Monte-Carlo Simulations and Option Pricing

The Euler-Maruyama method, the Milstein method and the stochastic Runge-Kutta method can be used to approximate a stochastic differential equation. Consider the geometric Brownian motion

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t$$

with $\mu = 0.06$ and $\sigma = 0.3$ and initial value $S_0 = 50$ for $t \in [0, 1]$.

a) Implement the Euler-Maruyama, the Milstein method and the stochastic Runge-Kutta method to approximate the SDE and plot some paths.

b) Compare the approximations $\hat{S}_T$ using all the methods to the exact solution S_T by computing the error $\widehat{\epsilon}$ defined by

$$\widehat{\epsilon}(h) := \frac{1}{N} \sum_{k=1}^{N} \left| S_{T,k} - \hat{S}_{T,k} \right|$$

for $N = 100$ different paths. Use the step sizes $h = 10^{-i}$ for $i = 2, 3, 4$ and thus try to estimate the rate of strong convergence in each method.

European Call-option:
In addition to the given parameter values of the geometric Brownian motion above, we set $K = 90$ (strike) and $r = 0.05$ (interest rate). Apply all the methods to approximate the European Call-option with the payoff

$$(S_T - K)^+,$$

and compare your results to the Black-Schloes solution for different N. What can you observe?

Asian-option:
Instead of the European Call-option we consider an Asian-option which has the payoff

$$\left(\frac{1}{T} \int_0^T S_T - K \right)^+ .$$

Price the Asian-option for the same parameter values using all the methods.

Two-dimensional Pricing

Call options on Max and Min:
Consider two correlated geometric Brownian motions

$$\mathrm{d}S_t^1 = \mu_1 S_t^1 \,\mathrm{d}t + \sigma_1 S_t^1 \,\mathrm{d}W_t^1$$

$$\mathrm{d}S_t^2 = \mu_1 S_t^2\,\mathrm{d}t + \sigma_1 S_t^2\,\mathrm{d}W_t^2,$$

with $\mathrm{d}W_t^1 \mathrm{d}W_t^2 = \rho dt$. Apply the Monte-Carlo method to compute the prices of Call-options on Max and Min with the payoff functions

$$\left(\max(S_T^1, S_T^2)\right)^+ \quad \text{and} \quad \left(\min(S_T^1, S_T^2)\right)^+,$$

respectively. You can freely choose one of those three methods above for the SDEs. Assume that $T = 1$, $S_0^1 = 100$, $S_0^2 = 105$, $\mu_1 = \mu_2 = 0, \sigma_1 = 0.2, \sigma_2 = 0.3$ and plot the corresponding prices for different values of the correlation, $\rho = -0.9$, $\rho = 0$, $\rho = 0.9$. By comparing the prices using the different correlations what can we conclude?

Heston Stochastic Volatility Model:
We know that the volatility in the financial market should not be a constant. There exists several stochastic volatility models for pricing the European option, the generalized Heston model is one of them and reads

$$\begin{aligned} \mathrm{d}S_t &= r_t S_t\,\mathrm{d}t + \sqrt{v_t} S_t\,\mathrm{d}W_t^1, \\ \mathrm{d}v_t &= \kappa(\theta - v_t)\,\mathrm{d}t + \sigma\sqrt{v_t}\,\mathrm{d}W_t^2, \end{aligned}$$

where the Cox-Ingersoll-Ross model is applied to describe stochastic volatility and the deterministic function

$$r_t = \frac{1}{100}\left(\sin(2\pi t) + t + 3\right)$$

is used for the time-dependent interest rate. In the generalized Heston model, the asset and volatility processes are allowed to be correlated through the correlated Brownian motions. Apply the Monte-Carlo method to price the European Call-option in the Heston model for $T = 3$, $\kappa = 2$, $v_0 = \theta = 0.04$, $\sigma = 0.1$, $\rho = -0.7, K = S_0 = 2$.

You must comment your code.