Remarks About Course

Certificate is based upon homework.

Review of Probability

Introduction to Probability by

Book: Grinstead + Snell.

Solve ~~this~~ exercises in this book for practice.

Key Topics: Random variables, probability distributions, moments, sums of random variables, Central Limit Thm.

Random Variables:

A <u>random variable</u> is

a function whose values are random, but distribution is known, through its probability distribution function.

Toy Examples

<u>Fair Coin</u>: (Discrete)

$$X_{coin}: \begin{cases} 1 & \text{head} \\ -1 & \text{tails} \end{cases}$$

$$P_{coin}(1) = P_{coin}(-1) = \frac{1}{2}.$$

<u>Sum of 2 dice</u>:

$$X_{dice} = \{2, 3, 4, \ldots, 12\}$$

X_{dice}	2	3	...	6	7	8	...	11	12
P_{dice}	$\frac{1}{36}$	$\frac{2}{36}$		$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$		$\frac{2}{36}$	$\frac{1}{36}$

Continuous Uniform Distribution

$X_{uniform}$ on open interval $(0, 1)$

Discrete Sum of Prob. $= 1$

Continuous: $\displaystyle\int_{-\infty}^{\infty} P_{uniform}(x)\,dx$

$$= \int_0^1 1\,dx = 1.$$

Normal Distribution is most fundamental example of continuous distribution.

Moments: X random variable,

1st Moment: (Mean / average / expectation)

$$E[X]$$

$$E[X_{coin}] = P_{coin}(1) \cdot 1 + P_{coin}(-1)(-1)$$
$$= \frac{1}{2}(1) + \frac{1}{2}(-1) = 0$$

$$E[X_{Dice}] = \frac{1}{36}(2) + \frac{2}{36}(3)$$
$$+ \cdots + \frac{1}{36}(12) = 7.$$

$$E[X_{unif}] = \int_0^1 P_{unif}(x) x \, dx$$
$$= \int_0^1 x \, dx = \frac{1}{2}.$$

2^{nd} Moments: $E[X^2]$.

$$E[X_{coin}^2] = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2$$
$$= 1.$$

ℓ^{th} Moment: $E[X^\ell]$.

Variance / Standard Deviation

The variance of X

$$E[(X - E[X])^2].$$

Critical Property of Expectation

X_1, X_2 random variables $\alpha \in \mathbb{R}$,

$$E[\alpha X_1 + X_2] = \alpha E[X_1] + E[X_2].$$

$$E[(X - E[X])^2]$$

$$= E[X^2 - 2E[X]X + E[X]^2]$$

$$= E[X^2] - 2E[X]E[X] + E[X]^2$$

$$= E[X^2] - E[X]^2.$$

$$=: Var(X) =: \sigma^2 =: \sigma_X^2$$

Standard deviation of X

is $\sqrt{Var(X)} = \sigma.$

Large std deviation $\sim$ distribution is more spread out.

Small std. deviation $\sim$ distribution is less spread out.

<u>Remark:</u> Normal distribution has mean 0 std. dev. 1.

Skewness:

$$\mathrm{Skew}(X) = E\left[\left(\frac{X - E[X]}{\sigma_X}\right)^3\right].$$

E.g. Skew of normal is 0

Skew of Roll of 2 dice is 0

Skewness ~ 0 reflects distribution balanced around mean.

Unfair Coin: $X_{coin} = \begin{cases} 1 & \text{Heads} \\ -1 & \text{Tails} \end{cases}$

$\boxed{p(1) = 3/4}$ $\quad p(-1) = 1/4$

$$\mathrm{Skew}(X_{coin}) = \frac{-3/4}{\left(\frac{\sqrt{3}}{2}\right)^3} < 0$$

$$E[X_{coin}] = \cancel{3/4} \; 1/2$$

Negative Skewness Distribution
favored above mean

Opposite for Positive.

~~**Kurtosis:**~~

$$\cancel{\text{Kurt}(X) = E}$$

~~E[X~~

Kurtosis: $\text{Kurt}(X) = E\left[\left(\frac{X - E[X]}{\sigma}\right)^4\right].$

Kurtosis of Normal Distribution is 3.

Excess Kurtosis: $\text{Kurt}(X) - 3.$

Positive Excess Kurtosis: Indicates more extreme behavior in distribution.

Negative Excess Kurtosis: Less ~~a~~ ~~weight~~ extreme behavior.

Relevant Example:

X = Random variable of daily return on a portfolio.

Mean: Positive mean indicates portfolio has been profitable on average.

Variance: ~~Indicates~~ Small variance indicates consistent returns.

Skewness: Negative Skewness ~ more days are profitable.

Excess Kurtosis: ~~Small~~ Negative excess ~~Kurto~~ Kurtosis ~ general outcome on day-to-day is consistent.

Correlation

X, Y are random variables.

Covariance of X, Y

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= Var(X) \text{ if } X = Y$$
$$= E[XY] - E[X]E[Y].$$

Correlation:

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \; \sigma_Y}$$

Note if $X = Y$, $\sigma_X = \sigma_Y$

$$Corr(X,Y) = \frac{Var(X)}{Var(X)} = 1.$$

$-1 \leq \text{Corr}(X,Y) \leq 1,$

$\text{Corr}(X,Y) \sim 1$ X and Y are "similar".

$\text{Corr}(X,Y) \sim -1$ X and Y are closer to "opposite".

Correlation near 0, distributions do not "have much in common".

Relevant Example: Diversified portfolios have less variance in returns.

The Central Limit Thm

Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent and identically distributed random variables

Let $\tilde{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$

Then

$$\sqrt{n}\left(\tilde{X}_n - \underset{\substack{\| \\ E[X_n]}}{E[X_1]}\right)$$

converges to a normal distribution with mean 0 and variance

$Var(X_1) = Var(X_2) = \cdots$.