7 Black-Scholes and Delta Hedging

Recall:

Call / Put: Strike price K

time now is 0 expiration at t.

$$C(t) = \operatorname{Max}(S(t) - K, 0)$$

$$P(t) = \operatorname{Max}(-S(t) + K, 0)$$

Black-Scholes Formula for

$$C(0) := E[C(t)]$$

$$P(0) := E[P(t)]$$

If we assume no risk-free interest ($r = 0$) and no drift

$$\sim S(t) = S(0) e^{-\frac{\sigma^2}{2}t + \sigma\sqrt{t}\,\mathcal{N}(0,1)}.$$

then

$$C(0) = S(0)\,\Phi(d_1) - K\,\Phi(d_2)$$

$$P(0) = -S(0)\,\Phi(-d_1) + K\,\Phi(-d_2).$$

Φ = c.d.f. of $\mathcal{N}(0,1)$

$$d_1 = \frac{\ln(S(0)/K) + \frac{\sigma^2}{2}t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}.$$

Black-Scholes with drift and risk-free interest rate.

Black-Scholes Assumption:

Stock end points is modeled

by

$N(0,1)$

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma\sqrt{t}\cancel{dt}}$$

Value @ time t.

Discounted price of $S(t)$ at time $t=0$ is

$$e^{-rt}S(t) = S(0)e^{(\mu - r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}N(0,1)}$$

a

Call option expires @ time t.

Want to find

$$C(0) = e^{-rt} E[\max[S(t)-K, 0]]$$

term discounts expected payoff @ expiration time t to time 0.

Overview of Risk-Neutral Pricing:

Assume:

Possible to continuously purchase/sell the underlying stock in fractional shares and this can be done in a manner that creates portfolios so that for each

$0 < T \leq t$

$$X(T) = C(T) = e^{-r(t-T)} E\left[\max(S(t)-K, 0) \,\middle|\, \text{@ time } 0 < T \leq t\right]$$

- The effect is that this portfolio neutralizes the drift term μ.

and $X(0)$ has to equal the value of same call option, but the stock is modeled to be risk-neutral with respect to the risk-free interest rate r.

i.e., we can actually assume

$$\Rightarrow \quad S(t) = S(0)\, e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma\sqrt{t}N(0,1)}.$$

$$C(0) = e^{-rt} E\left[(S(t) - K)^+\right].$$

$$= e^{-rt}\left(S(0)e^{rt}\Phi(d_1) - K\Phi(d_2)\right)$$

Algebra

$$= S(0)\,\Phi(d_1) - Ke^{-rt}\Phi(d_2)$$

<u>Call-Put Parity</u>:

$$C(t) - P(t) = S(t) - K$$

$$\Rightarrow C(0) - P(0) = S(0) - Ke^{-rt}.$$

$$\Rightarrow P(0) = S(0)\,\Phi(-d_1) + Ke^{-rt}\,\Phi(-d_2).$$

$$d_1 = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}.$$

Overview:

We want to find

$$E[\max(S(t) - K, 0)]$$

discounted to time 0.

Delta Hedging

Overview: Practical Method of protecting profits (especially if selling call/put options)

this is the discrete process when limited to a continuous process is the key trick in Black-Scholes option pricing.

First Objective:

Find rate of change of $C(0)$ with respect to $S(0)$.

☆ $C(0) = S(0)\,\Phi(d_1) - Ke^{-rt}\,\Phi(d_2)$

Let $\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$

$\left(\Phi(y) = \int_{-\infty}^{y} \varphi(z)\,dz \right)$.

Want to determine

$$\frac{\partial C(0)}{\partial S(0)}$$

$$= \Phi(d_1) + S(0)\frac{\partial \Phi(d_1)}{\partial S(0)} - Ke^{-rt}\frac{\partial \Phi(d_2)}{\partial S(0)}.$$

$$= \Phi(d_1) + S(0)\left[\varphi(d_1)\frac{\partial d_1}{\partial S(0)}\right] - Ke^{-rt}\left[\varphi(d_2)\frac{\partial d_2}{\partial S(0)}\right]$$

Recall: $d_2 = d_1 - \sigma\sqrt{t}$

$$\frac{\partial d_2}{\partial S(0)} = \frac{\partial d_1}{\partial S(0)}$$

$$= \Phi(d_1) + \frac{\partial d_1}{\partial S(0)}\left[\cancel{S(0)\varphi(d_1) - Ke^{-rt}\varphi(d_2)}\right] \quad 0$$

Algebra Magic: $S(0)\varphi(d_1) = Ke^{-rt}\varphi(d_2)$

$$= \Phi(d_1).$$

To find $\dfrac{\partial P(0)}{\partial S(0)}$

use Put-Call Parity:

$$C(0) - P(0) = S(0) - \underline{Ke^{-rt}}$$

$$\Phi(d_1) - \frac{\partial P(0)}{\partial S(0)} = 1 - 0$$

OR $\dfrac{\partial P(0)}{\partial S(0)} = \cancel{1 - \Phi(d_1)}$.

$\Phi(d_1) - 1$

Suppose we sell call 100 call options @ time 0 exp. @ time t for premium P.

@ Time 0

~~Expected~~ Portfolio value ~~in~~ @ time 0 is $100(P - C(0))$.

$$X(0) = 100(P - C(0))$$

$$\frac{\partial X(0)}{\partial S(0)} = -\frac{\partial C(0)}{\partial S(0)} \cdot 100$$

$$= -100 \cdot \Phi(d_1)$$

This says portfolio value is negatively correlated with changes in $S(0)$

We "Δ-hedge" by buying $100 \cdot \Phi(d_1)$ shares of stock:

$$X_{hedge}(0) = (P - C(0)) \cdot 100 + S(0) \frac{\partial C(0)}{\partial S(0)} \cdot 100$$

$$\frac{\partial X_{hedge}(0)}{\partial S(0)} = 0.$$