Task 1: Matrix representation of linear transformations

Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If $\boldsymbol{e}_1, \dots, \boldsymbol{e}_n$ are the canonical basis vectors of $\mathbb{R}^n$, then the standard matrix of T is given by

$$\mathrm{S} = \begin{bmatrix} T(\boldsymbol{e}_1) & T(\boldsymbol{e}_2) & \cdots & T(\boldsymbol{e}_n) \end{bmatrix}$$

a) Find the standard matrix for the following linear transformation:

$$T(x_1, x_2) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

b) Let $T_\theta \in \mathrm{End}(\mathbb{R}^2)$ be the linear transformation that rotates $\boldsymbol{x} \in \mathbb{R}^2$ by the angle θ around the origin. Find the standard matrix of T.

Subtask a:

$$T(\boldsymbol{e}_1) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad T(\boldsymbol{e}_2) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \implies \mathrm{S} = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

Subtask b: Let $f\colon \mathbb{R}^2 \to \mathbb{C}$ with

$$f(\boldsymbol{x}) = f(x_1, x_2) = x_1 + \mathrm{i}x_2$$

We see that f is bijective on $\mathbb{R}^2$, hence $\mathbb{C}$ and $\mathbb{R}^2$ are isomorphic under f with $f^{-1}(x_1 + \mathrm{i}x_2) = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{t}}$. Let $z \in \mathbb{C}$, then z has a polar-form $z = |z| \cdot \mathrm{e}^{\mathrm{i}\varphi}$. We want to show, that $f^{-1}(z)$ has angle θ to the x-axis. The standard-dot product in $\mathbb{R}^n$ has the following property:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\| \cdot \cos(\varphi)$$

We are interested in the angle of $f^{-1}(r\mathrm{e}^{\mathrm{i}\theta})$ to $\boldsymbol{e}_1$:

$$r\cos(\theta) = r\sqrt{\cos^2(\theta) + \sin^2(\theta)}\cos(\varphi) \iff \cos(\theta) = \cos(\varphi)$$

Since $\varphi \in [-\pi, \pi]$, we get $\theta = \varphi$. Notice that for $\zeta = \mathrm{e}^{\mathrm{i}\theta}$, we get:

$$z \cdot \zeta = |z|\mathrm{e}^{\mathrm{i}\varphi}\mathrm{e}^{\mathrm{i}\theta} = |z|\mathrm{e}^{\mathrm{i}(\varphi+\theta)}$$

Hence $z \in \mathbb{C}\colon |z| = 1$ act on $\mathbb{C}$ like a rotation of θ around the origin. We need R_θ to be linear. Let $z_1, z_2 \in \mathbb{C}$ and $\lambda, \mu \in \mathbb{R}$, then:

$$R_\theta(\lambda z_1 + \mu z_2) = (\lambda z_1 + \mu z_2)\mathrm{e}^{\mathrm{i}\theta} = \lambda z_1 \mathrm{e}^{\mathrm{i}\theta} + \mu z_2 \mathrm{e}^{\mathrm{i}\theta} = \lambda R_\theta(z_1) + \mu R_\theta(z_2)$$

Now let $R_\theta(z) = z\mathrm{e}^{\mathrm{i}\theta}$, then $\mathrm{S}_\theta = \begin{bmatrix} f^{-1}(R_\theta(f(\boldsymbol{e}_1))) & f^{-1}(R_\theta(f(\boldsymbol{e}_2))) \end{bmatrix}$

$$f^{-1}(R_\theta(f(\boldsymbol{e}_1))) = f^{-1}(\mathrm{e}^{\mathrm{i}\theta}) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad f^{-1}(R_\theta(f(\boldsymbol{e}_2))) = f^{-1}(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$$\implies \mathrm{S}_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, tail, two heads, "T_\theta"] \arrow[d, tail, two heads, "f"'] & \mathbb{R}^2 \\ \mathbb{C} \arrow[r, tail, two heads, "R_\theta"'] & \mathbb{C} \arrow[u, tail, two heads, "f^{-1}"'] \end{tikzcd}$$

Figure 1: Visualization for finding S_θ

Task 2: Distances between points and linear manifolds

a) Find the distance between the point $(-3, 1) = \boldsymbol{a}$ and the line $4x + 3y + 4 = 0$
b) Find the distance between the point $(3, 1, -2) = \boldsymbol{b}$ and the plane $x + 2y - 2z = 4$

We pursue a more general approach. Let $(V, \langle\cdot,\cdot\rangle)$ be an inner-product space, $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{R}$, with the induced norm $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$, and $U \subseteq V$ be a subspace of V. Let $\boldsymbol{x} \in V$, then we define the distance of $\boldsymbol{x}$ to U as

$$d(\boldsymbol{x}, U) = \inf_{\boldsymbol{u}\in U} \|\boldsymbol{x} - \boldsymbol{u}\|$$

We prove the following:

$$d(\boldsymbol{x}, U) = \|\boldsymbol{x} - \pi_U(\boldsymbol{x})\|$$

Where π_U is the orthogonal projection of $\boldsymbol{x}$ into U. Let $\boldsymbol{v} \in V$ and $\boldsymbol{u} \in U$, then

$$\begin{aligned}
&\|\boldsymbol{v} - \boldsymbol{u}\|^2 = \langle \boldsymbol{v} - \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}\rangle = \langle \boldsymbol{v} - \pi_U(\boldsymbol{v}) + \pi_U(\boldsymbol{v}) - \boldsymbol{u}, \boldsymbol{v} - \pi_U(\boldsymbol{v}) + \pi_U(\boldsymbol{v}) - \boldsymbol{u}\rangle \\
&= \langle \boldsymbol{v} - \pi_U(\boldsymbol{v}), \boldsymbol{v} - \pi_U(\boldsymbol{v})\rangle + \langle \pi_U(\boldsymbol{v}) - \boldsymbol{u}, \pi_U(\boldsymbol{v}) - \boldsymbol{u}\rangle = \|\boldsymbol{v} - \pi_U(\boldsymbol{v})\|^2 + \underbrace{\|\boldsymbol{u} - \pi_U(\boldsymbol{v})\|^2}_{\geq 0} \geq \|\boldsymbol{v} - \pi_U(\boldsymbol{v})\|^2
\end{aligned}$$

Thus $\forall \boldsymbol{u} \in U \colon \|\boldsymbol{v} - \boldsymbol{u}\| \geq \|\boldsymbol{v} - \pi_U(\boldsymbol{v})\|$.

Let $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ be a basis of U. Since the inner product in V is, at least, sesquilinear, π_U has the following properties:

$$\begin{aligned}
&\pi_U(\boldsymbol{x}) = \sum_{i=1}^{k} c_i \boldsymbol{u}_i \\
&\forall j = 1, \ldots, k \colon \langle \pi_U(\boldsymbol{x}) - \boldsymbol{x}, \boldsymbol{u}_j\rangle = 0 \iff \forall j = 1, \ldots, k \colon \langle \pi_U(\boldsymbol{x}), \boldsymbol{u}_j\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_j\rangle
\end{aligned}$$

We get the following linear system of equations:

$$\begin{aligned}
&\langle \pi_U(\boldsymbol{x}), \boldsymbol{u}_j\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_j\rangle \iff \left\langle \sum_{i=1}^{k} c_i \boldsymbol{u}_i, \boldsymbol{u}_j \right\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_j\rangle \\
&\iff \sum_{i=1}^{k} c_i \langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle = \langle \boldsymbol{x}, \boldsymbol{u}_j\rangle
\end{aligned}$$

We denote

$$\mathrm{Gram}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k) = \begin{bmatrix}
\langle \boldsymbol{u}_1, \boldsymbol{u}_1\rangle & \langle \boldsymbol{u}_1, \boldsymbol{u}_2\rangle & \cdots & \cdots & \langle \boldsymbol{u}_1, \boldsymbol{u}_k\rangle \\
\langle \boldsymbol{u}_2, \boldsymbol{u}_1\rangle & \langle \boldsymbol{u}_2, \boldsymbol{u}_2\rangle & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \langle \boldsymbol{u}_{k-1}, \boldsymbol{u}_k\rangle \\
\langle \boldsymbol{u}_k, \boldsymbol{u}_1\rangle & \langle \boldsymbol{u}_k, \boldsymbol{u}_2\rangle & \cdots & \cdots & \langle \boldsymbol{u}_k, \boldsymbol{u}_k\rangle
\end{bmatrix}$$

the Gramian-Matrix of $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$, then

$$\mathrm{Gram}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)\boldsymbol{c} = \begin{bmatrix} \langle \boldsymbol{x}, \boldsymbol{u}_1\rangle \\ \vdots \\ \langle \boldsymbol{x}, \boldsymbol{u}_k\rangle \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

Let $u_{ij} = \langle \boldsymbol{u}_i, \boldsymbol{u}_j\rangle$. By the properties of $\langle\cdot,\cdot\rangle$, we know $u_{ij} = \overline{u_{ji}}$, thus $\mathrm{Gram}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k)$ is hermitian. Let $\mathrm{U} = \begin{bmatrix} \boldsymbol{u}_1^{\mathsf{t}} & \cdots & \boldsymbol{u}_k^{\mathsf{t}} \end{bmatrix}^{\mathsf{t}}$. If $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}^{\mathsf{t}} \mathrm{P} \boldsymbol{v}$, then

$$\mathrm{Gram}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k) = \mathrm{U}^{\mathsf{t}}\mathrm{P}\mathrm{U}$$

Let $\boldsymbol{x} \in V \setminus \{\boldsymbol{0}\}: \mathrm{U}\boldsymbol{x} = \boldsymbol{0}$:

$$\boldsymbol{0} = \mathrm{U}\boldsymbol{x} \iff \boldsymbol{0} = \mathrm{U}^{\mathrm{t}}\mathrm{P}\mathrm{U}\boldsymbol{x} = \mathrm{Gram}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)\boldsymbol{x}$$

Notice that $\mathrm{U}\boldsymbol{x}$ can only be $\boldsymbol{0}$ for $\boldsymbol{x} \neq \boldsymbol{0}$, if the columns of U are linearly dependent. By requirement $\boldsymbol{u}_1, \dots, \boldsymbol{u}_k$ form a basis of U and are thus linearly independent, thusly $\mathrm{Gram}(\boldsymbol{u}_1, \dots, \boldsymbol{u}_k)$ is regular.

If $M = \boldsymbol{u} + U$ is a linear manifold, we apply the inverse translation $T_{\boldsymbol{u}}^{-1}(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{u}$ to make $T_{\boldsymbol{u}}^{-1}[M] = U$ a (linear) subspace of V. Notice that $T_{\boldsymbol{u}}^{-1}$ is an isometry, i.e. the following holds:

$$\begin{aligned}
&\|T_{\boldsymbol{u}}(\boldsymbol{x}) - T_{\boldsymbol{u}}(\boldsymbol{y})\| = \|\boldsymbol{x} + \boldsymbol{u} - \boldsymbol{y} - \boldsymbol{u}\| = \|\boldsymbol{x} - \boldsymbol{y}\| \\
&d(T_{\boldsymbol{u}}(\boldsymbol{x}), M) = \|T_{\boldsymbol{u}}(\boldsymbol{x}) - \pi_M(\boldsymbol{x})\| = \|T_{\boldsymbol{u}}(\boldsymbol{x}) - T_{\boldsymbol{u}}(\pi_U(\boldsymbol{x}))\| = \|\boldsymbol{x} - \pi_U(\boldsymbol{x})\| = d(\boldsymbol{x}, U)
\end{aligned}$$

Subtask a: We first need to find a subspace $\tilde{G}$ and $\boldsymbol{u} \in \mathbb{R}^2$, such that $G = \{(x, y) \in \mathbb{R}^2 : 4x + 3y + 4 = 0\} = \boldsymbol{u} + \tilde{G}$. We solve for y:

$$y = -\frac{4}{3}x - \frac{4}{3}$$

Thus:

$$G = \left\{\lambda \begin{bmatrix} 3 \\ -4 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{4}{3} \end{bmatrix} | \lambda \in \mathbb{R}\right\} = \mathrm{span}\left(\begin{bmatrix} 3 \\ -4 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ -\frac{4}{3} \end{bmatrix}$$

Notice $\mathrm{Gram}(\boldsymbol{u}_1) = \|\boldsymbol{u}_1\|^2 = 25$ and therefore

$$\pi_{\tilde{G}}(\boldsymbol{x}) = \frac{1}{25} \langle \boldsymbol{x}, \boldsymbol{u}_1 \rangle \boldsymbol{u}_1$$

Now:

$$\begin{aligned}
&T_{\boldsymbol{u}}^{-1}(\boldsymbol{x}) = \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{4}{3} \end{bmatrix} \implies T_{\boldsymbol{u}}^{-1}(\boldsymbol{a}) = \begin{bmatrix} -3 \\ \frac{7}{3} \end{bmatrix} \implies \pi_{\tilde{G}}(T_{\boldsymbol{u}}^{-1}(\boldsymbol{a})) = \frac{1}{25}\left(-9 - \frac{28}{3}\right)\begin{bmatrix} 3 \\ -4 \end{bmatrix} \\
&= \frac{55}{25}\frac{1}{3}\begin{bmatrix} -3 \\ 4 \end{bmatrix} = \frac{11}{15}\begin{bmatrix} -3 \\ 4 \end{bmatrix} \\
&\implies d(T_{\boldsymbol{u}}^{-1}(\boldsymbol{a}), \tilde{G}) = \left\|\frac{11}{15}\begin{bmatrix} -3 \\ 4 \end{bmatrix} - \begin{bmatrix} -3 \\ \frac{7}{3} \end{bmatrix}\right\| = \left\|\begin{bmatrix} \frac{4}{5} \\ \frac{3}{5} \end{bmatrix}\right\| = \frac{1}{5}\|\begin{bmatrix} 3 & 4 \end{bmatrix}^{\mathrm{t}}\| = 1
\end{aligned}$$

Subtask b: We again need to find a subspace $\tilde{E}$ and a vector $\boldsymbol{u}$, such that

$$E = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - 2z = 4\} = \boldsymbol{u} + \tilde{E}$$

We solve for z:

$$z = \frac{1}{2}x + y - 2 \implies E = \left\{\lambda \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} | \lambda, \mu \in \mathbb{R}\right\} = \mathrm{span}\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

We compute $\mathrm{Gram}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ and it's inverse:

$$\mathrm{Gram}(\boldsymbol{u}_1, \boldsymbol{u}_2) = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathrm{Gram}^{-1}(\boldsymbol{u}_1, \boldsymbol{u}_2) = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$$

Now:

$$\begin{aligned}
&T_{\boldsymbol{u}}^{-1}(\boldsymbol{b}) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \left\langle T_{\boldsymbol{u}}^{-1}(\boldsymbol{b}), \boldsymbol{u}_1 \right\rangle = 6 \qquad \left\langle T_{\boldsymbol{u}}^{-1}(\boldsymbol{b}), \boldsymbol{u}_2 \right\rangle = 1 \\
&\boldsymbol{c} = \frac{1}{9}\begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 6 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 11 \\ -1 \end{bmatrix} \implies \pi_{\tilde{E}}(T_{\boldsymbol{u}}^{-1}(\boldsymbol{b})) = \frac{11}{9}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 22 \\ -1 \\ 10 \end{bmatrix} \\
&\implies d(T_{\boldsymbol{u}}^{-1}(\boldsymbol{b}), \tilde{E}) = \left\|\frac{1}{9}\begin{bmatrix} 22 \\ -1 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}\right\| = \frac{1}{9}\left\|\begin{bmatrix} -5 \\ -10 \\ 10 \end{bmatrix}\right\| = \frac{1}{9}\sqrt{225} = \frac{15}{9} = \frac{5}{3}
\end{aligned}$$

Task 3: Matrix Rank

We are given the matrix

$$\mathrm{A}(r,s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r-2 & 2 \\ 0 & s-1 & r+2 \\ 0 & 0 & 3 \end{bmatrix}$$

Do there exist values for r, s, such that $\operatorname{rank}\mathrm{A}(r,s) = 1$ and $\operatorname{rank}\mathrm{A}(r,s) = 2$ respectively? If so, find them.

Since the first row $\boldsymbol{r}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ is linearly independent from the remaining three rows, we get that $\forall r, s \in \mathbb{R}\colon \operatorname{rank}\mathrm{A}(r,s) \geq 1$. Let

$$\mathrm{M}(r,s) = \begin{bmatrix} r-2 & 2 \\ s-1 & r+2 \end{bmatrix}$$

First let's analyze some special cases. If $r = 2$, then

$$\operatorname{rank}\mathrm{M}(2,s) = \begin{cases} 1 & s = 1 \\ 2 & s \neq 1 \end{cases}$$

If $r = -2$, then

$$\operatorname{rank}\mathrm{M}(-2,s) = \begin{cases} 1 & s = 1 \\ 2 & s \neq 1 \end{cases}$$

If $\operatorname{rank}\mathrm{M}(r,s) = 2$, the we can bring M into diagonal-form using elementary row-operations. Thus we can eliminate the fourth row $\boldsymbol{r}_4 = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$ of $\mathrm{A}(r,s)$, and hence $\operatorname{rank}\mathrm{A}(r,s) = 3$.

$$\begin{bmatrix} r-2 & 2 \\ s-1 & r+2 \end{bmatrix} \overset{\frac{1}{r-2}I,\, \frac{1}{s-1}II}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{2}{r-2} \\ 1 & \frac{r+2}{s-1} \end{bmatrix} \overset{II-I}{\rightsquigarrow} \begin{bmatrix} 1 & \frac{2}{r-2} \\ 0 & \frac{r+2}{s-1} - \frac{2}{r-2} \end{bmatrix}$$

We see that $\operatorname{rank}\mathrm{M}(r,s) = 1$ iff

$$\begin{aligned} &\frac{r+2}{s-1} - \frac{2}{r-2} = 0 \\ &\frac{r+2}{s-1} - \frac{2}{r-2} = \frac{r^2 - 4 - 2s + 2}{(s-1)(r-2)} = \frac{r^2 - 2s - 2}{(s-1)(r-2)} \overset{!}{=} 0 \implies r^2 - 2s - 2 = 0 \\ &r = \pm\sqrt{2}\sqrt{s+1} \qquad s = \frac{r^2}{2} - 1 \end{aligned}$$

Thus we get:

$$\operatorname{rank}\mathrm{M}(r,s) = \begin{cases} 1 & r = \pm 2 \wedge s = 1 \\ 1 & r = \pm\sqrt{2s+2} \wedge s \in \mathbb{R} \setminus \{1\} \\ 1 & s = \frac{r^2}{2} - 1 \wedge r \in \mathbb{R} \setminus \{2\} \\ 2 & \text{else} \end{cases}$$

Notice

$$\mathrm{M}(1,2) = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus for $r = 2$ and $s = 1$, we get $\operatorname{rank}\mathrm{A}(2,1) = 2$, otherwise $\operatorname{rank}\mathrm{A}(r,s)$ is 3. Hence there does not exists any pair $r, s \in \mathbb{R}$, such that $\operatorname{rank}\mathrm{A}(r,s) = 1$.

Task 4: Matrix Transformation

Let $T_{\mathrm{A}}\colon \mathbb{R}^n \to \mathbb{R}^m$ be a mapping of the form

$$T_{\mathrm{A}}(\boldsymbol{x}) = \mathrm{A}\boldsymbol{x}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $\mathrm{A} \in \mathbb{R}^{m\times n}$. Let $T_{\mathrm{A}}, T_{\mathrm{B}}\colon \mathbb{R}^n \to \mathbb{R}^m$ be two matrix transformations. Prove the following

$$\forall \boldsymbol{x} \in \mathbb{R}^n\colon T_{\mathrm{A}}(\boldsymbol{x}) = T_{\mathrm{B}}(\boldsymbol{x}) \implies \mathrm{A} = \mathrm{B}$$

Notice that a linear transformation is uniquely defined by it's image of the basis-vectors. Let $\boldsymbol{x} \in V$ and $T \in \mathrm{Hom}(V, W)$, V and W vector-spaces, then

$$T(\boldsymbol{x}) = T\left(\sum_{i=1}^{n} \lambda_i \boldsymbol{b}_i\right) = \sum_{i=1}^{n} \lambda_i T(\boldsymbol{b}_i)$$

Now:

$$\mathrm{A} = S_{\mathrm{A}} = \begin{bmatrix} T_{\mathrm{A}}(\boldsymbol{b}_1) & \cdots & T_{\mathrm{A}}(\boldsymbol{b}_n)\end{bmatrix} = \begin{bmatrix} T_{\mathrm{B}}(\boldsymbol{b}_1) & \cdots & T_{\mathrm{B}}(\boldsymbol{b}_n)\end{bmatrix} = S_{\mathrm{B}} = \mathrm{B}$$

Task 5: Properties of Euclidean Norm

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$, prove the following holds:

$$\left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^2} \right\| = \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|}$$

$$\begin{aligned}
\left(\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{x}\| \cdot \|\boldsymbol{y}\|}\right)^2 &= \langle \boldsymbol{x}-\boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}\rangle \frac{1}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2} = \frac{\langle \boldsymbol{x}, \boldsymbol{x}-\boldsymbol{y}\rangle - \langle \boldsymbol{y}, \boldsymbol{x}-\boldsymbol{y}\rangle}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2} \\
&= \frac{\langle \boldsymbol{x}, \boldsymbol{x}\rangle - \langle \boldsymbol{x}, \boldsymbol{y}\rangle - \langle \boldsymbol{y}, \boldsymbol{x}\rangle + \langle \boldsymbol{y}, \boldsymbol{y}\rangle}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2} = \frac{\|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2}
\end{aligned}$$

And:

$$\begin{aligned}
&\left(\left\| \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^2} \right\|\right)^2 = \left\langle \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^2}, \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|^2} - \frac{\boldsymbol{y}}{\|\boldsymbol{y}\|^2} \right\rangle = \left\langle \frac{\|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{y}}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2}, \frac{\|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{y}}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2} \right\rangle \\
&= \frac{1}{\|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{y}\|^4} \left\langle \|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{y}, \|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{x}\right\rangle = \frac{\|\boldsymbol{y}\|^2 \left\langle \boldsymbol{x}, \|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{y}\right\rangle - \|\boldsymbol{x}\|^2 \left\langle \boldsymbol{y}, \|\boldsymbol{y}\|^2\boldsymbol{x} - \|\boldsymbol{x}\|^2\boldsymbol{y}\right\rangle}{\|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{y}\|^4} \\
&= \frac{\|\boldsymbol{y}\|^2 \left\langle \boldsymbol{x}, \|\boldsymbol{y}\|^2\boldsymbol{x}\right\rangle - \|\boldsymbol{y}\|^2 \left\langle \boldsymbol{x}, \|\boldsymbol{x}\|^2\boldsymbol{y}\right\rangle - \|\boldsymbol{x}\|^2 \left\langle \boldsymbol{y}, \|\boldsymbol{y}\|^2\boldsymbol{x}\right\rangle + \|\boldsymbol{x}\|^2 \left\langle \boldsymbol{y}, \|\boldsymbol{y}\|^2\boldsymbol{y}\right\rangle}{\|\boldsymbol{y}\|^4 \cdot \|\boldsymbol{y}\|^4} \\
&= \frac{\|\boldsymbol{y}\|^4 \cdot \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2 \cdot \|\boldsymbol{x}\|^2 \langle \boldsymbol{x}, \boldsymbol{y}\rangle - \|\boldsymbol{y}\|^2 \cdot \|\boldsymbol{x}\|^2 \langle \boldsymbol{y}, \boldsymbol{x}\rangle + \|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{y}\|^2}{\|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{y}\|^4} \\
&= \frac{\|\boldsymbol{y}\|^4 \cdot \|\boldsymbol{x}\|^2 - 2\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2 \langle \boldsymbol{x}, \boldsymbol{y}\rangle + \|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{y}\|^2}{\|\boldsymbol{x}\|^4 \cdot \|\boldsymbol{x}\|^4} = \frac{\|\boldsymbol{y}\|^2 - 2\langle \boldsymbol{x}, \boldsymbol{y}\rangle + \|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2 \cdot \|\boldsymbol{y}\|^2}
\end{aligned}$$

Task 6: Linear Independence

Let V be a $\mathbb{K}$ vector-space and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_k \in V$, where $k \geq 2$. Prove that $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is linearly independent iff

$$\boldsymbol{v}_1 \neq \mathbf{0} \land \forall i = 1, \dots, k-1 \colon \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i) \subset \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i+1})$$

We use contraposition. If $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is linearly dependent, then

$$\boldsymbol{v}_1 = \mathbf{0} \lor \exists i \in \{1, \dots, k-1\} \colon \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i) = \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i+1})$$

Assume $\boldsymbol{v}_1 = \mathbf{0}$, then

$$\sum_{i=1}^{k} \lambda_i \boldsymbol{v}_i = \lambda_1 \boldsymbol{v}_1 + \mathbf{0} = \mathbf{0} \forall \lambda_1 \in \mathbb{R}$$

Thus $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is linearly dependent. Now let $\boldsymbol{v}_1 \neq \mathbf{0}$. Assume $\exists i \in \{1, \dots, k-1\}$ such that $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i)$ is linearly independent and:

$$U_i = \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_i) = \operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i+1})$$

Let $\boldsymbol{x} \in U_i \setminus \{\mathbf{0}\}$ and $\mu_{i+1} \neq 0$:

$$\begin{aligned}
\boldsymbol{x} = \sum_{l=1}^{i} \lambda_l \boldsymbol{v}_l = \sum_{l=1}^{i+1} \mu_l \boldsymbol{v}_l = \mu_{i+1} \boldsymbol{v}_{i+1} + \sum_{l=1}^{i} \mu_l \boldsymbol{v}_l \\
\iff \mathbf{0} = \mu_{i+1} \boldsymbol{v}_{i+1} + \sum_{l=1}^{i} (\mu_l - \lambda_l) \boldsymbol{v}_l \iff \mu_{i+1} \boldsymbol{v}_{i+1} = \sum_{l=1}^{i} (\lambda_l - \mu_l) \boldsymbol{v}_l \\
\iff \boldsymbol{v}_{i+1} = \sum_{l=1}^{i} \frac{\lambda_l - \mu_l}{\mu_{i+1}} \boldsymbol{v}_l
\end{aligned}$$

We found a linear combination for $\boldsymbol{v}_{i+1}$ in U_i, thus $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_{i+1})$ is not linearly independent, thus $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_k)$ is linearly dependent.