Exercise Sheet № 2

Task 2.1

Let (X, d) be a metric space. Prove that the following statements are equivalent:

i) (X, d) is compact
ii) For every family $(C_i)_{i\in I}$, where C_i is closed in (X, d) and $C_i \subseteq X$, such that $\forall J \in \mathcal{P}(I)\colon |J| \in \mathbb{N}$ with $\bigcap_{j\in J} C_j \neq \emptyset$, $\bigcap_{i\in I} C_i \neq \emptyset$ also holds.

Task 2.2

i) Prove, that every finite union of compact subsets in (X, d) is again compact
ii) Find an example of a metric space (X, d) such that the union of countably many compact subsets is not compact

Subtask i): Let $(A_i)_{i=0}^n$ be our family of compact sets whose union we want to build. Since A_i is compact:

$$\forall (C_{i,j})_{j\in I_i}\colon A_i \subseteq \bigcup_{j\in I_i} C_{i,j}\colon \exists J_i \subseteq I_i\colon |J_i| \in \mathbb{N} \wedge A_i \subseteq \bigcup_{j\in J_i} C_{i,j}$$

where $C_{i,j}$ are open in (X, d). Now :

$$\bigcup_{i=0}^n A_i \subseteq \bigcup_{i=0}^n \bigcup_{j\in I_i} C_{i,j} \Rightarrow \bigcup_{i=0}^n A_i \subseteq \bigcup_{i=0}^n \bigcup_{j\in J_i} C_{i,j}$$

We found a finite open-cover for the union for an arbitrary open cover for the union.

Subtask ii):

Let $A_i = [i - 1, i]$ for $i \in \mathbb{Z}$, then everyA_i is compact, but

$$\bigcup_{i\in\mathbb{Z}} A_i = \mathbb{R}$$

is not.

Task 2.3

Let (X, d) be a metric space and $C \subseteq X$. Prove that if C is totally bounded, then $\operatorname{cls} C$ is also totally bounded.

We call $C \subseteq X$ totally bounded if for every $r > 0$ we can find $x_1, \dots, x_n \in C$ such that

$$C \subseteq \bigcup_{i=1}^n \mathcal{B}_{\frac{r}{2}}(x_i)$$

We want to show, that $\operatorname{cls} C \subseteq \bigcup_{i=1}^n \mathcal{B}_r(x_i)$. Let $z \in \operatorname{cls} C$, then there exists $x \in C\colon d(z, x) < \frac{\varepsilon}{2}$, since $\forall x \in \operatorname{cls} C\colon \forall r > 0\colon \mathcal{B}_r(x) \cap C \neq \emptyset$. Now, since C is totally bounded, we can find x_i, such that $d(x, x_i) < \frac{\varepsilon}{2}$. Since d is a metric:

$$d(x_i, z) \le d(x_i, z) + d(z, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Task 2.4

Let (X, d) be a metric space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X.

i) Prove that $\mathcal{X} = \{x_n | n \in \mathbb{N}\} \cup \{x\}$ is compact if $\lim_{n\to\infty} x_n = x$. Is $\{x_n | n \in \mathbb{N}\}$ compact or relatively compact?
ii) Is $\{x_n | n \in \mathbb{N}\}$ relatively compact, if $\sup_{m,n\in\mathbb{N}} d(x_n, x_m) < \infty$?

Subtask i): We show that $\mathcal{X}$ is a sequentially compact space. Let $I \subseteq \mathbb{N}$ with $|I| = \aleph_0$. Since $(x_n)_{n\in\mathbb{N}}$ is convergent, we know

$$\forall \varepsilon > 0 \colon \exists N \in \mathbb{N} \colon n \geq N \Rightarrow d(x_n, x) < \varepsilon$$

Hence:

$$\forall \varepsilon > 0 \colon \exists N \in \mathbb{N} \colon \forall n \in I \colon n \geq N \colon d(x_n, x) < \varepsilon$$

Thus every partial sequence $(x_n)_{n\in I}$ converges to x, i.e. every sequence in $\mathcal{X}$ has a convergent partial sequence.