Exercise Sheet № 7

Given a matrix-representation $[\mathrm{A}]_B^B$, where $\mathrm{A} \in \mathbb{R}^{n\times n}$, in a basis B, we want to find $[\mathrm{A}]_C^C$ for another basis C. Then $[\mathrm{A}]_C^C = [\mathrm{I}]_B^C[\mathrm{A}]_B^B[\mathrm{I}]_C^B$, where $[\mathrm{I}]_C^B$ is the base-transition matrix from C to B. If C is the matrix corresponding to $[\mathrm{I}]_E^C$, where E is the canonical basis $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, and similarly B corresponds to $[\mathrm{I}]_E^B$, then $[\mathrm{I}]_C^B = [\mathrm{I}]_E^C[\mathrm{I}]_B^E$. Notice that $[\mathrm{I}]_B^E$ corresponds to B^{-1}. Thus

$$[\mathrm{A}]_C^C = \mathrm{B}\mathrm{C}^{-1}[\mathrm{A}]_B^B\mathrm{C}\mathrm{B}^{-1}$$

Task 35: Finding a corresponding matrix

A matrix A has eigenvalues -1 and 2 and associated eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix}1\\2\end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix}2\\3\end{bmatrix}$$

Find the matrix form of the linear transformation $L[\boldsymbol{u}] = \mathrm{A}\boldsymbol{u}$ in terms of

a) the canonical basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$
b) the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$
c) the basis

$$\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}\right\}$$

Recall the definition of a diagonalizable matrix A. We call A diagonalizable, iff there exists a regular matrix V and a diagonal matrix D, such that $\mathrm{A} = \mathrm{V}\mathrm{D}\mathrm{V}^{-1}$. We know from the lecture, that V has the eigenvector of A, and $\mathrm{D} = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where λ_i are the eigenvalues of A. Now we get $[\mathrm{A}]_E^E$ by simply computing the matrix product:

$$\mathrm{V} = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix} \implies \mathrm{V}^{-1} = \begin{bmatrix}-3 & 2\\2 & -1\end{bmatrix}$$
$$\implies \mathrm{A} = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}-3 & 2\\2 & -1\end{bmatrix} = \begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}\begin{bmatrix}3 & -2\\4 & -2\end{bmatrix} = \begin{bmatrix}11 & -6\\18 & -10\end{bmatrix}$$

Subtask b): Given $[\mathrm{A}]_E^E$, computing $[\mathrm{A}]_V^V$, where $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2\}$, is trivial, since:

$$[\mathrm{A}]_V^V = [\mathrm{I}]_E^V[\mathrm{A}]_E^E[\mathrm{I}]_V^E = \mathrm{V}^{-1}\mathrm{V}\mathrm{D}\mathrm{V}^{-1}\mathrm{V} = \mathrm{D}$$

Subtask c): Let

$$\mathrm{C} = \begin{bmatrix}1 & 3\\1 & 4\end{bmatrix} \implies \mathrm{C}^{-1} = \begin{bmatrix}4 & -3\\-1 & 4\end{bmatrix}$$

Hence

$$[\mathrm{A}]_C^C = [\mathrm{I}]_E^C[\mathrm{A}]_E^E[\mathrm{I}]_C^E = \mathrm{C}^{-1}\mathrm{A}\mathrm{C} = \begin{bmatrix}4 & -3\\-1 & 4\end{bmatrix}\begin{bmatrix}11 & -6\\16 & -10\end{bmatrix}\begin{bmatrix}1 & 3\\1 & 4\end{bmatrix} = \begin{bmatrix}-4 & -6\\3 & 5\end{bmatrix}$$

Task 36: Diagonalization

a) Diagonalize the following matrices

$$\mathrm{A} = \begin{bmatrix} 3 & -9 \\ 2 & -6 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 2 & 5 & 5 \\ 0 & 2 & 0 \\ 0 & -5 & -3 \end{bmatrix}$$

b) Find a real matrix M with $\mathrm{spec}(\mathrm{M}) = \{-1, 3\}$ and eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c) Find a real matrix M with $\mathrm{spec}(\mathrm{M}) = \{0, 2, -2\}$ and eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Subtask a):

$$\chi_{\mathrm{A}}(\lambda) = \begin{vmatrix} \lambda - 3 & 9 \\ -2 & \lambda + 6 \end{vmatrix} = (\lambda - 3)(\lambda + 6) + 18 = \lambda^2 + 3\lambda - 18 + 18 = \lambda(\lambda + 3)$$

Hence $\lambda_1 = 0$ and $\lambda_2 = -3$. Next we find $\ker(\mathrm{A})$:

$$\begin{bmatrix} 3 & -9 \\ 2 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \implies v_1 = 3v_2 \implies \boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

And $\ker(-3\mathrm{I}_2 - \mathrm{A})$:

$$-3\mathrm{I}_2 - \mathrm{A} = \begin{bmatrix} -6 & 9 \\ -2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \implies 2v_1 = 3v_2 \implies \boldsymbol{v}_2 = \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

Now:

$$\mathrm{V} = \begin{bmatrix} 3 & \frac{3}{2} \\ 1 & 1 \end{bmatrix} \implies \mathrm{V}^{-1} = \frac{2}{3}\begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 3 \end{bmatrix}$$
$$\mathrm{A} = \mathrm{V}\begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}\mathrm{V}^{-1}$$

For B:

$$\chi_{\mathrm{B}}(\lambda) = \begin{vmatrix} \lambda - 2 & -5 & -5 \\ 0 & \lambda - 2 & 0 \\ 0 & 5 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 2)(\lambda + 3)$$

Hence $\lambda_{1,2} = 2$ and $\lambda_3 = -3$.

$$2\mathrm{I} - \mathrm{B} = \begin{bmatrix} 0 & -5 & -5 \\ 0 & 0 & 0 \\ 0 & 5 & 5 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\implies v_2 = -v_3, v_1 \in \mathbb{R}$$

Notice that $\dim(\mathrm{im}(2\mathrm{I} - \mathrm{B})) = 1$, thus $\dim(\ker(2\mathrm{I} - \mathrm{B})) = 2$, hence we choose two linearly independent eigenvectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

For λ_3:

$$-3\mathrm{I}-\mathrm{B} = \begin{bmatrix} -5 & -5 & -5 \\ 0 & -5 & 0 \\ 0 & 5 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \implies v_2 = 0, v_1 = -v_3 \implies \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Now:

$$\mathrm{V} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \implies \mathrm{V}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathrm{B} = \mathrm{V} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \mathrm{V}^{-1}$$

Subtask b):

$$\mathrm{V} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \implies \mathrm{V}^{-1} = -\frac{1}{3} \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$$

$$\mathrm{M} = -\frac{1}{3} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}$$

Subtask c):

$$\mathrm{V} = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \implies \mathrm{V}^{-1} = \frac{1}{2} \begin{bmatrix} 4 & 6 & -2 \\ 3 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\mathrm{M} = \frac{1}{2} \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 4 & 6 & -2 \\ 3 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 6 & -2 \\ -2 & -2 & 0 \\ 6 & 6 & -4 \end{bmatrix}$$

Task 37: Complex Vectors

a) Let

$$\boldsymbol{u} = (1+\mathrm{i}, \mathrm{i}, 3-\mathrm{i}) \qquad \boldsymbol{v} = (1+\mathrm{i}, 2, 4\mathrm{i})$$

Find $\langle \boldsymbol{v}, \boldsymbol{u} \rangle$, $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, $\|\boldsymbol{u}\|$ and $\|\boldsymbol{v}\|$

b) Let

$$\boldsymbol{u} = (1+\mathrm{i}, 4, 3\mathrm{i}) \qquad \boldsymbol{v} = (3, -4\mathrm{i}, 2+3\mathrm{i}) \qquad \boldsymbol{w} = (1-\mathrm{i}, 4\mathrm{i}, 4-5\mathrm{i})$$

Compute $\overline{\langle \boldsymbol{u}, \overline{\boldsymbol{v}} \rangle - \overline{\langle \boldsymbol{w}, \boldsymbol{u} \rangle}}$ and $\overline{\langle \mathrm{i}\overline{\boldsymbol{u}}, \boldsymbol{w} \rangle + \langle \|\boldsymbol{u}\| \boldsymbol{v}, \boldsymbol{u} \rangle}$

Subtask a): Recall how the inner product on a complex vectorspace is defined:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^{\mathrm{t}} \overline{\boldsymbol{y}} \qquad \langle \boldsymbol{x}, \boldsymbol{y} \rangle = \overline{\langle \boldsymbol{y}, \boldsymbol{x} \rangle}$$

Hence:

$$\begin{aligned}
&\langle \boldsymbol{u}, \boldsymbol{v} \rangle = (1+\mathrm{i})(1-\mathrm{i}) + 2\mathrm{i} - (3-\mathrm{i})4\mathrm{i} = 2 + 2\mathrm{i} - 12\mathrm{i} - 4 = -2 - 10\mathrm{i} \\
&\implies \langle \boldsymbol{v}, \boldsymbol{u} \rangle = -2 + 10\mathrm{i} \\
&\|\boldsymbol{u}\| = \sqrt{\langle \boldsymbol{u}, \boldsymbol{u} \rangle} = \sqrt{(1+\mathrm{i})(1-\mathrm{i}) + \mathrm{i}(-\mathrm{i}) + (3-\mathrm{i})(3+\mathrm{i})} = \sqrt{2+1+10} = \sqrt{13} \\
&\|\boldsymbol{v}\| = \sqrt{(1+\mathrm{i})(1-\mathrm{i}) + 4 + 4\mathrm{i}(-4\mathrm{i})} = \sqrt{2+4+16} = \sqrt{22}
\end{aligned}$$

Subtask b):

$$\langle \boldsymbol{u}, \overline{\boldsymbol{v}} \rangle = \boldsymbol{u}^{\mathrm{t}} \boldsymbol{v} = (1+\mathrm{i})3 - 16\mathrm{i} + 3\mathrm{i}(2+3\mathrm{i}) = 3 + 3\mathrm{i} - 16\mathrm{i} + 6\mathrm{i} - 9$$

$$\begin{aligned}&= -6 - 7\mathrm{i}\\ \langle \boldsymbol{u}, \boldsymbol{w}\rangle &= \boldsymbol{u}^{\mathrm{t}}\overline{\boldsymbol{w}} = (1+\mathrm{i})(1+\mathrm{i}) - 16\mathrm{i} + 3\mathrm{i}(4+5\mathrm{i}) = 1 + 2\mathrm{i} - 1 - 16\mathrm{i} + 12\mathrm{i} - 15\\ &= -15 - 2\mathrm{i}\\ &\implies \overline{\langle \boldsymbol{u}, \overline{\boldsymbol{v}}\rangle - \overline{\langle \boldsymbol{w}, \boldsymbol{u}\rangle}} = \overline{-6 - 7\mathrm{i} + 15 + 2\mathrm{i}} = \overline{9 - 5\mathrm{i}} = 9 + 5\mathrm{i}\end{aligned}$$

Zu $\overline{\langle \overline{\mathrm{i}\boldsymbol{u}}, \boldsymbol{w}\rangle + \langle \|\boldsymbol{u}\|\boldsymbol{v}, \boldsymbol{u}\rangle}$:

$$\begin{aligned}&\|\boldsymbol{u}\| = \sqrt{(1+\mathrm{i})(1-\mathrm{i}) + 16 + 9} = \sqrt{27} = 3\sqrt{3}\\ &\langle \|\boldsymbol{u}\|\boldsymbol{v}, \boldsymbol{u}\rangle = 3\sqrt{3}(3(1-\mathrm{i}) - 16\mathrm{i} - 3\mathrm{i}(2+3\mathrm{i}))\\ &= 3\sqrt{3}(3 - 3\mathrm{i} - 16\mathrm{i} - 6\mathrm{i} + 9) = 3\sqrt{3}(12 - 25\mathrm{i}) = \sqrt{3}(36 - 75\mathrm{i})\\ &\mathrm{i}\boldsymbol{u} = (\mathrm{i} - 1, 4\mathrm{i}, -3) \implies \overline{\mathrm{i}\boldsymbol{u}} = (-1 - \mathrm{i}, -4\mathrm{i}, -3)\\ &\implies \langle \overline{\mathrm{i}\boldsymbol{u}}, \boldsymbol{w}\rangle = \overline{\mathrm{i}\boldsymbol{u}}^{\mathrm{t}}\overline{\boldsymbol{w}} = (-1-\mathrm{i})(1+\mathrm{i}) - 4\mathrm{i}\cdot(-4\mathrm{i}) - 3(4+5\mathrm{i})\\ &= -1 - \mathrm{i} - \mathrm{i} + 1 - 16 - 12 - 15\mathrm{i} = -28 - 17\mathrm{i}\\ &\implies \overline{\langle \overline{\mathrm{i}\boldsymbol{u}}, \boldsymbol{w}\rangle + \langle \|\boldsymbol{u}\|\boldsymbol{v}, \boldsymbol{u}\rangle} = -28 + 17\mathrm{i} + \sqrt{3}(36 + 75\mathrm{i})\end{aligned}$$

Task 38

Let A be a real 2×2 matrix with complex eigenvalues $\lambda \in \mathbb{C}\setminus\mathbb{R}$. If $\boldsymbol{x} = \mathfrak{Re}(\boldsymbol{x}) + \mathrm{i}\mathfrak{Im}(\boldsymbol{x})$ is an eigenvector of A corresponding to $\lambda = a + \mathrm{i}b$, prove that $\mathrm{P} = \begin{bmatrix}\mathfrak{Re}(\boldsymbol{x}) & \mathfrak{Im}(\boldsymbol{x})\end{bmatrix}$ is regular and

$$\mathrm{A} = \mathrm{P}\underbrace{\begin{bmatrix} a & -b\\ b & a\end{bmatrix}}_{=\mathrm{D}}\mathrm{P}^{-1}$$

Since $\mathrm{A} \in \mathbb{R}^{2\times 2}$, we know that $\overline{\boldsymbol{x}}$ is an eigenvector for $\overline{\lambda}$. We know that eigenvectors of distinct eigenvalues are linearly independent, thus

$$\mathfrak{Re}(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{x} + \overline{\boldsymbol{x}}) \qquad \mathfrak{Im}(\boldsymbol{x}) = \frac{1}{2\mathrm{i}}(\boldsymbol{x} - \overline{\boldsymbol{x}})$$

are also linearly independent, therefore P is regular. Now let $\boldsymbol{x} = \boldsymbol{v} + \mathrm{i}\boldsymbol{w}$, where $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$:

$$\mathrm{A}\boldsymbol{v} + \mathrm{i}\mathrm{A}\boldsymbol{w} = \mathrm{A}\boldsymbol{x} = (a + \mathrm{i}b)(\boldsymbol{v} + \mathrm{i}\boldsymbol{w}) = (a\boldsymbol{v} - b\boldsymbol{w}) + \mathrm{i}(a\boldsymbol{w} - b\boldsymbol{v})$$

Now:

$$\begin{gathered}\mathrm{A}\mathfrak{Re}(\boldsymbol{x}) = a\boldsymbol{v} - b\boldsymbol{w} \qquad \mathrm{A}\mathfrak{Im}(\boldsymbol{x}) = a\boldsymbol{w} - b\boldsymbol{v}\\ \begin{bmatrix}\boldsymbol{v} & \boldsymbol{w}\end{bmatrix}\begin{bmatrix} a & -b\\ b & a\end{bmatrix} = \begin{bmatrix} a\boldsymbol{v} + b\boldsymbol{w} & a\boldsymbol{w} - b\boldsymbol{v}\end{bmatrix}\end{gathered}$$

Thus $\mathrm{AP} = \mathrm{PD} \iff \mathrm{A} = \mathrm{PDP}^{-1}$.

Task 39

Suppose A has eigenvalue λ and eigenvector $\boldsymbol{v}$.

a) Let $\boldsymbol{b}$ be any vector. Prove that the matrix $\mathrm{B} = \mathrm{A} - \boldsymbol{v}\boldsymbol{b}^{\mathrm{t}}$ also has $\boldsymbol{v}$ as an eigenvector.

b) Prove that if $\mu \neq \lambda - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}$ is any other eigenvalue of A, then it is also an eigenvalue of B

Subtask a):

$$\mathrm{B}\boldsymbol{v} = \mathrm{A}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v} = \lambda\boldsymbol{v} - \boldsymbol{v}\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v} = \boldsymbol{v}(\lambda - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{v})$$

Subtask b): Let μ_i be the eigenvalues of A and wlog $\mu_1 = \lambda$, ν_i be the eigenvalues of B and wlog $\nu_1 = \lambda - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}$:

$$\mathrm{trace}(\mathrm{B}) = \sum_{i=1}^{n}\nu_i = \sum_{i=1}^{n}(a_{ii} - b_i v_i) = \sum_{i=1}^{n} a_{ii} - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{v} = \lambda - \boldsymbol{b}^{\mathrm{t}}\boldsymbol{v} + \sum_{i=2}^{n}\mu_i$$

Alternative method: Let $\boldsymbol{w}$ be an eigenvector of A corresponding to eigenvalue μ, now

$$\begin{aligned}\mathrm{B}(\boldsymbol{w}+c\boldsymbol{v}) &= \mathrm{A}\boldsymbol{w}+c\mathrm{A}\boldsymbol{v}-\boldsymbol{v}\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}-c\boldsymbol{v}\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}\\ &= \mu\boldsymbol{w}+\boldsymbol{v}(c\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}-c\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}) = \mu\boldsymbol{w}+\boldsymbol{v}(c(\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v})-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w})\end{aligned}$$

Choosing $c=\frac{\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}}{\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu}$ yields

$$\begin{aligned}\mathrm{B}(\boldsymbol{w}+c\boldsymbol{v}) &= \mu\boldsymbol{w}+\boldsymbol{v}\left(\frac{\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}(\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v})}{\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu}-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}\right)\\ &= \mu\boldsymbol{w}+\boldsymbol{v}\left(\frac{\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}(\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v})-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}(\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu)}{\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu}\right)\\ &= \mu\boldsymbol{w}+\boldsymbol{v}\left(\frac{\mu\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}}{\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu}\right) = \mu\boldsymbol{w}+\mu\boldsymbol{v}\frac{\boldsymbol{b}^{\mathrm{t}}\boldsymbol{w}}{\lambda-\boldsymbol{b}^{\mathrm{t}}\boldsymbol{v}-\mu} = \mu\boldsymbol{w}+c\mu\boldsymbol{v}\end{aligned}$$

Task 40: Simultaneously Diagonalizable

Two matrices $\mathrm{A},\mathrm{B}\in\mathbb{R}^{n\times n}$ are said to be simultaneously diagonalizable, if there exists a regular matrix S, such that both $\mathrm{S}^{-1}\mathrm{AS}$ and $\mathrm{S}^{-1}\mathrm{BS}$ are diagonal.

a) Show that if $\mathrm{A},\mathrm{B}\in\mathbb{R}^{n\times n}$ are simultaneously diagonalizable, then they commute
b) Assume that one of the matrices $\mathrm{A},\mathrm{B}\in\mathbb{R}^{n\times n}$ has no multiple eigenvalues, and A and B commute. Prove that A and S are simultaneously diagonalizable.

Subtask a): Let $\mathrm{spec}(\mathrm{A})=\{\lambda_1,\ldots,\lambda_n\}$ and $\mathrm{spec}(\mathrm{B})=\{\mu_1,\ldots,\mu_n\}$, then:

$$\begin{aligned}&\mathrm{A}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\lambda_i)\mathrm{S}^{-1}\qquad \mathrm{B}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{S}^{-1}\\ &\implies \mathrm{AB}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\lambda_i)\mathrm{S}^{-1}\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{S}^{-1}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\lambda_i)\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{S}^{-1}\\ &=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\lambda_i\mu_i)\mathrm{S}^{-1}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i\lambda_i)\mathrm{S}^{-1}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{diag}_{i=1}^{n}(\lambda_i)\mathrm{S}^{-1}\\ &=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{S}^{-1}\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\lambda_i)\mathrm{S}^{-1}=\mathrm{BA}\end{aligned}$$

Subtask b): Let $E_i=\ker(\lambda_i\mathrm{I}-\mathrm{A})$ be the eigenspaces of A. Since A only has eigenvalues of algebraic multiplicity 1, we know that $\forall i=1,\ldots,n\colon \dim(E_i)=1$. Furthermore $\forall\boldsymbol{v}\in E_i\colon \mathrm{A}\boldsymbol{v}\in E_i$, i.e. $\mathrm{A}E_i\subseteq E_i$. If $\mathrm{B}E_i\subseteq E_i$ for all i, then B is diagonalizable, as it's eigenspaces all have dimension one. Notice that $\mathrm{B}\boldsymbol{v}_i\in E_i\iff \mathrm{AB}\boldsymbol{v}_i\in E_i$ or rather $\mathrm{AB}\boldsymbol{v}_i=\lambda_i\mathrm{B}\boldsymbol{v}_i$, since $\forall\boldsymbol{v}\in E_i\colon \mathrm{A}\boldsymbol{v}=\lambda_i\boldsymbol{v}$:

$$\mathrm{AB}\boldsymbol{v}_i=\mathrm{BA}\boldsymbol{v}_i=\lambda_i\mathrm{B}\boldsymbol{v}_i$$

Now we showed that the eigenspaces F_i of B are identical to E_i, therefore B has n distinct, one-dimensional eigenspaces, thus B has the same-eigenvectors as A and therefore $\mathrm{B}=\mathrm{S}\,\mathrm{diag}_{i=1}^{n}(\mu_i)\mathrm{S}^{-1}$, i.e. A and B are simultaneously diagonalizable.