Exercise Sheet № 9

Task 47: Integrals as inner product

We equip $\mathbb{R}_3[x]$ with the inner product $\langle \cdot, \cdot \rangle : \mathbb{R}_3[x] \times \mathbb{R}_3[x] \to \mathbb{R}$

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x) \,\mathrm{d}x$$

Let $p(x) = 2x^3$ and $q(x) = 1 - x^3$.

a) Find $\|p\|$ and $\|q\|$

b) Find $\langle p, q \rangle$ and $\|p - q\|$

Subtask a):

$$\|p\|^2 = \langle p, p \rangle = \int_{-1}^{1} p^2 \,\mathrm{d}x = \int 4x^6 \,\mathrm{d}x = \frac{4}{7}x^7\Big|_{-1}^{1} = \frac{8}{7}$$

$$\|q\|^2 = \langle q, q \rangle = \int_{-1}^{1} q^2 \,\mathrm{d}x = \int_{-1}^{1} 1 - 2x^3 + x^6 \,\mathrm{d}x = 2 + \left(-\frac{1}{2}x^4 + \frac{1}{7}x^7\Big|_{-1}^{1}\right) = \frac{16}{7}$$

Subtask b):

$$\langle p, q \rangle = \int_{-1}^{1} 2x^3 - 2x^6 \,\mathrm{d}x = 2\left(\frac{1}{4}x^4 - \frac{1}{7}x^7\Big|_{-1}^{1}\right) = -\frac{4}{7}$$

$$\|p - q\|^2 = \langle p - q, p - q \rangle = \int_{-1}^{1} (x^3 - 1)^2 \,\mathrm{d}x = \int_{-1}^{1} 1 - 2x^3 + x^6 \,\mathrm{d}x = \frac{16}{7}$$

Task 48: Inner product identities

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $\boldsymbol{u}, \boldsymbol{v} \in V$. Prove the following:

a) $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \frac{1}{4}(\|\boldsymbol{u} + \boldsymbol{v}\|^2 - \|\boldsymbol{u} - \boldsymbol{v}\|^2)$

b) $\|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2$

Use a) to find the inner product on $\mathbb{R}^2$ corresponding to the norm $\|\boldsymbol{v}\| = \sqrt{v_1^2 - 3v_1v_2 + 5v_2^2}$

Subtask a):

$$\begin{aligned}
&\|\boldsymbol{u} + \boldsymbol{v}\|^2 - \|\boldsymbol{u} - \boldsymbol{v}\|^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle - \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{v} \rangle - \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \langle \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{v}, \boldsymbol{v} \rangle - \langle \boldsymbol{u}, \boldsymbol{u} \rangle + \langle \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle - \langle \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= 4 \langle \boldsymbol{u}, \boldsymbol{v} \rangle
\end{aligned}$$

To find $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ based on $\|\boldsymbol{v}\|$:

$$\begin{aligned}
&\langle \boldsymbol{v}, \boldsymbol{v} \rangle = v_1^2 - 3v_1v_2 + 5v_2^2 = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} v_1 \\ -3v_1 + 5v_1 \end{bmatrix} = \boldsymbol{v}^{\mathrm{t}} \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} \boldsymbol{v} \\
&\implies \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^{\mathrm{t}} \begin{bmatrix} 1 & 0 \\ -3 & 5 \end{bmatrix} \boldsymbol{v} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ -3v_1 + 5v_2 \end{bmatrix} \\
&= u_1v_1 - 3v_1u_2 + 5v_2u_2
\end{aligned}$$

Subtask b):

$$\begin{aligned}
&\|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle + \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{v} \rangle + \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} \rangle - \langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= \langle \boldsymbol{u} + \boldsymbol{v} + \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} \rangle + \langle \boldsymbol{u} + \boldsymbol{v} - \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{v} \rangle \\
&= \langle 2\boldsymbol{u}, \boldsymbol{u} \rangle + \langle 2\boldsymbol{v}, \boldsymbol{v} \rangle = 2 \langle \boldsymbol{u}, \boldsymbol{u} \rangle + 2 \langle \boldsymbol{v}, \boldsymbol{v} \rangle = 2\|\boldsymbol{u}\|^2 + 2\|\boldsymbol{v}\|^2
\end{aligned}$$

Task 49: Projection into onedimensional subspaces

We equip $\mathbb{R}^n$ with the euclidean inner product. The projection of a vector $\boldsymbol{u}$ onto some vector $\boldsymbol{a}$ is defined by

$$P_{\boldsymbol{a}}(\boldsymbol{u}) = \frac{\langle \boldsymbol{u}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

Prove or disprove the following statements:

a) If $\boldsymbol{a} \perp \boldsymbol{b}$, then for every non-zero $\boldsymbol{u}$ we have

$$P_{\boldsymbol{a}}(P_{\boldsymbol{b}}(\boldsymbol{u})) = \boldsymbol{0}$$

b) If $\boldsymbol{a}$ and $\boldsymbol{u}$ are non-zero, then

$$P_{\boldsymbol{a}}(P_{\boldsymbol{a}}(\boldsymbol{u})) = P_{\boldsymbol{a}}(\boldsymbol{u})$$

c) If $P_{\boldsymbol{a}}(\boldsymbol{u}) = P_{\boldsymbol{a}}(\boldsymbol{v})$ holds for some vector $\boldsymbol{a}$, then $\boldsymbol{u} = \boldsymbol{v}$.

Subtask a):

$$P_{\boldsymbol{a}}(P_{\boldsymbol{b}}(\boldsymbol{u})) = \frac{\langle \boldsymbol{u}, \boldsymbol{b} \rangle}{\|\boldsymbol{b}\|^2} P_{\boldsymbol{a}}(\boldsymbol{b}) = \frac{\langle \boldsymbol{u}, \boldsymbol{b} \rangle}{\|\boldsymbol{b}\|^2} \frac{\langle \boldsymbol{b}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|^2} \boldsymbol{a} = \boldsymbol{0}$$

Subtask b): Notice that $P_{\boldsymbol{a}}(\boldsymbol{a}) = \frac{\|\boldsymbol{a}\|^2}{\|\boldsymbol{a}\|^2} \boldsymbol{a} = \boldsymbol{a}$, thus:

$$P_{\boldsymbol{a}}(P_{\boldsymbol{a}}(\boldsymbol{u})) = \frac{\langle \boldsymbol{u}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|^2} P_{\boldsymbol{a}}(\boldsymbol{a}) = \frac{\langle \boldsymbol{u}, \boldsymbol{a} \rangle}{\|\boldsymbol{a}\|^2} \boldsymbol{a}$$

Subtask c): Let $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{a}^\perp$ and $\boldsymbol{u} \neq \boldsymbol{v}$:

$$P_{\boldsymbol{a}}(\boldsymbol{u}) = \boldsymbol{0} \qquad P_{\boldsymbol{a}}(\boldsymbol{v}) = \boldsymbol{0}$$

Task 50: Gram-Schmidt orthonormalization

We equip $\mathbb{R}^3$ with the euclidean inner product.

a) Let $\boldsymbol{u}_1 = (1,1,1)$, $\boldsymbol{u}_2 = (0,1,1)$, $\boldsymbol{u}_3 = (0,0,1)$. Orthonormalize them into a basis $\{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\} = Q$

b) Find the coordinates of $\boldsymbol{u} = (1,2,4)$ in the basis Q

Subtask a):

$$\begin{aligned}
\widetilde{\boldsymbol{q}}_1 &= \boldsymbol{u}_1 \qquad \|\widetilde{\boldsymbol{q}}_1\|_2 = \sqrt{3} \\
\widetilde{\boldsymbol{q}}_2 &= \boldsymbol{u}_2 - \langle \boldsymbol{u}_2, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \qquad \|\widetilde{\boldsymbol{q}}_2\|_2 = \frac{1}{3}\sqrt{6} \\
\widetilde{\boldsymbol{q}}_3 &= \boldsymbol{u}_3 - \langle \boldsymbol{u}_3, \boldsymbol{q}_1 \rangle \boldsymbol{q}_1 - \langle \boldsymbol{u}_3, \boldsymbol{q}_2 \rangle \boldsymbol{q}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 0 \\ -3 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \|\widetilde{\boldsymbol{q}}_3\|_2 = \frac{1}{2}\sqrt{2}
\end{aligned}$$

Subtask b): Since $\boldsymbol{u}$ is given in the canonical basis, we can simply find the inverse of $\mathrm{Q} = \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 & \boldsymbol{q}_3 \end{bmatrix}$ and compute $\mathrm{Q}^{-1}\boldsymbol{u}$. Notice that the columns of Q are orthonormal, thus $\boldsymbol{q}_i^{\mathrm{t}} \boldsymbol{q}_j = \delta_{ij}$, hence $\mathrm{Q}^{-1} = \mathrm{Q}^{\mathrm{t}}$

$$\mathrm{Q} = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies \mathrm{Q}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \implies \mathrm{Q}^{-1}\boldsymbol{q} = \begin{bmatrix} \frac{7}{\sqrt{3}} \\ \frac{2\sqrt{6}}{3} \\ \sqrt{2} \end{bmatrix}$$

Task 51

Let W be a finite-dimensional subspace of an inner product-space V. If $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r\}$ is an orthonormal basis for W and $\boldsymbol{v}$ is any vector in V, then the orthogonal projection of $\boldsymbol{v}$ into W satisfies

$$P_W(\boldsymbol{v}) = \sum_{i=1}^{r} \langle \boldsymbol{v}, \boldsymbol{w}_i \rangle \, \boldsymbol{w}_i$$

If $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r$ is an orthonormal family, then $\langle \boldsymbol{w}_i, \boldsymbol{w}_j \rangle = \delta_{ij}$. Hence the Gram-Matrix is $\operatorname{Gram}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r) = \mathrm{I}_r$. Then the projection $P_W(\boldsymbol{v})$ is given as

$$\sum_{i=1}^{r} \xi_i \boldsymbol{w}_i \qquad \boldsymbol{\xi} = \operatorname{Gram}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r)^{-1} \begin{bmatrix} \langle \boldsymbol{w}_1, \boldsymbol{r} \rangle \\ \vdots \\ \langle \boldsymbol{w}_r, \boldsymbol{r} \rangle \end{bmatrix}$$

$$\mathrm{G} = \operatorname{Gram}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r) = \mathrm{I}_r \implies \mathrm{G}^{-1} = \mathrm{I}_r$$

$$\implies \boldsymbol{\xi} = \begin{bmatrix} \langle \boldsymbol{w}_1, \boldsymbol{v} \rangle \\ \vdots \\ \langle \boldsymbol{w}_r, \boldsymbol{v} \rangle \end{bmatrix} \implies P_W(\boldsymbol{v}) = \sum_{i=1}^{n} \langle \boldsymbol{w}_i, \boldsymbol{v} \rangle \, \boldsymbol{w}_i$$

Task 52

If W is a subspace of a real inner product-space V, then the orthogonal complement of W is defined as $W^\perp = \{\boldsymbol{v} \in V \colon \langle \boldsymbol{v}, \boldsymbol{w} \rangle = 0 \forall \boldsymbol{w} \in W\}$

a) Let $\mathbb{R}^4$ have the euclidean inner product and assume that the subspace W is spanned by vectors $\{(0,1,-4,-1), (3,5,1,1)\}$. Express the vector $\boldsymbol{b} = (1,2,0,-2)$ in the form of $\boldsymbol{b} = \boldsymbol{b}_1 + \boldsymbol{b}_2$ where $\boldsymbol{b}_1 \in W$ and $\boldsymbol{b}_2 \in W^\perp$

b) Let W be the subspace of $\mathbb{R}^6$ spanned by

$$\begin{aligned} \boldsymbol{w}_1 &= (1,3,-2,0,2,0) \qquad & \boldsymbol{w}_2 &= (2,6,-5,-2,4,-3) \\ \boldsymbol{w}_3 &= (0,0,5,10,0,15) \qquad & \boldsymbol{w}_4 &= (2,6,0,8,4,18) \end{aligned}$$

Find a basis of $W^\perp$.

Subtask a): We begin by computing $P_W(\boldsymbol{b})$. Notice that $\boldsymbol{w}_1 \perp \boldsymbol{w}_2$, thus the gramian is of the following form:

$$\operatorname{Gram}(\boldsymbol{w}_1, \boldsymbol{w}_2) = \begin{bmatrix} 18 & 0 \\ 0 & 36 \end{bmatrix} \qquad \langle \boldsymbol{b}, \boldsymbol{w}_1 \rangle = 4 \qquad \langle \boldsymbol{b}, \boldsymbol{w}_2 \rangle = 3 + 10 - 2 = 11$$

$$\implies P_W(\boldsymbol{b}) = \frac{2}{9} \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix} + \frac{11}{36} \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \frac{8}{36} \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix} + \frac{11}{36} \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{36} \begin{bmatrix} 33 \\ 63 \\ -21 \\ 3 \end{bmatrix} = \boldsymbol{b}_1$$

To compute $P_{W^\perp}(\boldsymbol{b})$, we need a basis of $W^\perp$, thus we solve the following system of equations:

$$\begin{bmatrix} 0 & 1 & -4 & -1 \\ 3 & 5 & 1 & 1 \end{bmatrix} \boldsymbol{v} = \boldsymbol{0} \rightsquigarrow \begin{bmatrix} 3 & 5 & 1 & 1 \\ 0 & 1 & -4 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 0 & 21 & 6 \\ 0 & 1 & -4 & -1 \end{bmatrix}$$

$$\rightsquigarrow v_1 = -7v_3 - 2v_4 \qquad v_2 = 4v_3 + v_4$$

$$\implies W^\perp = \operatorname{span}\left(\begin{bmatrix} -7 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right) = \operatorname{span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$$

Now

$$\operatorname{Gram}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \begin{bmatrix} 66 & 18 \\ 18 & 6 \end{bmatrix} \implies \operatorname{Gram}(\boldsymbol{v}_1, \boldsymbol{v}_2)^{-1} = \frac{1}{72} \begin{bmatrix} 6 & -18 \\ -18 & 66 \end{bmatrix}$$

$$\langle \boldsymbol{v}_1, \boldsymbol{b} \rangle = 1 \qquad \langle \boldsymbol{v}_2, \boldsymbol{b} \rangle = -2$$

$$\mathrm{Gram}(\boldsymbol{v}_1, \boldsymbol{v}_2)^{-1} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 42 \\ -150 \end{bmatrix} \implies P_{W^\perp}(\boldsymbol{b}) = \frac{42}{72} \begin{bmatrix} -7 \\ 4 \\ 1 \\ 0 \end{bmatrix} - \frac{150}{72} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 6 \\ 18 \\ 42 \\ -150 \end{bmatrix} = \boldsymbol{b}_2$$

Computing $\boldsymbol{b}_1 + \boldsymbol{b}_2$:

$$\frac{1}{72} \begin{bmatrix} 66 \\ 126 \\ -42 \\ 6 \end{bmatrix} + \frac{1}{72} \begin{bmatrix} 6 \\ 18 \\ 42 \\ -150 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 72 \\ 144 \\ 0 \\ -144 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

Subtask b): We proceed similarly to subtask a):

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 0 & 0 & 0 & 8 & 0 & 18 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 9 \end{bmatrix}$$
$$\implies 4v_4 = -9v_6 \qquad v_3 = -2v_2 - 3v_6 = \frac{9}{2}v_6 - \frac{6}{2}v_6 = \frac{3}{2}v_6$$
$$v_1 = -3v_2 + 2v_3 - 2v_5 = -3v_2 + 3v_6 - 2v_5$$
$$\implies W^\perp = \mathrm{span}\left(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ \frac{3}{2} \\ -\frac{9}{4} \\ 0 \\ 1 \end{bmatrix} \right)$$