# 5 Stock Paths as Geometric Brownian Motion

**Last Time:**

n-step binomial tree model of a random walk

$S(0), S(1), \dots, S(n)$

where

$$S(n) = r_1 + \cdots + r_n$$

$$r_i = \begin{cases} u & \text{prob } p \\ d & 1-p \end{cases} \qquad \begin{matrix} u > 0 \\ d < 0 \end{matrix}$$

$$\frac{S(i)}{S(i-1)} = \begin{cases} 1+u & p \\ 1+d & 1-p \end{cases}.$$

## Notation:

Let $H(n) = \#$ of times walk moves up

$T(n) = \#$ of times walk moves down

$$\frac{S(n)}{S(0)} = \frac{S(1)}{S(0)} \cdot \frac{S(2)}{S(1)} \cdots \frac{S(n)}{S(n-1)}$$

$$= (1+u)^{H(n)} (1+d)^{T(n)}.$$

## Symmetric Random Walk

$$M_n = r_1 + \cdots + r_n$$

$$r_i = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$E[M_n] = 0$

$Var(M_n) = \sum^{n} Var(r_i) = n.$

## <u>Scaled Symmetric Walk</u>

Let $t > 0$, $t \in \mathbb{Q}$

Let $n \in \mathbb{N}_{>0}$ so that

$nt \in \mathbb{N}$.

$$W_n(t) := \frac{1}{\sqrt{n}} M_{nt}$$

Scaled Symmetric Walk with $nt$ steps by $\frac{1}{\sqrt{n}}$.

$E[W_n(t)] = 0$

$$Var(W_n(t)) = \frac{1}{n} Var(M_{nt}) = \frac{\cancel{n}t}{\cancel{n}} = t.$$

Central Limit Thm Application
(Good Exercise)

$$\lim_{n\to\infty} W_n(t) = N(0, \sqrt{t})$$
$$= \sqrt{t}\, N(0,1)$$

the normal distribution with mean $0$ and standard deviation $\sqrt{t}$.

<u>Problem:</u> Let $t \in \mathbb{Q}_{>0}$

$S(t)$ = distribution of stock path endpoints at time $t$ starting from time $0$,

<u>Assume:</u>

1) For each $n \in \mathbb{N}$, $nt \in \mathbb{N}$, there is a $nt$ binomial walk $S_n(t)$ approximating $S(t)$

2) $E[S(t) - S(0)]$

$= E[S_n(t) - S(0)] = 0$

(Remark: we are assuming a drift of 0, we'll go over how to include positive or negative drift).

3) There is $\sigma$ so that for each $n$,

$$\mathrm{Var}(S_n(t)) = \sigma^2 t.$$

4) At each step of $S_n(t)$ we assign probability of $\frac{1}{2}$ of moving up or down.

Thm. $S(t) = S(0)e^{-\frac{\sigma^2}{2}t + \sigma\sqrt{t}\,N(0,1)}$

where $N(0,1)$ = standard normal distribution.

Pf

$$S_n(t) - S(0) = r_{n,1} + r_{n,2} + \cdots + r_{n,tn}$$

$$\rightarrow r_{n,i} = \begin{cases} u_n & 1/2 \\ d_n & 1/2 \end{cases}$$

$$E[S_n(t) - S(0)] = 0$$

$$\parallel$$

$$\sum^{tn} E[r_{n,i}] = tn\, E[r_{n,i}]$$

So $E[r_{n,i}] = 0$

$$Var(S_n(t) - S(0))$$

$$= Var(S_n(t)) = \sigma^2 t$$

$$= \sum^{nt} Var(r_{n,i}) = nt\, Var(r_{n,i})$$

So, $Var(r_{n,i}) = \dfrac{\sigma^2}{n}$.

Some algebra

$$\Rightarrow \quad r_{n,i} = \begin{cases} \frac{\sigma}{\sqrt{n}} & 1/2 \\ -\frac{\sigma}{\sqrt{n}} & 1/2 \end{cases}$$

Let $H_n(t)$ = distribution # times $S_n(t)$ moves up.

$T_n(t)$ = distribution # times $S_n(t)$ moves down.

$$\frac{S_n(t)}{S(0)} = (1+u_n)^{H_n(t)} (1+d_n)^{T_n(t)}$$

$$= \left(1+\frac{\sigma}{\sqrt{n}}\right)^{H_n(t)} \left(1-\frac{\sigma}{\sqrt{n}}\right)^{T_n(t)}$$

- $H_n(t) + T_n(t) = nt$
- $H_n(t) - T_n(t) = \cancel{nt - M_{nt}}$ $M_{nt}$

$$H_n(t) = \frac{tn + \mu_n t}{2}$$

$$T_n(t) = \frac{tn - \mu_n t}{2}$$

$$\frac{S_n(t)}{S(0)} = \left(1 + \frac{\sigma}{\sqrt{n}}\right)^{\frac{tn + \mu_n t}{2}} \left(1 - \frac{\sigma}{\sqrt{n}}\right)^{\frac{tn - \mu_n t}{2}}$$

$$\ln\left(\frac{S_n(t)}{S(0)}\right) = \left(\frac{tn + \mu_n t}{2}\right) \underline{\ln\left(1 + \frac{\sigma}{\sqrt{n}}\right)}$$

$$+ \left(\frac{tn - \mu_n t}{2}\right) \underline{\ln\left(1 - \frac{\sigma}{\sqrt{n}}\right)}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3)$$

where $O(x^A)$ = error terms

so that $\forall \delta > 0$, $\lim_{x \to 0} \frac{O(x^A)}{x^{A+\delta}} = 0$.

$$\ln(1+x) = x - \frac{1}{2}x^2 + O(x^3).$$

$$\ln\left(\frac{S_n(t)}{S(0)}\right) = \left(\frac{t_n + \mu_{nt}}{2}\right)\left(\frac{\sigma}{\sqrt{n}} - \frac{1}{2}\frac{\sigma^2}{n} + O(n^{-\frac{3}{2}})\right)$$

$$+ \left(\frac{t_n - \mu_{nt}}{2}\right)\left(\frac{-\sigma}{\sqrt{n}} - \frac{1}{2}\frac{\sigma^2}{n} + O(n^{-\frac{3}{2}})\right)$$

$$= -\frac{t\sigma^2}{2} + \frac{\sigma}{\sqrt{n}}\mu_{nt} + O(n^{-1/2})$$
$$+ \mu_{nt} \cdot O(n^{-3/2})$$

$$\frac{1}{\sqrt{n}}\mu_{nt} = W_n(t)$$

$$\Rightarrow \mu_{nt} O(n^{-3/2}) = W_n(t) O(n^{-1})$$
$$\xrightarrow{\text{as}} 0 \quad \text{as } n \to \infty.$$

$$\lim_{n\to\infty} \ln\left(\frac{S_n(t)}{S(0)}\right) = \frac{-t\sigma^2}{2} + \lim_{n\to\infty} \sigma W_n(t)$$

$$= \frac{-t\sigma^2}{2} + \sigma\sqrt{t}\, N(0,1).$$

$$= \ln\left(\frac{S(t)}{S(0)}\right)$$

$$S(t) = S(0)e^{\frac{-t\sigma^2}{2} + \sigma\sqrt{t}\, N(0,1)}.$$

$\blacksquare$

## Drift and Interest Considerations

**Interest:** Time value of money in the future. $r$ = cont. compounded interest rate per year.

$\$X$ in $t$ amount of time is now $e^{rt}X$.

Distribution

Value of $S(t)$ at time 0

is $e^{-rt}S(t)$ =

$$S(t) = S(0)e^{-rt - \frac{\sigma^2}{2}t + \sigma\sqrt{t}N(0,1)}.$$

If $\mu$ = drift of log returns

Then

$$S(t) = S(0)e^{\mu t - rt - \frac{\sigma^2}{2}t + \sigma\sqrt{t}N(0,1)}$$

$$= S(0)e^{(\mu - r - \frac{\sigma^2}{2})t + \sigma\sqrt{t}N(0,1)}.$$

**Remark:** If $rt \notin \mathbb{N}$,

still define

$W_n(t)$ as follows:

Set $q = tn - \lfloor tn \rfloor$

$$W_n(t) = (1-q)\frac{1}{\sqrt{n}} M_{\lfloor nt \rfloor} + q\frac{1}{\sqrt{n}} M_{\lceil nt \rceil}.$$

$E[W_n(t)] = 0$

$\lim_{n\to\infty} Var(E[W_n(t)]) = t$

$$\lim_{n\to\infty} W_n(t) = N(0, \sqrt{t}).$$