Derivation of costate and investment in AK model

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- We can rewrite the dynamic optimization problem is as below

$$\max_{\{C,I,K\}} \mathbb{E}_t[V]$$

with two constraint:

$$K_{t+1} = K_t[1+\theta_2(\frac{I_t}{K_t})]^{\theta_1}\exp(B_{t+1}-B_t) \equiv f(K_t, I_t) \quad \forall t$$

$$C_t + I_t \leq \alpha K_t \quad \forall t$$

- Note that V is a function of Cs. Use the multiplier as MK_{t+1} and MC_t. [1]
- Write the lagrangian as below

$$\mathcal{L} = \mathbb{E}\Big[V + \sum_t \Big[MK_{t+1}(K_t[1+\theta_2(\frac{I}{K})]^{\theta_1}\exp(B_{t+1}-B_t) - K_{t+1}) + MC_t(\alpha K_t - C_t - I_t)\Big]|\mathfrak{U}_t\Big]$$

- The F.O.C on K_t will lead to the capital co-state. Firstly, take derivative of term $f(K_t, I_t)$ w.r.t K_t:

$$\begin{aligned}\frac{\partial f}{\partial k} =& \underbrace{[1+\theta_2(\frac{I_t}{K_t})]^{\theta_1}\exp(B_{t+1}-B_t)}_{K_{t+1}/K_t} + \theta_1\theta_2(-\frac{I_t}{K_t^2})K_t[1+\theta_2(\frac{I_t}{K_t})]^{\theta_1-1}\exp(B_{t+1}-B_t)\\ =&\frac{K_{t+1}}{K_t} - \theta_1\theta_2[1+\theta_2(\frac{I_t}{K_t})]^{\theta_1-1}\exp(B_{t+1}-B_t)\frac{I_t}{K_t}\end{aligned}$$

Now take derivative of $\mathcal{L}$ w.r.t K_t:

$$\begin{aligned}0 =& \mathbb{E}[MK_{t+1}\frac{\partial f(K_t, I_t)}{\partial K_t} - MK_t + \alpha MC_t|\mathfrak{U}_t]\\ =&\mathbb{E}[MK_{t+1}\frac{\partial f(K_t, I_t)}{\partial K_t}|\mathfrak{U}_t] - MK_t + \alpha MC_t\end{aligned}$$

thus we have

$$MU_t\mathbb{E}[\Big(\frac{S_{t+1}}{S_t}\Big)\Big(\frac{MK_{t+1}}{MU_{t+1}}\Big)\Big(\frac{K_{t+1}}{K_t} - \theta_1\theta_2[1+\theta_2(\frac{I_t}{K_t})]^{\theta_1-1}\exp(B_{t+1}-B_t)\frac{I_t}{K_t}\Big)|\mathfrak{U}_t] - MK_t + \alpha MC_t = 0$$

- The F.O.C on investment would be more straight forward: take derivative of $f(K_t, I_t)$ w.r.t I_t

$$\frac{\partial f(K_t, I_t)}{\partial I_t} = \theta_1\theta_2[1+\theta_2\Big(\frac{I_t}{K_t}\Big)]^{\theta_1-1}\exp(B_{t+1}-B_t)$$

[1] $MC_t = \frac{\partial V}{\partial C_t}$, you can get it from the F.O.C.

thus the derivative for I would be

$$\begin{aligned} 0 =& \mathbb{E}[\lambda_t \theta_1 \theta_2 [1+\theta_2 \Big(\frac{I_t}{K_t}\Big)]^{\theta_1 - 1} \exp(B_{t+1} - B_t) - \gamma_t | \mathfrak{U}_t] \\ =& \mathbb{E}[MK_{t+1} \theta_1 \theta_2 [1+\theta_2 \Big(\frac{I_t}{K_t}\Big)]^{\theta_1 - 1} \exp(B_{t+1} - B_t) - MC_t | \mathfrak{U}_t] \end{aligned}$$

thus

$$MU_t \mathbb{E}[\Big(\frac{S_{t+1}}{S_t}\Big)\Big(\frac{MK_{t+1}}{MU_{t+1}}\Big)\Big(\theta_1 \theta_2 [1+\theta_2 \Big(\frac{I_t}{K_t}\Big)]^{\theta_1 - 1} \exp(B_{t+1} - B_t)\Big) | \mathfrak{U}_t] - MC_t = 0$$