathrm{ht}(D_0)$ is maximal.
- Among the pairs $(\mathbf{R}, D)$ such that $B(\lambda, \mu_0, D, \mathbf{R})$ does not hold and $\mathrm{ht}(D) = \mathrm{ht}(D_0)$ choose $(\mathbf{R}_1, D_1)$ such that the set $Y(\mathbf{R}_1, D_1)$ is minimal.
- Among those pairs $(\mathbf{R}, D)$ for which $B(\lambda, \mu_0, D, \mathbf{R})$ does not hold, $\mathrm{ht}(D) = \mathrm{ht}(D_1)$ and $Y(\mathbf{R}, D) = Y(\mathbf{R}_1, D_1)$ choose $(\mathbf{R}_2, D_2)$ such that D_2 is of maximal length.

Since $Y(\mathbf{R}_2, D_2) \supseteq Y(\mathbf{R}'_2, D_2)$ if $\mathbf{R}_2 \succ \mathbf{R}'_2$, we can choose $\mathbf{R}_2$ to be minimal among the partitions $\mathbf{R} \in \mathrm{Conf}^1(\lambda, \mu_0 - D_2)$ for which $B(\lambda, \mu_0, D_2, \mathbf{R})$ does not hold.

By Lemma 2.33 there exists $\mathbf{Q} \in Y(\mathbf{R}_2, D_2)$ such that $A(\lambda, \mu_0, \mathbf{Q})$ does not hold. Let $\mathbf{P}$ be a minimal element in $Y(\mathbf{R}_2, D_2)$ for which $A(\lambda, \mu_0, \mathbf{P})$ does not hold. Then by Lemma 2.32 there must exist D' and $\mathbf{R}'$ with $\mathrm{ht}(D') = \mathrm{ht}(\lambda, \mu_0) \geq \mathrm{ht}(D_2)$ and $Y(\mathbf{R}', D') \prec \mathbf{P}$ such that $B(\lambda, \mu_0, D', \mathbf{R}')$ does not hold. We have $Y(\mathbf{R}', D') \prec \mathbf{P} \in Y(\mathbf{R}_2, D_2)$ and $\mathbf{P} \notin Y(\mathbf{R}', D')$, hence $Y(\mathbf{R}', D') \subset Y(\mathbf{R}_2, D_2)$. This contradicts the minimality of $Y(\mathbf{R}_1, D_1)$. It follows that $B(\lambda, \mu)$ and $A(\lambda, \mu)$ hold for all μ. Now Lemma 2.30 proves the first part.

For the second part, by Soergel's Hom formula (Theorem 1.6) and Theorem 2.9, it is enough to show that the set $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$ generates. It follows directly from the first part that, for $\nu \leq \lambda, \mu$ the subset $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$, with $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \nu)$, $\mathbf{Q} \in \mathrm{Conf}^1(\lambda, \nu)$, generates $\mathrm{Hom}_{\leq\nu}(B^I_\lambda, B^I_\mu)/\mathrm{Hom}_{<\nu}(B^I_\lambda, B^I_\mu)$. It is easy to see by induction on ν that the subset $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$, with $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \tau)$, $\mathbf{Q} \in \mathrm{Conf}^1(\lambda, \tau)$ and $\tau \leq \nu$ generates the subspace $\mathrm{Hom}_{\leq\nu}(B^I_\lambda, B^I_\mu)$.

The fact that $\mathbb{S}Bim^I$ is cellular with cellular basis $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}\}$ is immediate from the definitions. □

2.7 Bases of the Intersection Cohomology of Schubert Varieties

Let λ and μ be paths with $\lambda \leq \mu$ and let $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$. We define $F_{\mathbf{P}} := f_{\mathbf{P}}(1^{\otimes}) \in B^I_\mu$. It is a homogeneous element of degree $|\mathbf{P}| - \ell(\lambda)$.

Proposition 2.37. *For a fixed path μ, the set $\{F_{\mathbf{P}}\}$, where $\mathbf{P}$ runs over all the sets $\mathrm{Conf}^1(\lambda, \mu)$ with $\lambda \leq \mu$, is a R-basis of B^I_μ compatible with the support filtration.*

Proof. By comparing the graded dimensions, it is enough to check that the set $\{F_{\mathbf{P}}\}$, with $\mathbf{P} \in \mathrm{Conf}^1(\nu, \mu)$ and $\nu \leq \lambda$ generates $\Gamma^I_{\leq\lambda} B^I_\mu$.

By Lemma [Pat19, Lemma 4.7] and Theorem 2.36 we know that $\Gamma^I_{\leq\lambda} B^I_\mu$ is spanned by $\{f_{\mathbf{P}} \circ g_{\mathbf{Q}}(1^{\otimes})\}$, for $\mathbf{P} \in \mathrm{Conf}^1(\nu, \mu)$, $\mathbf{Q} \in \mathrm{Conf}^1(\nu, \lambda)$ and $\nu \leq \lambda$.

If D consists of a single box, then it follows by the explicit description f the morphisms given in Section §2.3 that $g_D(1^{\otimes}) = 1^{\otimes}$ (see also Remark 2.15. Then by degree reasons we have:

$$g_{\mathbf{Q}}(1^{\otimes}) = \begin{cases} 1^{\otimes} \in B^I_\nu & \text{if } |\mathbf{Q}| = \ell(\lambda) - \ell(\nu) \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

The proposition follows. □

The basis $\{F_{\mathbf{P}}\}$, as the basis $\{f_{\mathbf{P}}\}$, critically depends on the admissible order chosen. More generally, for any $o' \in \mathrm{Adm}(\mathbf{P})$ we can define $F_{\mathbf{P},o'} = f_{\mathbf{P},o'}(1^{\otimes})$. We have

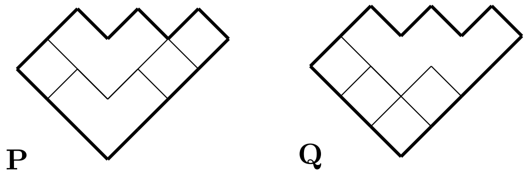
$$F_{\mathbf{P},o'} \in cF_{\mathbf{P}} + \mathrm{span}_R \langle F_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P} \rangle.$$

for some $c \in \mathbb{Q}^*$.

Let $\overline{F}_{\mathbf{P},o'}$ denote the element $1 \otimes F_{\mathbf{P},o} \in \mathbb{Q} \otimes_R B^I_\mu$. Then we have

$$\overline{F}_{\mathbf{P},o'} \in c\overline{F}_{\mathbf{P}} + \mathrm{span}_{\mathbb{Q}} \langle \overline{F}_{\mathbf{Q}} \mid \mathbf{Q} \prec \mathbf{P}, |\mathbf{Q}| = |\mathbf{P}| \rangle.$$

Example 2.38. The following is the smallest example of two Dyck partitions of type 1 $\mathbf{P}$ and $\mathbf{Q}$ with $|\mathbf{P}| = |\mathbf{Q}|$ and $\mathbf{P} \succ \mathbf{Q}$.



Lemma 2.39. *If the tableau corresponding to μ has only 2 rows (or only 2 columns) then there exists no Dyck partitions $\mathbf{P}, \mathbf{Q} \in \mathrm{Conf}^1(\lambda, \mu)$ such that $\mathbf{P} \succ \mathbf{Q}$ and $|\mathbf{P}| = |\mathbf{Q}|$.*

Proof. If the tableau of μ has only two rows, then every Dyck partition $\mathbf{P} \in \mathrm{Conf}^1(\lambda, \mu)$ for some $\lambda \leq \mu$ has at most one strip of length > 1. The proposition easily follows. □

Corollary 2.40. *If the tableau corresponding to μ has only 2 rows (or only 2 columns) the element $\overline{F}_{\mathbf{P},o} \in \overline{B}^I_\mu = IH(X_\mu, \mathbb{Q})$ does not depend on $o \in \mathrm{Adm}(\mathbf{P})$ up to a scalar.*

2.8 Comparison with the Schubert basis

For any $\lambda \in \Lambda_{n,i}$ the Schubert variety $X_\lambda \subseteq \mathrm{Gr}(i, n)$ is oriented and T-invariant, so it induces a class in the T-equivariant Borel-Moore homology (cf. [Bri00])

$$[X_\lambda] \in H^{BM}_{2\ell(\lambda),T}(\mathrm{Gr}(i, n), \mathbb{Q}).$$

The set $\{[X_\lambda]\}_{\lambda \in \Lambda_{n,i}}$ is a R-basis of $H^{BM}_{\bullet,T}(\mathrm{Gr}(i, n), \mathbb{Q})$. By taking the dual basis, we obtain a left R-basis $\{\mathcal{S}_\lambda\}_{\lambda \in \Lambda_{n,i}}$ of $H^\bullet_T(\mathrm{Gr}(i, n), \mathbb{Q})$, called the *Schubert basis.*

For $\mu \in \Lambda_{n,i}$ consider the embedding $i_\mu : X_\mu \hookrightarrow \mathrm{Gr}(i, n)$. Then $\{i^*_\mu \mathcal{S}_\lambda\}_{\lambda \leq \mu}$ is also a left R-basis of $H^\bullet_T(X_\mu, \mathbb{Q})$. By abuse of notation, we denote $i^*_\mu \mathcal{S}_\lambda$ simply by $\mathcal{S}_\lambda$. Using the canonical embedding $H^\bullet_T(X_\mu, \mathbb{Q})(\ell(\mu)) \subseteq IH^\bullet_T(X_\mu, \mathbb{Q}) = B^I_\mu$ we regard $\mathcal{S}_\lambda$ as an element of $(B^I_\mu)^{2\ell(\lambda) - \ell(\mu)}$.

Let $\mathbf{U}(\lambda)$ be the maximal Dyck partition in $\mathrm{Conf}^1(\lambda,\mu)$, i.e. the one which consists only of single boxes. In the previous section we introduced the basis $\{F_{\mathbf{P}}\}$ of B^I_μ. Our final goal is to show that $\mathcal{S}_\lambda$ can be obtained as the dual of the basis element $F_{\mathbf{U}(\lambda)}$.

Since $B^I_\mu \cong \mathbb{D}B^I_\mu$ and $\mathrm{End}^0(B^I_\mu)\cong \mathbb{Q}$ there exists a unique up to a scalar left invariant form on B^I_μ. Let $(\vec{I},\vec{J})$ be a translation pair such that $B^I_\mu \cong BS(\vec{I},\vec{J})$. Then by Lemma 1.18 we have $F_{\mathbf{U}(\lambda^{id})} = \Delta(\vec{I},\vec{J})$ up to lower terms. Define $\Delta(\mu) := F_{\mathbf{U}(\lambda^{id})}$. Notice that $\Delta(\mu)$ does not depend on the admissible order chosen for $\mathbf{U}(\lambda^{id})$ up to smaller terms.

Definition 2.41. We define $\langle -,-\rangle_\mu$ to be the unique invariant form on B^I_μ

$$\langle -,-\rangle_\mu : B^I_\mu \times B^I_\mu \in R$$

such that

$$\langle 1^\otimes, \Delta(\mu)\rangle_\mu = 1.$$

Recall by Proposition 1.24 that for any Dyck strip D the maps f_D and g_D are adjoint to each other.

If the Dyck strip C is a single box, we can give an alternative description of the map f_C. Assume $\mu = \lambda + C$, let j be the label of C (cf. Definition 2.3) and let s_j be the corresponding simple reflection. Let w_μ and w_λ the elements in W^I corresponding to μ and λ, so that we have $w_\mu = s_j w_\lambda$. The bimodule B^I_μ is a direct summand of $B_{s_j}B^I_\lambda$, and since $B_{s_j}B^I_\lambda$ is perverse this summand is uniquely determined. We denote by $\iota_\mu : B^I_\mu \to B_{s_j}B^I_\lambda$ and $\pi_\mu : B_{s_j}B^I_\lambda \to B^I_\mu$ respectively the inclusion and the projection of this summand, and by $e_\mu = \iota_\mu \circ \pi_\mu \in \mathrm{End}^0(B_s B^I_\lambda)$ the corresponding idempotent.

We can choose ι_μ so that $\iota_\mu(1^\otimes) = 1^\otimes$. This also forces $\pi_\mu(1^\otimes)=1^\otimes$. Since the vector space $\mathrm{Hom}^0(B_\mu, B_{s_j}B_\lambda)$ is one-dimensional, by Lemma 1.26 we have $\overline{\iota_\mu}(1^\otimes) = 1^\otimes$, and therefore $\overline{\iota_\mu} = \pi_\mu$.

Consider the following morphism

$$\phi := \pi_\mu \circ (f_{\smile} \otimes Id_{B^I_\lambda}) : R\otimes_R B^I_\lambda = B^I_\lambda \to B^I_\mu(1)$$

where $f_{\smile} : R \to B_{s_j}(1)$ is the morphism defined by $f_{\smile}(1) = \frac{1}{2}(\alpha_{s_j}\otimes 1 + 1\otimes \alpha_{s_j})$. The morphism ϕ is of degree 1, hence it is a scalar multiple of f_C. We have

$$\overline{\phi} = (\overline{f_{\smile}} \otimes Id_{B^I_\lambda})\circ \overline{\pi_\mu} = (f_{\frown}\otimes Id_{B^I_\lambda})\circ \iota_\mu$$

where $f_{\frown} : B_{s_j} \to R(1)$ is the map defined by $f\otimes g \mapsto fg$. It follows that $\overline{\phi}(1^\otimes) = 1^\otimes = g_C(1^\otimes)$, and we deduce $\overline{\phi} = g_C = \overline{f_C}$, hence $\phi = f_C$.

For a simple reflection s_j we simply write ∂_j for the Demazure operator ∂_{s_j}, i.e.

$$\partial_j(f) = \frac{f - s_j(f)}{x_j - x_{j+1}}.$$

Recall that the index i was fixed so that $\{s_i\} = S\setminus I$. The R-module structure on $H^\bullet_T(X_\mu,\mathbb{Q})$ equivariant Pieri's formula (see for example [KT03, Proposition 2]):

$$\mathcal{S}_\lambda \cdot h = w_\lambda(h)\mathcal{S}_\lambda + \partial_i(h) \sum_{\substack{C \text{ single box that} \\ \text{can be added to } \lambda}} \mathcal{S}_{\lambda+C} \tag{16}$$

There is a similar formula which holds in B^I_μ:

Proposition 2.42 (Equivariant Pieri's formula in intersection cohomology)**.** *Let* $\mathbf{P} \in \mathrm{Conf}^1(\nu,\mu)$*,* $o \in \mathrm{Adm}(\mathbf{P})$ *and* $h \in (\mathfrak{h}^*)^I$*. Then we have*

$$F_{\mathbf{P},o} \cdot h = w_\nu(h) F_{\mathbf{P},o} + \partial_i(h) \sum F_{\{C\}\cup\mathbf{P},(C,o)},$$

where $(C,o) \in \mathrm{Adm}(\{C\} \cup \mathbf{P})$ *is the order beginning with* C *and continuing as in* o*.*

Proof. If C is a Dyck strip we simply write F_C instead of $F_{\{C\}}$. We first show by induction on ν that

$$1^\otimes \cdot h = w_\nu(h) 1^\otimes + \partial_i(h) \sum_{\substack{C \text{ single box that} \\ \text{can be removed from } \nu}} F_C \in B^I_\nu. \tag{17}$$

Let D be a box that can be removed from ν and let $\lambda := \nu - D$. Let s_j be the simple reflection corresponding to the label of D. By the nil-Hecke relation [EW16, (5.2)] we have

$$\partial_j(w_\nu(h)) f_{\curvearrowleft}(1) = w_\nu(h) \otimes 1 - 1 \otimes w_\lambda(h) \in B_{s_j},$$

and by induction

$$\partial_j(w_\lambda(h))(f_{\curvearrowleft}(1) \otimes 1^\otimes) = w_\nu(h)1^\otimes - 1^\otimes \cdot h - \partial_i(h) \sum_{\substack{C \text{ single box that} \\ \text{can be removed from } \lambda}} 1 \otimes F_C \in B_{s_j} B^I_\lambda. \tag{18}$$

We have $F_C \in \Gamma^I_{\leq \lambda - C}(B^I_\lambda)$. Assume j is a valley for $\lambda - C$, or equivalently that C cannot be removed from ν. Then also $1 \otimes F_C \in \Gamma^I_{\leq\lambda-C}(B_{s_j} B^I_\lambda)$, thus $\pi_\nu(1 \otimes F_C) \subseteq \Gamma^I_{\leq \lambda - C}(B^I_\nu)$. Since $\Gamma^I_{\leq\lambda-C}(B^I_\nu)$ is concentrated in degrees $\geq -\ell(\nu) + 4$ we must have $\pi_\nu(1 \otimes F_C) = 0$.

Assume now that j is a valley for $\lambda - C$. In this case the boxes C and D are distant, and we have the following diagram.

$$\begin{tikzcd}
B_{\nu-C} \arrow[r, "\iota_{\nu-C}"] \arrow[d, "f_C"'] & B_{s_j} B^I_{\lambda-C} \arrow[d, "1 \otimes f_C"'] \\
B_\nu & B_{s_j} B^I_\lambda \arrow[l, "\pi_\nu"']
\end{tikzcd}$$

Since $\mathrm{Hom}^1(B_{\nu-C}, B_\nu) \cong \mathbb{Q}$, the diagram is commutative up to a scalar. However, by taking the flip, since

$$\pi_{\nu-C} \circ (1 \otimes g_C) \circ \iota_\nu(1^\otimes) = 1^\otimes = g_C(1^\otimes)$$

we see that $F_C = \pi_\nu(1 \otimes F_C) \in B^I_\nu$. We can now apply π_ν to (18) and get

$$\partial_j(w_\lambda(h))\pi_\nu(f_{\curvearrowleft}(1) \otimes 1^\otimes) = w_\nu(h)1^\otimes - 1^\otimes \cdot h - \partial_i(h) \sum_{\substack{C \neq D \text{ single box that} \\ \text{can be removed from } \nu}} F_C \in B^I_\nu.$$

Recall that $F_D = e_\nu(f_{\curvearrowleft}(1) \otimes 1^\otimes)$. To show (17), it remains to check that $\partial_j(w_\lambda(h)) = \partial_i(h)$. Consider the reflection $t := w_\lambda^{-1} s_j w_\lambda \in W = S_n$. It is not contained in W_I, so we can write $t = w s_i w^{-1}$ for some $w \in W_I$. Now $\partial_j(w_\lambda(h)) = \partial_i(w(h)) = \partial_i(h)$.

Finally, applying $f_{\mathbf{P},o}$ to (17) we obtain the original desired statement. □

Corollary 2.43. *Let* $\lambda, \nu \leq \mu$ *and* $\mathbf{P} \in \mathrm{Conf}^1(\nu,\mu)$*. We have*

$$\langle F_{\mathbf{P}}, \mathcal{S}_\lambda \rangle_\mu = \begin{cases} 1 & \textit{if } \mathbf{P} = \mathbf{U}(\lambda) \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. If C is a Dyck strip that can be removed from μ, then $g_C : B^I_\mu \to B^I_{\mu-C}$ is a morphism of (R, R^I)-bimodules, hence it also commutes with the $H^\bullet_T(\mathrm{Gr}(i,n),\mathbb{Q})$-action. By (15) it follows that for $\mathbf{P} \in \mathrm{Conf}^1(\nu,\mu)$ we have

$$g_{\mathbf{P},o}(\mathcal{S}_\lambda) = \begin{cases} \mathcal{S}_\lambda & \text{if } \mathbf{P} \text{ only consists of single boxes and } \lambda \leq \nu \\ 0 & \text{otherwise.} \end{cases}$$

From the adjointness, we have $\langle F_{\mathbf{P}}, \mathcal{S}_\lambda \rangle_\mu = \langle 1^\otimes, g_{\mathbf{P}}(\mathcal{S}_\lambda)\rangle_\nu$. It remains to show that $\langle 1^\otimes, \mathcal{S}_\lambda\rangle_\nu = \delta_{\lambda,\nu}$. We show this by induction on ν.

Recall that $\mathcal{S}_\lambda = 0$ in B^I_ν unless $\lambda \leq \nu$. If $\lambda < \nu$ by degree reasons follows $\langle 1^\otimes, \mathcal{S}_\lambda\rangle_\nu = 0$. Assume now $\lambda = \nu$ and let $h \in (\mathfrak{h}^*)^I$ be such that $\partial_i(h) = 1$. Let D be a Dyck strip that can be removed from ν. Then by (16) and (17) we have

$$\begin{aligned}\langle 1^\otimes, \mathcal{S}_\nu\rangle_\nu &= \langle 1^\otimes, \mathcal{S}_{\nu-D}\cdot h - w_{\nu-D}(h)\mathcal{S}_{\nu-D}\rangle_\nu = \langle 1^\otimes \cdot h - w_{\nu-D}(h)1^\otimes, \mathcal{S}_{\nu-D}\rangle_\nu = \\ &= \langle \sum_{\substack{C \text{ single box that} \\ \text{can be removed from } \nu}} F_C, \mathcal{S}_{\nu-D}\rangle_\nu = \sum_{\substack{C \text{ single box that} \\ \text{can be removed from } \nu}} \langle 1^\otimes, g_C(\mathcal{S}_{\nu-D})\rangle_{\nu-C}.\end{aligned}$$

Now, $g_C(\mathcal{S}_{\nu-D}) = 0 \in B^I_{\nu-C}$ unless $D = C$. We conclude because

$$\langle 1^\otimes, \mathcal{S}_\nu\rangle_\nu = \langle 1^\otimes, g_D(\mathcal{S}_{\nu-D})\rangle_{\nu-D} = \langle 1^\otimes, S_{\nu-D}\rangle_{\nu-D} = 1$$

by induction. □

Corollary 2.44. *Let $\{F^*_{\mathbf{P}}\}$ be the dual basis of $\{F_{\mathbf{P}}\}$ with respect to the form $\langle -,-\rangle_\mu$. Then $\mathcal{S}_\lambda = F^*_{\mathbf{U}(\lambda)}$, where $\mathbf{U}(\lambda) \in \mathrm{Conf}^1(\lambda,\mu)$ is the Dyck partition which consists only of single boxes.*

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