The lectures of chapter two are based on the textbook for STAT3025Q (Statistical Methods)

Probability and Statistics for Engineering and Sciences

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Jay L. Devore

CENGAGE Learning

Chapter 2

Probability

Section 1: Sample Spaces and Events

An experiment is any activity or process whose outcome is subject to uncertainty.

Examples:

- Tossing a coin once or several times.
- Selecting a card or cards from a deck.

The sample space of an experiment (S) is the set of all possible outcomes of that experiment.

Examples

- Check a single fuse to see whether it is defective. The sample space is $S = \{N, D\}$,

where N: not defective and D: defective. The braces are used to enclose the elements of a set.

- If three fuses are examined in sequence of N and D of length 3, then

$S = \{NNN, NND, NDN, NDD, DNN, DND, DDN, DDD\}$

- An experiment consists of observing the gender of the next child born at the local hospital, as

$S = \{M, F\}$.

- An experiment consists of observing the outcomes of tossing a pair of dice,

$S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\}$.

An event is any collection (subset) of outcomes contained in the sample space S. An event can be simple or compound.

- It is simple if it consists of exactly one outcome.
- It is compound if it consists of more than one outcome.

Note that:

The probability of an event A is represented as $P(A)$ and it is called unconditional probability of the event A.

Some Relations from Set Theory

An event is just a set, so relationships and results from elementary set theory can be used to study events.

1. The complement of an event A is "$\acute{A}$", is the set of all outcomes in S that are not contained in A.

2. The union of two events A and B is "$A \cup B$" which consists of all outcomes that are either in A or in B or in both events.

3. The intersection of two events A and B is "$A \cap B$" which consists of all outcomes that are in both events A and B.

Mutually exclusive (disjoint event)

If $A \cap B = \emptyset$ (the null event), then the events A and B are said to be mutually exclusive or disjoint.

Figure 2.1 (page 56) shows examples of Venn diagrams.

Note that:

- None of the events A and B occurs= $(A \cup B)' = \acute{A} \cap \acute{B}$.
- At least one of them does not occur= $(A \cap B)' = \acute{A} \cup \acute{B}$.

Exercise 2 (page 56)

Suppose that vehicles taking a particular freeway exit can turn right (R), turn left (L), or go straight (S). Consider observing the direction for each of three successive vehicles.

(a) List all outcomes in the event A that all three vehicles go in the same direction.

Answer

$A = \{RRR, LLL, SSS\}$.

(b) List all outcomes in the event B that all three vehicles take different directions.

Answer

$B = \{RLS, RSL, LRS, LSR, SRL, SLR\}$.

(c) List all outcomes in the event C that exactly two of the three vehicles turn right.

Answer

$C = \{RRL, RRS, RLR, RSR, LRR, SRR\}$.

(d) List all outcomes in the event D that exactly two vehicles go in the same direction.

Answer

$D = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL, SRS, SLS, RSS, LSS\}$.

(e) List outcomes in CUD and $C \cap D$.

Answer

$CUD = \{RRL, RRS, RLR, RSR, LRR, SRR, LLR, LLS, LRL, LSL, RLL, SLL, SSR, SSL, SRS, SLS, RSS, LSS\} = D$.

$C \cap D = \{RRL, RRS, RLR, RSR, LRR, SRR\} = C$.

Section 2: Axioms and Properties of Probability

Axiom 1: For any event A, $0 \leq P(A) \leq 1$.

Axiom 2: $P(S) = 1$.
Axiom 3: If $A_1, A_2, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup \ A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Proposition

$P(\emptyset) = 0$ where $\emptyset$ is the null set.

Note that:

- **Objective probabilities** that are stated after the outcomes of an event have been observed are said to be relative frequency probabilities.
- **Relative frequency probability:** Let $N(A)$ denote the number of replications on which A does occur. The relative frequency of occurrence of the event A in the sequence of N replications = $P(A) = N(A)/N$.
- **Subjective probability**: Since different observers may have different prior information, probability assignments may differ from individual to individual.

Propositions:

1. For any event A, $P(A) + P(\acute{A}) = 1$, thus $P(A) = 1 - P(\acute{A})$.
2. For any event A, $P(A) \leq 1$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ [If A and B are not disjoint].
4. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$[If A , B, and C are not disjoint].

Note that:

$P(A \cup B) = P(A) + P(A' \cap B) = P(A) + [P(B) - P(A \cap B)]$.
$P(A \cup B) = P(B) + P(A \cap B') = P(B) + [P(A) - P(A \cap B)]$.

Rule

Let $E_1, E_2, \ldots$ denote the corresponding simple events, each consisting of a single outcome. The probability of any compound event "A" is computed by adding together the $P(E_i)'s$ for all E_i's in A:

$$P(A) = \sum_{all\ E_i' \sin A} P(E_i)$$

Equally Likely Outcomes

In many experiments consisting of N outcomes, it is reasonable to assign equal probabilities to all N simple events. These include such obvious examples as tossing a fair coin or fair die once or twice (or any fixed number of times), or selecting one or several cards from a well-shuffled deck of 52. With $p = P(E_i)$ for every i,

$1 = \sum_{i=1}^{N} P(E_i) = \sum_{i=1}^{N} P = (N)(p) \rightarrow p = 1/N.$

If $N(A)$ denoting the number of outcomes contained in A, then $P(A) = \frac{N(A)}{N}$.

Exercise 12 (page 64)

Consider randomly selecting a student at a large university, and let A be the event that the selected student has a Visa card and B be the analogous event for Master card. Suppose that $P(A) = 0.6$ and $P(B) = 0.4$.

(a) Could it be the case that $P(A \cap B) = 0.5$? Why or why not?

Answer

No, this is not possible. Since event "$A \cap B$" is contained within event "B", then it must be $P(A \cap B) \leq P(B)$. However, using the given information $P(A \cap B) = 0.5 > P(B) = 0.4$.

(b) From now on, suppose that $P(A \cap B) = 0.3$. What is the probability that the selected student has at least one of these two types of cards?

Answer

$P(\text{At least one of these two types of cards}) = \mathrm{P(A \cup B) = P(A) + P(B) - P(A \cap B)} = 0.6 + 0.4 - 0.3 = 0.7.$

(c) What is the probability that the selected student has neither type of card?

Answer

$P(\textit{The selected student has neither type of card}) = P(A' \cap B') = P[(\mathrm{A \cup B})'] = 1 - P(\mathrm{A \cup B}) = 1 - 0.7 = 0.3.$

(d) Describe, in terms of A and B, the event that the selected student has a Visa card but not a Master card, and then calculate the probability of this event.

Answer

$P(\text{the selected student has a Visa card but not a MasterCard}) = P(A \cap B') = P(A) - P(\mathrm{A \cap B}) = 0.6 - 0.3 = 0.3.$

(e) Calculate the probability that the selected student has exactly one of the two types of cards.

Answer

$P(\text{Exactly one }) = P(\textit{at least one}) - P(\textit{both}) = P(\mathrm{A \cup B}) - P(\mathrm{A \cap B}) = 0.7 - 0.3 = 0.4$.

Note that:

$P(\text{exactly one}) = P(\textit{at least one}) - P(\textit{both}) = \mathrm{P(A \cup B) - P(A \cap B)}$. It can be also calculated as: $P(\text{exactly one}) = P(A \cap B') + P(A' \cap B)$.

Exercise 26 (page 66)

A certain system can experience three different types of defects. Let A_i ($i = 1,2,3$) denote the event that the system has a defect of type i. Suppose that

$P(A_1) = 0.12,\ P(A_2) = 0.07, P(A_3) = 0.05,$

$P(A_1 \cup A_2) = 0.13,\ P(A_1 \cup A_3) = 0.14, P(A_2 \cup A_3) = 0.10, P(A_1 \cap A_2 \cap A_3) = 0.01.$

(a) What is the probability that the system does not have a type 1 defect?

Answer

$P(\text{The probability that the system does not have a type 1 defect}) = P(A_1') = 1 - P(A_1) = 1 - 0.12 = 0.88.$

(b) What is the probability that the system has both type 1 and type 2 defects?

Answer

$\mathrm{P}(A_1 \cup A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2) - \mathrm{P}(A_1 \cap A_2)$

$P(\text{the system has both type 1 and type 2 defects}) = P(A_1 \cap A_2) = \mathrm{P}(A_1) + \mathrm{P}(A_2) - \mathrm{P}(A_1 \cup A_2) = 0.12 + 0.07 - 0.13 = 0.06.$

(c) What is the probability that the system has both type 1 and type 2 defects but not a type 3 defect?

Answer

$P(A_1 \cap A_2 \cap A_3') = P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3) = 0.06 - 0.01 = 0.05.$

(d) What is the probability that the system has at most two of these defects?

Answer

$P(\text{The system has at most two of these defects}) = 1 - \mathrm{P}(\text{all are defected}) = 1 - P(A_1 \cap A_2 \cap A_3) = 1 - 0.01 = 0.99.$

Section 3: Counting Techniques

The Product Rule for Ordered Pairs

If O_1 and O_2 are two objects, then the pair (O_1, O_2) is different from the pair (O_2, O_1).

Proposition

If the first element or object of an ordered pair can be selected in n_1 ways, and for each of these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $n_1 n_2$.

Exercise 29 (page 73)

As of April 2006, roughly 50 million.com web domain names were registered (e.g., yahoo.com).

a. How many domain names consisting of just two letters in sequence can be formed? How many domain names of length two are there if digits as well as letters are permitted as characters?

Answer
There are 26 letters, so allowing repeats there are $(26)(26) = 676$ possible 2-letter domain names.

Add in the 10 digits, and there are 36 characters available, so allowing repeats there are

$(36)(36) = 1296$ possible 2-character domain names.

b. How many domain names are there consisting of three letters in sequence? How many of this length are there if digits as well as letters are permitted as characters?

Answer
There are 26 letters, so allowing repeats there are $(26)(26)(26) = 17{,}576$ possible 3-letter domain names.
Add in the 10 digits, and there are 36 characters available, so allowing repeats there are

$(36)(36)(36)$ = $46{,}656$ possible 3-character domain names.

Exercise 31 (page 73)

The composer Beethoven wrote 9 symphonies, 5 piano concertos (music for piano and orchestra), and 32 piano sonatas (music for solo piano).

a. How many ways are there to play first a Beethoven symphony and then a Beethoven piano concerto?

Answer

There are (9)(5) = 45 ways to play first a Beethoven symphony and then a Beethoven piano concerto.

b. The manager of a radio station decides that on each successive evening (7 days per week), a Beethoven symphony will be played followed by a Beethoven piano concerto followed by a Beethoven piano sonata. For how many years could this policy be continued before exactly the same program would have to be repeated?

Answer

There are $(9)(5)(32) = 1440$ successive nights (almost 4 years) without repeating exactly the same program.

Permutations and Combinations

Consider a group of n distinct individuals or objects ("distinct" means that they are not the same).

Permutations

An ordered subset is called a permutation, where

$P_{k,n}$ (the number of permutations of size k that can be formed from n individuals)=

$n(n-1)(n-2)\ldots\left(n-(k-1)\right)=\frac{n!}{(n-k)!}$

Note that:

For any positive integer m, the factorial is $m!=m(m-1)(m-2)\ldots(2)(1)$ and $1!=1$

For example: The expression for $P_{3,7}$ can be rewritten with the aid of factorial notation.

$$P_{3,7}=(7)(6)(5)=\frac{7!}{(7-3)!}$$

For example: Suppose that a college of engineering has seven departments, which we denote by $a,b,c,d,e,f,$ and g. Each department has one representative on the college's student council. From these seven representatives, one is to be chosen chair, another is to be selected vice-chair, and a third will be secretary. How many ways are there to select the three officers?

Answer

The chair can be selected in any of $n_1=7$ ways. For each way of selecting the chair, there are $n_2=6$ ways to select the vice- and there are $n_3=5$ ways of choosing the secretary. This gives the number of permutations of size 3 that can be formed from 7 distinct individuals as

$P_{3,7}=(7)(6)(5)=210$

$$=\frac{7!}{(7-3)!}=\frac{7!}{4!}=\frac{7\times6\times5\times4!}{4!}=(7)(6)(5)=210.$$

Combinations (**i.e., unordered subsets**)

Refer to the student council scenario, and suppose that three of the seven representatives are to be selected to attend a statewide convention. The order of selection is not important; all that matters is which three get selected.

So, we are looking for $\binom{7}{3}$, the number of combinations of size 3 that can be formed from the 7 individuals.

Proposition

$$\binom{n}{k}=\frac{P_{k,n}}{k!}=\frac{n!}{k!\,(n-k)!}$$

$\binom{n}{n}=1$, $\binom{n}{0}=1$, and $\binom{n}{1}=n$

Exercise 33 (page 73)

Consider a Little League team that has 15 players on its roster.

a. How many ways are there to select 9 players for the starting lineup? (Order still matters).

Answer

$P_{k,n} = \dfrac{n!}{(n-k)!}$

$P_{9,15} = \frac{\mathbf{15}!}{(\mathbf{15-9})!} = 1{,}816{,}214{,}440.$

b. Suppose 5 of the 15 players are left-handed. How many ways are there to select 3 left-handed outfielders and have all 6 other positions occupied by right-handed players? (Order still matters).

Answer

Since 9 players are selected such that select 3 left-handed outfielders and have all 6 other positions occupied by right-handed players.

There are $P_{3,5} = \frac{\mathbf{5}!}{(\mathbf{5-3})!} = 60$ ways to choose three left-handers for the outfield and

$P_{6,10} = \frac{\mathbf{10}!}{(\mathbf{10-6})!} = 151{,}200$ ways to choose six right-handers for the other positions.

The total number of possibilities is = (60)(151,200) = 9,072,000.

Exercise 39 (page 74)

A box in a supply room contains 15 compact fluorescent lightbulbs, of which 5 are rated 13-watt, 6 are rated 18-watt, and 4 are rated 23-watt. Suppose that three of these bulbs are randomly selected.

a. What is the probability that exactly two of the selected bulbs are rated 23-watt?

Answer

The number of ways to select exactly 3 of total 15 $= \binom{15}{3} = \frac{15!}{3!12!} = 455.$

There are 4 of 23W bulbs available and $5 + 6 = 11$ of non-23W bulbs available.

The number of ways to select exactly two of the 23W bulbs (and, thus, exactly one of the non-23W bulbs)= $\binom{4}{2}\binom{11}{1} = \frac{4!}{2!2!} \times \frac{11!}{1!10!} = 6 \times 11 = 66$

the probability that exactly two of the selected bulbs are rated 23-watt=
$\frac{\text{The number of ways to select exactly two of the 23W bulbs}}{\text{The number of ways to select exactly 3 of total 15}} = \frac{66}{455} = 0.1450.$

b. What is the probability that all three of the bulbs have the same rating?

Answer

The number of ways to select 3 of 13W bulbs $= \binom{5}{3} = \frac{5!}{3!2!} = 10$

The number of ways to select 3 of 18W bulbs $= \binom{6}{3} = \frac{6!}{3!3!} = 20$

The number of ways to select 3 of 23W bulbs $= \binom{4}{3} = \frac{4!}{3!1!} = 4$

Thus, there are $10 + 20 + 4 = 34$ ways to select three bulbs of the same wattage, and so the probability of that= $34/455 = 0.075$.

c. What is the probability that one bulb of each type is selected?

Answer

The number of ways to obtain one of each type= $\binom{5}{1}\binom{6}{1}\binom{4}{1} = \frac{5!}{1!4!} \times \frac{6!}{1!5!} \times \frac{4!}{1!3!} = 120$ with probability $= 120/455 = 0.264$.

Section 4: Conditional Probability

The notation $P(A|B)$ to represent the conditional probability of the event A given that the event B has occurred and B is the "conditioning event."

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ and $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

Note that:

- $P(A|B) \neq P(B|A)$
- $P(A|B) \neq P(A)$
- $P(B|A) \neq P(B)$
- $P(B'|A) = 1 - P(B|A)$

Example

Assume the following table that includes different events and their corresponding probabilities:

Event	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	0.14	0.23	0.37	0.08	0.09	0.13	0.05

The previous events and their corresponding probabilities can also be represented in the following Venn diagram

- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.08}{0.23} = 0.348.$
- $P(A|B \cup C) = \frac{P[A \cap (B \cup C)]}{P(B \cup C)} = \frac{P[(A \cap B) \cup (A \cap C)]}{P(B \cup C)} = \frac{P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)}{P(B) + P(C) - P(B \cap C)} = \frac{0.08 + 0.09 - 0.05}{0.23 + 0.37 - 0.13} = \frac{0.12}{0.47} = 0.255.$
- $P(A|at\ least\ one\ of\ the\ events) = P(A|A \cup B \cup C) = \frac{P[A \cap (A \cup B \cup C)]}{P(A \cup B \cup C)} = \frac{P(A)}{P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)}$

$= \frac{0.14}{0.14 + 0.23 + 0.37 - 0.08 - 0.09 - 0.13 + 0.05} = \frac{0.14}{0.49} = 0.286.$

- $P(A \cup B|C) = \frac{P[(A \cup B) \cap C]}{P(C)} = \frac{P[(A \cap C) \cup P(B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P[(A \cap C) \cap (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P[(A \cap B \cap C]}{P(C)} = \frac{0.09 + 0.13 - 0.05}{0.37} = \frac{0.17}{0.37} = 0.459.$
- $P(A \cap B|C) = \frac{P[(A \cap B) \cap c]}{P(C)} = \frac{P(A \cap B \cap c)}{P(C)} = \frac{0.05}{0.37} = 0.1351.$

The Multiplication Rule

The definition of conditional probability yields the following result:

$P(A \cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)$

The extended formula for three events: $P(A_1 \cap A_2 \cap A_3) = P(A_3|A_1 \cap A_2) \times P(A_1 \cap A_2)$
$= P(A_3|A_1 \cap A_2) \times P(A_2|A_1) \times P(A_1)$

Note that:

The events $A_1, \dots, A_k$ are mutually exclusive if no two have any common outcomes.

The events are exhaustive if one A_i must occur, so that $A_1 \cup A_2 \cup \dots \cup A_k = S$

The Low of Total Probability

Define $A_1, \dots, A_k$ to be mutually exclusive and exhaustive events. Then for any other event B,
$P(B) = P(A_1 \cap B) + \cdots + P(A_k \cap B) = P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k) = \sum_{i=1}^{k} P(B|A_i)P(A_i)$.

Bayes' Theorem

Define $A_1, \dots, A_k$ to be mutually exclusive and exhaustive events with prior probabilities $P(A_i)$ $(i = 1,2, \dots, k)$. Then for any other event B for which $P(B) > 0$. The posterior probability of A_j given that B has occurred is

$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_i)P(A_i)} \quad j = 1,2, \dots, k.$

Exercise 45 (page 82)
The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying joint probability table gives the proportions of individuals in the various ethnic group–blood group combinations.

		Blood Group			
		O	A	B	AB
	1	0.082	0.106	0.008	0.004
Ethnic Group	2	0.135	0.141	0.018	0.006
	3	0.215	0.200	0.065	0.020

Suppose that an individual is randomly selected from the population, and define events by

$A =$ {type A selected}, $B = \{type\ B\ selected\}$, and $C = \{ethnic\ group\ 3\ selected\}$.

a. Calculate P$(A), P(C)$, and $P(A \cap C)$.

Answer

$P(\text{type } A \text{ selected}) = P(A) = 0.106 + 0.141 + 0.200 = 0.447$,

$P(ethnic\ group\ 3\ selected) = P(C) = 0.215 + 0.200 + 0.065 + 0.020 = 0.500$, and

$P(\text{type } A \text{ selected} \cap ethnic\ group\ 3\ selected) = P(A \cap C) = 0.200$.

b. Calculate both $P(A|C)$ and $P(C|A)$, and explain in context what each of these probabilities represents.

Answer

$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{0.200}{0.500} = 0.4$. If it is known that the individual came from ethnic group 3, the probability that he or she has Type A blood $= 0.40$.

$P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{0.200}{0.447} = 0.4474$. If a person has Type A blood, the probability that he or she is from ethnic group 3 $= 0.4474$.

c. If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

Answer

Define D = "ethnic group 1 selected" and

$B' = the\ selected\ individual\ does\ not\ have\ type\ B\ blood$

$$P(D|B') = \frac{P(D \cap B')}{P(B')} = \frac{0.192}{0.909} = 0.2112$$

where

$P(D \cap B') = 0.082 + 0.106 + 0.004 = 0.192$

$P(B') = 1 - P(B) = 1 - 0.091 = 0.909$ since $P(B) = 0.008 + 0.018 + 0.065 = 0.091$

So, if the selected individual does not have type B blood, the probability that he or she is from ethnic group $1 = 0.2112$.

Exercise 59 (page 84)

At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

a. What is the probability that the next customer will request plus gas and fill the tank [$P(A_2 \cap B)$]?

Answer

$P(A_2 \cap B) = P(B|A_2) \times P(A_2) = 0.60 \times 0.35 = 0.21$.

b. What is the probability that the next customer fills the tank?

Answer

By the law of total probability, $P(Fill\ tank) = P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) = 0.12 + 0.21 + 0.125 = 0.455$.

where

$P(A_1 \cap B) = P(B|A_1) \times P(A_1) = 0.30 \times 0.40 = 0.12$

$P(A_2 \cap B) = P(B|A_2) \times P(A_2) = 0.60 \times 0.35 = 0.21$

$P(A_3 \cap B) = P(B|A_3) \times P(A_3) = 0.50 \times 0.25 = 0.125$

c. If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?

Answer

Using Bayes' theorem:

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{0.12}{0.455} = 0.2637$$

$$P(A_2|B) = \frac{P(A_2 \cap B)}{P(B)} = \frac{0.21}{0.455} = 0.4615$$

$$P(A_3|B) = \frac{P(A_3 \cap B)}{P(B)} = \frac{0.125}{0.455} = 0.2747.$$

Section 5: Independence

If A and B are independent events, then the occurrence or nonoccurrence of one event has no bearing on the chance that the other will occur.

Thus, $P(A|B) = P(A)\ or\ P(B|A) = P(B)$.

The Multiplication Rule for $P(A \cap B)$

Proposition
A and B are independent iff:
$P(A \cap B) = P(A|B).P(B) = P(A).P(B)$

Note that:
If A and B are independent, then the complement of A and B are independent.

Independence of More Than Two Events

Events A_1,..,A_n are mutually independent if for every k $(k = 2,3, \dots, n)$ and every subset of indices i_1, i_2,… i_k, $P(A_{i_1} \cap A_{i_2} \cap \dots A_{i_k}) = P(A_{i_1}).P(A_{i_2}) \dots P(A_{i_k})$.

Note that:
Assume that A, B and C are three events, then:
Pairwise independent means that A and B are independent, and A and C are independent, and B and C are independent events. i.e.,

$P(A \cap\ B) = P(A) \times P(B)$

$P(A \cap \mathrm{C}) = P(A) \times P(C)$

$P(B \cap \mathrm{C}) = P(B) \times P(C)$

Note that:
Assume that A, B and C are three events, then A, B, and C are mutually independent iff the following two conditions hold:

1. $P(A \cap\ B) = P(A) \times P(B)$,
   $P(A \cap \mathrm{C}) = P(A) \times P(C)$,

   $P(B \cap \mathrm{C}) = P(B) \times P(C)$.

and

2. $P(A \cap \mathrm{B} \cap\ C) = P(A) \times P(B) \times P(C)$

Exercise 70 (page 89)

Let A be the event that the selected student has a Visa card, B be the analogous event for Master card, and let C be the event that the selected student has an American Express card.

In addition to,
$P(A) = 0.6, P(B) = 0.4$, $P(C) = 0.2$. Suppose that $P(A \cap B) = 0.3, P(A \cap C) = 0.15, P(B \cap C) = 0.1$, and $P(A \cap B \cap C) = 0.08$.

a. Show that A and B are dependent by using the definition of independence.
Answer
$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.4} = 0.75$ and $P(A) = 0.6$. Thus, $P(A|B) \neq P(A)$ which leads to A and B are not independent.

b. Show that A and B are dependent by verifying that the multiplication property does not hold.
Answer
$P(A \cap B) = 0.3$ and $P(A) \times P(B) = 0.6 \times 0.4 = 0.24$. Thus, $P(A \cap B) \neq P(A) \times P(B)$ which leads to A and B are dependent.

Exercise 71 (page 89)
An oil exploration company currently has two active projects, one in Asia and the other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppose that A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.7$.

a. If the Asian project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.
Answer
A and B are independent events→ A' and B' are independent events. Thus,
$P(\text{The European project is also not successful}|\text{the Asian project is not successful}) =$
$= P(B'|A') = P(B') = 1 - P(B) = 1 - 0.7 = 0.3.$

b. What is the probability that at least one of the two projects will be successful?
Answer
$P(\text{at least one of the two projects will be successful}) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A) \times P(B) = 0.4 + 0.7 - 0.4 \times 0.7 = 0.82.$

c. Given that at least one of the two projects is successful, what is the probability that only the Asian project is successful?
Answer
$P(\text{Only the Asian project is successful}|\text{at least one of the two projects is successful}) =$

$$= P(A \cap B'|A \cup B) = \frac{P[(A \cap B') \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A \cap B')}{P(A \cup B)} = \frac{P(A)P(B')}{P(A) + P(B) - P(A \cap B)}$$

$$= \frac{P(A)P(B')}{P(A)+P(B)-P(A)\times P(B)} = \frac{0.4\times 0.3}{0.4+0.7-(0.4\times 0.7)} = \frac{0.12}{0.82} = 0.146.$$

Exercise 78 (page 89)

A boiler has five identical relief valves. The probability that any particular valve will open on demand is 0.96. Assuming independent operation of the valves, calculate the following probabilities:

a. $P(at\ least\ one\ valve\ opens)$.

Answer

$P(at\ least\ one\ valve\ opens) = 1 - P(non\ open) = 1 - (0.04 \times 0.04 \times 0.04 \times 0.04 \times 0.04) = 1 - 0.04^5 = 0.99999.$

b. $P(at\ least\ one\ valve\ fails\ to\ open)$.

Answer

$P(at\ least\ one\ valve\ fails\ to\ open) = 1 - P(all\ open) = 1 - (0.96 \times 0.96 \times 0.96 \times 0.96 \times 0.96) = 1 - 0.96^5 = 0.1846.$

Exercise 84 (page 90)

Consider purchasing a system of audio components consisting of a receiver, a pair of speakers, and a CD player. Let A_1 be the event that the receiver functions properly throughout the warranty period, A_2 be the event that the speakers function properly throughout the warranty period, and A_3 be the event that the CD player functions properly throughout the warranty period. Suppose that these events are (mutually) independent with $P(A_1) = 0.95$, $P(A_2) = 0.98$, and $P(A_3) = 0.80$.

b. What is the probability that at least one component needs service during the warranty period?

Answer

$P(\text{at least one component needs service during the warranty period})$

$= 1 - P(All\ three\ components\ function\ properly\ throughout\ the\ warranty\ period)$

$= 1 - P(A_1 \cap A_2 \cap A_3) = 1 - [P(A_1) \times P(A_2) \times P(A_3)] = 1 - (0.95 \times 0.98 \times 0.80)$

$= 0.2552.$

d. What is the probability that only the receiver needs service during the warranty period?

Answer

$P(\text{only the receiver needs service during the warranty period}) = P(A_1' \cap A_2 \cap A_3) = P(A_1') \times P(A_2) \times P(A_3) = 0.05 \times 0.98 \times 0.80 = 0.0392.$

e. What is the probability that exactly one of the three components needs service during the warranty period?

Answer

$P(\text{exactly one of the three components needs service during the warranty period}) =$

$= P(A_1' \cap A_2 \cap A_3) + P(A_1 \cap A_2' \cap A_3) + P(A_1 \cap A_2 \cap A_3') =$

$P(A_1') \times P(A_2) \times P(A_3) + P(A_1) \times P(A_2') \times P(A_3) + P(A_1) \times P(A_2) \times P(A_3') =$

$(0.05 \times 0.98 \times 0.80) + (0.95 \times 0.02 \times 0.80) + (0.95 \times 0.98 \times 0.2) = 0.2406.$