Open systems in quantum informatics

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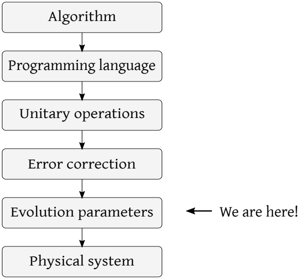
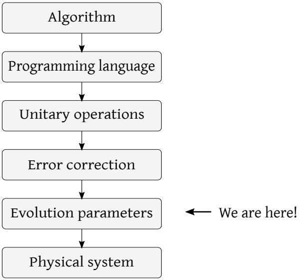
The main thesis of this dissertation was
"Proper selection and adaptation of models of quantum informatics systems allows for efficient quantum channel engineering".

The main original results of this work are:

- Numerically engineering control functions for a specific noise model.
- Showing that adding an ancilla to a system under decoherence, may allow us to achieve better fidelity of a unitary operation on the smaller system.
- Proposing an optimization scheme for finding control functions with spectral constraints.
- Studying the control functions in a setting in which the applications of the function causes a coupling to the environment.
- Introducing a new figure of merit for optimizing operations in open quantum systems.


Algorithm
Programming language
Unitary operations
Error correction
Evolution parameters
We are here!
Physical system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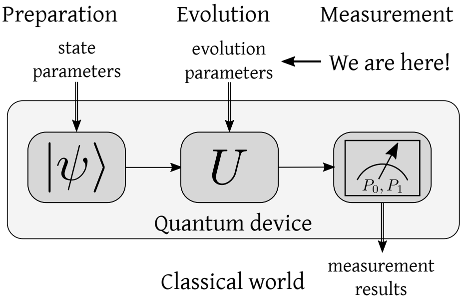
Preparation
Evolution
Measurement
state parameters
evolution parameters
We are here!
$|\psi\rangle$
U
P_0, P_1
Quantum device
Classical world
measurement results

Systems under consideration

Spin chains modeled by the Gorini-Kossakowski-Sudarshan master equation

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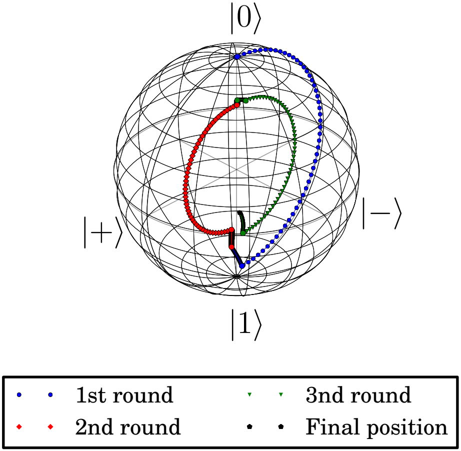
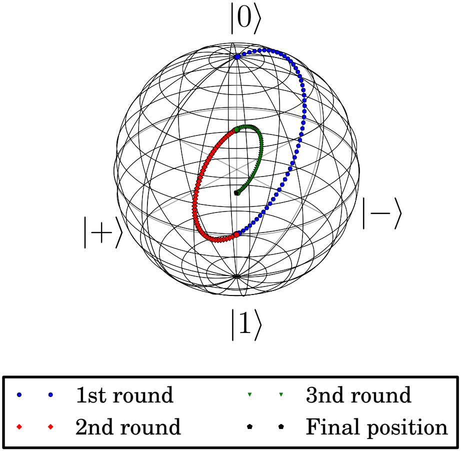
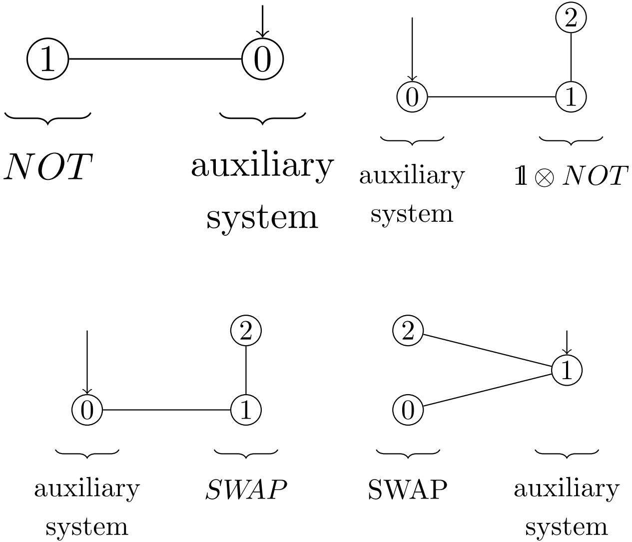
$$H_c(t) = b(t)\sigma_z^{(k)} + c(t)\sigma_x^{(k)}, \tag{4}$$

Example - single qubit







$$NOT : |0\rangle \mapsto |1\rangle\,,$$
$$|1\rangle \mapsto |0\rangle\,.$$

$$SWAP : |\phi\psi\rangle \mapsto |\psi\phi\rangle\,.$$

Smoothing effect

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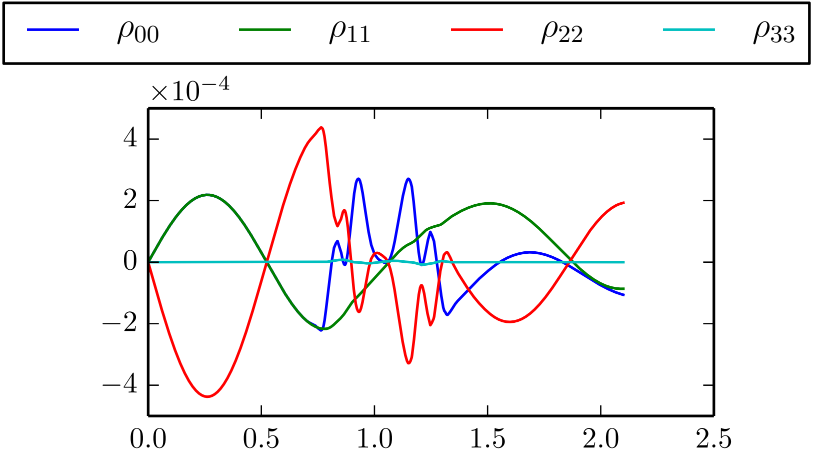
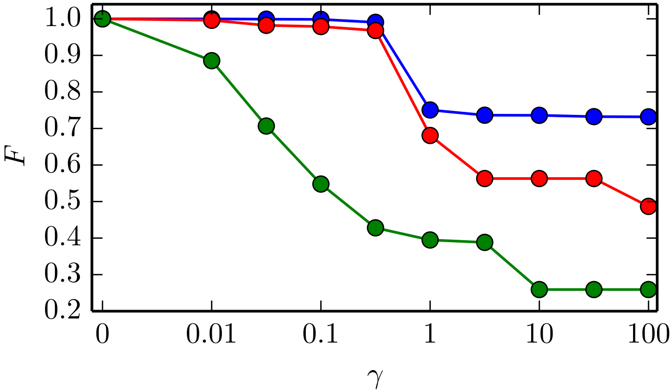
- Denote A the ancilla and S the system
- $\rho_S = \mathrm{Tr}_A(\rho)$
- $$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = \mathrm{Tr}_A() + \mathrm{Tr}_A(-\mathrm{i}[H_c(t), \rho])$$
- $$\mathrm{Tr}_A(-\mathrm{i}[H_c(t), \rho]) = 0 \tag{5}$$

Smoothing – numerics



Example of results



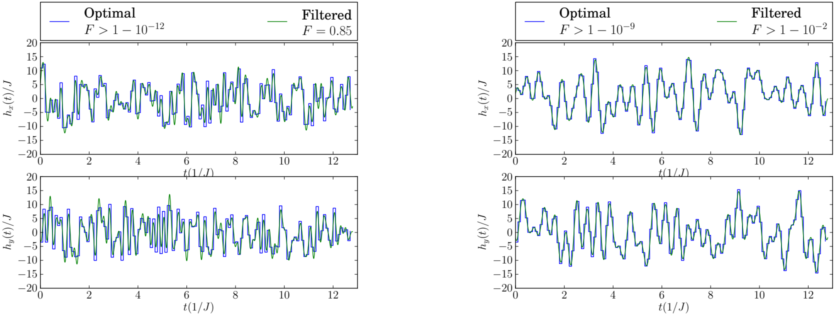
Constrained control pulses

- We impose additional constraints: L_1 norm or frequency constraints
- Modification of objective function

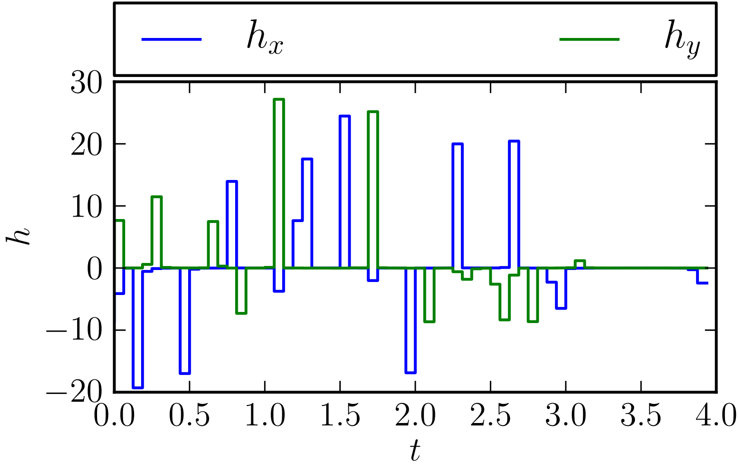
$$G = (1 - \mu)P - \mu F \tag{6}$$

- P - the L_1 norm of control pulses or the power in high frequencies

Example – frequency constraints



Example – L_1 norm constraints



$F > 0.99$

New figure of merit

Consider the fidelity function

$$F(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 \tag{7}$$

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$$F_{\mathrm{ch}}(\Psi; \sigma) = \inf_{\mathcal{Z}, \xi} F(|\xi\rangle\langle\xi|, (\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(|\xi\rangle\langle\xi|) \quad \sigma = \mathrm{Tr}_{\mathcal{Z}} |\xi\rangle\langle\xi| \tag{8}$$

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It can be shown that

$$F_{\mathrm{ch}}(\Phi; \sigma) = \sum_i |\mathrm{Tr}\sigma K_i|^2 \tag{9}$$

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Superfidelity

$$G(\rho, \sigma) = Tr\rho\sigma + \sqrt{1 - \mathrm{Tr}\rho^2}\sqrt{1 - \mathrm{Tr}\sigma^2} \tag{13}$$

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$$G(\rho, \sigma) \geq F(\rho, \sigma) \tag{14}$$

Channel superfidelity

$$G_{\mathrm{ch}}(\Psi;\sigma) = \inf_{\mathcal{Z},\xi} G((\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(|\xi\rangle\langle\xi|), (\Psi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Z})})(|\xi\rangle\langle\xi|) \quad \sigma = \operatorname{Tr}_{\mathcal{Z}} |\xi\rangle\langle\xi| \quad (15)$$

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We show that

$$G_{\mathrm{ch}}(\Phi, \Psi;\sigma) = \sum_{i,j} |\mathrm{Tr} B_j \sigma A_i^\dagger|^2 + \sqrt{1 - \sum_{i,j} A_j \sigma A_i^\dagger}\sqrt{1 - \sum_{i,j} B_j \sigma B_i^\dagger} \tag{16}$$

Sketch of the proof

We need to calculate quantities of the form

$$\mathrm{tr}\left(\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|)\left(\Psi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|) \tag{17}$$

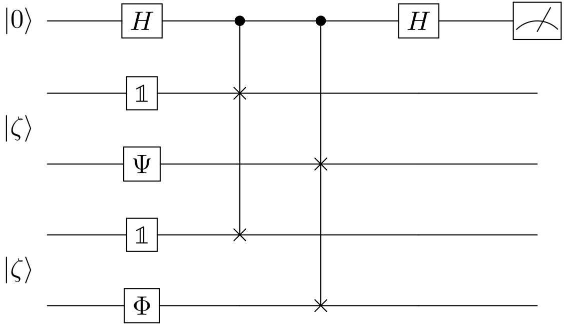
We get

$$\begin{aligned}\mathrm{tr}\left(\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|)\left(\Psi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|) =\\ \mathrm{tr}\,|\zeta\rangle\langle\zeta|\left(\Phi^\dagger \circ \Psi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|) =\\ \langle\zeta|\left(\Phi^\dagger \circ \Psi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|)\,|\zeta\rangle\,,\end{aligned} \tag{18}$$

All that remains is to use

$$|\zeta\rangle = \mathrm{res}(\sqrt{\sigma}U), \tag{19}$$

Quantum circuit for calculating channel superfidelity



Quantum circuit for measuring $\operatorname{tr}\left(\Phi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|)\left(\Psi \otimes \mathbb{1}_{\mathcal{L}(\mathcal{Z})}\right)(|\zeta\rangle\langle\zeta|) = 2p_0 - 1$, where p_0 is the probability of finding the top qubit in state $|0\rangle$. This allows direct estimation of the channel superfidelity.

Summary

- I studied how the interaction with the environment influences the dynamics of a single qubit
- I shown that it is possible to approximate a unitary evolution on an open system by performing the control on an ancilla
- I studied the possibility of optimizing control functions with restrictions on the L_1 norm and the functions' spectrum
- I introduced a new possibille figure of merit for quantum control opitmizations

These results show that proper selection and adaptation of models of quantum informatics systems allows for efficient quantum channel engineering

THANK YOU FOR YOUR ATTENTION

Response to reviewers' comments

Jan Bouda

Question

The central interest seems to be on the Nash equilibria of the game. Candidate notes on p. 673 that, due to the noise, the strategies in Table 2 are no longer Nash equilibrium. This does not seem to be straightforward — this is obviously true for $\gamma = 1$, but not for $\gamma = 0$. I would be curious whether candidate can comment on this further – do the Nash equilibria exist for various values of gamma. Do they change continuously with gamma?

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Answer

The main problem in the study of Nash equilibria in open quantum systems is that they depend not only on the players' strategies and the strength on the coupling, but also on the applied noise model. Consider for instance the depolarising channel, $\Phi(\rho) = frm[o]--/2$. Clearly, this channel helps neither of the players, as $\mathrm{Tr}\sigma_z = 0$. On the other hand, the spontaneous emission channel $\Phi(\rho) = |0\rangle\langle 0|$, clearly benefits one of the parties, as $\langle 0| \sigma_z |0\rangle = 1$ Hence, the answer is not straightforward and would require careful in depth study.

Jan Bouda

Question

Also, I'm a bit puzzled by p. 675 – „in each round, one player performs a series of unitary operations chosen randomly from a uniform distribution". I assume uniform distribution here is the Haar distribution, but the selecting a number of unitaries and applying them sequentially makes no sense, since they will only establish together one Haar-random unitary.

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Answer

This is not suppose to mean Haar random unitary, rather random Pauli matrix. Hence, each of the player's channels is a mixed unitary channel

$$\Phi(A) = \frac{1}{4}\sum_{i=0}^{3} \sigma_i A \sigma_i \tag{20}$$

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Answer

The introduced model of controlling and ancillary qubit is interesting, because we showed that the tracing out of the ancilla smooths the piecewise constant control functions.

Jan Bouda

Question

A bit puzzling here is the definition of channel fidelity (Def. 13) that only measures fidelity between input and output of a particular channel. On contrary, the superfidelity introduced in Def. 14 compares fidelity of outputs of two channels, and hence, compares to the fidelity as of Def. 13 if you fix one of the channels to be identity. Does this generalization bring complications in evaluation? What are the situations when you want to use it? (in addition to Corollary 1-3).

Jan Bouda

Proposition

Given two quantum channels $\Phi, \Psi : \mathrm{L}(\mathcal{X}) \to \mathrm{L}(\mathcal{X})$, an input state $\sigma \in \Omega(\mathcal{X})$ and its purification $|\xi\rangle \in \mathcal{X} \otimes \mathcal{Y}$, their channel fidelity

$$F_{\mathrm{ch}}(\Psi, \Phi; \sigma) = F((\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Y})})(|\xi\rangle\langle\xi|), (\Psi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Y})})(|\xi\rangle\langle\xi|)), \tag{21}$$

is given by

$$F_{\mathrm{ch}}(\Psi, \Phi; \sigma) = \left\| \left\{ A_k \sigma B_l^\dagger \right\}_{k,l} \right\|_1, \tag{22}$$

where A_k, B_l are the Kraus operators of channels Φ, Ψ.

Proof

Consider the Stinespring representation of the extended channels

$$(\Phi \otimes \mathbb{1}_{\mathrm{L}(\mathcal{Y})})(|\xi\rangle\langle\xi|_{\mathcal{X}\otimes\mathcal{Y}}) = \mathrm{Tr}_{\mathcal{Z}} \left(U_{\mathcal{X}\otimes\mathcal{Z}} \otimes frm[o]--_{\mathcal{Y}}\right) \left(|\xi\rangle\langle\xi|_{\mathcal{X}\otimes\mathcal{Y}} \otimes |0\rangle\langle 0|_{\mathcal{Z}}\right) \Big(U^{\dagger}_{\mathcal{X}\otimes\mathcal{Z}}$$
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Denote

$$|x\rangle = \left(U_{\mathcal{X}\otimes\mathcal{Z}} \otimes frm[o]--\mathcal{Y}\right)|\xi\rangle_{\mathcal{X}\otimes\mathcal{Y}}|0\rangle_{\mathcal{Z}}, \quad |y\rangle = \left(V_{\mathcal{X}\otimes\mathcal{Z}} \otimes frm[o]--\mathcal{Y}\right)|\xi\rangle_{\mathcal{X}\otimes\mathcal{Y}}| \tag{24}$$

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Hence

$$F_{\mathrm{ch}}(\Phi, \Psi; \sigma) = F(\mathrm{Tr}_{\mathcal{Z}} |x\rangle\langle x|, \mathrm{Tr}_{\mathcal{Z}} |y\rangle\langle y|) = \max_{W_{\mathcal{Z}}} |\langle y| \, frm[o]--_{\mathcal{X}\otimes\mathcal{Y}} \otimes W_{\mathcal{Z}} \, |x\rangle| \tag{25}$$

$$\langle y|\, frm[o]{-}{-}_{\mathcal{X}\otimes\mathcal{Y}\otimes} W_{\mathcal{Z}}\, |x\rangle = \mathrm{Tr}\left[frm[o]{-}{-}_{\mathcal{Y}} \otimes \left(V^{\dagger}_{\mathcal{X}\otimes\mathcal{Z}} \left(frm[o]{-}{-}_{\mathcal{X}} \otimes W_{\mathcal{Z}} \right) U_{\mathcal{X}\otimes} \right.\right. \tag{26}$$

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We note $\mathrm{Tr} A\Xi(B) = \mathrm{Tr}\Xi^{\dagger}(A)B$

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Recall

$$\| \mathrm{Tr}_{\mathcal{Y}} A_{\mathcal{X}\otimes\mathcal{Y}} \|_1 = \max_{W_{\mathcal{X}}} \left| \mathrm{Tr} \left(W_{\mathcal{X}} \otimes frm[o]{-}{-}_{\mathcal{Y}} \right) A \right| \tag{28}$$

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$$\max_{W_{\mathcal{Z}}} \left| \langle y| \, frm[o]{-}{-}_{\mathcal{X}\otimes\mathcal{Y}} \otimes W_{\mathcal{Z}} \, |x\rangle \right| = \max_{W_{\mathcal{Z}}} \left| \mathrm{Tr} \left(frm[o]{-}{-}_{\mathcal{X}} \otimes W_{\mathcal{Z}} \right) U_{\mathcal{X}\otimes\mathcal{Z}} \left(\sigma_{\mathcal{X}} \otimes |0\rangle \right.\right. \tag{27}$$

Recall

$$\left\| \mathrm{Tr}_{\mathcal{Y}} A_{\mathcal{X}\otimes\mathcal{Y}} \right\|_1 = \max_{W_{\mathcal{X}}} \left| \mathrm{Tr} \left(W_{\mathcal{X}} \otimes frm[o]{-}{-}_{\mathcal{Y}} \right) A \right| \tag{28}$$

Hence

$$\max_{W_{\mathcal{Z}}} \left| \langle y| \, frm[o]{-}{-}_{\mathcal{X}\otimes\mathcal{Y}} \otimes W_{\mathcal{Z}} \, |x\rangle \right| = \left\| \mathrm{Tr}_{\mathcal{X}} U_{\mathcal{X}\otimes\mathcal{Z}} \left(\sigma_{\mathcal{X}} \otimes |0\rangle\langle 0|_{\mathcal{Z}} \right) V^{\dagger}_{\mathcal{X}\otimes\mathcal{Z}} \right\|_1 \tag{29}$$

We calculate the k, l matrix element

$$\begin{aligned}
&\sum_i \langle i|_{\mathcal{X}} \langle k|_{\mathcal{Z}} U_{\mathcal{X}\otimes\mathcal{Z}} \left(\sigma_{\mathcal{X}} \otimes |0\rangle\langle 0|_{\mathcal{Z}}\right) V^{\dagger}_{\mathcal{X}\otimes\mathcal{Z}} |l\rangle_{\mathcal{X}} |i\rangle_{\mathcal{Z}} = \\
=&\sum_i \langle i|_{\mathcal{X}} \left(frm[o]--_{\mathcal{X}} \otimes \langle k|_{\mathcal{Z}}\right) U_{\mathcal{X}\otimes\mathcal{Z}} (frm[o]--_{\mathcal{X}} \otimes |0\rangle_{\mathcal{Z}}) \sigma_{\mathcal{X}} (frm[o]--_{\mathcal{X}} \otimes \langle 0|_{\mathcal{Z}} \\
&= \mathrm{Tr} A_k \sigma B_l^{\dagger}
\end{aligned} \tag{30}$$

We calculate the k, l matrix element

$$\sum_i \langle i|_{\mathcal{X}} \langle k|_{\mathcal{Z}} U_{\mathcal{X}\otimes\mathcal{Z}} (\sigma_{\mathcal{X}} \otimes |0\rangle\langle 0|_{\mathcal{Z}}) V^{\dagger}_{\mathcal{X}\otimes\mathcal{Z}} |l\rangle_{\mathcal{X}} |i\rangle_{\mathcal{Z}} =$$
$$= \sum_i \langle i|_{\mathcal{X}} (frm[o]--_{\mathcal{X}} \otimes \langle k|_{\mathcal{Z}}) U_{\mathcal{X}\otimes\mathcal{Z}} (frm[o]--_{\mathcal{X}} \otimes |0\rangle_{\mathcal{Z}}) \sigma_{\mathcal{X}} (frm[o]--_{\mathcal{X}} \otimes \langle 0|_{\mathcal{Z}}$$
$$= \mathrm{Tr} A_k \sigma B_l^{\dagger} \tag{30}$$

And thus

$$\max_{W_{\mathcal{Z}}} |\langle y| \, frm[o]--_{\mathcal{X}\otimes\mathcal{Y}} \otimes W_{\mathcal{Z}} |x\rangle| = \left\| \left\{ \mathrm{Tr} A_k \sigma B_l^{\dagger} \right\}_{k,l} \right\|_1 \tag{31}$$

$\square$

Jan Bouda

Question

I am not completely sure about the approach candidate took to test the properties of the superfidelity. Why are these particular examples interesting? Also, the question is whether you want to test how well superfidelity distinguishes two quantum channels, or rather how well it measures some quantity with well established operational meaning — such as the trace norm distance.

Jan Bouda

Question

I am not completely sure about the approach candidate took to test the properties of the superfidelity. Why are these particular examples interesting? Also, the question is whether you want to test how well superfidelity distinguishes two quantum channels, or rather how well it measures some quantity with well established operational meaning — such as the trace norm distance.

Answer

Our main goal here was to provide an upper bound on the fidelity of the output of two quantum channels. Our main goal was to study it in the setting of quantum control and its sensitivity to errors in the control functions.

Piotr Zawadzki

Question

A wider discussion of the numerics behind the simulations.

Piotr Zawadzki

$$J(\varepsilon) = \mathrm{tr}\{F_0(\rho(T))\}, \tag{32}$$

$$\mathrm{tr}\{F_0(\rho(T))\} = \frac{1}{2}||\rho(T) - \rho_\mathrm{T}||_\mathrm{F}^2, \tag{33}$$

$$\frac{\partial J}{\partial \varepsilon_k} = \mathrm{tr}\left\{-\mathrm{i}\lambda_k \left[\frac{\partial H(\varepsilon_k)}{\partial \varepsilon_k}, \rho_k\right]\right\} \Delta t_k, \tag{34}$$

Pontryagin's minimum principle states that a necessary condition for existence of a solution is existence of such λ that

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = -\mathrm{i}[H(\varepsilon(t)), \rho(t)] - \mathrm{i}L_\mathrm{D}[\rho(t)],\ t \in [0, T], \tag{35}$$

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = -\mathrm{i}[H(\varepsilon(t)), \lambda(t)] - \mathrm{i}L_\mathrm{D}^\dagger[\lambda(t)],\ t \in [0, T], \tag{36}$$

$$L_\mathrm{D}[A] = \mathrm{i}\sum_j \gamma_j (L_j A L_j^\dagger - \frac{1}{2}\{L_j^\dagger L_j, A\}), \tag{37}$$

$$\rho(0) = \rho_\mathrm{s}, \tag{38}$$

$$\lambda(T) = F_0'(\rho(T)). \tag{39}$$

Full derivation of the previous equation??

- The parameters ε are optimized using the BFGS algorithm
  Should I add a description of BFGS?
- The differential equations are solved using the `zvode` solver
- For high γ the system becomes stiff, in that case `ode` is used on an equivalent real system