3 Random Walks

Random Walks are used to model stock price movements.

S_0 = initial value

$S_1, S_2, \ldots, S_t, \ldots$

Random values,

Basic Random Walk:

$$S_t = Z_1 + Z_2 + \cdots + Z_t$$

where Z_i's are i.i.d.s,

in particular, $E[Z_i] = E[Z_j]$

$Var(z_i) = Var(z_j)$

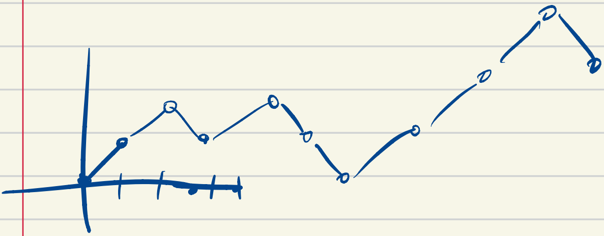
$Corr(z_i, z_j) = 0$, in particular,

$E[z_i z_j] = E[z_i]E[z_j]$ if

$i \neq j$.

Toy Example:

$$z_i = \begin{cases} 1 & p = \frac{1}{2} \\ -1 & p = \frac{1}{2} \end{cases}$$



If $S_t = z_1 + \cdots + z_t$

$$E[S_t] = E[z_1] + \cdots + E[z_t]$$
$$= t\,E[z_i].$$

$Var(S_t)$

$$= E\left[\left(z_1 + \cdots + z_t - t\,E[z_t]\right)^2\right]$$

$$= E\left[\left((z_1 - E[z_1]) + \cdots + (z_t - E[z_t])\right)^2\right]$$

$$= \sum_{i=1}^{t} E\left[(z_i - E[z_i])^2\right] + \underbrace{\cancel{\sum_{i \neq j} E\left[(z_i - E[z_i]) \times (z_j - E[z_j])\right]}}_{0}$$

$$= \sum_{i=1}^{t} Var(z_i)$$

$$= t\,Var(z_1).$$

$z_i - E[z_i] + z_j - E[z_j]$

if $i \neq j$ then uncorrelated

if Y_1, Y_2 uncorrelated

$$E[Y_1 Y_2] = E[Y_1]E[Y_2]$$

$$E[(z_i - E[z_i])(z_j - E[z_j])]$$

$i \neq j$

$$= E[(z_i - E[z_i])]\,E[(z_j - E[z_j])]$$

$$= 0 \times 0 = 0.$$

Normal ~~Generalized~~ Random Walks

$$S_t = r_1 + r_2 + \cdots + r_t$$

$$r_i = \mu + \sigma z_i .$$

where μ, σ constants

$z_i = N(0,1)$ standard normal distribution.

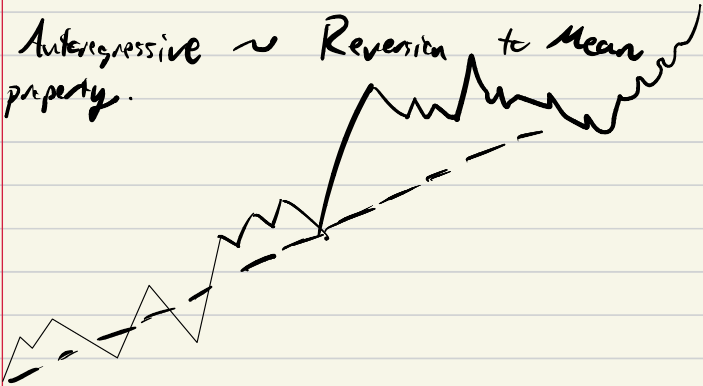
r_i = Normal distribution with mean μ and std. deviation σ.

$$E[S_t] = t\mu$$

$$Var(S_t) = t\sigma^2$$

Autoregressive and Moving Averages Random Walks

- Autoregressive ~ Reversion to Mean property.



$$S_t = r_1 + r_2 + \cdots + r_t$$

$$r_i = c_0 + c_1 r_{i-1} + \cdots + \cancel{r_p c_p}\ c_p r_{i-p} + \sigma z_i.$$

- p = # of lags.
- If $p=0$, we are back to a normal random walk.

- p = # of lags.
- Typically want $|c_1| + \cdots + |c_p| < 1$
- Assume a stationary condition:

$$E[r_i] = E[r_j] \quad \forall i, j$$
$$var(r_i) = Var(r_j).$$

- z_i = standard normal distribution

Closer look at AR(1) Random walk:

$$S_t = r_1 + \cdots + r_t$$

$$r_i = c_0 + c_1 r_{i-1} + \sigma z_i$$

Find $E[r_i]$:

$$E[r_i] = c_0 + c_1 \underbrace{E[r_{i-1}]}_{= E[r_i]} + \sigma \underbrace{\cancel{E}[z_i]}_{=0}$$

$$E[r_i](1-c_1) = c_0$$

or

$$E[r_i] = \frac{c_0}{1-c_1} =: \mu \; .$$

Rewrite $r_i = \mu + (\text{Extra stuff})$

$$r_i - \mu = c_0 + c_1 r_{i-1} + \sigma z_i - \frac{c_0}{1-c_1}$$

$$= \frac{c_0(1-c_1) - c_0}{1-c_1} + c_1 r_{i-1} + \sigma z_i$$

$$= \frac{-c_0 c_1}{1-c_1} + c_1 r_{i-1} + \sigma z_i$$

$$= c_1(r_{i-1} - \mu) + \sigma z_i$$

$$r_i = \mu + c_1(r_{i-1} - \mu) + \sigma z_i .$$

Noise.

$Var(r_i)$

$$= E[(r_i - \mu)^2]$$

$$= E[(c_1(r_{i-1} - \mu) + \sigma z_i)^2]$$

Uncorrelated

$$= E[(c_1(r_{i-1} - \mu))^2] + E[(\sigma z_i)^2]$$

$$= c_1^2 E[(r_{i-1} - \mu)^2] + \sigma^2 E[z_i^2]$$

$$= c_1^2 Var(r_{i-1}) + \sigma^2$$

$$= c_1^2 Var(r_i) + \sigma^2$$

$$Var(r_i)(1-c_1^2) = \sigma^2$$

$$\boxed{Var(r_i) = \frac{\sigma^2}{1-c_1^2}.}$$

$$E[S_t] = E[r_1 + \cdots + r_t]$$
$$= t\mu.$$

$$Var(S_t) = E[(S_t - E[S_t])^2]$$
$$= E[(r_1 + \cdots + r_t - t\mu)^2]$$
$$= E[((r_1-\mu) + (r_2-\mu) + \cdots (r_t-\mu))^2]$$

Algebra Magic

$$= \frac{\sigma^2}{1-c_1^2}\left[t + 2\sum_{i=1}^{t-1}(t-i)|c_1|^i\right]$$

$$r_i = \frac{1}{2} - \frac{1}{2}\left(r_{i-1} - \frac{1}{2}\right) + \sigma z_i .$$

Moving Average:

Moving average w/ q lags:

$MA(q)$:

$$S_t = r_1 + r_2 + \cdots + r_t .$$

Normal Random Walk: $r_i = \mu + \sigma z_i$

$$r_i = \mu + \left(\phi_1 z_{i-1} + \cdots + \phi_q z_{i-q}\right) + \sigma z_t$$

Allowing the past

q volatilities to influence r_i.

$$E[r_i] = \mu + \phi_1 \cancel{E[z_{t-1}]} + \cdots + \phi_q \cancel{E[z_{t-q}]} + \sigma \cancel{E[z_t]}$$

$$= \mu.$$

For $MA(1)$:

$$r_t = \mu + \phi_1 z_{t-1} + \sigma z_t.$$

$$Var(r_t) = E[(r_t - \mu)^2]$$

$$= E[(\phi_1 z_{t-1} + \sigma z_t)^2]$$

uncorrelated

$$= E[(\phi_1 z_{t-1})^2 + (\sigma z_t)^2]$$

$$= \phi_1^2 + \sigma^2 .$$

$$E[S_t] = t\mu .$$

$$Var(S_t) = E[(S_t - t\mu)^2]$$

$$= E[((r_1-\mu) + (r_2-\mu) + \cdots + (r_t-\mu))^2]$$

$$= \sum_{i=1}^{t} E[(r_i-\mu)^2] + \sum_{i \neq j} E[(r_i-\mu)(r_j-\mu)] .$$

☆ $r_i = \mu + \phi_1 z_{i-1} + \sigma z_i$

$r_i - \mu$ and $r_j - \mu$ are uncorrelated if $|i-j| \geq 2$

$$= t(\phi^2 + \overset{\sigma^2}{\cancel{\mu^2}}) + 2 \sum_{1 \le i \le t-1} E[(r_i - \mu)(r_{i+1} - \mu)]$$

$$E[(r_i - \mu)(r_{i+1} - \mu)]$$

$$= E[(\phi_1 z_{i-1} + \sigma z_i)(\phi_1 z_i + \sigma z_{i+1})]$$

$$= E[\sigma \phi_1 z_i^2] = \sigma \phi_1 .$$

$$= t(\phi_1^2 + \mu^2) + 2(t-1)\sigma\phi_1$$

$$\boxed{= (t-1)(\phi_1 + \sigma)^2 + \phi_1^2 + \sigma^2}$$

$$\text{ARMA}(p,q) = \text{Combine } \text{AR}(p) + \text{AR}(q)$$

$$S_t = r_1 + \cdots + r_t$$

$$r_i = \overbrace{(c_0 + c_1 r_{i-1} + \cdots + c_p r_{i-p})}^{\text{AR piece}}$$

$$+ \overbrace{(\phi_1 z_{i-1} + \cdots + \phi_q z_{i-q})}^{\text{MA piece w/ } \mu=0}$$

$$+ \underbrace{\sigma z_i}_{\text{Standard noise.}}$$