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Probability and Statistics for Engineering and Sciences

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Chapter 3

Discrete Random Variables and Probability

Section 1: Random Variables

A random variable is any rule that associates a number with each outcome in the sample space. Mathematically, it is a function whose domain is the sample space and whose range is the set of real numbers.

• Random because the observed value depends on which of the possible experimental outcomes results.

• Variable because different numerical values are possible depending on the possible experimental outcomes.

Example
The sample space of the experiment: tossing two coins is $\{HH,\ HT, TH, TT\}$.
Define $X =$ number of heads $\Rightarrow X\{HH\} = 2,\ X\{HT\} = 1,\ X\{TH\} = 1, X\{TT\} = 0$

Thus, $X = \{0, 1, 2\}$.

Note that:

- Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.
- There are two different types of random variables, discrete and continuous.
- A discrete random variable is a random variable whose possible values either constitute a finite set or can be listed in an infinite sequence in which there is a first element, a second element, and so on.
- Mathematically, the tools to study basic properties of discrete random variable are summation and differences.
- A random variable is continuous if both of the following apply:
  - Its set of possible values consists either of all numbers in a single interval on the number line such as $[0,25]$ or all numbers in a disjoint union of such intervals such as $[0,25] \cup [50,80]$.
  - No possible value of the variable has positive probability, $P(X = c) = 0$ for any constant value "c".
- The tools to study basic properties of continuous random variable mathematically are integrals and derivatives.

Section 2: Probability Distributions for Discrete Random Variables

Example

The Cal Poly Department of Statistics has a lab with six computers reserved for statistics majors. Let X denote the number of these computers that are in use at a particular time of day. Suppose that the probability distribution of X is as given in the following table.

x	0	1	2	3	4	5	6	Total
$p(x)$	0.05	0.10	0.15	0.25	0.20	0.15	0.10	1

Thus,

$P(X \leq 2) = P(X = 0\ or\ 1\ or\ 2) = P(0) + P(1) + P(2) = 0.05 + 0.10 + 0.15 = 0.30.$

$P(X \geq 3) = 1 - P(X \leq 2) = 1 - 0.30 = 0.70.$

$P(2 \leq X \leq 5) = P(2,3,4, or\ 5) = 0.15 + 0.25 + 0.20 + 0.15 = 0.75.$

$P(2 < X < 5) = P(X = 3\ or\ 4) = 0.25 + 0.20 = 0.45.$

A Parameter of a Probability Distribution

Assume that $p(x)$ depends on a quantity that can be assigned any value of possible values such that with each different value, a different probability distribution is determined. This quantity is called a parameter of the probability distribution. The family of probability distributions include collections of all probability distributions for different values of the parameter.

For example:

Consider the pmf of any Bernoulli random variable can be expressed in the form $p(1) = \alpha$ and $p(0) = 1 - \alpha$, where $0 < \alpha < 1$. Consider the following pmf that depends on the particular value of α such that each choice of α results in a different pmf:

$$p(x;\alpha) = \begin{cases} 1-\alpha & if\ x = 0 \\ \alpha & if\ x = 1 \\ 0 & otherwise \end{cases}$$

The Cumulative Distribution Function

The cumulative distribution function (cdf) $F(x)$ of a discrete random variable X with pmf $p(x)$ is defined for every number x as follows:

$$F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y)$$

For any two numbers a and b such that $a \leq b$, then

- $P(a \leq X \leq b) = F(b) - F(a-)$.
- $P(X = a) = F(a) - F(a-)$.
- $P(a < X < b) = P(a < X \leq b-) = F(b-) - F(a)$.

Example

A store carries flash drives with either 1 GB, 2 GB, 4 GB, 8 GB, or 16 GB of memory. The accompany table gives the distribution of $Y =$ the amount of memory in a particulsr drive:

y	1	2	4	8	16
$P(y)$	0.05	0.10	0.35	0.40	0.10

$F(1) = P(Y \leq 1) = P(Y = 1) = P(1) = 0.05$.

$F(2) = P(Y \leq 2) = P(Y = 1 \text{ or } 2) = P(1) + P(2) = 0.05 + 0.10 = 0.15$.

$F(4) = P(Y \leq 4) = P(Y = 1 \text{ or } 2 \text{ or } 4) = P(1) + P(2) + P(4) = 0.05 + 0.10 + 0.35 = 0.50$.

$F(8) = P(Y \leq 8) = P(Y = 1 \text{ or } 2 \text{ or } 4 \text{ or } 8) = P(1) + P(2) + P(4) + P(8) = 0.05 + 0.10 + 0.35 + 0.40 = 0.90$.

$F(16) = P(Y \leq 16) = P(Y = 1 \text{ or } 2 \text{ or } 4 \text{ or } 8 \text{ or } 16) = P(1) + P(2) + P(4) + P(8) + P(16) = 0.05 + 0.10 + 0.35 + 0.40 + 0.10 = 1$.

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$F(2.7) = P(Y \leq 2.7) = P(Y \leq 2) = F(2) = 0.15$

$F(7.999) = P(Y \leq 7.999) = P(Y \leq 4) = F(4) = 0.5$

$F(25) = 1$

$$F(y) = \begin{cases} 0 & y < 1 \\ 0.05 & 1 \le y < 2 \\ 0.15 & 2 \le y < 4 \\ 0.50 & 4 \le y < 8 \\ 0.90 & 8 \le y < 16 \\ 1 & 16 \le y \end{cases}$$

A graph of this cdf is shown in Figure 3.5 (page 105).

Note that:

For X a discrete rv, the graph of $F(x)$ is called a step function will have a jump at every possible value of X and will be flat between possible values.

Exercise 23 (page 108)

A branch of a certain bank in New York City has six ATMs. Let X represent the number of machines in use at a particular time of day. The cdf of X is as follows:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.06 & 0 \le x < 1 \\ 0.19 & 1 \le x < 2 \\ 0.39 & 2 \le x < 3 \\ 0.67 & 3 \le x < 4 \\ 0.92 & 4 \le x < 5 \\ 0.97 & 5 \le x < 6 \\ 1 & 6 \le x \end{cases}$$

Calculate the following probabilities directly from the cdf:

a. $p(2)$.

Answer

$P(2) = P(X = 2) = F(2) - F(1) = 0.39 - 0.19 = 0.20.$

b. $P(X > 3)$.

Answer

$P(X > 3) = 1 - P(X \le 3) = 1 - F(3) = 1 - 0.67 = 0.33.$

c. $P(2 \le X \le 5)$.

Answer

$P(2 \le X \le 5) = F(5) - F(1) = 0.97 - 0.19 = 0.78.$

d. $P(2 < X < 5)$.

Answer

$P(2 < X < 5) = P(2 < X \le 4) = F(4) - F(2) = 0.92 - 0.39 = 0.53.$

Exercise 25 (page 109)

Starting at a fixed time, the gender of each newborn child was observed at a certain hospital until a boy (B) is born. let $X =$ the number of girls born before the experiment terminates. $With\ p = P(B)$ and $1 - p = P(G)$, what is the pmf of X?

Answer

$P(0) = P(X = 0) = P(B\ first) = p$

$P(1) = P(X = 1) = P(G\ first, then\ B) = (1-p)p$

$P(2) = P(X = 2) = P(G, G, B) = (1-p)(1-p)p = (1-p)^2 p$

$P(3) = P(X = 3) = P(G, G, G, B) = (1-p)(1-p)(1-p)p = (1-p)^3 p$

.

Thus, pmf of X is $P(X = x) = \begin{cases} (1-p)^x p, & x = 0,1,2, \ldots.. \\ 0 & otherwise \end{cases}$

Note that:

This pmf is called a geometric distribution (as it will be shown in Section 5).

Section 3: Expected Values

Let X be a discrete random variable with set of possible values D and pmf "$p(x)$". The expected value (mean value) of X is denoted by $E(X)$ [μ_X or μ] is:

$$E(X) = \mu_X = \mu = \sum_{x \in D} x.p(x)$$

The Expected Value of a Function

Sometimes interest will focus on $E[h(X)]$ rather than on just $E(X)$.

Proposition

Let X be a discrete random variable with set of possible values D and pmf "$p(x)$". The expected value of function of X h(X) is denoted by $E[h(X)] = \mu_{h(X)}$ and be calculated as follows:

$$E[h(X)] = \mu_{h(X)} = \sum_{x \in D} h(x).p(x)$$

Rules of Expected Value

Proposition

$E(aX + b) = a.E(X) + b$

Note that:

$E(aX) = aE(X)$.

$E(X + b) = E(X) + b$.

$E(b) = b$.

The Variance of *X*

Let X have pmf "$p(x)$" and expected value μ. Then variance of X is:

$V(X) = \sigma_X^2 = \sum_{x \in D}(x - \mu)^2.p(x) \Rightarrow \sigma_X = \sqrt{\sigma_X^2}$ [standard deviation (SD) of X]

The quantity $h(X) = (X - \mu)^2$ is the squared deviation of X from its mean, and σ^2 is the expected squared deviation.

A Shortcut Formula for $\boldsymbol{\sigma^2}$

$V(X) = \sigma^2 = \sum_{x \in D} x^2 . p(x) - \mu^2 = E(X^2) - [E(X)]^2$.

Rules of Variance

Proposition

$V(aX + b) = \sigma_{aX+b}^2 = a^2 \sigma_X^2$ and $\sigma_{aX+b} = |a| \sigma_X$.

Note that:

- $V(X + b) = \sigma_{X+b}^2 = \sigma_X^2 \;\Rightarrow\; \sigma_{X+b} = \sigma_X$.
- $V(aX) = \sigma_{aX}^2 = a^2 \sigma_X^2 \quad \Rightarrow\; \sigma_{aX} = |a| \sigma_X$.

Thus, adding or subtracting a constant to a random variable does not impact variability.

Exercise

The pmf of the amount of memory X (GB) in a purchased flash drive as:

x	1	2	4	8	16
$p(x)$	0.05	0.10	0.35	0.40	0.10

Compute the following:

a. $E(2X+3)$.

Answer

$E(2X+3) = 2E(X) + 3$

$E(X) = \sum_{i=1}^{5} x_i \times p(x_i) = 1 \times 0.05 + 2 \times 0.10 + 4 \times 0.35 + 8 \times 0.40 + 16 \times 0.10 =$

$6.45\ GB$.

$E(2X+3) = 2E(X) + 3 = (2 \times 6.45) + 3 = 15.9\ GB$.

b. $V(2X+3)$.

Answer

$V(2X+3) = V(2X) = 4V(X)$

$E(X^2) = \sum_{i=1}^{3} x_i^2 \times p(x_i) = 1^2 \times 0.05 + 2^2 \times 0.10 + 4^2 \times 0.35 + 8^2 \times 0.40 + 16^2 \times 0.10 =$

57.25.

Using the shortcut formula $V(X) = E(X^2) - \big(E(X)\big)^2 = 57.25 - 6.45^2 = 15.6475\ GB^2$.

$V(2X+3) = V(2X) = 4V(X) = 4 \times 15.6475 = 62.59 GB^2$.

Exercise 33 (page 116)

Let X be a Bernoulli random variable with pmf such that $p(1) = p$ and $p(0) = 1 - p$.

a. Compute $E(X^2)$.

Answer

$E(X^2) = 0^2 \times (1-p) + 1^2 \times p = p$.

b. Show that $V(X) = p(1-p)$.

Answer

$E(X) = 0 \times (1-p) + 1 \times p = p$.

$V(X) = E(X^2) - E(X)^2 = p - p^2 = p(1-p)$.

c. Compute $E(X^{79})$.

Answer
$E(X^{79}) = 0^{79} \times (1-p) + 1^{79} \times p = p$.

Exercise 37 (page 117)
Assume that a random variable X has the following pmf:

$$P(x) = \begin{cases} \frac{1}{n} & x = 0,1,2,\dots,n \\ 0 & otherwise \end{cases}$$

[Hint: The sum of the first n positive integers is $n(n+1)/2$, whereas the sum of their squares is $n(n+1)(2n+1)/6$.]

a. Compute $E(X)$.

Answer
$E(X) = \sum_{i=1}^{n} x_i P(x_i) = \sum_{i=1}^{n} x_i \left(\frac{1}{n}\right) = \left(\frac{1}{n}\right) \sum_{i=1}^{n} x_i = \left(\frac{1}{n}\right)\left(\frac{n(n+1)}{2}\right) = \frac{n+1}{2}$.

b. Compute $V(X)$ using the shortcut formula.

Answer
$V(X) = E(X^2) - E(X)^2 = \sum_{i=1}^{n} x_i^2 P(x_i) - \left(\frac{n+1}{2}\right)^2 = \sum_{i=1}^{n} x_i^2 \left(\frac{1}{n}\right) - \left(\frac{n+1}{2}\right)^2 =$
$\left(\frac{1}{n}\right) \sum_{i=1}^{n} x_i^2 - \left(\frac{n+1}{2}\right)^2 = \left(\frac{1}{n}\right)\left[\frac{n(n+1)(2n+1)}{6}\right] - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$.

Section 4: The Binomial Probability Distribution

The following requirements should be satisfied:

1. The experiment consists of a sequence of n smaller experiments called trials, where n is fixed in advance of the experiment.

2. Each trial can result in one of the same two possible outcomes (dichotomous trials), which we generically denote by success (S) and failure (F).

3. The trials are independent, so that the outcome on any particular trial does not influence the outcome on any other trial.

4. The probability of success $P(S) = p$ is constant from trial to trial.

Note that:

- If sampling is without replacement, the experiment will not yield independent trials.
- If sampling is with replacement, the experiment will yield independent trials.
- Consider sampling without replacement from a dichotomous population of size N. If the sample size (number of trials) n is at most 5% of the population size, the experiment can be analyzed as though it were a binomial experiment.

The Binomial Random Variable and Distribution

The binomial random variable X =the number of successes among n trial and its pmf depends on two parameters n and p has the following form:

$$b(x; n, p) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0,1,2, \dots, n \\ 0 & otherwise \end{cases}$$

Using Binomial Tables:

Even for a relatively small value of n, the computation of binomial probabilities can be tedious. **Appendix Table A.1 tabulates the cdf $\boldsymbol{F(x) = P(X \leq x)}$ for $\boldsymbol{n = 5, 10, 15, 20, 25}$** in the combination with selected values of p.
For $X \sim Bin(n, p), the\ cdf$ will be denoted by:

$$B(x; n, p) = P(X \leq x) = \sum_{y=0}^{x} b(y; n, p) \quad x = 0,1, \dots, n$$

The Mean and Variance of $\boldsymbol{X}$

If $X \sim Bin(n, p)$, then:

$E(X) = np,$

$V(X) = \sigma_X^2 = np(1-p) = nqp \Rightarrow \sigma_X = \sqrt{npq}$ and $q = 1 - p$.

Exercise 50 (page 123)

A particular telephone number is used to receive both voice calls and fax messages. Suppose that 25% of the incoming calls involve fax messages, and consider a sample of 25 incoming calls. Using the binomial table, what is the probability that:

a. At most 6 of the calls involve a fax message?

Answer

$P(X \leq 6) = B(6; 25, .25) = 0.561.$

b. Exactly 6 of the calls involve a fax message?

Answer

$P(X = 6) = B(6; 25, .25) - B(5; 25, .25) = 0.561 - 0.378 = 0.183.$

Note that:

Using the pmf: $b(x; n, p) = \binom{n}{x}(p^x)(1-p)^{n-x} \Rightarrow$

$b(6; 25{,}0.25) = \binom{25}{6}(0.25^6)(1-.25)^{25-6} = \frac{25!}{6!\times(25-6)!}(0.25^6)(1-.25)^{19} = 0.1828.$

c. At least 6 of the calls involve a fax message?

Answer

$P(X \geq 6) = 1 - P(X < 6) = 1 - P(X \leq 5) = 1 - B(5; 25, .25) = 1 - 0\,.378 = 0.622.$

d. More than 6 of the calls involve a fax message?

Answer

$P(X > 6) = 1 - P(X \leq 6) = 1 - 0.561 = 0.439.$

e. What is the expected number of calls among the 25 that involve a fax message?

Answer

$E(X) = np = 25 \times 0.25 = 6.25.$

f. What is the standard deviation of the number among the 25 calls that involve a fax message?

Answer

$V(X) = np(1-p) = 25 \times 0.25 \times 0.75 = 4.6875$

The standard deviation = $\sqrt{V(X)} = \sqrt{4.6875} = 2.1651$

Section 3: Hypergeometric and Negative Binomial Distributions

Hypergeometric Distribution

The assumptions leading to the hypergeometric distribution are as follows:

1. The population or set to be sampled consists of N individuals, objects, or elements (a finite population).

2. Each individual can be characterized as a success (S) or a failure (F), and there are M successes in the population.

3. A sample of n individuals is selected without replacement in such a way that each subset of size n is equally likely to be chosen.

Proposition

If X is the number of $S's$ in a completely random sample of size n drawn from a population consisting of M of $S's$ and $(N-M)$ of $F's$, then the probability distribution of X, called the hypergeometric distribution and can be expressed as:

$P(X=x)=h(x;n,M,N)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$ where $\max(0,n-N+M)\leq x\leq\min(n,M)$.

Proposition

If $X \sim h(x;n,M,N)$, then:

$E(X)=np$,

$V(X)=\left(\frac{N-n}{N-1}\right)(np)(1-p)$.

Note that:

- The proportion of S's in the population $(p)=\frac{M}{N}$.
- The means of the binomial and hypergeometric random variable's are equal which is np.
- The variances of the two random variables differ by the factor finite population correction factor= $\frac{N-n}{N-1} < 1$ (it is approximately 1 when n is small relative to N). So, the hypergeometric variable has smaller variance than does the binomial random variable.

Exercise 72 (page 130)

A personnel director interviewing 11 senior engineers for four job openings has scheduled six interviews for the first day and five for the second day of interviewing. Assume that the candidates are interviewed in random order.

a. What is the probability that x of the top four candidates are interviewed on the first day?

Answer

$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, x is an integer s.t. $\max(0, n - N + M) \leq x \leq \min(n, M)$,

$N = 11$, $M = 4$, $n = 6$ selected for the first day's interviews.

$P(x \text{ of the top four candidates are interviewed on the first day}) = h(x; 6,4,11) = \frac{\binom{4}{x}\binom{11-4}{6-x}}{\binom{11}{6}}$

$\max(0,6 - 11 + 4) \leq x \leq \min(6,4) \Rightarrow \max(0, -1) \leq x \leq \min(6,4) \Rightarrow 0 \leq x \leq 4 \Rightarrow \mathrm{x} = 0,1,2,3,4.$

b. How many of the top four candidates can be expected to be interviewed on the first day?

Answer

$E(X) = n\left(\frac{M}{N}\right) = 6\left(\frac{4}{11}\right) = 2.1818.$

Exercise 73 (page 131)

Twenty pairs of individuals playing in a bridge tournament have been seeded $1, \ldots, 20$. In the first part of the tournament, the 20 are randomly divided into 10 east–west pairs and 10 north–south pairs.

a. If a random sample of size 10 pairs is drawn from the population, what is the probability that x of the top 10 pairs end up playing east–west?

Answer

$P(X = x) = h(x; n, M, N) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, x is an integer s.t. $\max(0, n - N + M) \leq x \leq \min(n, M)$,

$N = 20$, $M = 10$, a sample of $n = 10$ pairs is drawn from $N = 20$

$P(X = x) = h(x; 10{,}10{,}20) = \frac{\binom{10}{x}\binom{20-10}{10-x}}{\binom{20}{10}}$, $x = 0,1, \ldots, 10$.

b. If there are $2n$ pairs are randomly divided into n east–west pairs and n north–south pairs and a random sample of size n pairs is drawn from the population, what is the pmf of $X =$ the number among the top n pairs who end up playing east–west? What are $E(X)$ and $V(X)$?

Answer

$N = 2n$, $M = n$, and a sample of size n

$P(X = x) = h(x; n, M, N) = h(x, n, n, 2n) = \frac{\binom{n}{x}\binom{2n-n}{n-x}}{\binom{2n}{n}} = \frac{\binom{n}{x}\binom{n}{n-x}}{\binom{2n}{n}}$, $x = 0,1, \ldots, n$

$E(X) = n\left(\frac{M}{N}\right) = n\left(\frac{n}{2n}\right) = \frac{n}{2}$.

$V(X) = \left(\frac{N-n}{N-1}\right) n \left(\frac{M}{N}\right)\left(1 - \frac{M}{N}\right) = \left(\frac{2n-n}{2n-1}\right) n \left(\frac{n}{2n}\right)\left(1 - \frac{n}{2n}\right) = \left(\frac{n}{2n-1}\right) n \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \left(\frac{1}{4}\right)\left(\frac{n^2}{2n-1}\right)$.

The Negative Binomial Distribution

The following conditions should be satisfied:

1.The experiment consists of a sequence of independent trials.

2. Each individual can be characterized as a success (S) or a failure (F).

3. The probability of success is constant from trial to trial, so $P(S\ on\ trial\ i) = p$ for $i = 1, 2, 3, \ldots$

4. The experiment continues (trials are performed) until a total of r successes have been observed, where r is a specified positive integer.

The random variable of interest is:

$X = the\ number\ of\ failures\ that\ precede\ the\ rth\ success$

Question

Why X is called a negative binomial random variable?

Answer

In contrast to the binomial random variable, the number of successes is fixed and the number of trials is random.

The pmf has the following form:

$nb(x; r, p) = \binom{x+r-1}{r-1} p^r (1-p)^x \quad x = 0,1,2, \ldots$

$E(X) = \frac{r(1-p)}{p}$.

$V(X) = \frac{r(1-p)}{p^2}$.

Note that:

In the special case $r = 1$ is called the geometric distribution.

$nb(x; 1, p) = \binom{x+1-1}{1-1} p(1-p)^x = p(1-p)^x \quad x = 0,1,2, \ldots$

Exercise 75 (page 131)

The probability that a randomly selected box of a certain type of cereal has a particular prize is 0.2. Suppose you purchase box after box until you have obtained two of these prizes.

a. What is the probability that you purchase x boxes that do not have the desired prize?

Answer

Let X = the number of boxes that do not contain a prize until you find 2 prizes.

Thus, $X \sim nb(x; 2, 0.2)$ with $r = 2$ and $p = 0.2 \Rightarrow$

$nb(x; r, p) = \binom{x+r-1}{r-1} p^r (1-p)^x, x = 0,1,2, \ldots$

$nb(x; 2,0.2) = \binom{x+2-1}{2-1} 0.2^2 (1-0.2)^x = (x+1)(0.2^2)(0.8)^x, x = 0,1,2, \ldots$

b. What is the probability that you purchase four boxes?

Answer

$P(4\ boxes\ purchased) = P(2\ boxes\ without\ prizes) = P(X = 2) = (2+1)(0.2^2)(0.8)^2 = 0.0768.$

c. What is the probability that you purchase at most four boxes?

Answer

$P(at\ most\ 4\ boxes\ purchased) =$

$P(2\ boxes\ without\ prizes) or P(1\ boxes\ without\ prizes) or P(0\ boxes\ without\ prizes) =$

$(P(X=0) + P(X+1) + P(X=2) = (0+1)(0.2^2)(0.8)^0 + (1+1)(0.2^2)(0.8)^1 +$

$(2+1)(0.2^2)(0.8)^2 = 0.1808.$

d. How many boxes without the desired prize do you expect to purchase? How many boxes do you expect to purchase?

Answer

The expected boxes to purchase without the desired prize $= E(X) = \frac{r(1-p)}{p} = \frac{2(1-0.2)}{0.2} = 8.$

The expected boxes to purchase without and with the desired prize $= 8$ (from previous step)$+2$ (with desired prize) $= 10.$

Section 6: The Poisson Probability Distribution

A discrete random variable X follows a Poisson distribution with parameter $\mu > 0$ if the pmf of X has the form:

$p(x;\mu) = \frac{e^{-\mu}.\mu^{x}}{x!} \quad x = 0,1,2,\ldots$

where $\mu = E(X)$ and "e" represents the base of the natural logarithm system; its numerical value is approximately 2.71828.

$E(X) = \mu$.

$V(X) = \mu$.

The Poisson Distribution as a Limit

Proposition

Suppose that in the binomial pmf $b(x;n,p)$ and let $n \Rightarrow \infty$ and $p \Rightarrow 0$ such that np approaches a value μ, then $b(x;n,p) \Rightarrow p(x;\mu)$ where $\mu = np$.

As a rule of thumb, this approximation can safely be applied if $n > 50$ and $np < 5$.

Note that: Using Poisson Table

Appendix Table A.2 exhibits the cdf $F(x;\mu)$for $\mu = 0.1, 0.2, \ldots 1, 2, \ldots 10, 15, 20.$

For example, if $\mu = 2$, then:

- $P(X \leq 3) = F(3;2) = 0.857$.
- $P(X = 3) = F(3;2) - F(2;2) = 0.857 - 0.677 = 0.180$.

The Poisson Process

It represents the occurrence of events over time and the parameter α specifies the rate for the process.

let $P_k(t) = \frac{(e^{-\alpha t})(\alpha t)^k}{k!}$ denote the probability that k events will be observed during any particular time interval of length t.

The number of events during a time interval of length t is a Poisson random variable with parameter $\mu = \alpha t$ and the expected number during a unit interval of time is α.

Exercise 80 (page 135)

Let X be the number of material anomalies occurring in a particular region of an aircraft gas-turbine disk. Proposes a Poisson distribution for X. Suppose that $\mu = 4$.

a. Compute both $P(X \leq 4)$ and $P(X < 4)$ using the suitable statistics table.

Answer

Using Appendix Table A.2:

$P(X \leq 4) = F(4; 4) = 0.629.$

$P(X < 4) = P(X \leq 3) = F(3; 4) = 0.433.$

b. Compute $P(4 \leq X \leq 8)$ using the suitable statistics table.

Answer

$P(4 \leq X \leq 8) = P(X \leq 8) - P(X \leq 3) = F(8; 4) - F(3; 4) = 0.979 - 0.433 = 0.546.$

c. Compute $P(8 \leq X)$ using the suitable statistics table.

Answer

Compute $P(8 \leq X) = 1 - P(X < 8) = 1 - P(X \leq 7) = 1 - F(7; 4) = 1 - 0.949 = 0.051.$

d. Calculate $E(X)$ and standard deviation of X.

Answer

$E(X) = \mu = 4$

Standard deviation of $X = \sqrt{\mu} = \sqrt{4} = 2.$

Exercise 84 (page 135)

The Centers for Disease Control and Prevention reported in 2012 that 1 in 88 American children had been diagnosed with an autism spectrum disorder (ASD).

a. If a random sample of 200 American children is selected, what are the expected value and standard deviation of the number who have been diagnosed with ASD?

Answer

$n = 200$ and $p = 1/88 = 0.0114$

The expected value $(\mu) = np = 200 \times 0.0114 = 2.28$

The standard deviation= $\sqrt{np(1-p)} = \sqrt{200 \times 0.0114 \times (1 - 0.0114)} = 1.5013.$

b. Referring back to (a), calculate the approximate probability that at least 2 children in the sample have been diagnosed with ASD?

Answer

$n = 200 > 50$ and $np = 2.28 < 5 \Rightarrow$ Using Poisson as approximation for binomial:

$P(X = x) = \frac{e^{-\mu}\mu^x}{x!}$ where $\mu = 2.28$

$P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X = 0) + P(X = 1)] =$

$1 - \left[\frac{e^{-2.28}2.28^0}{0!} + \frac{e^{-2.28}2.28^1}{1!}\right] = 1 - [0.1023 + 0.2332] = 0.6645.$

c. If the sample size is 352, what is the approximate probability that fewer than 5 of the selected children have been diagnosed with ASD?

Answer

$\mu = np = 352 \times (1/88) = 4$

$P(X < 5) = P(X \leq 4) = F(4; 4) = 0.629.$

Exercise 87 (page 136)

The number of requests for assistance received by a towing service is a Poisson process with rate $\alpha = 4$ per hour.

a. Compute the probability that exactly ten requests are received during a particular 2 −hour period.

Answer

$\mu = \alpha t = 4 \times 2 = 8$

$P(X = 10) = \frac{e^{-\alpha t}\alpha t^k}{k!} = \frac{e^{-8}8^{10}}{10!} = 0.099.$

b. If the operators of the towing service take a 30 −min break for lunch, what is the probability that they do not miss any calls for assistance?

Answer

$\mu = \alpha t = 4 \times 0.5 = 2$

$P(X = 0) = \frac{e^{-\alpha t}\alpha t^k}{k!} = \frac{e^{-2}2^0}{0!} = 0.135.$

c. Using the given information in part (b), how many calls would you expect during their break?

Answer

$E(X) = \alpha t = 4 \times 0.5 = 2.$