Exercise Sheet № 11

Task 11.2: Examples of adjoint operators

Determine the adjoint operator to

i) $T\colon \ell^1(\mathbb{R}) \to \ell^1(\mathbb{R})$ where

$$(\xi_k)_{k\in\mathbb{N}} \mapsto \left(\sum_{k=1}^{\infty} \xi_k, 0, 0, \dots\right)$$

ii) $S\colon \ell^2(\mathbb{R}) \to \ell^2(\mathbb{R})$ where

$$(Tx)_k = \frac{1}{k^2}\sum_{j=1}^{k} \xi_j$$

iii) $f\colon X \to \mathbb{R}$, where X is a real normed space

Subtask i): Let $y^* \in \ell^1(\mathbb{R})^* \simeq \ell^\infty(\mathbb{R})$ and $x \in \ell^1(\mathbb{R})$

$$\langle T^*y^*, x\rangle_X = \langle y^*, Tx\rangle_Y = \sum_{k\in\mathbb{N}} y_K (Tx)_k = \sum_{k\in\mathbb{N}} \delta_k y_k \sum_{j\in\mathbb{N}} \xi_j = y_1 \sum_{k\in\mathbb{N}} \xi_k$$

$$\implies y^* \overset{T^*}{\mapsto} (y_1, y_1, \dots)$$

Subtask ii): Let $y^* \in \ell^2(\mathbb{R})^* \simeq \ell^2(\mathbb{R})$ and e_l be the j-th unit vector in $\ell^2(\mathbb{R})$:

$$\langle S^*y^*, e_l\rangle_X = \langle y^*, Te_l\rangle_Y = \sum_{k\in\mathbb{N}} y_k \frac{1}{k^2}\sum_{j=1}^{k} e_{l,k}$$

$$k < l \implies e_{l,k} = 0 \qquad k \geq l \implies \sum_{j=1}^{k} e_{l,k} = 1$$

$$\implies \sum_{k\in\mathbb{N}} y_k \frac{1}{k^2}\sum_{j=1}^{k} e_{k,l} = \sum_{k=l}^{\infty} \frac{y_k}{k^2}$$

$$\implies y^* \overset{T^*}{\mapsto} \left(\sum_{k=1}^{\infty}\frac{y_k}{k^2}, \sum_{k=2}^{\infty}\frac{y_k}{k^2}, \dots\right)$$

Subtask iii):

Task 11.3: Arithmetic with adjoint operators

Let X, Y and Z be normed spaces and $T, S \in L(X,Y)$ and $R \in L(Y,Z)$. Prove the following statements

i) $(T+S)^* = T^* + S^*$

Subtask i): Let $y* \in X^*$ and $x \in X$:

$$\langle (T+S)^*y^*, x\rangle_X = \langle y^*, (T+S)x\rangle_Y = \langle y^*, Tx\rangle_Y + \langle y^*, Sx\rangle_Y = \langle T^*y^*, x\rangle_X + \langle S^*y^*, x\rangle_X$$