Exercise Sheet № 7

Task 7.1

Assume X is separable and Y is isometrically isomorphic to X. Prove that Y is also separable.

Since X is separable, there exists a countable dense subset $A \subseteq X$. Since X and Y are isometrically isomorphic, there exists an isomorphism $T\colon X \to Y$ that is also an isometry, so $\|x\|_X = \|Tx\|_Y$. Let $B = T[A]$. Since A is dense in X we know

$$\forall x \in X\colon \exists (x_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}\colon \lim_{n\to\infty} x_n = x \in X$$

Since T is an isometry we have for $\varepsilon > 0$ and $N \in \mathbb{N}$ sufficiently large, such that $n \geq N \implies \|x_n - x\| < \varepsilon$:

$$\|Tx_n - Tx\|_Y = \|x_n - x\|_X < \varepsilon$$

Thus B is a countable subset of Y with $\operatorname{cls}(B) = Y$, i.e. Y is separable.

Task 7.2

Prove that $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}_0(\mathbb{R})$ are isomorphic. *Hint*: For every $x = (\xi_k)_{k\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, set $\ell(x) = \lim_{k\to\infty} \xi_k$ and define the sequence $Tx = (\eta_k)_{k\in\mathbb{N}}$ via

$$\eta_k = \begin{cases} \ell(x) & k = 0 \\ \xi_{k-1} - \ell(x) & k \geq 1 \end{cases}$$

and prove T is an isomorphism.

Recall the definition of $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}_0(\mathbb{R})$:

$$\mathfrak{c}(\mathbb{R}) = \{x \in \mathbb{R}^{\mathbb{N}}\colon x \text{ converges}\}$$
$$\mathfrak{c}_0(\mathbb{R}) = \left\{(\xi_k)_{k\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})\colon \lim_{k\to\infty} \xi_k = 0\right\}$$

We equip $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}_0(\mathbb{R})$ with $\|\cdot\|\infty$ and first show, that $Tx \in \mathfrak{c}_0(\mathbb{R})$. Let $x = (\xi_k)_{k\in\mathbb{N}}$:

$$Tx = (\ell(x), \xi_1 - \ell(x), \xi_2 - \ell(x), \dots) \implies \lim_{k\to\infty} \eta_k = \ell(x) - \ell(x) = 0$$

Next we prove T is linear. Let $x, y \in \mathfrak{c}(\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$, where $x = (\xi_k)_{k\in\mathbb{N}}$ and $y = (\upsilon_k)_{k\in\mathbb{N}}$. Notice that

$$\ell(\lambda x + \mu y) = \lim_{k\to\infty} \lambda\xi_k + \mu\upsilon_k = \lambda \lim_{k\to\infty} \xi_k + \mu \lim_{k\to\infty} \upsilon_k = \lambda\ell(x) + \mu\ell(y)$$

Hence:

$$\begin{aligned} T(\lambda x + \mu y) &= (\ell(\lambda x + \mu y), \lambda\xi_1 + \mu\upsilon_1 - \ell(\lambda x + \mu y), \dots) \\ &= \lambda(\ell(x), \xi_1 - \ell(x), \dots) + \mu(\ell(y), \upsilon_1 - \ell(y)) = \lambda Tx + \mu Ty \end{aligned}$$

Next we prove that T is bounded:

$$\|\ell(x)\| \leq \sup_{k\in\mathbb{N}} |\xi_k| = \|x\|_\infty \qquad |\xi_n - \ell(x)| \leq \sup_{k\in\mathbb{N}} |\xi_k| = \|x\|_\infty \implies \|Tx\|_\infty \leq \|x\|_\infty$$

Hence T is bounded and thus continuous. As a last step, we prove T is a bijection. For injectivity, we show $\ker T = \{0\}$. Let $\|Tx\|_\infty = 0$, then $\ell(x) = 0$ and $\eta_k = \xi_k - \ell(x) = 0$ for all $k \in \mathbb{N}$, thus $x = 0$, which means T has a trivial kernel and is thus injective. For surjectivity, let $y \in \mathfrak{c}_0(\mathbb{R})$ with $y = (\upsilon_k)_{k\in\mathbb{N}}$. Let $\xi_k = \upsilon_{k+1} + \upsilon_1$. Since $\lim_{k\to\infty} \upsilon_k = 0$, we know that $\lim_{k\to\infty} \xi_k = \upsilon_1$. Now let $x = (\xi_k)_{k\in\mathbb{N}}$ and

$$Tx = (\ell(x), x_1 - \ell(x), x_2 - \ell(x), \dots) = (y_1, y_2 + y_1 - y_1, y_3 + y_1 - y_1, \dots) = (y_1, y_2, y_3, \dots) = y$$

Hence T is surjective and thus bijective. Since $\mathfrak{c}(\mathbb{R})$ and $\mathfrak{c}_0(\mathbb{R})$ are Banach-spaces, we know that T^{-1} is a bound linear operator by the theorem of the continuous inverse. Hence T is an isomorphism.

Task 7.4

For every $x \in X$ we define a map $T_x \colon \mathcal{L}(X, \mathbb{R}) \to \mathbb{R}$ via $\forall f \in \mathcal{L}(X, \mathbb{R}) \colon T_x(f) = f(x)$.

i) We equip $\mathcal{L}(X, \mathbb{R})$ with the operator-norm $\| \cdot \|_O$ and let $x \in X$. Prove T_x is linear and continuous

Subtask i): Let $f, g \in \mathcal{L}(X, \mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$:

$$T_x(\lambda f + \mu g) = (\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x) = \lambda T_x(f) + \mu T_x(g)$$

Notice that for any T_x, x is fixed, hence:

$$|T_x(f)| = |f(x)| \leq \|f\|_O \cdot \|x\|_X$$

Hence T is bounded and thus continuous.