Exercise Sheet № 10

Task 10.1

Find an example of a non-empty, open, convex subset C of a real normed vectorspace X and $x_0 \in X \setminus C$, such that there doesn't exist $x^* \in X^*$ with

$$\sup_{x \in C} \langle x^*, x \rangle < \langle x^*, x_0 \rangle$$

Is this a contradiction to the separation theorem of Hahn-Banach?

We set $X = \mathbb{R}$ and $C = (0,1)$. Notice that trivially, C is non-empty, open and convex considering the standard topology on $\mathbb{R}$ induced by the norm $|\cdot| \colon R \to \mathbb{R}_0^+$. Let $(x_k)_{k\in\mathbb{N}} \in \mathfrak{c}_0(C)$, $x_0 = 0$ and $x^* \in \mathbb{R}^+$ be arbitrary. If we set $y_k = \langle x^*, x_k \rangle$, then notice that $\lim_{k\to\infty} y_k = 0$ and hence

$$\sup_{x \in C} \langle x^*, x \rangle \geq 0$$

However, notice that $\langle x^*, x_0 \rangle = 0$. Now

$$\sup_{x \in C} \langle x^*, x \rangle \geq 0 < 0 = \langle x^*, x_0 \rangle$$

is a false statement for arbitrary $x^* \in \mathbb{R}^*$, hence we proved the statement.

Recall the statement of Hahn-Banach for a real normed space X, $A \subseteq X$ non-empty, open and convex, with $x_0 \in X \setminus A$. Then there exists $x^* \in X^*$ such that

$$\forall x \in A\colon \ \langle x^*, x \rangle_X < \langle x^*, x_0 \rangle_X$$

Our example isn't a contradiction to this statement, since we can take $x^* \in \mathbb{R}^*$ such that $\langle x^*, x \rangle < 0$, then $\langle x^*, x \rangle < 0 = \langle x^*, x_0 \rangle$. Only in the limit does this statement fail. Let $x^* = -1$ and $x_k = \frac{1}{k+1} \xrightarrow{k\to\infty} 0$. Then $(x_k)_{k\in\mathbb{N}} \in \mathfrak{c}_0(C)$. Notice that $\langle x^*, x_K \rangle = -\frac{1}{k+1} \xrightarrow{k\to\infty} 0$ but $\langle x^*, x_k \rangle < 0$.

Task 10.2

We consider the action of taking the limit on $\mathfrak{c}(\mathbb{R})$ by defining $f \colon \mathfrak{c}(\mathbb{R}) \to \mathbb{R}$ with $(x_k)_{k\in\mathbb{N}} \mapsto \lim_{k\to\infty} x_k$. We equip $\mathfrak{c}(\mathbb{R})$ with $\|\cdot\|_\infty$.

i) Prove $f \in \mathfrak{c}(\mathbb{R})^*$ and find $\|f\|_{\mathfrak{c}(R)^*}$

Subtask i): We first prove that f is linear. Let $x, y \in \mathfrak{c}(\mathbb{R})$ with $x = (\xi_k)_{k\in\mathbb{N}}$ and $y = (\eta_k)_{k\in\mathbb{N}}$, and $\lambda, \mu \in \mathbb{R}$:

$$f(\lambda x + \mu y) = \lim_{k\to\infty} \lambda \xi_k + \mu \eta_k = \lambda \lim_{k\to\infty} \xi_k + \mu \lim_{k\to\infty} \eta_k = \lambda f(x) + \mu f(y)$$

If f is bounded, then f is also continuous. Let $(x_k)_{k\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$, then $0 \leq \|x\|_\infty = M < \infty$. Now:

$$|f(x)| = \left| \lim_{k\to\infty} x_k \right| \leq M = \|x\|_\infty$$

Thus f is bounded and therefore continuous. At last, let $\xi_k = 1$, then $(\xi_k)_{k\in\mathbb{N}} \in \mathfrak{c}(\mathbb{R})$ and $\|\xi\|_k = 1$. Hence

$$\|f\|_{\mathfrak{c}(\mathbb{R})^*} = \sup_{\|x\|_\infty = 1} f(x) = 1$$