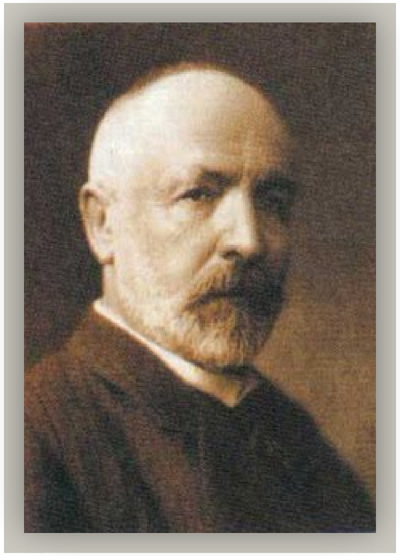
By Luke Pederson

Countable and Uncountable Sets in Discrete Math

History

- Georg Cantor (1845-1918) was the key mathematician in the development of Set Theory (Epp 2004)



- Cantor established and verified many important properties of Sets like One-To-One Correspondence, as well as defining Infinite sets
- Cantor developed a theorem called, strangely enough, Cantor's Theorem that was used for proving properties of both Infinite and Finite sets
- "In elementary set theory, Cantor's theorem states that, for any set A, the set of all subsets of A (the power set of A) has a strictly greater cardinality than A itself." (Wikipedia 2011)

Set Structure

- A Set is a well defined collection of elements.
- We use:
  - Upper case letters for referring to a **set**
  - Lower case letters for **elements** of a set
  - **Brackets { }** to contain the elements in the set
    - ex. A = { a, b }
- A Sub-Set is a set where all of its elements are contained within another set.
  - Ex. A = { 1 , 2 } and B = { 0, 1, 2, 3, 4 } , so A is a sub-set of B since all its elements are within B

Set Notation

- There are two main ways to mathematically define a set:

  - A = { 1, 2, 3, 4}
  - A = { y is an Integer | 1 < y < 5 }

Equivalently:

A = { y € $\mathbb{Z}$ | 1 < y < 5 }

- Here, $\mathbb{Z}$ is in the <u>Universal Set of Integers</u> so ‘ **y** ‘ is an element “€” of **Integers**.

Universal Sets

- **Universal Sets** are used to hold all potential elements for a given set.
- **Cardinality** is the number of elements in a set, denoted: C (“Set”)
  - Ex. Let A = { a, b, c }
  - So, C (A) = 3

Common Universal Sets

$\mathbf{N}$ = the set of natural numbers
$\mathbf{Q}$ = the set of rational numbers
$\mathbf{R}$ = the set of real numbers
$\mathbf{P}$ = the set of prime numbers
$\mathbf{Z}$ = the set of integers
$\mathbf{E}$ = the set of even integers
$\mathbf{O}$ = the set of odd integers

Power Sets

- A Power Set is a set of all possible sub-sets, denoted: P(Set)
  - Meaning a sub-set could contain: 0, 1, 2, … etc. elements

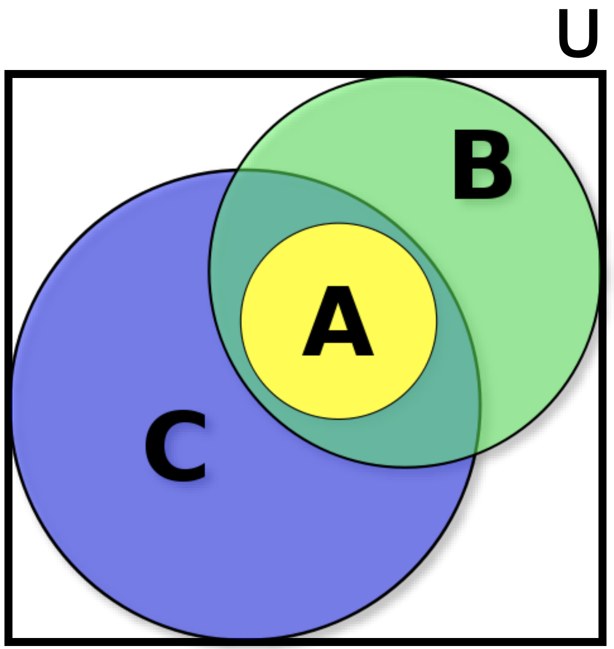
  $\varnothing$

- The Null Set (   ) represents the possibility of no elements in a set, an empty set.

- Ex. $\varnothing$
  - Let A = { a , b }
  - P (A) = {   , { a } , { b } , { a , b } }
    - The Null Set appears whenever we produce a Power Set as a possibility of having a subset of no elements

Venn Diagrams

- Venn Diagrams are useful in visualizing relations between sets.
- Here we have 3 sets: A, B, and C.
  - B and C have overlapping elements.
  - A is a subset of both B and C.
  - All 3 of the sets are within the <u>Universe</u> of possible elements, denoted as "U"

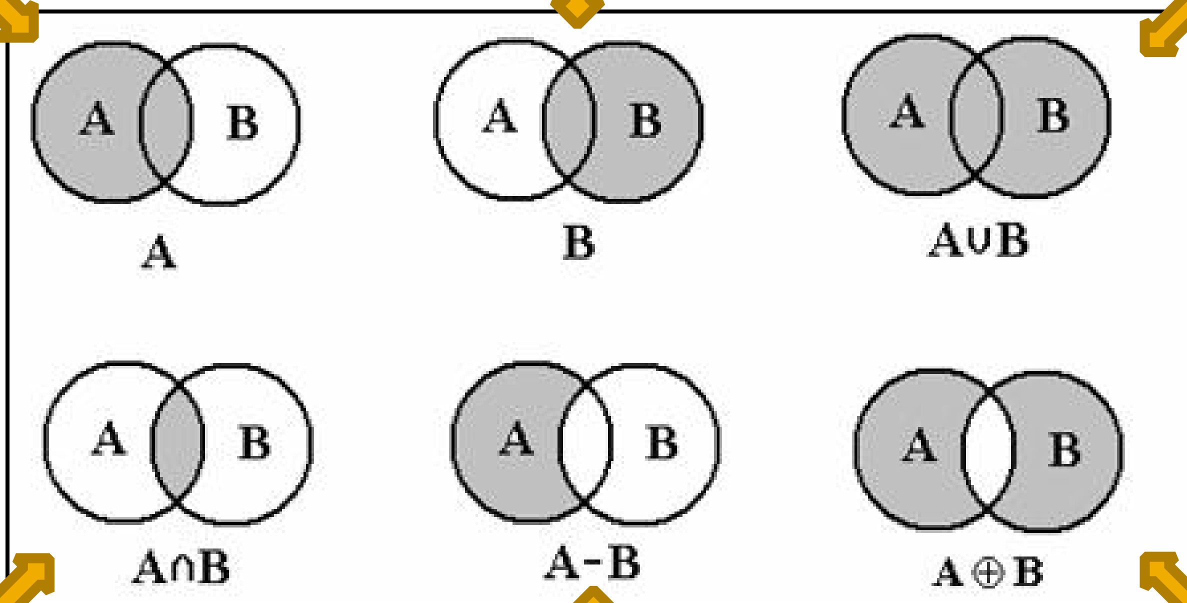


Venn Diagram Relations

Only set **B**

Only set **A**

A union **B**



A intersect **B**

A minus **B**

Either **A** or **B**, but not both.

Countable vs Uncountable

- All Finite sets are countable
  - You can count all the elements in the set
- Infinite sets can be countable, if they follow certain rules.
  - These sets are called “Countably Infinite”
- Countable doesn’t mean Countably Infinite
  - Finite sets can be Countable
  - Infinite set can be Countably Infinite

Countable and Uncountable Sets

- Some mathematically proven Countably Infinite sets are:
  - Integers, Rational, and Natural numbers

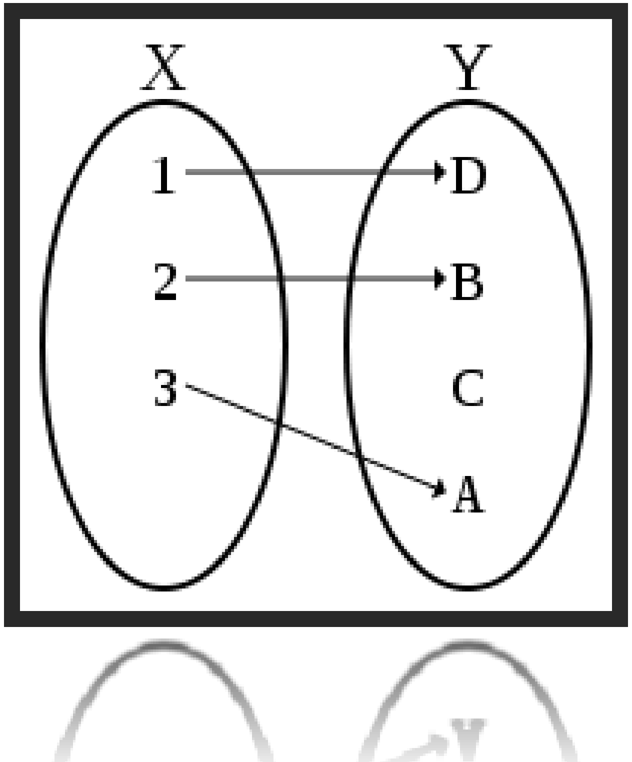
- Some mathematically proven Uncountable Infinite sets are:
  - Real and Irrational numbers

Countable

- A finite set is considered **Countable** if there is a <u>Injective Function</u> between the elements of set Y to the *Set of Natural numbers.*

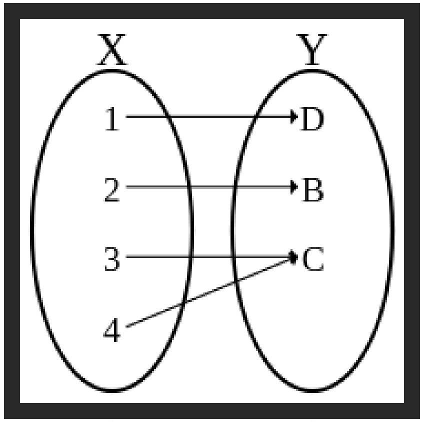
Injective means: every element in Set (X) corresponds to an element in the co-domain set (Y) <u>by at **most** one element</u>.

Another way of putting this is to say the set has the same **Cardinality** as a sub-set of the *Set of Natural numbers*.

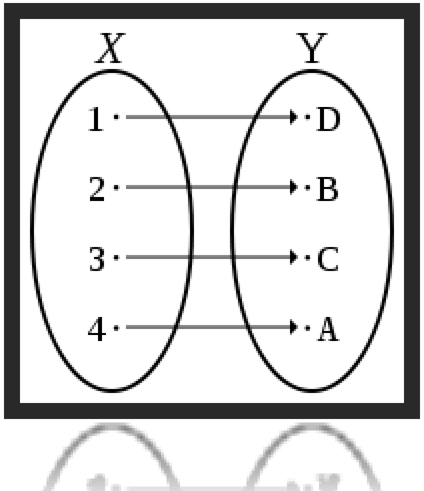


Countably Infinite

- An Infinite set is considered **Countably Infinite** if there is, in addition to being Injective, a <u>Surjective Function</u> between set (Y) to the *Set of Natural numbers.*



Surjective means: every element of set (Y) <u>has at **least** one</u> corresponding element in set (X).



If a set is Surjective and Injective, it must be **Bijective** with the *Set of Natural Numbers* (aka. One-to-one correspondence) making the set Countably Infinite

Uncountable

- A set (X) is Uncountable or Uncountably Infinite if:
  - There is **no** <u>Injective function</u> from set (X) to the *Set of Natural Numbers*
  - There is **no** <u>Surjective function</u> from set (X) to the *Set of Natural Numbers*
  - The cardinality of set (X) is <u>greater than</u> the cardinality of the *Set of Natural Numbers*

Other Properties

- Any sub-set of a countable set is countable. (PlanetMath 2010)
- The Cartesian Product of two countable sets is countable. (Wikipedia 2011)
- The Union of two countable sets is countable. (PlanetMath 2009)
- The set of all finite sub-sets of the *Set of Natural Numbers* is countable. (Wikipedia 2011)

Example Proofs

"**Cantor's theorem** states that, for any set *A*, the set of all subsets of *A* (the power set of *A*) has a strictly greater cardinality than *A* itself." (Wikipedia 2011)

-Proof of Cantor's Theorem-

Consider a function $F : X \to \mathcal{P}(X)$ from a set X to its power set. Then we define the set $Z \subseteq X$ as follows:

$$Z = \{x \in X \mid x \notin F(x)$$

Suppose that F is a bijection. Then there must exist an $x \in X$ such that $F(x) = Z$. Then we have the following contradiction:

$$x \in Z \Leftrightarrow x \notin F(x) \Leftrightarrow x \notin$$

Hence, F cannot be a bijection between X and $\mathcal{P}(X)$.

-Proof of Integers and Rational numbers are Countably Infinite-

Proposition: The integers $\mathbb{Z}$ are countable and the rational numbers $\mathbb{Q}$ are countable.

Proof: The integers $\mathbb{Z}$ are countable because the function $f : \mathbb{Z} \to \mathbb{N}$ given by $f(n) = 2^n$ if n is non-negative and $f(n) = 3^{|n|}$ if n is negative is an injective function. The rational numbers $\mathbb{Q}$ are countable because the function $g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ given by $g(m,n) = m / (n + 1)$ is a surjection from the countable set $\mathbb{Z} \times \mathbb{N}$ to the rationals $\mathbb{Q}$.

Set Theory Application

- Set theory has deep connections with most branches of Mathematics and has had major applications throughout history.

- Perhaps the most notable usage of Set Theory currently is its correlation with Programming for electronics.

- Advancing consumer technologies are about more speed and small size.

- Set Theory plays a key role in increasing the efficiency of programs for electronic equipment.

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