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Exchange Option Pricing under Stochastic Volatility and Jump-Diffusion Dynamics

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Introduction

Exchange Options

- An **exchange option** is a contract that grants the holder the right, but not the obligation, to exchange one risky asset for another.
- The payoff of the European exchange option at maturity T is given by

$$\left(S_{1,T} - S_{2,T}\right)^{+},$$

where $x^{+} = \max\{x, 0\}$ and $S_{1,t}$ and $S_{2,t}$ are the prices of the two risky assets at time t.
- The exchange option is a special type of **spread option** which has payoff function

$$\left(S_{1,T} - S_{2,T} - K\right)^{+},$$

where K is called the strike price.

Related Work on Exchange Options

- **European exchange options**
  - (Pure Diffusion) Margrabe (1978)
  - (Stochastic Volatility) Antonelli and Scarlatti (2010) and Alòs and Rheinlander (2017)
  - (Jump-Diffusion) Cheang and Chiarella (2011), Caldana et al. (2015), and Cufaro-Petroni and Sabino (2018)
- **American exchange options**
  - (Pure Diffusion) Bjerskund and Stensland (1993) and Carr (1995)
  - (Jump-Diffusion) Cheang and Chiarella (2011)

Main Contributions

Under the assumption that asset prices are modelled with **stochastic volatility jump-diffusion (SVJD)** dynamics:

- We demonstrate the use of the Geman et al. (1995) change-of-numéraire technique to obtain a (probabilistic) representation of European exchange option prices similar to the original Margrabe formula;
- Using the lower bound approximation of Caldana and Fusai (2013) for European spread options, we obtain an alternative representation for the price of European exchange options in terms of the joint characteristic function of log-prices; and
- We extend the probabilistic analysis of American exchange options by Cheang and Chiarella (2011) to the case of stochastic volatility and jump-diffusion dynamics.

Main Contributions

This presentation is a condensed version of the paper **Representation of Exchange Option Prices under Stochastic Volatility Jump-Diffusion Dynamics** published in *Quantitative Finance*.

Details of results indicated with a ★ can be found in this paper.

A Stochastic Volatility and Jump-Diffusion (SVJD) Model for Stock Prices

Notation and Some Assumptions

- We consider the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\mathbb{P}$ is the objective market measure and $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the filtration (satisfying usual conditions) generated by all stochastic processes included in the model specification.
- Let $\{S_{1,t}\}$ and $\{S_{2,t}\}$ denote the price processes of two stocks that pay (constant) continuously compounded dividend yields q_1 and q_2, respectively.
- Let $S_{i,t-} \equiv \lim_{u\to t^-} S_{i,u}$ denote the price of stock i immediately before time t. If a jump occurs at time t, then $S_{i,t-}$ represents the *pre-jump price* of stock i.
- We assume the existence of a money market account which earns interest at the (constant) risk-free rate $r > 0$.

Asset Price Dynamics

We assume that stock prices evolve according to

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t-}} = (\mu_i - \lambda_i \kappa_i)\,\mathrm{d}t + \sqrt{v_{i,t}}\,\mathrm{d}W_{i,t} + \int_{\mathbb{R}} (e^{y_i} - 1) p(\mathrm{d}y_i, \mathrm{d}t) \tag{1}$$

$$\mathrm{d}v_{i,t} = \xi_i(\eta_i - v_{i,t})\,\mathrm{d}t + \sigma_i \sqrt{v_{i,t}}\,\mathrm{d}Z_{i,t} \tag{2}$$

where $\mu_i, \lambda_i, \sigma_i, \eta_i, \xi_i > 0$ are constants, $\{W_{i,t}\}$ and $\{Z_{i,t}\}$ are standard $\mathbb{P}$-Wiener processes, $p(\mathrm{d}y_i, \mathrm{d}t)$ is the counting measure with $\mathbb{P}$-local characteristics $(\lambda_i, m_{\mathbb{P}}(\mathrm{d}y_i))$, and $\kappa_i \equiv \mathbb{E}_{\mathbb{P}}[e^{Y_i} - 1]$ is the mean jump size.

Note that the counting measures and marks are independent of each other and of the Wiener processes enumerated above.

Correlation Structure

	$W_{1,t}$	$W_{2,t}$	$Z_{1,t}$	$Z_{2,t}$
$W_{1,t}$	1	ρ_w	ρ_{wz_1}	0
$W_{2,t}$	ρ_w	1	0	ρ_{wz_2}
$Z_{1,t}$	ρ_{wz_1}	0	1	ρ_z
$Z_{2,t}$	0	ρ_{wz_2}	ρ_z	1

Table: Correlation structure (denoted by $\mathbf{\Sigma}$) of the $\mathbb{P}$-Wiener processes in the SVJD model (1) and (2). It is assumed that $|\rho_w| \neq 1, |\rho_z| \neq 1$.

Parameter Assumptions

Assumption. Assume that $Z_{1,t}$ and $Z_{2,t}$ are uncorrelated (i.e. $\rho_z = 0$). Furthermore, assume that the coefficients ξ_i, η_i, and σ_i are positive and satisfy $2\xi_i\eta_i \geq \sigma_i^2$, for $i = 1, 2$. Lastly, assume that $-1 < \rho_{wz_i} < \min\left\{\xi_i/\sigma_i, 1\right\}$, for $i = 1, 2$.

- The condition $2\xi_i\eta_i \geq \sigma_i^2$ ensures that $\{v_{i,t}\}$ does not hit zero or explode in finite time under $\mathbb{P}$.
- The condition on ρ_{wz_i} ensures that the variance processes are a.s. positive and finite under any measure equivalent to $\mathbb{P}$ as defined by the Radon-Nikodým derivative to be discussed later.

Asset Price Dynamics (contd.)

Using Itô's Lemma for jump-diffusion processes (see Runggaldier, 2003; Øksendal and Sulem, 2007), equation (1) admits a solution

$$S_{i,t} = S_{i,0} \exp\left\{ (\mu_i - \lambda_i \kappa_i)t - \frac{1}{2}\int_0^t v_{i,s}\,\mathrm{d}s + \int_0^t \sqrt{v_{i,s}}\,\mathrm{d}W_{i,s} + \sum_{n=1}^{N_{i,t}} Y_{i,n} \right\}. \quad (3)$$

Furthermore, the dynamics of the discounted yield process $\{\tilde{S}_{i,t}\}$, where $\tilde{S}_{i,t} \equiv e^{-(r-q_i)t} S_{i,t}$, is given by

$$\begin{aligned} \mathrm{d}\tilde{S}_{i,t} &= (\mu_i + q_i - r - \lambda_i \kappa_i)\tilde{S}_{i,t-}\,\mathrm{d}t + \sqrt{v_{i,t}}\tilde{S}_{i,t-}\,\mathrm{d}W_{i,t} \\ &\quad + \tilde{S}_{i,t-}\int_{\mathbb{R}} (e^{y_i} - 1) p(\mathrm{d}y_i, \mathrm{d}t). \end{aligned} \quad (4)$$

A Radon-Nikodým Derivative (Notation)

- Let $\mathbf{B}_t = (W_{1,t}, W_{2,t}, Z_{1,t}, Z_{2,t})^\top$ be a vector of standard $\mathbb{P}$-Wiener processes with correlation matrix $\boldsymbol{\Sigma}$;
- Let $Q_{1,t} = \sum_{n=1}^{N_{1,t}} Y_{1,n}$ and $Q_{2,t} = \sum_{n=1}^{N_{2,t}} Y_{2,n}$ be compound Poisson processes;
- Let $\boldsymbol{\theta}_t = (\psi_{1,t}, \psi_{2,t}, \zeta_{1,t}, \zeta_{2,t})^\top$ be a vector of real-valued adapted processes.

A Radon-Nikodým Derivative

Proposition. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a probability space. Let L_t be given by the equation

$$
\begin{aligned}
L_t = \exp &\left\{ -\int_0^t (\boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}_s)^\top \, \mathrm{d}\mathbf{B}_s - \frac{1}{2}\int_0^t \boldsymbol{\theta}_s^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\theta}_s \, \mathrm{d}s \right\} \\
&\times \exp\left\{ \sum_{n=1}^{N_{1,t}} (\gamma_1 Y_{1,n} + \nu_1) - \lambda_1 t \left(e^{\nu_1}\mathbb{E}_{\mathbb{P}}(e^{\gamma_1 Y_1}) - 1 \right) \right\} \\
&\times \exp\left\{ \sum_{n=1}^{N_{2,t}} (\gamma_2 Y_{2,n} + \nu_2) - \lambda_2 t \left(e^{\nu_2}\mathbb{E}_{\mathbb{P}}(e^{\gamma_2 Y_2}) - 1 \right) \right\}
\end{aligned} \tag{5}
$$

and suppose that $\{L_t\}$ is a strict $\mathbb{P}$-martingale such that $\mathbb{E}_{\mathbb{P}}[L_t] = 1$.

A Radon-Nikodým Derivative (contd.)

Then L_T is the Radon-Nikodým derivative of some probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and the following hold:

1. $W_{i,t}$ and $Z_{i,t}$ have drift $-\psi_{i,t}$ and $-\zeta_{i,t}$, respectively for $i = 1, 2$, under $\mathbb{Q}$;
2. the compound Poisson process $Q_{i,t}$ has a new intensity rate $\tilde{\lambda}_i = \lambda_i e^{\nu_i} \mathbb{E}_{\mathbb{P}}[e^{\gamma_i Y_i}]$, $i = 1, 2$, under $\mathbb{Q}$; and
3. the mgf of jump sizes under $\mathbb{Q}$ is given by

$$M_{\mathbb{Q},Y_i}(u) = M_{\mathbb{P},Y_i}(u + \gamma_i)/M_{\mathbb{P},Y_i}(\gamma_i), \qquad i = 1, 2.$$

Asset Price Dynamics under $\mathbb{Q}$

Let $\mathbb{Q} \sim \mathbb{P}$ be a prob. measure under which $\{e^{-(r-q_i)t} S_{i,t}\}_{t \geq 0}$ are $\mathbb{Q}$-martingales (i.e. $\mathbb{Q}$ is the risk-neutral measure). ★

Under $\mathbb{Q}$, the dynamics of the stock prices and instantaneous variances are given by the SDEs

$$\frac{\mathrm{d}S_{i,t}}{S_{i,t-}} = (r - q_i - \tilde{\lambda}_i \tilde{\kappa}_i)\,\mathrm{d}t + \sqrt{v_{i,t}}\,\mathrm{d}\tilde{W}_{i,t} + \int_{\mathbb{R}} (e^{y_i} - 1) p(\mathrm{d}y_i, \mathrm{d}t) \tag{6}$$

$$\mathrm{d}v_{i,t} = \left[\xi_i \eta_i - (\xi_i + \Lambda_i) v_{i,t}\right] \mathrm{d}t + \sigma_i \sqrt{v_{i,t}}\,\mathrm{d}\tilde{Z}_{i,t}, \tag{7}$$

where $\tilde{\kappa}_i \equiv \mathbb{E}_{\mathbb{Q}}[e^{Y_i} - 1]$, $\Lambda_i \geq 0$ is constant, and $\{\tilde{W}_{i,t}\}$ and $\{\tilde{Z}_{i,t}\}$ are standard $\mathbb{Q}$-Wiener processes.

Asset Price Dynamics under $\mathbb{Q}$ (contd.)

The solution $S_{i,t}$, for $0 < t \leq T$, to equation (6) is given by

$$S_{i,t} = S_{i,0} \exp\left\{ (r - q_i - \tilde{\lambda}_i \tilde{\kappa}_i)t - \frac{1}{2}\int_0^t v_{i,s}\,\mathrm{d}s + \int_0^t \sqrt{v_{i,s}}\,\mathrm{d}\tilde{W}_{i,s} + \sum_{n=1}^{N_{i,t}} Y_{i,n} \right\}. \tag{8}$$

An Integro-Partial Differential Equation (IPDE) for Exchange Option Prices

European and American Exchange Option Prices

The European and American exchange option prices, respectively denoted by $C_t^E \equiv C_t^E(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$ and $C_t^A \equiv C_t^A(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t})$, are given as $\mathbb{Q}$-expectations by

$$C_t^E = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[(S_{1,T} - S_{2,T})^+ \middle| S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t} \right] \tag{9}$$

$$C_t^A = \operatorname*{ess\,sup}_{\tau \in [t,T]} \mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)}(S_{1,\tau} - S_{2,\tau})^+ \middle| S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t} \right], \tag{10}$$

where τ is a $\{\mathcal{F}_t\}$-stopping time (Bjerskund and Stensland, 1993; Cheang and Chiarella, 2011).

For either option, let C_{t-} denote the exchange option price *prior to any jumps in* $S_{1,t}$ *and* $S_{2,t}$ *at time* t.

An Assumption on Differentiability

Assumption. For $t \in [0, T]$, the functions C_t^E and C_t^A are assumed to be (at least) twice-continuously-differentiable in s_1, s_2, v_1, and v_2 and first-order continuously-differentiable in t.

An IPDE for Exchange Option Prices

Proposition. Suppose the differentiability assumption holds. Given the that asset prices and variances have $\mathbb{Q}$-dynamics given by equations (6) and (7), the exchange option price C_t satisfies the IPDE

$$\begin{aligned} rC_{t-} = \mathcal{L}[C_{t-}] &+ \tilde{\lambda}_1 \mathbb{E}_{\mathbb{Q}}^{Y_1} \left[C_t \left(S_{1,t-} e^{Y_1}, S_{2,t-}, v_{1,t}, v_{2,t} \right) - C_{t-} \right] \\ &+ \tilde{\lambda}_2 \mathbb{E}_{\mathbb{Q}}^{Y_2} \left[C_t \left(S_{1,t-}, S_{2,t-} e^{Y_2}, v_{1,t}, v_{2,t} \right) - C_{t-} \right], \end{aligned} \tag{11}$$

where $\mathbb{E}_{\mathbb{Q}}^{Y_i}[\cdot]$ denotes the expectation with respect to the random variable Y_i $(i = 1, 2)$ under $\mathbb{Q}$,

An IPDE for Exchange Option Prices (contd.)

and the differential operator $\mathcal{L}$ is defined by

$$\mathcal{L}[f] = \frac{\partial f}{\partial t} + \sum_{i=1}^{2}(r - q_i - \tilde{\lambda}_i\tilde{\kappa}_i)S_{i,t-}\frac{\partial f}{\partial s_i} + \sum_{i=1}^{2}[\xi_i\eta_i - (\xi_i + \Lambda_i)v_{i,t}]\frac{\partial f}{\partial v_i}$$
$$+ \frac{1}{2}\sum_{i=1}^{2} v_{i,t}S_{i,t-}^2\frac{\partial^2 f}{\partial s_i^2} + \frac{1}{2}\sum_{i=1}^{2}\sigma_i^2 v_{i,t}\frac{\partial^2 f}{\partial v_i^2} + \rho_w\sqrt{v_{1,t}v_{2,t}}S_{1,t-}S_{2,t-}\frac{\partial^2 f}{\partial s_1\partial s_2}$$
$$+ \rho_{wz_1}\sigma_1 v_{1,t}S_{1,t-}\frac{\partial^2 f}{\partial s_1\partial v_1} + \rho_{wz_2}\sigma_2 v_{2,t}S_{2,t-}\frac{\partial^2 f}{\partial s_2\partial v_2}$$
$$+ \rho_z\sigma_1\sigma_2\sqrt{v_{1,t}v_{2,t}}\frac{\partial^2 f}{\partial v_1\partial v_2}. \tag{12}$$

An IPDE for Exchange Option Prices

- For the European exchange option, the terminal condition for the IPDE is $C_T^E = (S_{1,T} - S_{2,T})^+$.
- In the case of the American exchange option, additional conditions, namely the early exercise boundary condition and smooth-pasting conditions, must be specified given the early exercise boundary of the option.

Representation of the European Exchange Option Price: A Change-of-Numéraire Approach

Rewriting the European Exchange Option Price

Let $\mathcal{A}_0 = \{S_{1,T} > S_{2,T}\}$ be the event that the option is in-the-money at time T (as viewed from time 0). It follows that

$$\begin{aligned} C_0^E &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\left(S_{1,T} - S_{2,T}\right)^+\right] \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}[S_{1,T}\mathbf{1}_{\mathcal{A}_0}] - e^{-rT}\mathbb{E}_{\mathbb{Q}}[S_{2,T}\mathbf{1}_{\mathcal{A}_0}], \end{aligned} \tag{13}$$

where $\mathbf{1}_{(\cdot)}$ is the indicator function.

With the explicit formula (8) for $S_{i,T}$, we can further write

$$C_0^E = S_{1,0}e^{-q_1 T}\mathbb{E}_{\mathbb{Q}}[U_{1,T}\mathbf{1}_{\mathcal{A}_0}] - S_{2,0}e^{-q_2 T}\mathbb{E}_{\mathbb{Q}}[U_{2,T}\mathbf{1}_{\mathcal{A}_0}], \tag{14}$$

where

$$U_{i,T} = \exp\left\{-\frac{1}{2}\int_0^T v_{i,t}\,\mathrm{d}t + \int_0^T \sqrt{v_{i,t}}\,\mathrm{d}\tilde{W}_{i,t} - \tilde{\lambda}_i\tilde{\kappa}_i T + \sum_{n=1}^{N_{i,T}} Y_{i,n}\right\}. \tag{15}$$

European Exchange Option Price

In terms of $\hat{\mathbb{Q}}_1$ and $\hat{\mathbb{Q}}_2$, the price of the European exchange option may be written as

$$C_0^E = S_{1,0} e^{-q_1 T} \hat{\mathbb{Q}}_1(\mathcal{A}_0) - S_{2,0} e^{-q_2 T} \hat{\mathbb{Q}}_2(\mathcal{A}_0). \tag{16}$$

where with equation (8), the event $\mathcal{A}_0$ may be rewritten as

$$\left\{ \mathfrak{A}_{0,T} > \ln\left(\frac{S_{2,0}}{S_{1,0}}\right) - \left(q_2 - q_1 - \tilde{\lambda}_1 \tilde{\kappa}_1 + \tilde{\lambda}_2 \tilde{\kappa}_2\right) T \right\},$$

and $\mathfrak{A}_{0,T}$ is the r.v.

$$\begin{aligned} \mathfrak{A}_{0,T} = -\frac{1}{2} \int_0^T (v_{1,t} - v_{2,t}) \, \mathrm{d}t + \int_0^T \sqrt{v_{1,t}} \, \mathrm{d}\tilde{W}_{1,t} - \int_0^T \sqrt{v_{2,t}} \, \mathrm{d}\tilde{W}_{2,t} \\ + \sum_{n=1}^{N_{1,T}} Y_{1,n} - \sum_{n=1}^{N_{2,T}} Y_{2,n}. \end{aligned}$$

Some Remarks

- Similar to the original Margrabe (1978) formula, our characterization of the European exchange option price under SVJD dynamics is also independent of the risk-free interest rate r.
- The probabilities $\hat{\mathbb{Q}}_1(\mathcal{A}_0)$ and $\hat{\mathbb{Q}}_2(\mathcal{A}_0)$ may be simulated given the new properties of the Wiener processes after a transition in probability measure from $\mathbb{Q}$ to $\hat{\mathbb{Q}}_1$ or $\hat{\mathbb{Q}}_2$.
- In the presence of stochastic volatilities $v_{1,t}$ and $v_{2,t}$, a series expression for the European exchange option price similar to those obtained in Cheang and Chiarella (2011) and Caldana et al. (2015) cannot be obtained.

Representation of the European Exchange Option Price via Fourier Transform

Introduction

- Pricing European *spread options* via Fourier inversion (i.e. representing prices in terms of the **joint characteristic function (cf)** of the underlying price processes) has been analyzed by Dempster and Hong (2002), Hurd and Zhou (2010), and Alfeus and Schlögl (2019).
- Caldana and Fusai (2013) proposed a *lower bound approximation* for the spread option price in terms of the joint cf of log prices. This lower bound provides the *exact price* of the European exchange option by setting $K = 0$.
- This section applies the Caldana and Fusai (2013) result and uses the joint cf of log prices under the SVJD model to obtain a representation of European exchange option prices similar to equation (16).

Joint Characteristic Function of Log Prices

- **Some notation:**
  - Let $X_t = (X_{1,t}, X_{2,t})^\top$ be the vector of log prices $X_{i,t} = \ln S_{i,t}$
  - Let $\phi_{X_{T|t}}$ denote the joint cf of X_T conditional on $\mathcal{F}_t$ under $\mathbb{Q}$,

$$\phi_{X_{T|t}}(u_1, u_2) = \mathbb{E}_{\mathbb{Q}}\left[e^{iu_1 X_{1,T} + iu_2 X_{2,T}} \middle| X_{1,t} = x_1, X_{2,t} = x_2, v_{1,t} = v_1, v_{2,t} = v_2 \right].$$

  We let ϕ_{X_t} denote the joint cf given the current values of log prices and instantaneous variances.

- The joint cf of log prices under the SVJD specification can be obtained following the techniques used by Cont and Tankov (2004) and Cane and Olivares (2014). ★
- An analytic formula for the joint cf of log prices under the SVJD model is available if we assume that $\rho_w = \rho_z = 0$.[1]

[1]Specifically, this assumption is required to preserve the affine structure of the joint cf.

Caldana and Fusai (2013) Lower Bound

Proposition. A lower bound for the time $t \in [0, T)$ price of a European spread option with strike price K is given by

$$C_{K,t}^{\alpha,k} = \left\{ \frac{e^{-r(T-t)-\delta k}}{\pi} \int_0^\infty e^{-izk} \psi_{T|t}(z, \delta, \alpha) \, \mathrm{d}z \right\}^+, \tag{17}$$

where

$$\begin{aligned} \psi_{T|t}(z, \delta, \alpha) = \frac{e^{i(z-i\delta) \ln \phi_{X_{T|t}}(0,-i\alpha)}}{iz+\delta} \Big[& \phi_{X_{T|t}} \left(z - i\delta - i, -\alpha(z - i\delta) \right) \\ & - \phi_{X_{T|t}} \left(z - i\delta, -\alpha(z - i\delta) - i \right) \\ & - K \phi_{X_{T|t}} \left(z - i\delta, -\alpha(z - i\delta) \right) \Big] \end{aligned}$$

Caldana and Fusai (2013) Lower Bound (contd.)

and

$$\phi_{X_T|t}(u_1, u_2) = \mathbb{E}_{\mathbb{Q}}\left[e^{iu_1 X_{1,T} + iu_2 X_{2,T}} \middle| \mathcal{F}_t \right]$$

$$\alpha = \frac{F_2(t, T)}{F_2(t, T) + K}$$

$$k = \ln\left[F_2(t, T) + K\right],$$

and $F_2(t, T)$ is the time t forward price of asset 2 with for delivery at time T.

Remarks

- The parameter δ governs the exponentially decaying term $e^{-\delta k}$ to ensure square-integrability in the negative k-axis.
- This result considers the approximate (sub-optimal) exercise policy

$$A_t = \left\{ S_{1,T} \geq \frac{e^k S_{2,T}^{\alpha}}{\mathbb{E}_{\mathbb{Q}}[S_{2,T}^{\alpha} | \mathcal{F}_t]} \right\}$$

  (Bjerskund and Stensland, 2011).
- If $K = 0$, then A_t coincides with the true exercise strategy $B = \{S_{1,T} \geq S_{2,T}\}$ for the exchange option.
- Thus, if $K = 0$ in the preceding proposition, what results is the price of the European exchange option at time t.

European Exchange Option Price

Proposition. The time t European exchange option price is given by

$$\begin{aligned} C_t^E = {} & S_{1,t} e^{-q_1(T-t)} \int_0^\infty \frac{1}{\pi(iz+\delta)} \phi^{(1)}_{X_T|t}(iz+\delta, -iz-\delta)\,\mathrm{d}z \\ & - S_{2,T} e^{-q_2(T-t)} \int_0^\infty \frac{1}{\pi(iz+\delta)} \phi^{(2)}_{X_T|t}(iz+\delta, -iz-\delta)\,\mathrm{d}z, \end{aligned} \tag{18}$$

where $\phi^{(i)}_{X_T|t}(u_1, u_2) = \mathbb{E}_{\hat{\mathbb{Q}}_i}[e^{iu_1 X_{1,T} + iu_2 X_{2,T}} | \mathcal{F}_t]$, $i = 1, 2$ is the joint conditional cf of $X_T = (X_{1,T}, X_{2,T})^\top$ under $\hat{\mathbb{Q}}_i$, and $\hat{\mathbb{Q}}_1$ and $\hat{\mathbb{Q}}_2$ are the probability measures equivalent to $\mathbb{Q}$ (discussed earlier).

Remarks

- The above calculations therefore show that the Caldana and Fusai (2013) result, when applied to European exchange options, allows for a decomposition similar to equation (16), which was obtained via the change-of-numéraire technique.
- This therefore presents the possibility that the probabilities $\hat{\mathbb{Q}}_1(\mathcal{A}_0)$ and $\hat{\mathbb{Q}}_2(\mathcal{A}_0)$ (the probability of the option being in-the-money under the alternative measures $\hat{\mathbb{Q}}_1$ and $\hat{\mathbb{Q}}_2$) may be computed using Fourier inversion in equation (18).

Representation of the American Exchange Option Price

Early Exercise Boundary

The decision to exercise an American exchange option is determined by the **early exercise boundary** $B_t \equiv B(v_{1,t}, v_{2,t}, t)$, which is unknown.[2]

The boundary splits $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$ into two regions, the **stopping region** $\mathcal{S}$ and the **continuation region** $\mathcal{C}$, defined as (Broadie and Detemple, 1997)

$$\mathcal{S} = \left\{ (S_{1,t}, S_{2,t}) \in \mathbb{R}_+^2 : S_{1,t} \geq B(v_{1,2}, v_{2,t}, t) S_{2,t} \right\}$$
$$\mathcal{C} = \left\{ (S_{1,t}, S_{2,t}) \in \mathbb{R}_+^2 : S_{1,t} < B(v_{1,2}, v_{2,t}, t) S_{2,t} \right\}.$$

The term 'early exercise boundary' may also refer to the line $s_1 = B_t s_2$ dividing the stopping and continuation regions. It is known that $B_t \geq 1$.

[2] The dependence of the boundary on the instantaneous variance has been established by Touzi (1999) for the single-asset American option.

Early Exercise Boundary

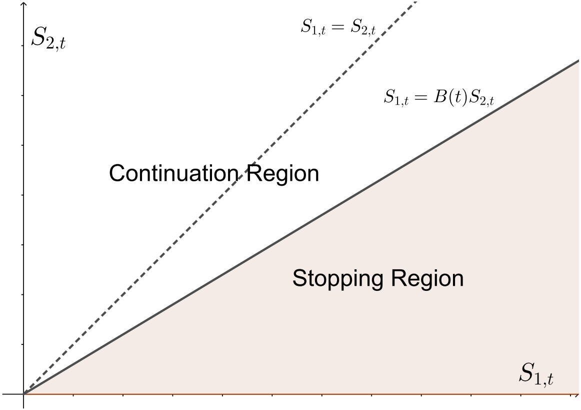

Figure: The early exercise boundary and the continuation and stopping regions for the American exchange option, adapted from Broadie and Detemple (1997); Cheang and Chiarella (2011).

A Free-Boundary Problem

The American exchange option price is the solution to the IPDE

$$\begin{aligned} rC^A_{t-} = \mathcal{L}[C^A_{t-}] &+ \tilde{\lambda}_1 \mathbb{E}^{Y_1}_{\mathbb{Q}} \left[C^A_t \left(S_{1,t-}e^{Y_1}, S_{2,t-}, v_{1,t}, v_{2,t} \right) - C^A_{t-} \right] \\ &+ \tilde{\lambda}_2 \mathbb{E}^{Y_2}_{\mathbb{Q}} \left[C^A_t \left(S_{1,t-}, S_{2,t-}e^{Y_2}, v_{1,t}, v_{2,t} \right) - C^A_{t-} \right], \end{aligned} \tag{19}$$

over the *restricted domain* $0 < S_{1,t} < B(v_{1,t}, v_{2,t}, t)S_{2,t}$, $S_{2,t} > 0$, $v_{1,t} > 0$, $v_{2,t} > 0$, and $0 \leq t < T$.

Terminal and boundary conditions for the IPDE are given by

$$\begin{aligned} C^A_T(S_{1,T}, S_{2,T}, v_{1,T}, v_{2,T}) &= (S_{1,T} - S_{2,T})^+ \\ C^A_t(0, S_{2,t}, v_{1,t}, v_{2,t}) &= 0, \\ C^A_t(S_{1,t}, 0, v_{1,t}, v_{2,t}) &= S_{1,t}. \end{aligned} \tag{20}$$

Value-Matching and Smooth-Pasting Conditions

The IPDE is also supplemented by the **value-matching condition**

$$C_t^A = S_{1,t} - S_{2,t}, \qquad \text{for } S_{1,t} \geq B(v_{1,t}, v_{2,t}, t) S_{2,t} \text{ and } S_{2,t} > 0 \tag{21}$$

and **smooth-pasting conditions** (Pham, 1997; Cheang and Chiarella, 2011)

$$\begin{aligned}
\lim_{s_1 \to B(v_1,v_2,t)s_2} \frac{\partial C_t^A}{\partial s_1}(s_1, s_2, v_1, v_2) &= 1 \\
\lim_{s_1 \to B(v_1,v_2,t)s_2} \frac{\partial C_t^A}{\partial s_2}(s_1, s_2, v_1, v_2) &= -1 \\
\lim_{s_1 \to B(v_1,v_2,t)s_2} \frac{\partial C_t^A}{\partial v_1}(s_1, s_2, v_1, v_2) &= 0 \\
\lim_{s_1 \to B(v_1,v_2,t)s_2} \frac{\partial C_t^A}{\partial v_2}(s_1, s_2, v_1, v_2) &= 0 \\
\lim_{s_1 \to B(v_1,v_2,t)s_2} \frac{\partial C_t^A}{\partial t}(s_1, s_2, v_1, v_2) &= 0.
\end{aligned} \tag{22}$$

Early Exercise Representation

Proposition. The price of the American exchange option admits the representation

$$C_t^A(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}) = C_t^E(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}) + C_t^P(S_{1,t}, S_{2,t}, v_{1,t}, v_{2,t}), \quad (23)$$

where C_t^E is the price of the corresponding European exchange option and C_t^P is the early exercise premium of the American exchange option.

Early Exercise Representation (contd.)

The early exercise premium is given by

$$
\begin{aligned}
C_t^P = & \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left[\left(q_1 S_{1,u-} - q_2 S_{2,u-} \right) \mathbf{1}_{\mathcal{A}_u} \middle| \mathcal{F}_t \right] \mathrm{d}u \\
& - \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \tilde{\lambda}_1 \mathbb{E}_{\mathbb{Q}}^{Y_1} \left[C_u^A \left(S_{1,u-} e^{Y_1}, S_{2,u-}, v_{1,u}, v_{2,u} \right) \right. \right. \\
& \qquad \left. \left. - \left(S_{1,u-} e^{Y_1} - S_{2,u-} \right) \right] \mathbf{1}_{\mathcal{A}_{1,u}} \middle| \mathcal{F}_t \right\} \mathrm{d}u \\
& - \int_t^T e^{-r(u-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \tilde{\lambda}_2 \mathbb{E}_{\mathbb{Q}}^{Y_2} \left[C_u^A \left(S_{1,u-}, S_{2,u-} e^{Y_2}, v_{1,u}, v_{2,u} \right) \right. \right. \\
& \qquad \left. \left. - \left(S_{1,u-} - S_{2,u-} e^{Y_2} \right) \right] \mathbf{1}_{\mathcal{A}_{2,u}} \middle| \mathcal{F}_t \right\} \mathrm{d}u,
\end{aligned}
\tag{24}
$$

Early Exercise Representation (contd.)

where the events $\mathcal{A}_u$, $\mathcal{A}_{1,u}$, and $\mathcal{A}_{2,u}$, for $t \leq u \leq T$, are defined as

$$\begin{aligned}
\mathcal{A}_u &= \left\{(S_{1,u-}, S_{2,u-}) \in \mathcal{S}\right\} = \{S_{1,u-} \geq B_u S_{2,u-}\} \\
\mathcal{A}_{1,u} &= \mathcal{A}_u \cap \left\{S_{1,u-} e^{Y_1} < B(v_{1,u}, v_{2,u}, u) S_{2,u-}\right\} = \left\{B_u \leq \frac{S_{1,u-}}{S_{2,u-}} < B_u e^{-Y_1}\right\} \\
\mathcal{A}_{2,u} &= \mathcal{A}_u \cap \left\{S_{1,u-} < B(v_{1,u}, v_{2,u}, u) S_{2,u-} e^{Y_2}\right\} = \left\{B_u \leq \frac{S_{1,u-}}{S_{2,u-}} < B_u e^{Y_2}\right\}.
\end{aligned} \tag{25}$$

Here, $B_u = B(v_{1,u}, v_{2,u}, u)$ is the early exercise boundary at time u.

Linked System of Integral Equations

The value-matching condition dictates that

$$C_t^A(B_t S_{2,t}, S_{2,t}, v_{1,t}, v_{2,t}) = S_{2,t}\left(B_t - 1\right).$$

$$S_{2,t}(B_t - 1) = C_t^E(B_t S_{2,t}, S_{2,t}, v_{1,t}, v_{2,t}) + C_t^P(B_t S_{2,t}, S_{2,t}, v_{1,t}, v_{2,t}). \tag{26}$$

This equation, however, must solved as a linked system in conjunction with equation (23), since this equation involves the (yet unknown) American exchange option price C_t^A.

Conclusion and Future Work

Summary and Conclusions

- We have provided an extension to the results of Margrabe (1978) and Cheang and Chiarella (2011) for exchange options to consider the case where, aside from the presence of jumps, asset prices are also driven by a stochastic volatility process.
- Representations for European exchange option prices were derived using two methods: (1) a change-of-numéraire approach and (2) a Fourier inversion approach. We showed that representations from these methods have similar forms.
- We also demonstrated that the American exchange option price can also be represented as the sum of the price of the corresponding European exchange option price and an early exercise premium.
- It was also shown that the early exercise premium can be decomposed into a premium on the diffusive component of asset prices and a premium owing to the possibility of jumps back into the continuation region right before exercise.

Future Outlook and Direction

- The representations we obtained may be numerically evaluated via Monte Carlo simulation, numerical integration, or fast Fourier transform methods (see Hurd and Zhou, 2010; Caldana and Fusai, 2013; Cane and Olivares, 2014, for example).
- Meanwhile, extensions to the numerical methods proposed by Chiarella et al. (2009) and Chiarella and Ziveyi (2011) may be considered in providing a numerical solution for the exchange option IPDE.
- The efficacy of these methods in implementing our exchange option price representations is a topic for further research.

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Any questions?

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