Exercise Sheet № 9

Task 9.1

Let X be a finite-dimensional vector space and $\boldsymbol{f}: X \to X$ be a diffeomorphism. Further let $\boldsymbol{g}: X \to X$ be a $\mathcal{C}^1$ mapping, and $A \subseteq X$ be compact, such that $\boldsymbol{g}|_B = \mathbf{0}$, where $B = X \setminus A$. Proof there exists $\varepsilon > 0$, such that forall $\lambda \in \mathcal{B}_\varepsilon(0)$ the mapping $\boldsymbol{f} + \lambda\boldsymbol{g}: X \to X$ is a diffeomorphism.

Let $\boldsymbol{h} = \boldsymbol{f} + \lambda\boldsymbol{g}$. By requirement we get that $\boldsymbol{h}$ is continuously differentiable. Since $\boldsymbol{f}$ is a diffeomorphism, $\forall \boldsymbol{x} \in X: \det \mathrm{D}\boldsymbol{f}(\boldsymbol{x}) \neq 0$. Further if $\|\mathrm{D}\boldsymbol{f}(\boldsymbol{x}) - \lambda \mathrm{D}\boldsymbol{g}(\boldsymbol{x})\| < \delta$ for some $\delta > 0$ on A. Since the set of invertible matrices is open, we get that $\det \mathrm{D}\lambda\boldsymbol{g}(\boldsymbol{x}) \neq 0$, on A, thus $\lambda\boldsymbol{g}$ is a local diffeomorphism. We only need to prove now, that for small enough ε, we get that $\lambda\boldsymbol{g}$ is injective. Since A is compact, $\exists C > 0: \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \geq C\|\boldsymbol{x} - \boldsymbol{y}\|$ for $\boldsymbol{x}, \boldsymbol{y} \in A$. Since $\mathrm{D}\boldsymbol{h}$ is small, there exists $\varepsilon\lambda > 0$, such that $\|(\lambda\boldsymbol{g} - \boldsymbol{f})(\boldsymbol{x}) - (\lambda\boldsymbol{g} - \boldsymbol{f})(\boldsymbol{y})\| \leq \frac{C}{2}\|\boldsymbol{x} - \boldsymbol{y}\|$ for $\boldsymbol{x}, \boldsymbol{y} \in A$. It follows:

$$\|\lambda\boldsymbol{g}(\boldsymbol{x}) - \lambda\boldsymbol{g}(\boldsymbol{y})\| \geq \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| - \|(\lambda\boldsymbol{g} - \boldsymbol{f})(\boldsymbol{x}) - (\lambda\boldsymbol{g} - \boldsymbol{f})(\boldsymbol{y})\| \geq \frac{C}{2}\|\boldsymbol{x} - \boldsymbol{y}\|$$

Since $\forall \boldsymbol{x}, \boldsymbol{y} \in A$ we get $\|\lambda\boldsymbol{g}(\boldsymbol{x}) - \lambda\boldsymbol{g}(\boldsymbol{y})\| \geq \frac{C}{2}\|\boldsymbol{x} - \boldsymbol{y}\|$, i.e. for $\boldsymbol{x} \neq \boldsymbol{y}$ it follows $\lambda\boldsymbol{g}(\boldsymbol{x}) \neq \lambda\boldsymbol{g}(\boldsymbol{y})$, thus $\lambda\boldsymbol{g}$ is injective.

Task 9.2: Special Linear Group

Let $\mathrm{SL}(n, \mathbb{R}) = \{\mathrm{A} \in \mathbb{R}^{n\times n}: \det(\mathrm{A}) = 1\}$. Find a representation of $T_{\mathrm{I}}\mathrm{SL}(n, \mathbb{R})$, the tangent-space of $\mathrm{SL}(n, \mathbb{R})$ in I. Furthermore, show that for $\mathrm{A} \in \mathbb{R}^{n\times n}$ with $\mathrm{tr}(\mathrm{A}) = 0$, $\boldsymbol{\gamma}(t) = \exp(t\mathrm{A})$ is a curve in $\mathrm{SL}(n, \mathbb{R})$ with $\boldsymbol{\gamma}'(0) = \mathrm{A}$.