Exercise Sheet № 6

Task 28: Eigenvalues and -vectors

a) Find the eigenvalues and -vectors of

$$\mathrm{A} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$$

b) Find the eigenvalues and -vectors of $1_{n\times n}$

Subtask a):

$$\chi_{\mathrm{A}}(\lambda) = \begin{vmatrix} \lambda-1 & -2 & 0 \\ 0 & \lambda-1 & 2 \\ -2 & -2 & \lambda+1 \end{vmatrix} = (\lambda-1)(\lambda^2+3)+8 = \lambda^3+3\lambda-\lambda^2-3+8$$
$$= \lambda^3-\lambda^2+3\lambda+5$$
$$\lambda_1 = -1 \qquad \frac{\chi_{\mathrm{A}}(\lambda)}{(\lambda+1)} = \lambda^2-2\lambda+2 \implies \lambda_{2,3} = 1 \pm 2\mathrm{i}$$

Notice

$$\overline{\mathrm{A}\boldsymbol{v}_2} = \overline{\lambda_2 \boldsymbol{v}_2} \iff \mathrm{A}\overline{\boldsymbol{v}_2} = \lambda_3 \overline{\boldsymbol{v}_2}$$

Hence $\boldsymbol{v}_2 = \overline{\boldsymbol{v}_3}$. Now:

$$\lambda_1 \mathrm{I} - \mathrm{A} = \begin{bmatrix} -2 & -2 & 0 \\ 0 & -2 & 2 \\ -2 & -2 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies \boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 \mathrm{I} - \mathrm{A} = \begin{bmatrix} 2\mathrm{i} & -2 & 0 \\ 0 & 2\mathrm{i} & 2 \\ -2 & -2 & 2+2\mathrm{i} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \mathrm{i} & 0 \\ 0 & 1 & -\mathrm{i} \\ 1 & 1 & -1-\mathrm{i} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \mathrm{i} & 0 \\ 0 & 1 & -\mathrm{i} \\ 0 & 1-\mathrm{i} & -1-\mathrm{i} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \mathrm{i} & 0 \\ 0 & 1 & -\mathrm{i} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\implies \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ \mathrm{i} \\ 1 \end{bmatrix} \implies \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ -\mathrm{i} \\ 1 \end{bmatrix}$$

Subtask b):

Assume $\boldsymbol{x}$ is an eigenvector of 1, then

$$1\boldsymbol{x} = \lambda \boldsymbol{x} \implies \begin{bmatrix} x_1 + x_2 \cdots + x_n \\ \vdots \\ x_1 + x_2 + \cdots + x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} \iff \begin{bmatrix} x_1(\lambda-1) + x_2 + \cdots + x_n \\ \vdots \\ x_1 + x_2 + \cdots + x_n(\lambda-1) \end{bmatrix} = \boldsymbol{0}$$

Notice that for $\lambda = 0$ we get the space

$$W = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \right\}$$

Notice that the geometric multiplicity of 0 is $\dim(W)$ and hence we get the lower bound $\dim W$ for the algebraic multiplicity. Furthermore $\boldsymbol{0} \in W$, hence $W \neq \emptyset$. Notice that $W = \ker \mathrm{I}_{n\times n}$. Since all rows of $1_{n\times n}$ are linearly dependent, we know that $\operatorname{rank} 1_{n\times n} = 1$. With the dimension-formula[1] we thus get $\dim W = n-1$.

For $\lambda = n$ we get $(n-1)x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} x_j$ for all i and thus $x_1 = x_2 = \cdots = x_n$. Let $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = \cdots = x_n\}$. Notice that $W \cap V = \{\boldsymbol{0}\}$ and $\dim V = 1$. Since $1_{n\times n}$ is symmetric, it's eigenspaces are

[1] Let $L \in \operatorname{Hom}(V, W)$, then $V/\ker(L) \simeq \operatorname{im}(L) \implies \operatorname{rank}(L) = \dim(V) - \dim(\ker(L))$

orthogonal[2]. Independent of orthogonality, we know that

$$\bigoplus_{i=0}^{r} \mathrm{Eig}(\lambda_i) \simeq \mathbb{R}^n$$

Hence we found all eigenspaces of $1_{n\times n}$.

Task 29: Eigenvalues

a) Suppose A satisfies $\mathrm{A}^4 = \mathrm{I}$. Prove that $\mathrm{spec}(\mathrm{A}) \subseteq \{1, -1, \mathrm{i}, -\mathrm{i}\}$. Give an example of a real matrix that has all four numbers as eigenvalues.
b) Show that if all row sums of A are equal to 1, then $1 \in \mathrm{spec}(\mathrm{A})$. Suppose all the column-sums of A are equal to 1. Does the same result hold?

Subtask a):

$$\boldsymbol{x} = \mathrm{I}\boldsymbol{x} = \mathrm{A}^4\boldsymbol{x} = \lambda^4\boldsymbol{x}$$

Thus the only possible eigenvalues of A are complex numbers $\lambda \in \mathbb{C}$, such that $\lambda^4 = 1$, i.e. $\lambda \in \{1, -1, \mathrm{i}, -\mathrm{i}\}$. Thus $\mathrm{spec}(\mathrm{A}) \subseteq \{1, -1, \mathrm{i}, -\mathrm{i}\}$. Notice that

$$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 \implies \lambda_{1,2} = \pm\mathrm{i}$$

Then the following matrix has spectrum $\{1, -1, \mathrm{i}, -\mathrm{i}\}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Subtask b): Let $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^n$:

$$\mathrm{A}\mathbf{1} = \begin{bmatrix} a_{11} + a_{12} + \cdots + a_{1n} \\ \vdots \\ a_{n1} + a_{n2} + \cdots + a_{nn} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \mathbf{1}$$

Thus $1 \in \mathrm{spec}(\mathrm{A})$.

Yes, since $\mathrm{spec}(\mathrm{A}) = \mathrm{spec}(\mathrm{A}^{\mathsf{t}})$, as $\det(\lambda\mathrm{I} - \mathrm{A}) = \det((\lambda\mathrm{I} - \mathrm{A})^{\mathsf{t}}) = \det(\lambda\mathrm{I} - \mathrm{A}^{\mathsf{t}})$.

[2] This condition is not strictly required.

Task 30: Eigenvalues of operators

Let $\mathcal{D}^2\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathcal{C}^\infty(\mathbb{R})$ be the operator with $\mathcal{D}f = f''$.

a) Show that for $\omega > 0$, $\sin(\sqrt{\omega}x)$ and $\cos(\sqrt{\omega}x)$ are eigenvectors of $\mathcal{D}^2$ and find their corresponding eigenvalues.

b) Show that for $\omega > 0$, $\sinh(\sqrt{\omega}x)$ and $\cosh(\sqrt{\omega}x)$ are eigenvectors of $\mathcal{D}^2$ and find their corresponding eigenvalues.

Subtask a):

$$\mathcal{D}^2 \sin(\sqrt{\omega}x) = -\omega \sin(\sqrt{\omega}x) \qquad \mathcal{D}^2 \cos(\sqrt{\omega}x) = -\omega \cos(\sqrt{\omega}x)$$

Subtask b):

$$\mathcal{D}^2 \sinh(\sqrt{\omega}x) = \omega \sinh(\sqrt{\omega}x) \qquad \mathcal{D}^2 \cosh(\sqrt{\omega}x) = \omega \cosh(\sqrt{\omega}x)$$

Task 31: Trace

Let

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

a) Prove that the characteristic equation of A can be expressed as

$$\lambda^2 - \mathrm{trace}(\mathrm{A})\lambda + \det(\mathrm{A}) = 0$$

b) Verify that

$$\mathrm{A}^2 - \mathrm{trace}(\mathrm{A})\mathrm{A} + \det(\mathrm{A})\mathrm{I}_2 = 0_{2\times 2}$$

c) If A is regular, prove

$$\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})}(\mathrm{trace}(\mathrm{A})\mathrm{I}_2 - \mathrm{A})$$

Subtask a):

$$\chi_{\mathrm{A}}(\lambda) = (\lambda - a)(\lambda - d) - cb = \lambda^2 - \lambda \underbrace{(a+d)}_{=\mathrm{trace}(\mathrm{A})} + \underbrace{ad - cb}_{=\det(\mathrm{A})}$$

Subtask b):

$$\mathrm{A}^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix}$$

$$\implies \mathrm{A}^2 - \mathrm{tr}(\mathrm{A})\mathrm{A} + \det(\mathrm{A})\mathrm{I}_2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix} - \begin{bmatrix} (a+d)a & (a+d)b \\ (a+d)c & (a+d)d \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc - a^2 - ad + ad - bc & ab + bd - ab + bd \\ ac + cd - ac - cd & d^2 + bc - ad - d^2 + ad - bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Subtask c): Let $\det(\mathrm{A}) \neq 0$:

$$\mathrm{A}^{-1} = \frac{1}{ad - bc}((a+d)\mathrm{I}_2 - \mathrm{A}) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\implies \mathrm{A}\mathrm{A}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} ad - bc & bd - bd \\ ac - ac & ad - bc \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = \mathrm{I}_2$$

Task 32: Characteristic Polynomial of degree 3

a) Find the explicit formula $-\lambda^3 + a\lambda^2 - b\lambda + c$ for the characteristic polynomial $\det(A - \lambda I_3)$ of a general 3×3 matrix.
b) If A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$, prove that

$$\begin{aligned}\lambda_1 + \lambda_2 + \lambda_3 &= \operatorname{trace}(A)\\ b &= \lambda_1\lambda_2 + \lambda_1\lambda_2 + \lambda_2\lambda_3\\ \det(A) &= \lambda_1\lambda_2\lambda_3\end{aligned}$$

Subtask a): With Laplacian expansion we get:

$$\begin{aligned}\chi_A(\lambda) &= -\lambda^3 + \lambda^2\left(a_{0,0} + a_{1,1} + a_{2,2}\right) + \lambda\left(-a_{0,0}a_{1,1} - a_{0,0}a_{2,2} + a_{0,1}a_{1,0} + a_{0,2}a_{2,0} - a_{1,1}a_{2,2} + a_{1,2}a_{2,1}\right)\\ &+ a_{0,0}a_{1,1}a_{2,2} - a_{0,0}a_{1,2}a_{2,1} - a_{0,1}a_{1,0}a_{2,2} + a_{0,1}a_{1,2}a_{2,0} + a_{0,2}a_{1,0}a_{2,1} - a_{0,2}a_{1,1}a_{2,0}\\ &= -\lambda^3 + \lambda^2\operatorname{trace}(A) + \lambda\left(-a_{0,0}a_{1,1} - a_{0,0}a_{2,2} + a_{0,1}a_{1,0} + a_{0,2}a_{2,0} - a_{1,1}a_{2,2} + a_{1,2}a_{2,1}\right) + \det(A)\end{aligned}$$

Subtask b): Recall that $\chi_A(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$:

$$\begin{aligned}-\chi_A(\lambda) &= -(\lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1\lambda_2)(\lambda - \lambda_3)\\ &= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2) - \lambda\lambda_1\lambda_2 + \lambda_3\lambda^2 - \lambda\lambda_3(\lambda_1 + \lambda_2) + \lambda_1\lambda_2\lambda_3\\ &= -\lambda^3 + \lambda^2(\lambda_1 + \lambda_2 + \lambda_3) - \lambda(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) + \lambda_1\lambda_2\lambda_3\end{aligned}$$

Comparing coefficients yields $\operatorname{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$, $b = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $\det(A) = \lambda_1\lambda_2\lambda_3$.

Task 33: Similar Matrices

Let A and B be similar, i.e. $\exists S \in GL(n, \mathbb{R})$ such that $B = S^{-1}AS$.
a) Prove that $\operatorname{spec}(A) = \operatorname{spec}(B)$
b) How are the eigenvectors of A and B related?

Subtask a): Recall

$$1 = \det(I) = \det(SS^{-1}) = \det(S)\det(S^{-1}) \iff \det(S^{-1}) = \frac{1}{\det(S)}$$

Hence:

$$\begin{aligned}&\lambda I - B = S^{-1}(\lambda S - AS) = S^{-1}(\lambda I - A)S\\ &\chi_B(\lambda) = \det(\lambda I - B) = \det(S^{-1}(\lambda I - A)S) = \det(S^{-1})\det(\lambda I - A)\det(S)\\ &= \frac{\det(S)}{\det(S)}\det(\lambda I - A) = \det(\lambda I - A) = \chi_A(\lambda)\end{aligned}$$

Thus if $\chi_A(\mu) = 0$ then $\chi_B(\mu) = \chi_A(\mu) = 0$, i.e. $\operatorname{spec}(A) = \operatorname{spec}(B)$.

Subtask b): Let $\boldsymbol{v}$ be an eigenvector of B:

$$B\boldsymbol{v} = \lambda\boldsymbol{v} \iff S^{-1}AS\boldsymbol{v} = \lambda\boldsymbol{v} \iff A\boldsymbol{v} = \lambda S\boldsymbol{v}S^{-1}$$

Task 34: Orthogonal Matrices

A matrix $\mathrm{Q} \in \mathbb{R}^{n\times n}$ is called orthogonal, if $\mathrm{Q}^{\mathrm{t}}\mathrm{Q} = \mathrm{I} = \mathrm{Q}\mathrm{Q}^{\mathrm{t}}$. Let Q be an orthogonal matrix.

a) Prove that for every non-zero eigenvalue λ, $\frac{1}{\lambda}$ is also an eigenvalue

b) Prove that $\forall\lambda \in \mathrm{spec}(\mathrm{Q}) \subseteq \mathbb{C}$: $|\lambda| = 1$

c) Suppose $\boldsymbol{v} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}$ is an eigenvector of Q corresponding to a non-real eigenvalue. Prove that $\boldsymbol{x}^{\mathrm{t}}\boldsymbol{y} = 0$ and $\|\boldsymbol{x}\| = \|\boldsymbol{y}\|$.

Subtask a): Let $\lambda \in \mathrm{spec}(\mathrm{Q})$. Since Q is a real matrix, we know that $\overline{\lambda}$ is also an eigenvalue. Let $\lambda = \mathrm{e}^{\mathrm{i}\varphi}$:

$$\overline{\lambda} = \mathrm{e}^{-\mathrm{i}\varphi} = \frac{1}{\mathrm{e}^{\mathrm{i}\varphi}} = \frac{1}{\lambda}$$

Thus $\frac{1}{\lambda} \in \mathrm{spec}(\mathrm{Q})$.

Subtask b): Let $\boldsymbol{x} \in \mathbb{R}^n$ and $\mathrm{Q}\boldsymbol{v} = \lambda\boldsymbol{v}$:

$$\begin{aligned} \|\mathrm{Q}\boldsymbol{x}\| &= \sqrt{\langle \mathrm{Q}\boldsymbol{x}, \mathrm{Q}\boldsymbol{x}\rangle} = \sqrt{\boldsymbol{x}^{\mathrm{t}}\mathrm{Q}^{\mathrm{t}}\mathrm{Q}\boldsymbol{x}} = \sqrt{\boldsymbol{x}^{\mathrm{t}}\boldsymbol{x}} = \|\boldsymbol{x}\| \\ \implies &\|\boldsymbol{v}\| = \|\mathrm{Q}\boldsymbol{v}\| = |\lambda|\|\boldsymbol{v}\| \iff |\lambda| = 1 \end{aligned}$$

Subtask c):

Let $\lambda, \mu \in \mathrm{spec}(\mathrm{Q}) \setminus \mathbb{R}$ and $\mu \neq \lambda$. Further let $\mathrm{Q}\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\mathrm{Q}\boldsymbol{y} = \mu\boldsymbol{y}$:

$$\boldsymbol{x}^*\boldsymbol{y} = \boldsymbol{x}^*\mathrm{Q}^*\mathrm{Q}\boldsymbol{y} = (\mathrm{Q}\boldsymbol{x})^*(\mathrm{Q}\boldsymbol{y}) = \overline{\lambda}\mu\boldsymbol{x}^*\boldsymbol{y}$$

This is only the case if $\boldsymbol{x}^*\boldsymbol{y} = 0$ or $\overline{\lambda}\mu = 1$:

$$\overline{\lambda}\mu = 1 \iff \mu = \frac{1}{\overline{\lambda}} = \lambda$$

Thus $\ker(\lambda\mathrm{I} - \mathrm{Q}) \perp \ker(\lambda\mathrm{I} - \mathrm{Q})$. Let $\mathrm{Q}\boldsymbol{v} = \lambda\boldsymbol{v}$ with $\lambda \in \mathrm{spec}(\mathrm{Q}) \setminus \mathbb{R}$ and $\boldsymbol{v} = \boldsymbol{x} + \mathrm{i}\boldsymbol{y}$:

$$\begin{aligned} 0 &= \langle \boldsymbol{x}, \overline{\boldsymbol{x}}\rangle = (\boldsymbol{x}^{\mathrm{t}} - \mathrm{i}\boldsymbol{y}^{\mathrm{t}})(\boldsymbol{x} - \mathrm{i}\boldsymbol{y}) \\ &= \boldsymbol{x}^{\mathrm{t}}\boldsymbol{x} - \mathrm{i}\boldsymbol{x}^{\mathrm{t}}\boldsymbol{y} - \mathrm{i}\boldsymbol{y}^{\mathrm{t}}\boldsymbol{x} - \boldsymbol{y}^{\mathrm{t}}\boldsymbol{y} = \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 - 2\mathrm{i}\boldsymbol{x}^{\mathrm{t}}\boldsymbol{y} \end{aligned}$$

Notice that for $z \in \mathbb{C}$ we know $z = 0 \iff \mathfrak{Re}(z) = 0 \wedge \mathfrak{Im}(z) = 0$. Hence $\boldsymbol{x}^{\mathrm{t}}\boldsymbol{y} = 0$ and thus $\langle \boldsymbol{v}, \overline{\boldsymbol{v}}\rangle = \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 = 0$, therefore $\|\boldsymbol{x}\|^2 = \|\boldsymbol{y}\|^2$.