Exercise Sheet № 6

Task 6.1: Operator Norm

Let $T \in L(X,Y)$. Prove the following

i)

$$\|T\|_O = \sup_{\substack{x\in X \\ \|x\|_X<1}} \|Tx\|_Y = \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X}$$

ii) Ist $X \neq \{0\}$, gilt

$$\|T\|_O = \sup_{\substack{x\in X \\ \|x\|_X=1}} \|Tx\|_Y$$

iii)

$$\|T\|_O = \inf\{\alpha \in (0,\infty) \colon \forall x \in X \colon \|Tx\|_Y \leq \alpha \|x\|_X\}$$

iv) $\forall x \in X \colon \|Tx\|_Y \leq \|T\|_O \cdot \|x\|_X$

Subtask i): Let $U_X = U_1(0)$.

$$\|T\|_O = \sup_{x\in U_X} \|Tx\|_Y = \sup_{\|x\|\leq 1} \|Tx\|_Y \geq \sup_{\|x\|=1} \|Tx\|_Y = \sup_{x\neq 0} \left\| T\frac{x}{\|x\|_X} \right\|_Y \geq \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

$$\|T\|_O = \sup_{\|x\|_X\leq 1} \|Tx\|_Y \leq \sup_{0<\|x\|\leq 1} \frac{1}{\|x\|_X}\|Tx\|_X \leq \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

Subtask iii): Let $\alpha \in (0,\infty)$, such that $\forall x \in X \colon \|Tx\|_Y \leq \alpha\|x\|_X$. For $x = 0$ it follows $0 \leq \alpha \cdot 0$. Let $x \neq 0$, then

$$\forall x \in X\setminus\{0\} \colon \frac{\|Tx\|_Y}{\|x\|_X} \leq \alpha \implies \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} \leq \alpha \implies \|T\|_O \leq \alpha$$

Hence:

$$\inf\{\alpha \in (0,\infty) \colon \forall x \in X \colon \|Tx\|_Y \leq \alpha\|x\|_X\} = \inf\{\alpha \in (0,\infty) \colon \|T\|_O \leq \alpha\} = \|T\|_O$$

Task 6.2: Continuity of Operators

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed, real, vectorspaces. Define $G \colon L(X,Y) \times X \to Y$ with $(T,x) \mapsto Tx$. Prove that G is continuous.

Task 6.3

We define $T \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$, where $Tx \colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ with $t \mapsto tx(t)$.

i) Prove T is a linear continuous operator
ii) Compute $\|T\|$
iii) Prove or disprove that the image of T is closed

Subtask i: For linearity, let $x, y \in \mathcal{C}([0,1])$ and $\lambda, \mu \in \mathbb{R}$:

$$T(\lambda x + \mu y) = t(\lambda x(t) + \mu y(t)) = \lambda t x(t) + \mu t y(t) = \lambda T(x) + \mu T(y)$$

Let $(x_n)_{n\in\mathbb{N}}$ be convergent with $x \in \mathcal{C}([0,1])$:

$$\|Tx_n - Tx\| = \sup_{t\in[0,1]} |t(x(t) - x_n(t))| = \sup_{t\in[0,1]} |t| \cdot |x(t) - x_n(t)| \leq \sup_{t\in[0,1]} \|x(t) - x_n(t)\| < \varepsilon$$

Thus $\lim_{n\to\infty} Tx_n = Tx$ and henceforth T is continuous.

Subtask ii: Let $x \in \mathcal{C}([0,1])$ with $\|x\|_\infty = 1$. We propose that $\|T\| = 1$. Notice that $\|T\| \leq 1$, since for $x \in \mathcal{C}([0,1])$ with $\|x\|_\infty = 1$, we have $\|Tx\|_\infty \leq \|t\|_\infty = 1$. For $x = 1$ we found a vectors such that $\|Tx\|_\infty = 1$.