6 Options

(European)
A [1] **call option** is a contract between a buyer and seller that gives buyer the right, but not the obligation, to purchase underlying asset at a predetermined price (strike price) at an agreed upon future date (expiration date).

Put Option: Same idea except buyer of option can sell ....

E.g.

Call option 1

- Stock currently trading @ 100
- Strike price of 110.
- Option Expires in 1 year
- In 1-year stock market value of 120.

Contract owner buys stock from contract seller for 110, sells on market for 120 keeps 10 profit.

Call Option 2

Same example but after 1 year stock mkt value is

105. Contract is worthless and owner of contract does not use it.

Stock S, strike price K.

t = time of expiration

Value of call option @ time t

If $S(t) \geq K$

Value is $S(t) - K$.

If $S(t) \leq K$

Value is 0.

OR

$$\text{Max}\,(S(t) - K, 0)$$
$$= (S(t) - K)^+ .$$

Put Option: Strike K,

expires @ time t,

If $S(t) \le K$

then value is

$$-S(t) + K$$

If $S(t) \ge K$, value is

0.

Value is

$$\left(-S(t) + K\right)^+$$

Option Terminology / Notation

- Strike Price: K agreed upon buy (~~opt~~ call) or sell (put) in future
- Spot Price: Price of underlying Stock
- Expiration Date:
- Time to Expiration: t

(Current time is t and Expiration date is T then time to expiration is $T-t$)

- Out of Money Option

Call: Spot Price $\leq K$

Put: Spot Price > K

- In the Money Option

Call: Spot Price > K

Put: Spot Price < K.

Premium: The price a buyer pays to own option contract.

~~Ca~~ Profit & Loss of Call Option (Buyer)

Strike = K

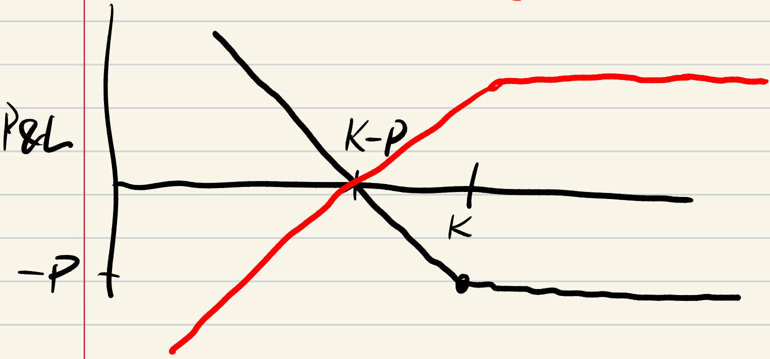
Premium = P

P&L Call (Seller)

Rmk. Option contracts (typically) sold in multiples of 100.

Selling options can have huge risk if not managed correctly.

P & L Puts (Buyer) (Seller)



Question:

~~What~~

What is the expected value of a call/put option?

Assume $t > 0$

$S(t)$ = distribution of stock end points is modeled by the standard Black-Scholes / Geometric Brownian Motion

$$S(t) = S(0)e^{-\frac{\sigma^2}{2}t + \sqrt{t}\,\sigma\,\mathcal{N}(0,1)}$$

where $\sigma =$ yearly std. dev. of log-returns.

Let $K > 0$ Strike Price.

Want to find

$$C(0) = E[\operatorname{Max}(S(t) - K), 0]$$

where $t =$ time to expiration.

Thm (Black-Scholes Option Prices)

$$\star\ C(0) = S(0)\,\Phi(d_1) - K\,\Phi(d_2)$$

$$P(0) = -S(0)\,\Phi(-d_1) + K\,\Phi(-d_2).$$

$$\Phi(y) = \mathrm{Prob}(\mathcal{N}(0,1) \leq y)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-z^2/2}\, dz$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \frac{\sigma^2}{2} t}{\sqrt{t}\,\sigma}$$

$$d_2 = \frac{\ln\left(\frac{S(0)}{K}\right) - \frac{\sigma^2}{2} t}{\sqrt{t}\,\sigma}$$

$$= d_1 - \sqrt{t}\,\sigma.$$

Proof of Call Option Price

$$E[(S(t)-K)^+]$$

Let $\mathbb{1}_{S(t)-K\geq 0} = \begin{cases} 1 & \text{if } S(t)\geq K \\ 0 & \text{else} \end{cases}$

$$E[(S(t)-K)^+]$$

$$= E[(S(t)-K)\mathbb{1}_{S(t)-K\geq 0}].$$

$$= \underbrace{E[S(t)\mathbb{1}_{S(t)-K\geq 0}]}^{S(t)\,\Phi(d_1)}$$

$$\underbrace{- K E[\mathbb{1}_{S(t)-K\geq 0}]}_{-K\,\Phi(d_2)}$$

$$E[\mathbb{1}_{S(t)-K\geq 0}]$$

$$S(t)-K\geq 0$$

$$S(0)e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}\mathcal{N}(0,1)} \geq K$$

$$\Leftrightarrow -\frac{\sigma^2}{2}t+\sigma\sqrt{t}\mathcal{N}(0,1)\geq \ln(K/S(0))$$

$$\Leftrightarrow \mathcal{N}(0,1) \geq \frac{\frac{\sigma^2}{2}t+\ln(K/S_0)}{\sigma\sqrt{t}}$$

$$= \frac{\frac{\sigma^2}{2}t-\ln(S_0/K)}{\sigma\sqrt{t}}$$

$$=: -d_2$$

$$\mathbb{1}_{S(t)-K\geq 0} = \mathbb{1}_{\mathcal{N}(0,1)\geq d_2}$$

$$\mathbb{E}\left[\mathbb{1}_{\mathcal{N}(0,1)\geq -d_2}\right]$$

$$= \Phi(+d_2)$$

<u>Other piece:</u>

$$\mathbb{E}\left[S(t)\cdot\mathbb{1}_{S(t)-K\geq 0}\right]$$

$$= \mathbb{E}\left[S(0)\,e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}\,\mathcal{N}(0,1)}\,\mathbb{1}_{\substack{\mathcal{N}(0,1)\\ \geq d_2}}\right]$$

$$= S(0)\,\mathbb{E}\left[e^{-\frac{\sigma^2}{2}+\sigma\sqrt{t}\,\mathcal{N}(0,1)}\,\mathbb{1}_{\mathcal{N}(0,1)\geq -d_2}\right]$$

$$= \frac{S(0)}{\sqrt{2\pi}}\int_{-d_2}^{\infty} e^{-\frac{\sigma^2}{2}t+\sigma\sqrt{t}\,z}\,e^{-\frac{z^2}{2}}\,dz$$

exponent:

$$-\frac{z^2}{2} + \sigma\sqrt{t}\, z - \frac{\sigma^2}{2} t$$

$$= \frac{-1}{2}\left(z^2 - 2\sigma\sqrt{t}\, z + \sigma^2 t\right)$$

$$= \frac{-1}{2}\left(z - \sigma\sqrt{t}\right)^2$$

$$= \frac{S(0)}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{t})^2} dz$$

$$= \frac{S(0)}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{t}}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$d_2 + \sigma\sqrt{t} = d_1$$

$$= \frac{S(0)}{\sqrt{2\pi}} \int_{-\infty}^{d_2+\sigma\sqrt{t}} e^{-\frac{1}{2}u^2} \, du$$

$$= S(0)\, \Phi(d_1). \qquad \blacksquare$$

Call - Put Parity

$C(t) - P(t)$ = Pay off of Call − Pay off of Put

$$= (S(t)-K)^+ - (-S(t)+K)^+$$

If $S(t) \geq K$

$$C(t)-P(t) = S(t) - K - 0$$

If $S(t) \leq K$

$$C(t) - P(t) = 0 - (-S(t)+K)$$

$$= S(t) - K.$$

So

$$C(t) - P(t) = S(t) - K.$$

OR

$$C(t) = P(t) + S(t) - K.$$

$$E[C(t)] = E[P(t)] + E[S(t)] - K$$

$$\begin{array}{ccccccc} \| & & \| & & \| & & \\ C(0) & = & P(0) & + & S(0) & - & K. \end{array}$$

$$\boxed{C(0) = P(0) + S(0) - K}$$