2. Correlation and Covariance Matrices.

Toy Example

Investor has \$10,000 to invest in 2 stocks, S_1, S_2.

Assume initial price of stocks S_1 and S_2 at time $t=0$ is $S_1(0) = S_2(0) = 100$

The price of S_1 and S_2 at time $t=1$, has probability distribution

$$S_1(1) = \begin{cases} 120 & \text{prob. } 3/5 \\ 90 & \text{prob } 2/5 \end{cases}$$

$$S_2(1) = \begin{cases} 110 & 4/5 \\ 90 & 1/5 \end{cases}$$

X_1 = R.O.I. all in on S_1.

$$X_1 = \begin{cases} \dfrac{(120-100)\times \cancel{100}}{100 \times \cancel{100}} = \overset{2/5}{\cancel{1/5}} & p = \frac{3}{5} \\[2ex] \dfrac{90-100}{100} = -1/10 & p = \frac{2}{5} \end{cases}$$

$$E[X_1] = \frac{2}{\cancel{5}\,10}\left(\frac{3}{5}\right) + \frac{-1}{10}\left(\frac{2}{5}\right)$$

$$= \frac{4}{50} \quad \text{OR} \quad 8\% \text{ R.O.I.}$$

X_2 = R.O.I. all in on S_2

$$E[X_2] = \frac{4}{5}\cdot\frac{1}{10} + \frac{1}{5}\cdot\left(\frac{-1}{10}\right)$$

$$= \frac{3}{50} \quad \text{OR} \quad 6\% \text{ R.O.I.}$$

$\sigma_{X_1}, \sigma_{X_2}$ std. deviations of X_1, X_2

$$\sigma_{X_1} = \sqrt{E[X_1^2] - E[X_1]^2}$$

$$= \sqrt{\frac{50}{500} - \left(\frac{4}{50}\right)^2}$$

$$\sim 0.147 \quad \text{OR} \quad 14.7\%$$

$\sigma_{X_2} = 0.08$ OR 8%

$X_3 =$ R.O.I. of 50% of funds allocated to S_1 and S_2

$$X_3 = \begin{cases} \text{Both go up} & \frac{1}{2} \cdot \frac{2}{10} + \frac{1}{2} \frac{1}{10} & \overset{\text{prob.}}{\frac{3}{5} \cdot \frac{4}{5}} \\ S_1 \uparrow S_2 \downarrow & \frac{1}{2} \frac{2}{10} + \frac{1}{2} \cdot \frac{-1}{10} & \frac{3}{5} \cdot \frac{1}{5} \\ S_1 \downarrow S_2 \uparrow & \frac{1}{2}\left(\frac{-1}{10}\right) + \frac{1}{2} \cdot \frac{1}{10} & \frac{2}{5} \cdot \frac{4}{5} \\ S_1 \downarrow S_2 \downarrow & \frac{1}{2}\left(\frac{-1}{10}\right) + \frac{1}{2} \cdot \frac{-1}{10} & \frac{2}{5} \cdot \frac{1}{5} \end{cases}$$

$$E[X_3] = \frac{1}{2} E[X_1] + \frac{1}{2} E[X_2]$$

$= 0.07$ or 7%

$$\sigma_{X_3}^2 = E[X_3^2] - E[X_3]^2$$

$\sigma_{X_3} \sim 0.0837$ or

8.37%

More generally, suppose

$w_1, w_2 \geq 0$, $w_1 + w_2 = 1$

are the weights of investment

in S_1 and S_2

$P(0) = \$10{,}000$ $\quad P(1) =$ value at $t=1$

$$P(1) = \left(\cancel{1} + w_1 \frac{S_1(1) - S_1(0)}{S_1(0)} + w_2\left(\frac{S_2(1) - S_2(0)}{S_2(0)}\right)\right) \cdot \cancel{P(0)}$$

R.O.I. with weights w_1, w_2

$$\frac{P(1)-P(c)}{P(c)} = w_1 \frac{S_1(1)-S_1(c)}{S_1(c)} + w_2 \frac{S_2(1)-S_2(c)}{S_2(c)}$$

$$X_3 = w_1 X_1 + w_2 X_2$$

$$\sigma_{X_3}^2 = E[X_3^2] - E[X_3]^2$$

$$= E[(w_1 X_1 + w_2 X_2)^2] - (w_1 E[X_1] + w_2 E[X_2])^2$$

$$= w_1^2 Var(X_1) + w_2^2 Var(X_2)$$

$$+ 2 w_1 w_2 Cov(X_1, X_2)$$

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1]E[X_2].$$

$$= [w_1, w_2] \begin{pmatrix} Var(X_1) & Var(X_1, X_2) \\ Var(X_1, X_2) & Var(X_2) \end{pmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

<u>More general picture</u>:

$X_1, X_2, \ldots, X_t$ are the probability distributions on returns of stocks $S_1, S_2, \ldots, S_t$

Y investment account in $S_1, \ldots, S_t$ with weights $w_1, w_2, \ldots, w_t$ where $\underline{\sum w_i = 1}$, $\underline{w_i} \geq 0$.

Then

$$\sigma_Y^2 = Var(Y) = \boxed{\vec{w}^T Cov(X_1, \dots, X_t)\, \vec{w}}$$

where $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_t \end{bmatrix}$

$$Cov(X_1, \dots, X_t) = \left(Cov(X_i, X_j)\right)_{1 \le i, j \le t.}$$

<u>Takeaway</u> Quadratic optimization allows one to find weights that minimize volatility of a portfolio subject to

predetermined constraints and volatility models.

Efficient Frontier

$S_1, \dots, S_t$ stocks $X_1, \dots, X_t$

R.O.I. random variables.

Consider all portfolios of investments in $S_1, \dots, S_t$.

A portfolio Y (distribution of return)

<u>dominates</u> a portfolio Z

if $E[Y] \geq E[Z]$

and $\sigma_Y \leq \sigma_Z$.

The <u>efficient frontier</u> is collection of all portfolios not dominated by another.

<u>Key Fact</u>

If $\vec{w}_1$, $\vec{w}_2$ distinct weight vectorter so that the portfolios with weights $\vec{w}_1$ + $\vec{w}_2$ have minimum possible variance, then any other weight $\vec{w}_3$ that also minimizes variance is of the form

$$\vec{w}_3 = c\,\vec{w}_1 + (1-c)\,\vec{w}_2$$

for some $c \in \mathbb{R}$.