A Numerical Approach to Pricing Exchange Options under Stochastic Volatility and Jump-Diffusion Dynamics

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Introduction

Introduction

- An **exchange option** gives the holder the right, but not the obligation, to exchange one risky asset for another. The payoff of a European exchange option at maturity T is

$$C(T) = (S_1(T) - S_2(T))^+ \triangleq \max\{S_1(T) - S_2(T), 0\}.$$

- The **put-call transformation** technique involves "factoring out" one of the risky assets and establishing an equivalence to an ordinary put or call option on the ratio of the two assets.
  - The term was introduced by Bjerskund and Stensland [1993] in the context of pricing American exchange options
  - Also known as the *dual market method* [Fajardo and Mordecki, 2006]

- The technique requires a transition to an equivalent martingale measure corresponding to **one of the risky assets as the numéraire.**

- This technique takes advantage of the **homogeneity of the payoff** function.

Introduction

- The put-call transformation is the method used by Margrabe [1978] to price *European* exchange options.

- The technique was also used to price European exchange options under more complex asset price dynamics:
  - Fajardo and Mordecki [2006]: Lévy processes (also addressed perpetual American options)
  - Cheang et al. [2006]: Jump-diffusion
  - Antonelli and Scarlatti [2010]; Alòs and Rheinlander [2017]: Stochastic volatility

Introduction

- More on European exchange option pricing:
  - Kim and Park [2017] and Parischa and Goel [2021] under stochastic volatility dynamics
  - Cheang and Chiarella [2011], Caldana et al. [2015], Cufaro-Petroni and Sabino [2020], and Ma et al. [2020] under jump-diffusion dynamics
  - Cheang and Garces [2020] under SVJD dynamics

- American exchange option pricing: Cheang and Chiarella [2011] under jump-diffusion dynamics and Cheang and Garces [2020] under SVJD dynamics

Main Contributions and Results

1. Apply the put-call transformation technique to pricing European and American exchange options when the underlying assets are driven by stochastic volatility and jump-diffusion (SVJD) dynamics.

2. Investigate properties of the finite-maturity American exchange option and its early exercise features under the SVJD dynamics.

3. Propose a numerical solution based on the Method of Lines (MOL) to solve the exchange option pricing IPDE, taking advantage of the simplification from the put-call transformation.

The Proportional Stochastic Volatility and Jump-Diffusion Model

Model Specification

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which we define

- Standard Brownian motions W_1, W_2, and Z, with $\mathrm{d}W_1\,\mathrm{d}W_2 = \rho_w\,\mathrm{d}t$ and $\mathrm{d}W_j\,\mathrm{d}Z = \rho_j\,\mathrm{d}t$, $j = 1, 2$.
  - Let $\mathbf{B} = (W_1, W_2, Z)^\top$ and $\mathbf{\Sigma}$ be the associated correlation matrix.
- Poisson random measures $p_j(\mathrm{d}y, \mathrm{d}t)$ defined on the mark space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with $\mathbb{P}$-local characteristics $(\lambda_j, m_{\mathbb{P}}^{(j)}(\mathrm{d}y))$.
  - Let N_j be the associated Poisson counting process.
  - We assume that N_1 and N_2 are independent of the Wiener processes and of each other.

We fix a finite time horizon $T > 0$.

We equip the prob. space with the filtration $\{\mathcal{F}_t\}_{0 \le t \le T}$ generated by $\mathbf{B}(t)$, $p_1(\mathrm{d}y, \mathrm{d}t)$, and $p_2(\mathrm{d}y, \mathrm{d}t)$, satisfying the usual conditions.

The Proportional SVJD Model

We assume that the market consists of two risky assets S_1 and S_2 and a risk-free asset M satisfying the equations

$$\frac{\mathrm{d}S_j(t)}{S_j(t)} = (\mu_j - \lambda_j\kappa_j)\,\mathrm{d}t + \sigma_j\sqrt{v(t)}\,\mathrm{d}W_j(t) + \int_{\mathbb{R}} (e^y - 1)\,p_j(\mathrm{d}y, \mathrm{d}t) \tag{1}$$

$$\mathrm{d}v(t) = \xi\left(\eta - v(t)\right)\mathrm{d}t + \omega\sqrt{v(t)}\,\mathrm{d}Z(t) \tag{2}$$

$$\mathrm{d}M(t) = rM(t)\,\mathrm{d}t, \tag{3}$$

with $S_j(0), v(0) > 0$ are known constants and $M(0) = 1$.

Here, $\kappa_j \equiv \mathbb{E}_{\mathbb{P}}[e^{Y_j} - 1] = \int_{\mathbb{R}}(e^y - 1)m_{\mathbb{P}}^{(j)}(\mathrm{d}y)$ and r, μ_j, σ_j, ξ, η, and ω are positive constants.

We assume that $2\xi\eta \geq \omega^2$ (Feller condition) and $-1 < \rho_j < \min\left\{\xi/\omega, 1\right\}$, $j = 1, 2$.

A Radon-Nikodým Derivative

Proposition. Suppose $\boldsymbol{\theta}(t) = \left(\psi_1(t), \psi_2(t), \zeta(t)\right)^\top$ is a vector of $\mathcal{F}_t$-predictable processes and let $\gamma_1, \gamma_2, \nu_1, \nu_2$ be constants. Define the process $\{L(t)\}$ by

$$
\begin{aligned}
L(t) = \exp\left\{-\int_0^t \left(\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}(s)\right)^\top \mathrm{d}\mathbf{B}(s) - \frac{1}{2}\int_0^t \boldsymbol{\theta}(s)^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}(s)\,\mathrm{d}s\right\} \\
\times \exp\left\{\sum_{n=1}^{N_1(t)} (\gamma_1 Y_{1,n} + \nu_1) - \lambda_1 t\left(e^{\nu_1}\mathbb{E}_{\mathbb{P}}(e^{\gamma_1 Y_1}) - 1\right)\right\} \\
\times \exp\left\{\sum_{n=1}^{N_2(t)} (\gamma_2 Y_{2,n} + \nu_2) - \lambda_2 t\left(e^{\nu_2}\mathbb{E}_{\mathbb{P}}(e^{\gamma_2 Y_2}) - 1\right)\right\}
\end{aligned}
\tag{4}
$$

and suppose that $\{L(t)\}$ is a strictly positive $\mathbb{P}$-martingale and $\mathbb{E}_{\mathbb{P}}[L(T)] = 1$.

A Radon-Nikodým Derivative (contd.)

Then $L(T)$ is the Radon-Nikodým derivative of some probability measure $\hat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ (on $\mathcal{F} = \mathcal{F}_T$) and the following hold:

1. Under $\hat{\mathbb{Q}}$, the vector process $\mathbf{B}(t)$ has drift $-\boldsymbol{\theta}(t)$;
2. The Poisson process $N_j(t)$ has a new intensity $\tilde{\lambda}_j = \lambda_j e^{\nu_j} \mathbb{E}_{\mathbb{P}}[e^{\gamma_j Y_j}]$, $j = 1, 2$ under $\hat{\mathbb{Q}}$; and
3. The moment generating function of jump sizes random variable Y_j under $\hat{\mathbb{Q}}$ is given by $M_{\hat{\mathbb{Q}},Y_j}(u) = M_{\mathbb{P},Y_j}(u + \gamma_j)/M_{\mathbb{P},Y_j}(\gamma_j)$, $j = 1, 2$.

This proposition is standard. See e.g. Runggaldier [2003, Theorem 2.4] and Cheang and Teh [2014, Theorem 1] for a proof. This is a specialized version of the Girsanov Theorem for Itô-Lévy processes [see e.g. Øksendal and Sulem, 2019, Theorem 1.33].

An Equivalent Martingale Measure (EMM)

- We set the second asset yield ratio $S_2(t)e^{q_2 t}$ as the numéraire instead of $M(t)$.
- We define $\hat{\mathbb{Q}}$ to be the measure, given by $\mathrm{d}\hat{\mathbb{Q}} = L(T)\,\mathrm{d}\mathbb{P}$, under which the processes $\tilde{S}$ and $\tilde{M}$, given by

$$\tilde{S}(t) = \frac{S_1(t)e^{q_1 t}}{S_2(t)e^{q_2 t}}, \qquad \tilde{M}(t) = \frac{M(t)}{S_2(t)e^{q_2 t}},$$

  are martingales. We shall refer to $\tilde{S}$ as the **asset yield ratio** of S_1 and S_2.
- The second asset yield process is also taken to be our discounting factor.

An Equivalent Martingale Measure (EMM)

If we choose $\{\psi_1(t)\}$, $\{\psi_2(t)\}$, and $\{\zeta(t)\}$ as

$$\psi_1(t) = \frac{\mu_1 + q_1 - r - \rho_w \sigma_1 \sigma_1 v(t) - \lambda_1 \kappa_1 + \tilde{\lambda}_1 \tilde{\kappa}_1}{\sigma_1 \sqrt{v(t)}} \tag{5}$$

$$\psi_2(t) = \frac{\mu_2 + q_2 - r - \sigma_2^2 v(t) - \lambda_2 \kappa_2 - \tilde{\lambda}_2 \tilde{\kappa}_2^-}{\sigma_2 \sqrt{v(t)}} \tag{6}$$

$$\zeta(t) = \frac{\Lambda}{\omega} \sqrt{v(t)} \qquad \text{for some constant } \Lambda \geq 0, \tag{7}$$

where $\tilde{\kappa}_1 = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{Y_1} - 1]$ and $\tilde{\kappa}_2^- = \mathbb{E}_{\hat{\mathbb{Q}}}[e^{-Y_2} - 1]$, then $\tilde{S}$ and $\tilde{M}$ are $\hat{\mathbb{Q}}$-martingales on $[0, T]$.

We assume that $\gamma_1, \gamma_2, \nu_1, \nu_2$ are constant to preserve the time-homogeneity of the intensity and the jump size distribution.

An Equivalent Martingale Measure (EMM)

Under $\hat{\mathbb{Q}}$, $\tilde{S}$ and v evolve according to the equations

$$\mathrm{d}\tilde{S}(t) = -\tilde{S}(t)\left(\tilde{\lambda}_1\tilde{\kappa}_1 + \tilde{\lambda}_2\tilde{\kappa}_2^-\right)\mathrm{d}t + \sigma\sqrt{v(t)}\tilde{S}(t)\,\mathrm{d}\bar{W}(t) \tag{8}$$

$$+ \int_{\mathbb{R}} \left(e^{y_1} - 1\right)\tilde{S}(t)p(\mathrm{d}y_1, \mathrm{d}t) + \int_{\mathbb{R}} \left(e^{-y_2} - 1\right)\tilde{S}(t)p(\mathrm{d}y_2, \mathrm{d}t)$$

$$\mathrm{d}v(t) = \left[\xi\eta - (\xi + \Lambda)v(t)\right]\mathrm{d}t + \omega\sqrt{v(t)}\,\mathrm{d}\bar{Z}(t), \tag{9}$$

where $\bar{W}$ and $\bar{Z}$ are Wiener processes under $\hat{\mathbb{Q}}$ and $\sigma = \sigma_1^2 + \sigma_2^2 - 2\rho_w\sigma_1\sigma_2$. Note that $\mathbb{E}_{\hat{\mathbb{Q}}}[\mathrm{d}\bar{W}\,\mathrm{d}\bar{Z}] = [(\sigma_1\rho_1 - \sigma_2\rho_2)/\sigma]\,\mathrm{d}t$.

Exchange Option Pricing Integro-Partial Differential Equations (IPDEs)

European Exchange Option IPDE

- Denote by $C(t) = C(t, S_1(t), S_2(t), v(t))$ the price of a European exchange option whose terminal payoff is given by $C(T) = \big(S_1(T) - S_2(T)\big)^+$. A rearrangement of terms yields

$$\frac{C(T)}{S_2(T)e^{q_2 T}} = e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+.$$

- Let $\tilde{C}(t)$ be the discounted European exchange option price. Then in the absence of arbitrage opportunities, $\tilde{C}(t)$ must be given by

$$\tilde{C}(t) = \mathbb{E}_{\hat{\mathbb{Q}}}[\tilde{C}(T)|\mathcal{F}_t] = e^{-q_1 T}\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+ \middle| \mathcal{F}_t\right]. \tag{10}$$

European Exchange Option IPDE

- We represent by the process $\tilde{V}(t, \tilde{S}(t), v(t))$ the discounted European exchange option price,

$$\tilde{V}(t, \tilde{S}(t), v(t)) = e^{-q_1 T} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\left(\tilde{S}(T) - e^{(q_1 - q_2)T} \right)^+ \middle| \mathcal{F}_t \right]. \tag{11}$$

- By taking the second stock's yield process as the numéraire asset, *the exchange option pricing problem is equivalent to pricing a European call option on the asset yield price ratio $\tilde{S}(t)$ with maturity date T and strike price $e^{(q_1 - q_2)T}$*.

- For $t \in [0, T]$, we assume that $\tilde{V}(t, \tilde{s}, v)$ is (at least) twice-differentiable in $\tilde{s}$ and v and differentiable in t with continuous partial derivatives.

European Exchange Option IPDE

Proposition. The price at time $t \in [0, T)$ of the European exchange option is given by

$$C(t, S_1(t), S_2(t), v(t)) = S_2(t)e^{q_2 t}\tilde{V}(t, \tilde{S}(t), v(t)), \tag{12}$$

where $\tilde{V}$ is the solution of the terminal value problem

$$0 = \frac{\partial \tilde{V}}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[\tilde{V}(t, \tilde{S}(t), v(t))\right] \tag{13}$$

$$\tilde{V}(T) = e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+, \tag{14}$$

for $(t, \tilde{S}(t), \tilde{v}(t)) \in [0, T] \times \mathbb{R}_+^2$, with $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$.

European Exchange Option IPDE (contd.)

Here, the IPDE operator $\mathcal{L}_{\tilde{s},v}$ defined as

$$\begin{aligned}\mathcal{L}_{\tilde{s},v}\left[\tilde{V}(t,\tilde{S},v)\right] = &-\tilde{S}\left(\tilde{\lambda}_1\tilde{\kappa}_1+\tilde{\lambda}_2\tilde{\kappa}_2^-\right)\frac{\partial\tilde{V}}{\partial\tilde{s}}+\left[\xi\eta-(\xi+\Lambda)v\right]\frac{\partial\tilde{V}}{\partial v}\\ &+\frac{1}{2}\sigma^2 v\tilde{S}^2\frac{\partial^2\tilde{V}}{\partial\tilde{s}^2}+\frac{1}{2}\omega^2 v\frac{\partial^2\tilde{V}}{\partial v^2}+\omega(\sigma_1\rho_1-\sigma_2\rho_2)v\tilde{S}\frac{\partial^2\tilde{V}}{\partial\tilde{s}\partial v}\\ &+\tilde{\lambda}_1\mathbb{E}_{\hat{\mathbb{Q}}}^{Y_1}\left[\tilde{V}\left(t,\tilde{S}e^{Y_1},v\right)-\tilde{V}(t,\tilde{S},v)\right]\\ &+\tilde{\lambda}_2\mathbb{E}_{\hat{\mathbb{Q}}}^{Y_2}\left[\tilde{V}\left(t,\tilde{S}e^{-Y_2},v\right)-\tilde{V}(t,\tilde{S},v)\right],\end{aligned} \tag{15}$$

where $\mathbb{E}_{\hat{\mathbb{Q}}}^{Y_i}$ is the expectation with respect to the r.v. Y_i $(i=1,2)$ under the measure $\hat{\mathbb{Q}}$. Note that all partial derivatives are evaluated at $(t,\tilde{S}(t),v(t))$.

American Exchange Option Price

- The continuation and stopping regions, denoted by $\mathcal{C}$ and $\mathcal{S}$, respectively, on the domain $[0,T] \times \mathbb{R}^2_+$ of IPDE (13) are given by

$$\begin{aligned} \mathcal{S} &= \left\{ (t, \tilde{S}, v) \in [0,T] \times \mathbb{R}^2_+ : \tilde{S} \geq B(t,v) e^{(q_1 - q_2)t} \right\} \\ \mathcal{C} &= \left\{ (t, \tilde{S}, v) \in [0,T] \times \mathbb{R}^2_+ : \tilde{S} < B(t,v) e^{(q_1 - q_2)t} \right\}. \end{aligned} \tag{16}$$

- The line $s_1 = B(t,v) s_2$ on the $s_1 s_2$-plane is known as the **early exercise boundary**.
- $\tilde{V}^A$ satisfies IPDE (13) for $(t, \tilde{S}, v) \in \mathcal{C}$.
- $\tilde{V}^A$ is known analytically in $\mathcal{S}$,

$$\tilde{V}^A(t, \tilde{S}(t), v(t)) = e^{-q_1 t} (\tilde{S}(t) - e^{(q_1 - q_2)t}), \qquad (t, \tilde{S}, v) \in \mathcal{S}.$$

American Exchange Option Price

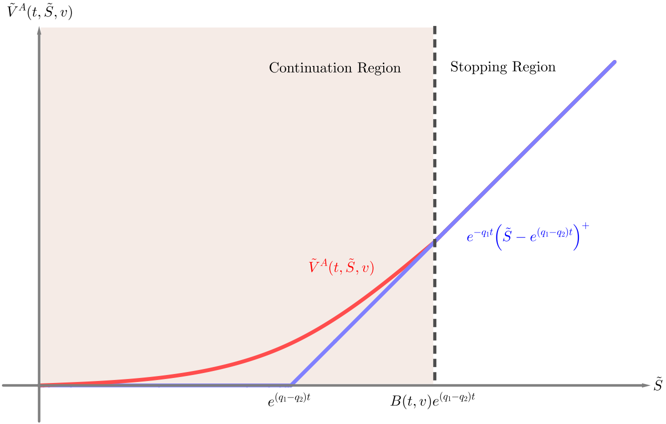


Figure: The continuation and stopping regions and the early exercise boundary

American Exchange Option Price

- Value-matching condition:

$$\tilde{V}^A(t, A(t,v), v(t)) = e^{-q_1 t}\left(A(t,v) - e^{(q_1-q_2)t}\right), \tag{17}$$

where $A(t,v) = B(t,v)e^{(q_1-q_2)t}$

- Smooth-pasting conditions:

$$\begin{aligned}
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial \tilde{s}}(t, \tilde{S}(t), v(t)) &= e^{-q_1 t} \\
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial v}(t, \tilde{S}(t), v(t)) &= 0 \\
\lim_{\tilde{S}\to A(t,v)} \frac{\partial \tilde{V}^A}{\partial t}(t, \tilde{S}(t), v(t)) &= -q_1 e^{-q_1 t}\tilde{S}(t) + q_2 e^{-q_2 t}.
\end{aligned} \tag{18}$$

American Exchange Option Price

Therefore, $\tilde{V}^A$ is the solution of

$$0 = \frac{\partial \tilde{V}^A}{\partial t} + \mathcal{L}_{\tilde{s},v}\left[\tilde{V}^A(t, \tilde{S}(t), v(t))\right]$$

over the domain $0 \leq t \leq T$, $0 < \tilde{S} < A(t, v)$, $0 < v < \infty$ subject to the boundary conditions

$$\begin{aligned} \tilde{V}(T, \tilde{S}(T), v(T)) &= e^{-q_1 T}\left(\tilde{S}(T) - e^{(q_1 - q_2)T}\right)^+ \\ \tilde{V}(t, 0, v(t)) &= 0, \end{aligned} \tag{19}$$

value-matching condition (17) and smooth-pasting condition (18).

Limit of the Early Exercise Boundary

Proposition. Suppose G_1 and G_2 are the probability density functions implied by $m^{(1)}_{\hat{\mathbb{Q}}}(\mathrm{d}y)$ and $m^{(2)}_{\hat{\mathbb{Q}}}$, resp. The limit $B(T^-, v) \equiv \lim_{t \to T^-} B(t, v)$ is a solution of the equation

$$B(T^-, v) = \max\left\{1, \frac{q_2 + \tilde{\lambda}_1 \int_{-\infty}^{-\ln B(T^-,v)} G_1(y)\,\mathrm{d}y + \tilde{\lambda}_2 \int_{\ln B(T^-,v)}^{\infty} G_2(y)\,\mathrm{d}y}{q_1 + \tilde{\lambda}_1 \int_{-\infty}^{-\ln B(T^-,v)} e^{y} G_1(y)\,\mathrm{d}y + \tilde{\lambda}_2 \int_{\ln B(T^-,v)}^{\infty} e^{-y} G_2(y)\,\mathrm{d}y}\right\}. \quad (20)$$

The implicit equation has a unique positive root solution if $q_1 > 0$.

It is known that $B(T, v) = 1$ for all $v > 0$.

Method of Lines (MOL) Solution of the Exchange Option Pricing IPDEs

Introduction

- The MOL involves discretizing the I/PDE in all but one variable resulting to a sequence or system of ODEs in the remaining continuous variable [Schiesser and Griffiths, 2009; Meyer, 2015].

- It is useful for approximating American option prices as the algorithm can be easily adjusted to accommodate unknown free boundaries.

- The option delta and gamma are calculated as part of the algorithm with no additional computational cost.

- This type of MOL has been applied to pricing single-asset American options by various researchers under a variety of asset price dynamics: Meyer and van der Hoek [1997], Meyer [1998], Chiarella et al. [2009], Adolfsson et al. [2013], Chiarella and Ziveyi [2013], Kang and Meyer [2014], [Chiarella and Ziveyi, 2014], Chiarella et al. [2016].

Some Simplifying Notation

- We transform the terminal value problem into an initial value problem involving the time to maturity $\tau = T - t$.
- Let $V(\tau, s, v)$ denote the discounted exchange option price, where $s = \tilde{S}(T - \tau)$ and $v = v(T - \tau)$.

The Exchange Option IPDE

The IPDE for the discounted European exchange option price can be written as

$$
\begin{aligned}
0 = {} & \frac{1}{2}\sigma^2 v s^2 \frac{\partial^2 V}{\partial s^2} + \frac{1}{2}\omega^2 v \frac{\partial^2 V}{\partial v^2} + \omega(\sigma_1\rho_1 - \sigma_2\rho_2) v s \frac{\partial^2 V}{\partial s \partial v} \\
& - \left(\tilde{\lambda}_1 \tilde{\kappa}_1 + \tilde{\lambda}_2 \tilde{\kappa}_2^{-}\right) s \frac{\partial V}{\partial s} + \left[\xi\eta - (\xi + \Lambda) v\right] \frac{\partial V}{\partial v} - (\tilde{\lambda}_1 + \tilde{\lambda}_2) V \\
& - \frac{\partial V}{\partial \tau} + \tilde{\lambda}_1 \int_{\mathbb{R}} V(\tau, s e^{y}, v) G_1(y)\,\mathrm{d}y + \tilde{\lambda}_2 \int_{\mathbb{R}} V(\tau, s e^{-y}, v) G_2(y)\,\mathrm{d}y,
\end{aligned} \tag{21}
$$

for $(\tau, s, v) \in [0, T] \times (0, \infty)^2$, subject to the initial condition

$$
V(0, s, v) = e^{-q_1 T} \left(s - e^{(q_1 - q_2)T}\right)^{+}. \tag{22}
$$

and boundary conditions

$$
\lim_{s \to 0^+} V(\tau, s, v) = 0, \quad \lim_{s \to \infty} \frac{\partial^2 V}{\partial s^2} = 0, \quad \lim_{v \to \infty} \frac{\partial V}{\partial v} = 0. \tag{23}
$$

Domain Truncation and Additional Notation

- The infinite domains for s and v are truncated to $[0, s_J]$ and $[0, v_M]$ for some suitably chosen s_J and v_M.

- Partitioning the computational domain:
  - $[0, T]$: $0 = \tau_0 < \tau_1 < \cdots < \tau_N = T$ with uniform width $\Delta\tau$
  - $[0, s_J]$: $0 = s_0 < s_1 < \cdots < s_J$ with uniform width Δs
  - $[0, v_M]$: $0 = v_0 < v_1 < \cdots < v_M$ with uniform width Δv

- Denote by $V_{n,m}(s)$ the approximate solution of the IPDE at time "line" $\tau = \tau_n$ and variance "line" $v = v_m$

- Let $\mathscr{V}_{n,m}(s) = V'_{n,m}(s)$ denote the approximation of the option delta.

Finite Difference Approximations

- Time derivative [Meyer and van der Hoek, 1997]:

$$\frac{\partial V_{n,m}}{\partial \tau} \approx \begin{cases} \frac{1}{\Delta\tau}(V_{n,m}(s) - V_{n-1,m}(s)) + O(\Delta\tau) & \text{if } n = 1, 2, \\ \frac{3}{2\Delta\tau}(V_{n,m}(s) - V_{n-1,m}(s)) & \\ \qquad -\frac{1}{2\Delta\tau}(V_{n-1,m}(s) - V_{n-2,m}(s)) + O((\Delta\tau)^2) & \text{if } n \geq 3. \end{cases}$$

- Second-order derivatives (central difference):

$$\frac{\partial^2 V_{n,m}}{\partial v^2} \approx \frac{V_{n,m+1}(s) - 2V_{n,m}(s) + V_{n,m-1}(s)}{(\Delta v)^2} + O((\Delta v)^2),$$

$$\frac{\partial^2 V_{n,m}}{\partial s \partial v} \approx \frac{\mathscr{V}_{n,m+1}(s) - \mathscr{V}_{n,m-1}(s)}{2\Delta v} + O((\Delta v)^2).$$

Finite Difference Approximations

- $\frac{\partial V_{n,m}}{\partial v}$ (upwinding scheme) [Ikonen and Toivanen, 2007; Chiarella et al., 2009]:

$$
\begin{aligned}
&[\xi\eta - (\xi+\Lambda)v] \frac{\partial V_{n,m}}{\partial v} \\
&\quad \approx \max\left\{\xi\eta - (\xi+\Lambda)v_m, 0\right\} \left[\frac{V_{n,m+1}(s) - V_{n,m}(s)}{\Delta v}\right] \\
&\qquad + \min\left\{\xi\eta - (\xi+\Lambda)v_m, 0\right\} \left[\frac{V_{n,m}(s) - V_{n,m-1}(s)}{\Delta v}\right] + O(\Delta v).
\end{aligned}
$$

- At $m = M$, we approximate $\lim_{v\to\infty} \frac{\partial V_{n,m}}{\partial v} = 0$ by $\frac{\partial V(\tau_n, s, v_M)}{\partial v} = 0$. With a forward difference approximation, we have $V_{n,M+1}(s) = V_{n,M}(s)$ and $\mathscr{V}_{n,M+1}(s) = \mathscr{V}_{n,M}(s)$.

Approximation of Integral Terms

- We assume that $Y_1 \sim N(\alpha_{j_1}, \beta_{j_1}^2)$ and $Y_2 \sim N(\alpha_{j_2}, \beta_{j_2}^2)$ under $\hat{\mathbb{Q}}$.
- The integral terms, denoted by $I_1(\tau, s, v_m)$ and $I_2(\tau, s, v_m)$, are approximated by

$$\begin{aligned} I_1(\tau_n, s, v_m) &= -\tilde{\lambda}_1 \int_{\mathbb{R}} V(\tau_n, se^y, v_m) G_1(y)\,\mathrm{d}y \\ &\approx -\frac{\tilde{\lambda}_1}{\sqrt{\pi}} \sum_{l=1}^{L} \varrho_l^H V_{n,m}(se^{\sqrt{2}\beta_{j_1} z_l^H + \alpha_{j_1}}), \\ I_2(\tau_n, s, v_m) &= -\tilde{\lambda}_2 \int_{\mathbb{R}} V(\tau_n, se^{-y}, v_m) G_2(y)\,\mathrm{d}y \\ &\approx -\frac{\tilde{\lambda}_2}{\sqrt{\pi}} \sum_{l=1}^{L} \varrho_l^H V_{n,m}(se^{-\sqrt{2}\beta_{j_2} z_l^H - \alpha_{j_2}}), \end{aligned}$$

where ϱ_l^H and z_l^H are the weights and abscissas of the Gauss-Hermite quadrature scheme with L integration points.

MOL Approximation of IPDE (21)

The MOL approximation of equation (21) at $\tau = \tau_n$ and $v = v_m$ is

$$\begin{aligned} a(s, v_m)V''_{n,m}(s) + b(s, v_m)V'_{n,m}(s) - c(\tau_n, s, v_m)V_{n,m}(s) \\ = F(\tau_n, s, v_m) + I_1(\tau_n, s, v_m) + I_2(\tau_n, s, v_m), \end{aligned} \tag{24}$$

where the coefficients a, b, c, and F are obtained by comparing (24) and (21) after substituting the finite difference and integral term approximations.

Because the RHS of (24) depends the whole option price profile (in I_1 and I_2) and option prices and deltas at the next variance line (in F), we employ a *nested two-level iterative scheme*.

- Outer level: **Integral term iteration** (with iteration counter k')
- Inner level: **Variance line iteration** (with iteration counter k)

Overview of the MOL Algorithm (Part 1)

Algorithm 1: MOL Algorithm (Part 1)

Compute initial values $V_{0,m}(s)$ and $\mathscr{V}_{0,m}(s)$ for all m and s;

for $n = 1$ **to** N **do**

Set $V_{n,\cdot}^{k'-1}(\cdot) = V_{n-1,\cdot}(\cdot)$ and $\mathscr{V}_{n,\cdot}^{k'-1}(\cdot) = \mathscr{V}_{n-1,\cdot}(\cdot)$;

`/* Commence integral term iterations */`

while *Error* $>$ *Tolerance* **do**

Compute $I_1(\tau_n, s, v_m)$ and $I_2(\tau_n, s, v_m)$ using $V_{n,m}^{k'-1}(s)$;

[Variance line iterations];

Set $V_{n,\cdot}^{k'}(\cdot)$ and $\mathscr{V}_{n,\cdot}^{k'}(\cdot)$ to be the final output of the variance line iterations;

Compute error and check against convergence criterion;

Update $V_{n,\cdot}^{k'-1}(\cdot) \leftarrow V_{n,\cdot}^{k'}(\cdot)$ and $\mathscr{V}_{n,\cdot}^{k'-1}(\cdot) \leftarrow \mathscr{V}_{n,\cdot}^{k'}(\cdot)$;

Set $V_{n,\cdot}(\cdot) = V_{n,\cdot}^{k'}(\cdot)$ and $\mathscr{V}_{n,\cdot}(\cdot) = \mathscr{V}_{n,\cdot}^{k'}(\cdot)$

Variance Line Iterations

- In the kth variance line iteration, we approximate $V^k_{n,m+1}(s)$ and $\mathscr{V}^k_{n,m+1}(s)$ appearing in $F(\tau_n, s, v_m)$ by $V^{k-1}_{n,m+1}(s)$ and $\mathscr{V}^{k-1}_{n,m+1}(s)$.
- Then in each iteration, for each m, we solve (24) by solving the corresponding system of first-order ODEs

$$\begin{aligned} V'^k_{n,m}(s) &= \mathscr{V}^k_{n,m}(s), \\ \mathscr{V}'^k_{n,m}(s) &= C(\tau_n, s, v_m)V^k_{n,m}(s) + D(\tau_n, s, v_m)\mathscr{V}^k_{n,m}(s) + g(\tau_n, s, v_m), \end{aligned} \tag{25}$$

where $C = c/a$, $D = -b/a$, and $g = (F + I_1 + I_2)/a$.

Riccati Transform Approach

- Equation (25) is solved using the Riccati transform [see e.g. Meyer and van der Hoek, 1997; Meyer, 2015],

$$V_{n,m}^k(s) = R(s)\mathscr{V}_{n,m}^k(s) + w(s), \tag{26}$$

  where R and w are solutions of the initial value problem

$$\begin{aligned} R'(s) &= 1 - D(\tau_n, s, v_m)R(s) - C(\tau_n, s, v_m)R^2(s), \qquad & R(0) = 0 \\ w'(s) &= -C(\tau_n, s, v_m)R(s)w(s) - g(\tau_n, s, v_m)R(s), \qquad & w(0) = 0. \end{aligned} \tag{27}$$

- **Forward Sweep:** We solve (27) using the trapezoidal rule in s starting from $s_0 = 0$.
- **Reverse Sweep:** Starting with terminal condition $\mathscr{V}_{n,m}^k(s_J) = 0$, we integrate

$$\begin{aligned} \mathscr{V}'^k_{n,m}(s) = &\left[C(\tau_n, s, v_m)R(s) + D(\tau_n, s, v_m)\right]\mathscr{V}_{n,m}^k(s) \\ &+ C(\tau_n, s, v_m)w(s) + g(\tau_n, s, v_m) \end{aligned}$$

  using the trapezoidal rule. $V_{n,m}^k(s)$ is calculated using (26).

Overview of the MOL Algorithm (Part 2)

Algorithm 2: MOL Algorithm (Part 2) - Variance Line Iterations

Set $V^{k-1}_{n,\cdot}(\cdot) = V^{k'-1}_{n,\cdot}(\cdot)$ and $\mathscr{V}^{k-1}_{n,\cdot}(\cdot) = \mathscr{V}^{k'-1}_{n,\cdot}(\cdot)$;

`/* Commence variance line iterations */`

while *Error* $>$ *Tolerance* **do**

- **for** $m = 0$ **to** M **do**
  - Forward sweep: Solve for $R(\cdot)$ and $w(\cdot)$ in (27) using the trapezoidal rule;
  - Reverse sweep: Compute $\mathscr{V}^{k}_{n,m}(\cdot)$ using the trapezoidal rule and $V^{k}_{n,m}(\cdot)$ using (26);
- Compute error and check against convergence criterion;
- Update $V^{k-1}_{n,\cdot}(\cdot) \leftarrow V^{k}_{n,\cdot}(\cdot)$ and $\mathscr{V}^{k-1}_{n,\cdot}(\cdot) \leftarrow \mathscr{V}^{k}_{n,\cdot}(\cdot)$;

`/* Return to current integral term iteration (see Part 1) */`

Convergence Criterion

- The variance line iterations terminate once the convergence criterion

$$\max_{0\leq m\leq M}\left\{\max_{0\leq j\leq J}\left|V_{n,m}^{k}(s_j)-V_{n,m}^{k-1}(s_j)\right|\right\}<10^{-8} \tag{28}$$

  is satisfied.

- The same convergence criterion is applied in the integral term iterations with $V_{n,m}^{k'}(s)$ and $V_{n,m}^{k'-1}(s)$.

- The option delta $\mathscr{V}$ can also be incorporated into the convergence criterion if desired.

MOL for American Exchange Options

- The discounted American exchange option price $V(\tau, s, v)$ also satisfies (21) with the same initial condition but over over $0 < v < \infty$, $0 \leq \tau \leq T$, and $0 < s < A(\tau, v)$.

- Boundary conditions:

$$\lim_{s \to 0^+} V(\tau, s, v) = 0, \qquad \lim_{v \to \infty} \frac{\partial V}{\partial v} = 0,$$

Value-matching and smooth-pasting conditions:

$$V(\tau, A(\tau, v), v) = e^{-q_1(T-\tau)} \left(A(\tau, v) - e^{(q_2 - q_1)t} \right)$$

$$\lim_{s \to A(\tau, v)} \frac{\partial V}{\partial s} = e^{-q_1(T-t)}.$$

- For $s \geq A(\tau, v)$, the option price is known analytically:

$$V(\tau, s, v) = e^{-q_1(T-\tau)} (s - e^{(q_1 - q_2)(T-\tau)}).$$

MOL for American Exchange Options

- The MOL approximation of the American exchange option at $\tau = \tau_n$ and $v = v_m$ is also given by equation (24),

$$a(s, v_m)V''_{n,m}(s) + b(s, v_m)V'_{n,m}(s) - c(\tau_n, s, v_m)V_{n,m}(s)$$
$$= F(\tau_n, s, v_m) + I_1(\tau_n, s, v_m) + I_2(\tau_n, s, v_m),$$

- For each n and m, we also need to solve for $A_{n,m} = A(\tau_n, v_m)$, the approximation of the unknown boundary at $\tau = \tau_n$ and $v = v_m$.

- A suitable initial condition for the early exercise boundary is

$$A_{0,m} = A(0, v_m) = B(0^+, v_m)e^{(q_1 - q_2)T} \qquad \forall m,$$

where $B(0^+, v_m)$ is a solution of equation (20).

MOL for American Exchange Options

- At each step of the trapezoidal rule in the forward sweep, we monitor the sign of the function

$$\phi(s_j) = R(s_j)e^{-q_1(T-\tau_n)} + w(s_j) - e^{-q_1(T-\tau_n)}\left(s_j - e^{(q_1-q_2)(T-\tau_n)}\right) \tag{29}$$

  is monitored. We note that $\phi(s_0) = e^{-q_2(T-\tau_n)} > 0$.

- The forward sweep stops at the index j^* at which ϕ first becomes negative; i.e. j^* is the first index such that $\phi(s_{j^*-1})\phi(s_{j^*}) < 0$.

- We approximate $A^k_{n,m}$ to be the zero of the cubic spline interpolant through the points $\{(s_i, \phi(s_i))\}_{i=j^*-2}^{j^*+1}$ that occurs in between s_{j^*-1} and s_{j^*}.

MOL for American Exchange Options

- Once $A^k_{n,m}$ has been determined, the reverse sweep for $\mathscr{V}^k_{n,m}(s)$ is then carried out over the interval $[0, A^k_{n,m}]$.

- Since $A^k_{n,m}$ is not part of the regular mesh, the trapezoidal rule is first applied over $[s_{j^*-1}, A^k_{n,m}]$, where the required values of R, w, and the coefficients are linearly interpolated from values at s_{j^*-1} and s_{j^*}.

- Once $\mathscr{V}^k_{n,m}(s_{j^*-1})$ has been calculated, the reverse sweep can continue over $[0, s_{j^*-1}]$ along the regular mesh. $V^k_{n,m}(s_j)$ can then be calculated using the Riccati transform equation.

- For $j \geq j^*$, $V^k_{n,m}(s_j)$ and $\mathscr{V}^k_{n,m}(s_j)$ are known analytically.

Numerical Illustrations and Discussion

Parameter Values Used

Table: Values of parameters in the SVJD model, upper bounds s_J and v_M and mesh sizes for the spatial computational domain used for numerical experiments.

Asset Price		**Stoch. Vol.**		**Jumps**		**Mesh Sizes**	
T	0.50	ξ	2.00	$\tilde{\lambda}_1$	5.00	s_J	4.00
q_1	0.05	η	0.56	β_{j_1}	0.20	v_M	2.00
q_2	0.03	Λ	0.00	α_{j_1}	0.00	J	140
σ_1	0.50	ω	0.40	$\tilde{\lambda}_2$	2.00	M	25
σ_2	0.30	ρ_1	-0.50	β_{j_2}	0.20		
ρ_w	0.50	ρ_2	-0.50	α_{j_2}	0.00		
				L	20		

Comparisons and Benchmarking

- **Benchmarking:** Projected successive overrelaxation (PSOR) method (with a very fine partition) on the corresponding linear complementarity problem [Ikonen and Toivanen, 2007; Chiarella et al., 2009].
  - Reference prices are calculated with $N = 500$, $J = 250$, and $M = 50$.

- **Comparisons:** PSOR (with a spatial mesh size similar to that used in the MOL) and the least squares Monte Carlo (LSMC) method [Longstaff and Schwartz, 2001].
  - To ensure convergence in the PSOR approach, we use $N = 150$ time steps.
  - For PSOR, we also considered three time discretization schemes: Rannacher, implicit Euler, and Crank-Nicholson.
  - The LSMC method was implemented with an Euler-Maruyama discretization of the SDEs for S_1, S_2, and v, with $N = 1000$ time steps.

Numerical Results - Comparison Metrics

Table: Approximation errors (root mean square relative difference for option prices and percentage difference for early exercise boundary estimates) and computational times (average per time step) of MOL, PSOR, and LSMC prices relative to references prices.

Method	**RMSRD**	**EEB Error (%)**	**Time per N (s)**
MOL ($N = 20$)	0.0224	-0.7583	8.79
MOL ($N = 50$)	0.0218	-0.9513	6.92
MOL ($N = 100$)	0.0217	-1.2822	5.20
PSOR - Ran	0.0614	1.7004	3.52
PSOR - IE	0.0617	2.1552	3.92
PSOR - CN	0.0614	1.7096	3.32
LSMC	0.3502		3241.68
Reference			20.22

Note: All quantities are calculated at option inception $\tau = T$ (or $t = 0$) and at $v = 0.56$.

MOL Approximation of the European Exchange Option

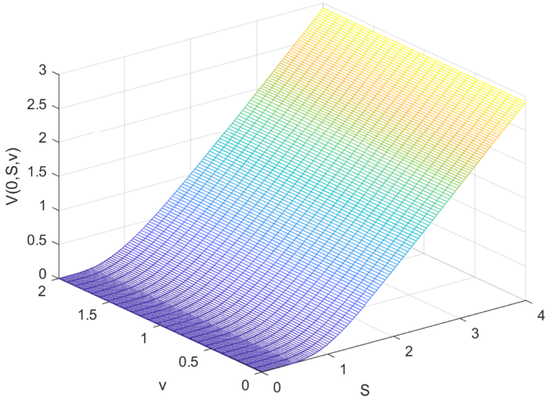


Figure: MOL approximation of the discounted European exchange option price at $\tau = T$.

MOL Approximation of the American Exchange Option

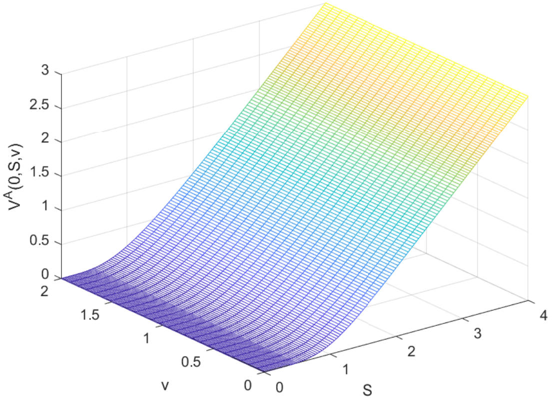


Figure: MOL approximation of the discounted American exchange option price at $\tau = T$.

MOL Approximation of the Early Exercise Premium

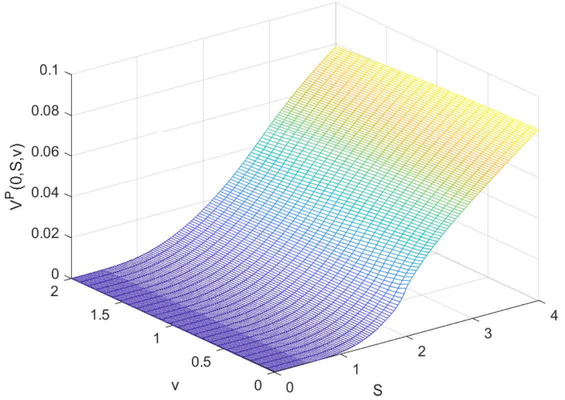


Figure: MOL approximation of the discounted early exercise premium at $\tau = T$.

MOL Approximation of the Early Exercise Boundary

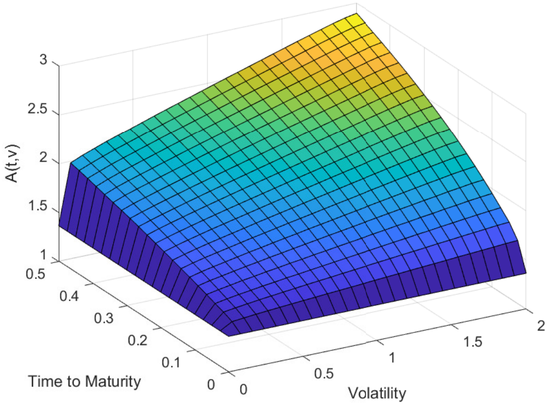
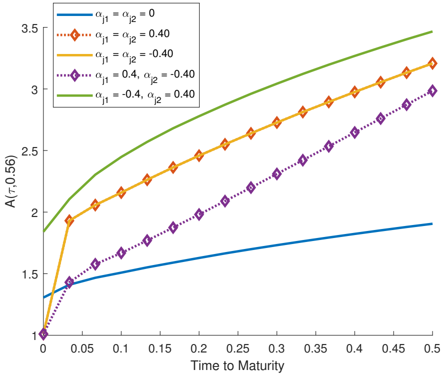
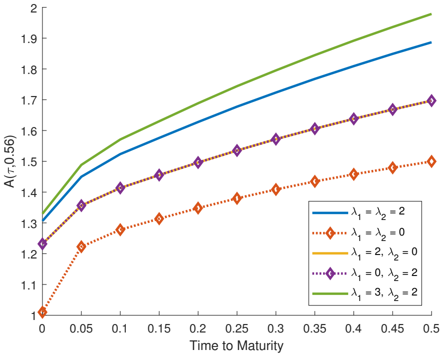


Figure: MOL approximation of the early exercise boundary surface.

Impact of Jumps on the Early Exercise Boundary



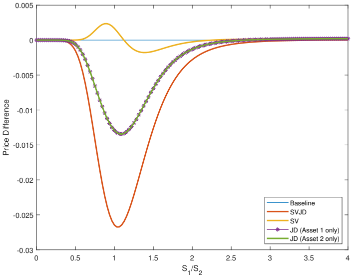
(a) α_{j_1} and α_{j_2}



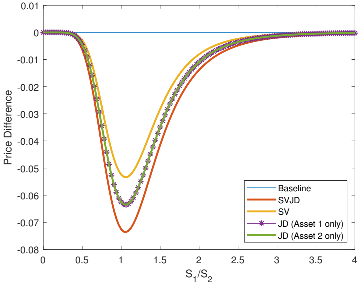
(b) $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$

Figure: Effect of changes in key jump parameters parameters on the early exercise boundary when $v = 0.56$.

Impact of Jumps and SV on Exchange Option Prices



(a) $\rho_w = 0.5$



(b) $\rho_w = -0.5$

Figure: Comparison of discounted European exchange option prices at $t = 0$ and $v = 0.56$ generated under pure diffusion, stochastic volatility, SVJD (jumps in asset 1 only), SVJD (jumps in asset 2 only), and SVJD (jumps in both assets).

Summary

- Using Bjerskund and Stensland's (1993) put-call transformation technique, we are able to simplify the main problem into a call option on the asset yield ratio.
- Given the reduction in dimensions, we then formulated a method of lines algorithm to numerically solve the option pricing IPDE.
- The MOL performs more efficiently and more accurately compared to the PSOR approach and the Longstaff and Schwartz [2001] Monte Carlo approach.
- Jump intensities and the jump size density mean have substantial impact on option prices as it is able to shift the early exercise boundary curves upward and generate the largest price differences relative to the baseline.

For Further Study

- For which asset price model specifications will the put-call transformation produce the intended simplification?
- Is it also applicable to n-asset option pricing problems, $n \geq 3$?
- We also aim to see how the MOL can be extended when pricing is done in higher dimensions and under general asset price dynamics.
- A formal convergence analysis of the MOL is also an issue to be investigated in future work.

Thank you very much!

Further details on the results shown in this presentation can be found in these papers:

1. A Put-Call Transformation of the Exchange Option Pricing Problem under Stochastic Volatility and Jump-Diffusion Dynamics [arXiv:2002.10194]

2. A Numerical Approach to Pricing Exchange Options under Stochastic Volatility and Jump-Diffusion Dynamics, *Quantitative Finance* [DOI:10.1080/14697688.2021.1926534, arXiv:2106.07362]

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