Aufgabe 53. *Gegeben sind einige Abbildungen, welche auf Linearität überprüft werden sollen.*

Zu (a): Seien $\lambda, \mu \in \mathbb{R}$ und $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$
\begin{aligned}
&f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = \sqrt{(\lambda x_1 + \mu x)^2 + (\lambda y_2 + \mu y_2)^2} \\
&= \sqrt{\lambda^2(x_1^2 + y_1^2) + \mu^2(x_2^2 + y_2^2) + 2(\lambda x_1 x_2 + \mu y_1 y_2)} \\
&\lambda f(x_1, y_1) + \mu f(x_2, y_2) = \lambda\sqrt{x_1^2 + y_1^2} + \mu\sqrt{x_2^2 + y_2^2} \\
&\Rightarrow f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) \neq \lambda f(x_1, y_1) + \mu f(x_2, y_2)
\end{aligned}
$$

Zu (b): Seien $\lambda, \mu \in \mathbb{R}$ und $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$
\begin{aligned}
&f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = (\lambda x_1 + \mu x_2)(\lambda y_1 + \mu y_2) \\
&= \lambda^2 x_1 y_1 + \lambda\mu x_1 y_2 + \lambda\mu x_2 y_1 + \mu^2 x_2 y_2 \\
&\lambda f(x_1, y_1) + \mu f(x_2, y_2) = \lambda x_1 y_2 + \mu x_2 y_2 \\
&\Rightarrow f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) \neq \lambda f(x_1, y_1) + \mu f(x_2, y_2)
\end{aligned}
$$

Zu (c): Seien $\lambda, \mu \in \mathbb{R}$ und $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$

$$
\begin{aligned}
&f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = (\lambda x_1 + \mu x_2, \lambda x_1 + \mu x_2) \\
&\lambda f(x_1, y_1) + \mu f(x_2, y_2) = \lambda(x_1, x_1) + \mu(x_2, x_2) = (\lambda x_1, \lambda x_1) + (\mu x_2, \mu x_2) = (\lambda x_1 + \mu x_2, \lambda x_1 + \mu x_2) \\
&\Rightarrow f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = \lambda f(x_1, y_1) + \mu f(x_1, y_1)
\end{aligned}
$$

Zu (d): Seien $\lambda, \mu \in \mathbb{Z}_3$ und $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_3^2$

Da $[0]_3^3 = [0]_3 \cdot [0]_3 \cdot [0]_3 = [0 \cdot 0 \cdot 0]_3 = [0]_3$ und $[1]^3 = [1]_3$ und $[2]_3^3 = [8]_3 = [2]_3$ können wir f umschreiben als $f(x_1, x_2) = x_1 + x_2$:

$$
\begin{aligned}
&f(\lambda x_1 + \mu x_1, \lambda y_1 + \mu y_2) = \lambda x_1 + \mu x_2 + \lambda y_1 + \mu y_2 = \lambda(x_1 + y_1) + \mu(x_2 + y_2) \\
&\lambda f(x_1, y_1) + \mu f(x_2, y_2) = \lambda(x_1 + y_1) + \mu(x_2 + y_2) \\
&\Rightarrow f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) = \lambda f(x_1, y_1) + \mu f(x_2, y_2)
\end{aligned}
$$

Zu (e) und (f) siehe Übungsblatt 7.

Zu (e): Seien $\lambda, \mu \in \mathbb{R}$ und $z_1, z_2 \in \mathbb{C}$

$$
\begin{aligned}
&f(\lambda z_1 + \mu z_2) = \overline{\lambda z_1 + \mu z_2} = \overline{\lambda z_1} + \overline{\mu z_2} = \lambda\overline{z_1} + \mu\overline{z_2} \\
&\lambda f(z_1) + \mu f(z_2) = \lambda\overline{z_1} + \mu\overline{z_2} \\
&\Rightarrow f(\lambda z_1 + \mu z_2) = \lambda f(z_1) + \mu f(z_2)
\end{aligned}
$$

Zu (f): Seien $\lambda, \mu \in \mathbb{C}$ und $z_1, z_2 \in \mathbb{C}$

$$
\begin{aligned}
&f(\lambda z_1 + \mu z_2) = \overline{\lambda z_1 + \mu z_2} = \overline{\lambda z_1} + \overline{\mu z_2} = \overline{\lambda} \cdot \overline{z_1} + \overline{\mu} \cdot \overline{z_2} \\
&\lambda f(z_1) + \mu f(z_2) = \lambda\overline{z_1} + \mu\overline{z_2} \\
&\Rightarrow f(\lambda z_1 + \mu z_2) \neq \lambda f(z_1) + \mu f(z_2)
\end{aligned}
$$

Seien $p_1(x) = \sum_{k=0}^{n_1} a_k x^k$ und $p_2(x) = \sum_{k=0}^{n_2} b_k x^k$, mit $p_1, p_2 \in \mathbb{R}[x]$ und $\lambda, \mu \in \mathbb{R}$. Dann ist $\lambda p_1(x) = \sum_{k=0}^{n_1} \lambda a_k x^k$ und $\mu p_2(x) = \sum_{k=0}^{n_1} \mu b_k x^k$ und weiter:

$$
\begin{aligned}
&\lambda p(x - x_0) = \sum_{k=0}^{n_1} (x - x_0)^k \lambda a_k \\
&\lambda p_1(x - x_0) + \mu p_2(x - x_0) = \sum_{k=0}^{\max(n_1, n_2)} (x - x_0)^k (\lambda\alpha_k + \mu\beta_k) = \sigma(x - x_0) \\
&\alpha_k = \begin{cases} a_k & k \leq n_1 \\ 0 & k > n_1 \end{cases} \qquad \beta_k = \begin{cases} b_k & k \leq n_2 \\ 0 & k > n_2 \end{cases}
\end{aligned}
$$

Zu (g):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = \sigma(x-1) = \lambda p_1(x-1)+\mu p_2(x-1)\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = \lambda p_1(x-1)+\mu p_2(x-1)\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) = \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$

Zu (h):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = \sigma(x)-1 = \lambda p_1(x)+\mu p_2(x)-1\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = \lambda(p_1(x)-1)+\mu(p_2(x)-1)\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) \neq \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$

Zu (i):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = \sigma(x)-\sigma(1)\\
&= \lambda p_1(x)+\mu p_2(x)-\lambda p_1(1)-\mu p_2(1) = \lambda(p_1(x)-p_1(1))+\mu(p_2(x)-p_2(1))\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = \lambda(p_1(x)-p_1(1))+\mu(p_2(x)-p_2(1))\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) = \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$

Zu (j):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = x^2\sigma(x) = x^2(\lambda p_1(x)+\mu p_2(x))\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = x^2\lambda p_1(x)+x^2\mu p_2(x) = x^2(\lambda p_1(x)+\mu p_2(x))\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) = \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$

Zu (k):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = \sigma(x^2) = \lambda p_1(x^2)+\mu p_2(x^2)\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = \lambda p_1(x^2)+\mu p_2(x^2)\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) = \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$

Zu (l):

$$\begin{aligned}
&f(\lambda p_1(x)+\mu p_2(x)) = f(\sigma(x)) = \sigma^2(x) = \lambda^2 p_1^2(x)+\mu^2 p_2^2(x)+2\lambda\mu p_1(x)p_2(x)\\
&\lambda f(p_1(x)) + \mu f(p_2(x)) = \lambda p_1^2(x)+\mu p_2^2(x)\\
&\Rightarrow f(\lambda p_1(x)+\mu p_2(x)) \neq \lambda f(p_1(x)) + \mu f(p_2(x))
\end{aligned}$$