~~4~~ 4 Binomial Trees and Random Walks

1-step model:

S_0 $\qquad\qquad$ S_1

$\frac{S_1}{S_0}$ = Return

$S_1 = \left(\frac{S_1}{S_0}\right) \cdot S_0$

$S_1/S_0 > \cancel{0}\ 1$ positive return

$S_1/S_0 < 1$ negative return

1-step model

$$\frac{S_1}{S_0} = \begin{cases} 1+u & \text{prob } p \\ 1+d & \text{prob } 1-p \end{cases}$$

where $u > 0$ $d < 0$

are the up and down factors.

$$K = \frac{S_1}{S_0} - 1 = \begin{cases} u & \text{prob } p \\ d & \text{prob } 1-p \end{cases}$$

$$E[K] = pu + (1-p)d$$

$$Var(K) = pu^2 + (1-p)d^2 - (pu + (1-p)d)^2.$$

Case of $p = 1/2$

$$K = \begin{cases} u & \text{prob } 1/2 \\ d & \text{prob } 1/2 \end{cases}$$

$$E[K] = \mu = \frac{u+d}{2}$$

$$\mathrm{Var}(K) = \frac{u^2+d^2}{2} - \left(\frac{u+d}{2}\right)^2$$

$$4\,\mathrm{Var}(K) = 2(u^2+d^2) - (u+d)^2$$
$$= u^2 - 2ud + d^2$$
$$= (u-d)^2$$

$$\mathrm{Var}(K) = \left(\frac{u-d}{2}\right)^2.$$

$$\sigma = \sqrt{\operatorname{Var}(K)} = \frac{u-d}{2}$$

$$\mu = \frac{u+d}{2}$$

$$K = \begin{cases} u & 1/2 \\ d & 1/2 \end{cases} = \begin{cases} \mu+\sigma \\ \mu-\sigma \end{cases}$$

$$\begin{tikzcd}
 & S_0(1+u) \\
S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & S_0(1+d)
\end{tikzcd}$$

2-step binomial tree model

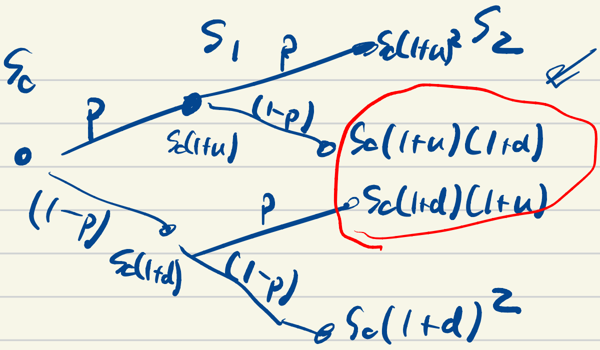
$$K = \begin{cases} u & p \\ d & 1-p \end{cases}$$

$S_0, \quad S_1, \quad S_2$

$$S_2/S_0 = S_1/S_0 \cdot S_2/S_1$$

Assume independent

$$= \begin{cases} (1+u)^2 & p^2 \\ (1+u)(1+d) & 2p(1-p) \\ (1+d)^2 & (1-p)^2 \end{cases}$$



$$E\left[\frac{S_2}{S_0}\right] = p^2(1+u)^2 + 2p(1-p)(1+u)(1+d) + (1-p)^2(1+d)^2$$

$$= \big(p(1+u) + (1-p)(1+d)\big)^2.$$

More generally:

N-step binomial tree model.

$$S_0,\ S_1,\ \ldots,\ S_N$$

$$\frac{S_N}{S_0} = \frac{S_1}{S_0}\cdot\frac{S_2}{S_1}\cdots\frac{S_N}{\cancel{S_0}\ S_{N-1}}$$

$$= \left\{ \cancel{\binom{N}{i}}(1+u)^i(1+d)^{N-i} \quad \text{prob } \binom{N}{i}p^i(1-p)^{N-i} \right\}$$

How to "discover" Brownian Motion in stock movements.

Want to understand the distribution of a stock path from time 0 to time $\underset{\underline{1}}{\underset{\parallel}{t}}$.

Assume (continuous) distribution of stock paths is a limit of discrete N-step binomial tree models and for each N the N-step model has up factor and down factor probabilities of $\frac{1}{2}$.

Fix $N >> 0$

$\tau = t/N$ $\left(\tau = 1/N \text{ if } t = 1 \right)$

$$\underline{K_N} = \begin{cases} u_N & 1/2 \\ d_N & 1/2 \end{cases}$$

Because products naturally appear in expected values / variance of Binomial trees, we'll consider the distribution of $\ln\left(S_t / S_0 \right)$.

$$R_N = \begin{cases} \ln(1+u_N) & \tfrac{1}{2} \\ \ln(1+d_N) & \tfrac{1}{2} \end{cases}$$

$$S_t/S_0 = \frac{S_{t/N}}{S_0} \cdot \frac{S_{2t/N}}{S_{t/N}} \cdot \ldots\ldots \cdot \frac{S_{tN/N}}{S_{\frac{t(N-1)}{N}}}$$

$$\ln(S_t/S_0) = \sum \ln\left(\frac{S_{it/N}}{S_{(i-1)t/N}}\right)$$

$$= \sum R_N(i)$$

Set $\mu = E\left[\ln(S_t/S_0)\right]$

$$\sigma = \mathrm{std}\left(\ln(S_t/S_0)\right)$$

$$\mu = E\left[\sum k_N(i)\right]$$

$$= N\, E[k_N]$$

OR $$E[k_N] = \mu / N$$

$$\sigma^2 = Var\left(\ln\left(S_t / S_0\right)\right)$$

$$= E\left[\left(\sum k_N(i)\right)^2\right] - E\left[\sum k(i)\right]^2$$

~~all are~~ i.i.d

$$= N\, Var(k_N)$$

OR $$Var(k_N) = \sigma^2 / N$$

$t = 1$

time goes from 0 to t

$$k_N = \begin{cases} \ln(1+u) & 1/2 \\ \ln(1+d) & 1/2 \end{cases}$$

$$= \begin{cases} \frac{\mu}{N} + \frac{\sigma}{\sqrt{N}} & 1/2 \\ \frac{\mu}{N} - \frac{\sigma}{\sqrt{N}} & 1/2 \end{cases}$$

We'll introduce 2 more distributions:

$$\boxed{\xi_N = \begin{cases} 1/\sqrt{N} & 1/2 \\ -1/\sqrt{N} & 1/2 \end{cases}}$$

$$\underline{W_N} = \xi_N(1) + \cdots + \underline{\xi_N(N)}.$$

$$\underline{W_{N,i}} = \underbrace{\xi_N(1) + \cdots + \xi_N(i)}_{\text{i.i.d}}.$$

$$k_N = \underline{\frac{\mu}{N} + \sigma \xi_N}$$

$$\ln\left(\frac{S_t}{S_c}\right)$$

$$= \sum^{N} k_N(i)$$

$$= \sum \frac{\mu}{N} + \sigma \xi_N(i)$$

$$= \underline{\mu} + \underline{\sigma}\,\underline{W_N}$$

$$\frac{S_t}{S_c} \approx e^{\mu + \sigma W_N}$$

Where W_N = sum of N-random draws of $\begin{cases} \frac{1}{\sqrt{N}} & \frac{1}{2} \\ -\frac{1}{\sqrt{N}} & \frac{1}{2} \end{cases}$

What is

$$\lim_{N\to\infty} e^{\mu+\sigma W_N} \ ?$$

Only care about

$$\lim_{N\to\infty} W_N \ ?$$

Limit distribution exists,

denoted by W_t