A Put-Call Transformation of the Exchange Option Problem under Stochastic Volatility and Jump-Diffusion Dynamics

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- This technique takes advantage of the **homogeneity of the payoff** function and requires a transition to an equivalent martingale measure corresponding to **one of the risky assets as the numéraire.**

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- Some extensions under more complex asset price models that use this technique are:
  - Fajardo and Mordecki (2006): Lévy processes (also addressed perpetual American options)
  - Antonelli and Scarlatti (2010): Stochastic volatility (via correlation expansions)
  - Alòs and Rheinlander (2017): Stochastic volatility (via the Clark-Ocone formula)

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  - (Stochastic Volatility) Antonelli and Scarlatti (2010) and Alòs and Rheinlander (2017)
  - (Jump-Diffusion) Cheang and Chiarella (2011), Caldana et al. (2015), and Cufaro-Petroni and Sabino (2018)
  - (SVJD) Cheang and Garces (2020)

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  - (SVJD) Cheang and Garces (2020)
- **American exchange options**
  - (Pure Diffusion) Bjerskund and Stensland (1993) and Carr (1995)
  - (Jump-Diffusion) Cheang and Chiarella (2011)
  - (SVJD) Cheang and Garces (2020)

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3. Derive integral representations of European and American exchange option prices in terms of the transition density function of the underlying state variables (i.e. asset prices and stochastic volatility).

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Further details on the results shown in this presentation can be found in our **arXiv preprint**:

https://arxiv.org/abs/2002.10194

The Stochastic Volatility and Jump-Diffusion Model Specification

Model Specification

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we define

- Wiener processes W_1, W_2, and Z, with $\mathrm{d}W_1\,\mathrm{d}W_2 = \rho_w\,\mathrm{d}t$ and $\mathrm{d}W_j\,\mathrm{d}Z = \rho_j\,\mathrm{d}t$, $j = 1, 2$.

  Let $\boldsymbol{B} = (W_1, W_2, Z)^\top$ and $\boldsymbol{\Sigma}$ be the associated correlation matrix.

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- Poisson random measures $p(\mathrm{d}y_j, \mathrm{d}t)$ with $\mathbb{P}$-local characteristics $(\lambda_j, m_{\mathbb{P}}(\mathrm{d}y_j))$. Let N_j be the associated Poisson counting process.

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We consider a finite time horizon $T > 0$ representing the expiry of the exchange options.

We also consider the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ generated by $\mathbf{B}(t)$ and $N_j(t)$, augmented with the $\mathbb{P}$-null sets.

The Proportional SVJD Model

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We assume that the market consists of two risky assets S_1 and S_2 and a risk-free asset M satisfying the equations

$$\frac{\mathrm{d}S_j(t)}{S_j(t)} = (\mu_j - \lambda_j\kappa_j)\,\mathrm{d}t + \sigma_j\sqrt{v(t)}\,\mathrm{d}W_j(t) + \int_{\mathbb{R}} (e^{y_j} - 1)\, p(\mathrm{d}y_j, \mathrm{d}t) \tag{1}$$

$$\mathrm{d}v(t) = \xi\left(\eta - v(t)\right)\mathrm{d}t + \omega\sqrt{v(t)}\,\mathrm{d}Z(t) \tag{2}$$

$$\mathrm{d}M(t) = rM(t)\,\mathrm{d}t, \tag{3}$$

with $S_j(0), v(0) > 0$ and $M(0) = 1$. Here, $\kappa_j \equiv \mathbb{E}_{\mathbb{P}}[e^{Y_j} - 1] = \int_{\mathbb{R}}(e^{y_j} - 1)m_{\mathbb{P}}(\mathrm{d}y_j)$ is the mean jump size of the price of asset j, and r, μ_j, σ_j, ξ, η, and ω are positive constants.

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We assume that the variance process parameters and correlations satisfy $2\xi\eta \geq \omega^2$ (Feller condition) and $-1 < \rho_j < \min\left\{\xi/\omega, 1\right\}$, $j = 1, 2$.

A Radon-Nikodým Derivative

Proposition. Suppose $\boldsymbol{\theta}(t) = \big(\psi_1(t), \psi_2(t), \zeta(t)\big)^\top$ is a vector of $\mathcal{F}_t$-adapted processes and let $\gamma_1, \gamma_2, \nu_1, \nu_2$ be constants. Define the process $\{L_t\}$ by

$$\begin{aligned} L(t) = \exp&\left\{ -\int_0^t \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}(s)\right)^\top \mathrm{d}\mathbf{B}(s) - \frac{1}{2}\int_0^t \boldsymbol{\theta}(s)^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}(s)\,\mathrm{d}s \right\} \\ &\times \exp\left\{ \sum_{n=1}^{N_1(t)} (\gamma_1 Y_{1,n} + \nu_1) - \lambda_1 t\left(e^{\nu_1}\mathbb{E}_{\mathbb{P}}(e^{\gamma_1 Y_1}) - 1\right) \right\} \\ &\times \exp\left\{ \sum_{n=1}^{N_2(t)} (\gamma_2 Y_{2,n} + \nu_2) - \lambda_2 t\left(e^{\nu_2}\mathbb{E}_{\mathbb{P}}(e^{\gamma_2 Y_2}) - 1\right) \right\} \end{aligned} \tag{4}$$

s.t. $\{L(t)\}$ is a strictly positive $\mathbb{P}$-martingale and $\mathbb{E}_{\mathbb{P}}[L(T)] = 1$.

A Radon-Nikodým Derivative (contd.)

Then $L(T)$ is the Radon-Nikodým derivative of some probability measure $\hat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ and the following hold:

1. Under $\hat{\mathbb{Q}}$, the vector process $\mathbf{B}(t)$ has drift $-\boldsymbol{\theta}(t)$;
2. The Poisson process $N_j(t)$ has a new intensity $\tilde{\lambda}_j = \lambda_j e^{\nu_j} \mathbb{E}_{\mathbb{P}}[e^{\gamma_j Y_j}]$, $j = 1, 2$ under $\hat{\mathbb{Q}}$; and
3. The moment generating function of jump sizes random variable Y_j under $\hat{\mathbb{Q}}$ is given by $M_{\hat{\mathbb{Q}}, Y_j}(u) = M_{\mathbb{P}, Y_j}(u + \gamma_j)/M_{\mathbb{P}, Y_j}(\gamma_j)$, $j = 1, 2$.

An Equivalent Martingale Measure (EMM)

- Instead of taking $M(t)$ as the numéraire, we suppose that the second asset yield ratio $S_2(t)e^{q_2 t}$ is the numéraire.
- We define $\hat{\mathbb{Q}}$ to be the measure, given by $\mathrm{d}\hat{\mathbb{Q}} = L(T)\,\mathrm{d}\mathbb{P}$, under which the processes $\tilde{S}$ and $\tilde{M}$, given by

$$\tilde{S}(t) = \frac{S_1(t)e^{q_1 t}}{S_2(t)e^{q_2 t}}, \qquad \tilde{M}(t) = \frac{M(t)}{S_2(t)e^{q_2 t}},$$

  are martingales. We shall refer to $\tilde{S}$ as the **asset yield ratio** of S_1 and S_2.
- The process $X(t)$ divided by $S_2(t)e^{q_2 t}$ shall be referred to as the **discounted value** of the process.

An Equivalent Martingale Measure (EMM)

If we choose $\{\psi_1(t)\}$, $\{\psi_2(t)\}$, and $\{\zeta(t)\}$ as

$$\psi_1(t) = \frac{\mu_1 + q_1 - r - \rho_w \sigma_1 \sigma_1 v(t) - \lambda_1 \kappa_1 + \tilde{\lambda}_1 \tilde{\kappa}_1}{\sigma_1 \sqrt{v(t)}} \tag{5}$$

$$\psi_2(t) = \frac{\mu_2 + q_2 - r - \sigma_2^2 v(t) - \lambda_2 \kappa_2 - \tilde{\lambda}_2 \tilde{\kappa}_2^-}{\sigma_2 \sqrt{v(t)}} \tag{6}$$

$$\zeta(t) = \frac{\Lambda}{\omega} \sqrt{v(t)} \qquad \text{for some constant } \Lambda \geq 0, \tag{7}$$

where $\tilde{\kappa}_1 = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{Y_1} - 1]$ and $\tilde{\kappa}_2^- = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-Y_2} - 1]$, then $\tilde{S}$ and $\tilde{M}$ are $\hat{\mathbb{Q}}$-martingales on $[0, T]$.

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We assume that $\gamma_1, \gamma_2, \nu_1, \nu_2$ are constant to preserve the time-homogeneity of the intensity and the jump size distribution.

An Equivalent Martingale Measure (EMM)

Under $\hat{\mathbb{Q}}$, $\tilde{S}$ and v evolve according to the equations

$$d\tilde{S}(t) = -\tilde{S}(t)\left(\tilde{\lambda}_1\tilde{\kappa}_1 + \tilde{\lambda}_2\tilde{\kappa}_2^-\right)dt + \sigma\sqrt{v(t)}\tilde{S}(t)\,d\bar{W}(t) + \int_{\mathbb{R}}\left(e^{y_1}-1\right)\tilde{S}(t)p(dy_1,dt) + \int_{\mathbb{R}}\left(e^{-y_2}-1\right)\tilde{S}(t)p(dy_2,dt) \tag{8}$$

$$dv(t) = \left[\xi\eta - (\xi+\Lambda)v(t)\right]dt + \omega\sqrt{v(t)}\,d\bar{Z}(t), \tag{9}$$

where $\bar{W}$ and $\bar{Z}$ are Wiener processes under $\hat{\mathbb{Q}}$ and $\sigma = \sigma_1^2 + \sigma_2^2 - 2\rho_w\sigma_1\sigma_2$. Note that $\mathbb{E}_{\hat{\mathbb{Q}}}[d\bar{W}\,d\bar{Z}] = [(\sigma_1\rho_1 - \sigma_2\rho_2)/\sigma]\,dt$.

Exchange Option Pricing Integro-Partial Differential Equation (IPDE)

European Exchange Option IPDE

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- Denote by $C(t) = C(t, S_1(t), S_2(t), v(t))$ the price of a European exchange option whose terminal payoff is given by $C(T) = \left(S_1(T) - S_2(T)\right)^+$. A rearrangement of terms yields

$$\frac{C(T)}{S_2(T)e^{q_2 T}} = e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+.$$

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$$\frac{C(T)}{S_2(T)e^{q_2 T}} = e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+ .$$

- Let $\tilde{C}(t)$ be the discounted European exchange option price. Then in the absence of arbitrage opportunities, $\tilde{C}(t)$ must be given by

$$\tilde{C}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[\tilde{C}(T)|\mathcal{F}_t] = e^{-q_1 T}\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+\middle|\mathcal{F}_t\right]. \qquad (10)$$

European Exchange Option IPDE

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- Since the terminal payoff is variable only in the asset yield ratio, we thus represent by the process $\tilde{V}(t, \tilde{S}(t), v(t)) \equiv \tilde{C}\left(t, S_1(t), S_2(t), v(t)\right)$ the discounted European exchange option price and so

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- By taking the second stock's yield process as the numéraire asset, *the exchange option pricing problem is equivalent to pricing a European call option on the asset yield price ratio $\tilde{S}(t)$ with maturity date T and strike price $e^{(q_1 - q_2)T}$*.

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- For $t \in [0, T]$, we assume that $\tilde{V}(t, \tilde{s}, v)$ is (at least) twice-differentiable in $\tilde{s}$ and v and differentiable in t with continuous partial derivatives.

European Exchange Option IPDE

Proposition. The price at time $t \in [0, T)$ of the European exchange option is given by

$$C(t, S_1(t), S_2(t), v(t)) = S_2(t) e^{q_2 t} \tilde{V}(t, \tilde{S}(t), v(t)), \tag{12}$$

where $\tilde{V}$ is the solution of the terminal value problem

$$0 = \frac{\partial \tilde{V}}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[\tilde{V}(t, \tilde{S}(t), v(t))\right] \tag{13}$$

$$\tilde{V}(T) = e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+, \tag{14}$$

for $(t, \tilde{S}(t), \tilde{v}(t)) \in [0, T] \times \mathbb{R}_+^2$, with $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$.

European Exchange Option IPDE (contd.)

Here, the IPDE operator $\mathcal{L}_{\tilde{s},v}$ defined as

$$\begin{aligned}\mathcal{L}_{\tilde{s},v}\left[\tilde{V}(t,\tilde{S},v)\right] &= -\tilde{S}\left(\tilde{\lambda}_1\tilde{\kappa}_1+\tilde{\lambda}_2\tilde{\kappa}_2^-\right)\frac{\partial \tilde{V}}{\partial \tilde{s}}+\left[\xi\eta-(\xi+\Lambda)v\right]\frac{\partial \tilde{V}}{\partial v}\\ &\quad+\frac{1}{2}\sigma^2 v\tilde{S}^2\frac{\partial^2\tilde{V}}{\partial \tilde{s}^2}+\frac{1}{2}\omega^2 v\frac{\partial^2\tilde{V}}{\partial v^2}+\omega(\sigma_1\rho_1-\sigma_2\rho_2)v\tilde{S}\frac{\partial^2\tilde{V}}{\partial\tilde{s}\partial v}\\ &\quad+\tilde{\lambda}_1\mathbb{E}^{Y_1}_{\hat{\mathbb{Q}}}\left[\tilde{V}\left(t,\tilde{S}e^{Y_1},v\right)-\tilde{V}(t,\tilde{S},v)\right]\\ &\quad+\tilde{\lambda}_2\mathbb{E}^{Y_2}_{\hat{\mathbb{Q}}}\left[\tilde{V}\left(t,\tilde{S}e^{-Y_2},v\right)-\tilde{V}(t,\tilde{S},v)\right],\end{aligned} \tag{15}$$

where $\mathbb{E}^{Y_i}_{\hat{\mathbb{Q}}}$ is the expectation with respect to the r.v. Y_i $(i=1,2)$ under the measure $\hat{\mathbb{Q}}$. Note that all partial derivatives are evaluated at $(t,\tilde{S}(t),v(t))$.

American Exchange Option IPDE

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- After a rearrangement of terms, standard theory on American option pricing (see e.g. Myneni, 1992) allows us to write the discounted American exchange option price $\tilde{V}^A(t, \tilde{S}(t), v(t))$ as

$$\begin{aligned} \tilde{V}^A(t, \tilde{S}(t), v(t)) &\equiv \frac{C^A(t, S_1(t), S_2(t), v(t)}{S_2(t)e^{q_2 t}} \\ &= \operatorname*{ess\,sup}_{u \in [t,T]} e^{-q_1 u} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\left(\tilde{S}(u) - e^{(q_1 - q_2)u} \right)^+ \middle| \mathcal{F}_t \right], \end{aligned} \tag{16}$$

where the supremum is taken over all $\hat{\mathbb{Q}}$-stopping times $u \in [t, T]$.

American Exchange Option Price

- The associated continuation and stopping regions, denoted by $\mathcal{C}$ and $\mathcal{S}$, respectively, that divide the domain $[0, T] \times \mathbb{R}_+^2$ of IPDE (13) are given by

$$\begin{aligned} \mathcal{S} &= \left\{ (t, \tilde{S}, v) \in [0, T] \times \mathbb{R}_+^2 : \tilde{S} \geq B(t, v) e^{(q_1 - q_2)t} \right\} \\ \mathcal{C} &= \left\{ (t, \tilde{S}, v) \in [0, T] \times \mathbb{R}_+^2 : \tilde{S} < B(t, v) e^{(q_1 - q_2)t} \right\}. \end{aligned} \tag{17}$$

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- The line $s_1 = B(t, v)s_2$ on the $s_1 s_2$-plane is known as the **early exercise boundary**.
- It is known that in the continuation region the American exchange option behaves like its live European counterpart, and so $\tilde{V}^A$ satisfies IPDE (13) for $(t, \tilde{S}, v) \in \mathcal{C}$.

American Exchange Option Price

Let $A(t,v) = B(t,v)e^{(q_1-q_2)t}$. The associated value-matching condition is

$$\tilde{V}^A(t, A(t,v), v(t)) = e^{-q_1 t}\left(A(t,v) - e^{(q_1-q_2)t}\right), \tag{18}$$

and the smooth-pasting conditions are

$$\begin{aligned}
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial \tilde{s}}(t, \tilde{S}(t), v(t)) &= e^{-q_1 t} \\
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial v}(t, \tilde{S}(t), v(t)) &= 0 \\
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial t}(t, \tilde{S}(t), v(t)) &= -q_1 e^{-q_1 t}\tilde{S}(t) + q_2 e^{-q_2 t}.
\end{aligned} \tag{19}$$

American Exchange Option Price

Therefore, $\tilde{V}^A$ is the solution of

$$0 = \frac{\partial \tilde{V}^A}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[\tilde{V}^A(t, \tilde{S}(t), v(t))\right]$$

over the domain $0 \leq t \leq T$, $0 < \tilde{S} < A(t, v)$, $0 < v < \infty$ subject to the boundary conditions

$$\begin{aligned} \tilde{V}(T, \tilde{S}(T), v(T)) &= e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+ \\ \tilde{V}(t, 0, v(t)) &= 0, \end{aligned} \tag{20}$$

value-matching condition (18) and smooth-pasting condition (19).

Inhomogeneous IPDE for $\tilde{V}^A$

Proposition. $\tilde{V}^A(t, \tilde{S}, v)$ is a solution to the inhomogeneous IPDE

$$0 = \frac{\partial \tilde{V}^A}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[\tilde{V}^A(t, \tilde{S}(t), v(t))\right] + \Xi(t, \tilde{S}(t), v(t)), \tag{21}$$

where the inhomogeneous term Ξ is given by

$$\begin{aligned}
&\Xi(t, \tilde{S}(t), v(t)) \\
&= \left(q_1 e^{-q_1 t}\tilde{S}(t) - q_2 e^{-q_2 t}\right)\mathbf{1}(\mathcal{A}(t)) \\
&\quad - \tilde{\lambda}_1 \mathbf{1}(\mathcal{A}(t)) \int_{-\infty}^{b(t,\tilde{S}(t),v(t))} \left[\tilde{V}^A\left(t, \tilde{S}(t)e^y, v(t)\right) - \left(e^{-q_1 t}\tilde{S}(t)e^y - e^{-q_2 t}\right)\right] G_1(y)\,\mathrm{d}y \\
&\quad - \tilde{\lambda}_2 \mathbf{1}(\mathcal{A}(t)) \int_{-b(t,\tilde{S}(t),v(t))}^{\infty} \left[\tilde{V}^A\left(t, \tilde{S}(t)e^{-y}, v(t)\right) - \left(e^{-q_1 t}\tilde{S}(t)e^{-y} - e^{-q_2 t}\right)\right] G_2(y)\,\mathrm{d}y.
\end{aligned} \tag{22}$$

Inhomogeneous IPDE for $\tilde{V}^A$ (contd.)

Here, $\mathcal{A}(t) = \{(\tilde{S}(t), v(t)) \in \mathcal{S}(t)\}$, G_1 and G_2 are the pdfs of Y_1 and Y_2, respectively, under $\hat{\mathbb{Q}}$, and

$$b(t, \tilde{S}(t), v(t)) = \ln \left[\frac{B(t, v(t)) e^{(q_1 - q_2)t}}{\tilde{S}(t)} \right].$$

This equation is to be solved for $(t, \tilde{S}(t), v(t)) \in [0, T] \times \mathbb{R}_+^2$, subject to terminal and boundary conditions (20).

Limit of the Early Exercise Boundary at Maturity

Limit of the Early Exercise Boundary

Proposition. The limit $B(T^-, v) \equiv \lim_{t \to T^-} B(t, v)$ is a solution of the equation

$$B(T^-, v) = \max\left\{1, \frac{q_2 + \tilde{\lambda}_1 \int_{-\infty}^{-\ln B(T^-,v)} G_1(y)\,\mathrm{d}y + \tilde{\lambda}_2 \int_{\ln B(T^-,v)}^{\infty} G_2(y)\,\mathrm{d}y}{q_1 + \tilde{\lambda}_1 \int_{-\infty}^{-\ln B(T^-,v)} e^{y} G_1(y)\,\mathrm{d}y + \tilde{\lambda}_2 \int_{\ln B(T^-,v)}^{\infty} e^{-y} G_2(y)\,\mathrm{d}y}\right\}. \tag{23}$$

The implicit equation has a unique positive root solution if $q_1 > 0$.

This can be proved by adapting the arguments used by Chiarella and Ziogas (2009) to an inhomogeneous version of the IPDE derived in the previous section.

It is known that $B(T, v) = 1$ for all $v > 0$.

Continuity of the Early Exercise Boundary

Proposition. Suppose $q_1 > 0$. For any fixed $v \in (0, \infty)$, $B(t, v)$ is continuous at maturity $t = T$ if

$$q_1 \geq q_2 + \tilde{\lambda}_1 \int_{-\infty}^{0} (1 - e^{y}) G_1(y) \, \mathrm{d}y + \tilde{\lambda}_2 \int_{0}^{\infty} (1 - e^{-y}) G_2(y) \, \mathrm{d}y. \tag{24}$$

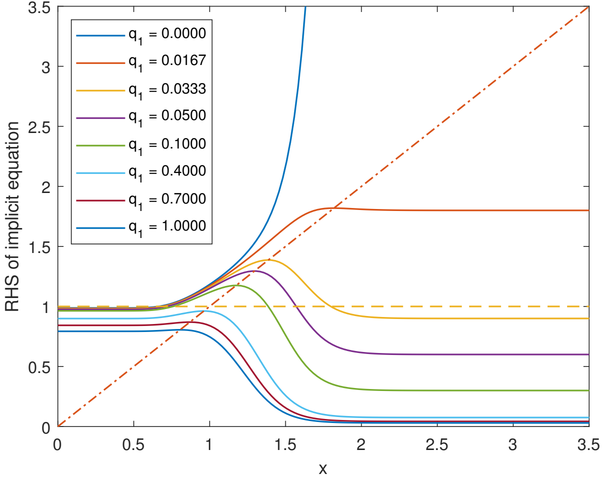

Figure: Behavior of the implicit equation part of (23) with respect to q_1 when jumps are normally distributed. Solid lines represent the right-hand side of (23) and their intersection with the dash-dotted 45° line represents the solution x^* of (23). The horizontal dashed line indicates the position x^* relative to unity.

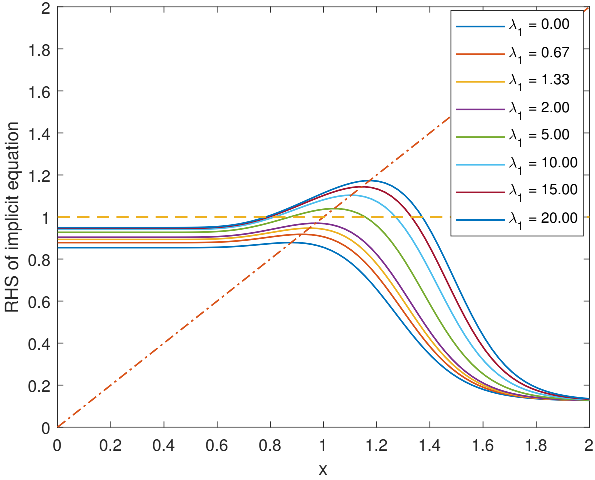

Figure: Behavior of the implicit equation part of (23) with respect to $\tilde{\lambda}_1$ when jumps are normally distributed. The dashed and dash-dotted lines function in a way similar to that for Figure 1.

The Transition Density Function of $\tilde{S}$ and v

The Transition Density Function under $\hat{\mathbb{Q}}$

The Transition Density Function under $\hat{\mathbb{Q}}$

Let $Q(T, s_T, v_T; t, s, v)$ denote the joint transition density function of $(\tilde{S}, v)$ under the probability measure $\hat{\mathbb{Q}}$:

$$Q(T, \tilde{s}_T, v_T; t, \tilde{s}, v) = \hat{\mathbb{Q}}\left(\tilde{S}(T) = \tilde{s}_T, v(T) = v_T \,\middle|\, \tilde{S}(t) = \tilde{s}, v(t) = v\right).$$

The Transition Density Function under $\hat{\mathbb{Q}}$

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$$Q(T, \tilde{s}_T, v_T; t, \tilde{s}, v) = \hat{\mathbb{Q}}\left(\tilde{S}(T) = \tilde{s}_T, v(T) = v_T \middle| \tilde{S}(t) = \tilde{s}, v(t) = v \right).$$

Then Q satisfies the backward equation

$$\begin{aligned} &\frac{\partial Q}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[Q(T, \tilde{s}_T, v_T; t, \tilde{s}, v)\right] = 0 \\ &Q(T, \tilde{s}_T, v_T; T, \tilde{s}, v) = \delta(\tilde{s} - \tilde{s}_T)\delta(v - v_T), \end{aligned}$$

for $(t, \tilde{s}, v) \in [0, T] \times \mathbb{R}_+^2$. Here $\delta(\cdot)$ is the Dirac delta function.

The Transition Density Function under $\hat{\mathbb{Q}}$

The Transition Density Function under $\hat{\mathbb{Q}}$

We consider a change of variables $x = \ln \tilde{s}$ and define H by

$$H(T, x_T, v_T; t, x, v) = Q(T, e^{x_T}, v_t; t, e^x, v).$$

The Transition Density Function under $\hat{\mathbb{Q}}$

We consider a change of variables $x = \ln \tilde{s}$ and define H by

$$H(T, x_T, v_T; t, x, v) = Q(T, e^{x_T}, v_t; t, e^x, v).$$

Then H is the solution of the IPDE

$$\begin{aligned}
0 = \frac{\partial H}{\partial t} &- \left(\tilde{\lambda}_1 \tilde{\kappa}_1 + \tilde{\lambda}_2 \tilde{\kappa}_2^- + \frac{1}{2}\sigma^2 v\right) \frac{\partial H}{\partial x} + \left[\xi\eta - (\xi + \Lambda) v\right] \frac{\partial H}{\partial v} \\
&+ \frac{1}{2}\sigma^2 v \frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\omega^2 v \frac{\partial^2 H}{\partial v^2} + \omega\left(\sigma_1\rho_1 - \sigma_2\rho_2\right) v \frac{\partial^2 H}{\partial x \partial v} \\
&+ \tilde{\lambda}_1 \mathbb{E}_{\hat{\mathbb{Q}}}^{Y_1}\left[H(t, x + Y_1, v) - H(t, x, v)\right] \\
&+ \tilde{\lambda}_2 \mathbb{E}_{\hat{\mathbb{Q}}}^{Y_2}\left[H(t, x - Y_2, v) - H(t, x, v)\right],
\end{aligned} \tag{25}$$

for $(t, x, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}_+$, subject to the terminal condition

$$H(T, x_T, v_T; T, x, v) = \delta(x - x_T)\delta(v - v_T).$$

Solving for the Transition Density Function

As in Cheang et al. (2013), we can solve equation (25) using Fourier and Laplace integral transforms.

Let $\hat{H}(t, \phi, v)$ denote the Fourier transform of $H(t, x, v)$ with respect to x,

$$\hat{H}(t, \phi, v) = \mathscr{F}_x \left\{ H(t, x, v) \right\} (\phi) = \int_{-\infty}^{\infty} e^{i\phi x} H(t, x, v) \, \mathrm{d}x. \tag{26}$$

Let $\bar{H}(t, \phi, \vartheta)$ be the Laplace transform of $\hat{H}(t, \phi, v)$ with respect to v,

$$\bar{H}(t, \phi, \vartheta) = \mathscr{L}_v \left\{ \hat{H}(t, \phi, v) \right\} (\vartheta) = \int_{0}^{\infty} e^{-\vartheta v} \hat{H}(t, \phi, v) \, \mathrm{d}v. \tag{27}$$

Solving for the Transition Density Function

Some assumptions on the behavior of H and $\hat{H}$ are necessary to ensure that all integral transforms are well-defined.

Assumption.

1. As $x \to \pm\infty$, $H(t, x, v) \to 0$, $\partial H/\partial x \to 0$, and $\partial H/\partial v \to 0$.
2. As $v \to +\infty$, $e^{-\vartheta v}\hat{H}(t, \phi, v) \to 0$ and $e^{-\vartheta v}\partial\hat{H}/\partial v \to 0$.

Solving for the Transition Density Function

Proposition. The transition density function $H(t,x,v)$ is given by

$$H(t,x,v) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\tilde{\lambda}_1(T-t))^m(\tilde{\lambda}_2(T-t))^n e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)(T-t)}}{m!n!}$$
$$\times \mathbb{E}_{\hat{\mathbb{Q}}}^{(m,n)}\Bigg[\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{-i\phi\left[x - x_T - \tilde{\lambda}_1\tilde{\kappa}_1(T-t) - \tilde{\lambda}_2\tilde{\kappa}_2^-(T-t)\right]\right\}$$
$$\times \exp\left\{-i\phi\left(\Upsilon_{1,m} - \Upsilon_{2,n}\right)\right\} h(T-t,\phi,v;v_T)\,\mathrm{d}\phi\Bigg]. \tag{28}$$

Here, $\Upsilon_{1,m}$ and $\Upsilon_{1,n}$ are given by $\Upsilon_{1,m} = \sum_{k=1}^{m} Y_{1,k}$ and $\Upsilon_{2,n} = \sum_{l=1}^{n} Y_{2,l}$, where $\{Y_{1,1},\ldots,Y_{1,m}\}$ and $\{Y_{2,1},\ldots,Y_{2,n}\}$ are collections of i.i.d. random variables sampled from populations with $\hat{\mathbb{Q}}$-density functions $G_1(y)$ and $G_2(y)$, respectively, of Y_1 and Y_2, and $\mathbb{E}_{\hat{\mathbb{Q}}}^{(m,n)}[\cdot]$ is the expectation operator with respect to $\Upsilon_{1,m}$ and $\Upsilon_{2,n}$ only.

Solving for the Transition Density Function (contd.)

Furthermore, we have

$$h(\tau, \phi, v; v_T) = \exp\left\{ \frac{(\Theta - F)}{\omega^2}(v - v_T + \alpha\tau) \right\} \frac{2F e^{F\tau}}{\omega^2(e^{F\tau} - 1)} \left[\frac{v_T e^{F\tau}}{v} \right]^{\frac{\alpha}{\omega^2} - \frac{1}{2}} \times \exp\left\{ -\frac{2F(v_T e^{F\tau} + v)}{\omega^2(e^{F\tau} - 1)} \right\} \times I_{\frac{2\alpha}{\omega^2} - 1}\left(\frac{4F\sqrt{v_T v e^{F\tau}}}{\omega^2(e^{F\tau} - 1)} \right), \tag{29}$$

where $I_k(u)$ is the modified Bessel function of the first kind

$$I_k(u) = \sum_{n=0}^{\infty} \frac{(u/2)^{2n+k}}{n!\Gamma(n + k + 1)}, \tag{30}$$

Solving for the Transition Density Function (contd.)

and

$$
\begin{aligned}
F = F(\phi) &\equiv \sqrt{\Theta^2(\phi) - \omega^2 \varepsilon(\phi)} \\
\alpha &\equiv \xi\eta \\
\Theta = \Theta(\phi) &\equiv \xi + \Lambda + i\phi\omega\left(\sigma_1\rho_1 - \sigma_2\rho_2\right) \\
\varepsilon = \varepsilon(\phi) &\equiv \sigma^2\left(i\phi - \phi^2\right)
\end{aligned}
$$

Integral Representation of Exchange Option Prices

European Exchange Option

In terms of the transition density function H, the discounted European exchange option price $\tilde{V}(t, \tilde{s}, v)$ can be written as

$$\begin{aligned} \tilde{V}(t, \tilde{s}, v) = e^{-q_1 T} \int_{-\infty}^{\infty} \int_{0}^{\infty} \left(e^{x_T} - e^{(q_1 - q_2) T} \right)^{+} \\ \times H(T, x_T, v_T; t, \ln \tilde{s}, v) \, \mathrm{d}v_T \, \mathrm{d}x_T . \end{aligned} \tag{31}$$

Evaluating these integrals and multiplying the result by $S_2 e^{q_2 t}$ yields an explicit formula for $C(t, S_1, S_2, v)$.

European Exchange Option

Proposition. The price of a European exchange option is given by

$$C(t, S_1, S_2, v) = S_1 e^{-q_1(T-t)}\hat{Q}_1 - S_2 e^{-q_2(T-t)}\hat{Q}_2, \tag{32}$$

where

$$\begin{aligned}\hat{Q}_1 = &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\tilde{\lambda}_1(T-t))^m(\tilde{\lambda}_2(T-t))^n e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)(T-t)}}{m!n!} \\ &\times \mathbb{E}_{\hat{\mathbb{Q}}}^{(m,n)}\Bigg[e^{-(\tilde{\lambda}_1\tilde{\kappa}_1+\tilde{\lambda}_2\tilde{\kappa}_2^-)(T-t)}e^{\Upsilon_{1,m}-\Upsilon_{2,n}} \\ &\times P_1^E\left(T-t, \tilde{s}e^{-(\tilde{\lambda}_1\tilde{\kappa}_1+\tilde{\lambda}_2\tilde{\kappa}_2^-)(T-t)}e^{\Upsilon_{1,m}-\Upsilon_{2,n}}, v; (q_1-q_2)T\right)\Bigg]\end{aligned} \tag{33}$$

European Exchange Option (contd.)

$$
\begin{aligned}
\hat{Q}_2 = & \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\tilde{\lambda}_1(T-t))^m (\tilde{\lambda}_2(T-t))^n e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)(T-t)}}{m!n!} \\
& \times \mathbb{E}_{\hat{\mathbb{Q}}}^{(m,n)} \left[P_2^E \left(T-t, \tilde{s} e^{-(\tilde{\lambda}_1 \tilde{\kappa}_1 + \tilde{\lambda}_2 \tilde{\kappa}_2^-)(T-t)} e^{\Upsilon_{1,m} - \Upsilon_{2,n}}, v; (q_1 - q_2) T \right) \right].
\end{aligned} \tag{34}
$$

Here, P_1^E and P_2^E are given by

$$
\begin{aligned}
P_1^E(\tau, z, v; K) &= \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{f_1(\tau, z, v; \phi) e^{-i\phi K} - f_1(\tau, z, v; -\phi) e^{i\phi K}}{i\phi} \, \mathrm{d}\phi \\
P_2^E(\tau, z, v; K) &= \frac{1}{2} + \frac{1}{2\pi} \int_0^{\infty} \frac{f(\tau, z, v; \phi) e^{-i\phi K} - f(\tau, z, v; -\phi) e^{i\phi K}}{i\phi} \, \mathrm{d}\phi,
\end{aligned} \tag{35}
$$

European Exchange Option (contd.)

where,

$$
\begin{aligned}
f(\tau, z, v; \phi) &= \exp\left\{ i\phi \ln z + B(\tau, -\phi) + D(\tau, -\phi) v \right\} \\
B(\tau, \phi) &= \frac{\alpha}{\omega^2} \left\{ (\Theta + \digamma)\tau - 2 \ln \left(\frac{1 - \chi e^{\digamma \tau}}{1 - \chi} \right) \right\} \\
D(\tau, \phi) &= \frac{\Theta + \digamma}{\omega^2} \left(\frac{1 - e^{\digamma \tau}}{1 - \chi e^{\digamma \tau}} \right),
\end{aligned}
\tag{36}
$$

with $\chi = (\Theta + \digamma)/(\Theta - \digamma)$,

European Exchange Option (contd.)

and

$$
\begin{aligned}
f_1(\tau, z, v; \phi) &= \exp\left\{ i\phi \ln z + B_1(\tau, -\phi) + D_1(\tau, -\phi) \right\} \\
B_1(\tau, \phi) &= \frac{\alpha}{\omega^2} \left\{ (\Theta_1 + F_1)\tau - 2 \ln \left[\frac{1 - \chi_1 e^{F_1 \tau}}{1 - \chi_1} \right] \right\} \\
D_1(\tau, \phi) &= \frac{\Theta_1 + F_1}{\omega^2} \left[\frac{1 - e^{F_1 \tau}}{1 - \chi_1 e^{F_1 \tau}} \right],
\end{aligned} \tag{37}
$$

with $\Theta_1(\phi) \equiv \Theta(\phi - i)$, $F_1(\phi) \equiv F(\phi - i)$, and $\chi_1(\phi) = \chi(\phi - i)$.

Early Exercise Premium

The discounted American exchange option price $\tilde{V}^A$ can be represented as

$$\tilde{V}^A(t, \tilde{s}, v) = \tilde{V}(t, \tilde{s}, v) + \tilde{V}^p(t, \tilde{s}, v), \tag{38}$$

where $\tilde{V}$ is the discounted price of the corresponding European exchange option and $\tilde{V}^p$ is the (discounted) **early exercise premium**.

The early exercise premium can be further decomposed into

$$\tilde{V}^P(t, \tilde{s}, v) = \tilde{V}_D^P(t, \tilde{s}, v) - \tilde{\lambda}_1 \tilde{V}_{J_1}^P(t, \tilde{s}, v) - \tilde{\lambda}_2 \tilde{V}_{J_2}^P(t, \tilde{s}, v), \tag{39}$$

where $\tilde{V}_D^P$ is the premium arising from the diffusion component and $\tilde{V}_{J_i}^P$ is the *rebalancing cost* due to sudden jumps in the price of asset i, $i = 1, 2$.

Early Exercise Premium

Each component of the early exercise premium can be written in terms of $H(t, \tilde{s}, v)$ as

$$\tilde{V}_D^P(t, \tilde{s}, v) = \int_t^T \int_0^\infty \int_{\ln A(u,v_u)}^{\infty} e^{-q_1 u}\left(q_1 e^{x_u} - q_2 e^{(q_1 - q_2)u}\right) \times H(u, x_u, v_u; t, \ln \tilde{s}, v)\, \mathrm{d}x_u\, \mathrm{d}v_u\, \mathrm{d}t$$

$$\tilde{V}_{J_1}^P(t, \tilde{s}, v) = \int_t^T \int_0^\infty \int_{\ln A(u,v_u)}^{\infty} \int_{-\infty}^{\ln A(u,v_u) - x_u} \left(\tilde{V}^A(u, e^{x_u + y}, v_u) - \left(e^{-q_1 u} e^{x_u + y} - e^{-q_2 u}\right)\right) \times G_1(y) H(u, x_u, v_u; t, \ln \tilde{s}, v)\, \mathrm{d}y\, \mathrm{d}x_u\, \mathrm{d}v_u\, \mathrm{d}u$$

$$\tilde{V}_{J_2}^P(t, \tilde{s}, v) = \int_t^T \int_0^\infty \int_{\ln A(u,v_u)}^{\infty} \int_{x_u - \ln A(u,v_u)}^{\infty} \left(\tilde{V}^A(u, e^{x_u + y}, v_u) - \left(e^{-q_1 u} e^{x_u + y} - e^{-q_2 u}\right)\right) \times G_2(y) H(u, x_u, v_u; t, \ln \tilde{s}, v)\, \mathrm{d}y\, \mathrm{d}x_u\, \mathrm{d}v_u\, \mathrm{d}u$$

American Exchange Option

In the presence of jumps, the early exercise boundary $A(t, v)$ must be solved jointly with the American exchange option price.

Proposition. The discounted American exchange option $\tilde{V}^A(t, \tilde{s}, v)$ and the critical asset yield ratio $B(t, v)$ are the solution of the linked system of integral equations

$$\begin{aligned} \tilde{V}^A(t, \tilde{s}, v) &= \tilde{V}(t, \tilde{s}, v) + \tilde{V}^P(t, \tilde{s}, v) \\ e^{-q_1 t}\left(A(t, v) - e^{(q_1 - q_2)t}\right) &= \tilde{V}(t, A(t, v), v) + \tilde{V}^P(t, A(t, v), v), \end{aligned} \tag{40}$$

where $A(t, v) = B(t, v)e^{(q_1 - q_2)t}$, $\tilde{V}(t, \tilde{s}, v)$ is the price of the European exchange option and $\tilde{V}^P(t, \tilde{s}, v)$ is the early exercise premium.

Next Steps, Summary and Conclusion

Integral Representations of Exchange Option Prices

Integral Representations of Exchange Option Prices

- We can obtain integral representations of option prices in terms of the joint transition density function of $\tilde{S}$ and v.

Integral Representations of Exchange Option Prices

- We can obtain integral representations of option prices in terms of the joint transition density function of $\tilde{S}$ and v.
- The corresponding backward equation of the transition density function can then be solved using Fourier and Laplace integral transforms as was done by Cheang et al. (2013) for the American call option under SVJD dynamics.

For Further Study

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- The integral representations can be evaluated using numerical integration (see e.g. Chiarella and Ziveyi, 2014).

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- The IPDEs derived in this analysis can also be solved numerically via finite difference schemes and the componentwise splitting method (Ikonen and Toivanen, 2007; Chiarella et al., 2009).

For Further Study

- The integral representations can be evaluated using numerical integration (see e.g. Chiarella and Ziveyi, 2014).
- The IPDEs derived in this analysis can also be solved numerically via finite difference schemes and the componentwise splitting method (Ikonen and Toivanen, 2007; Chiarella et al., 2009).
- We are currently working on an implementation of the method of lines (MOL) to solve the IPDEs arising from the put-call transformation.

Acknowledgments

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Thank you very much!

Further details on the results shown in this presentation can be found in our **arXiv preprint**:

https://arxiv.org/abs/2002.10194

We hope to see you during the Live Discussion!