Exercise Sheet № 4

Task 17: Determinants I

Let $(\mathbb{F}, +, \cdot)$ be a field. For $x \in \mathbb{F}$ we define

$$\mathrm{A}_n = [(x-1)\delta_{ij} + 1]_{i,j=1,\ldots,n} \in \mathbb{F}^{n\times n}$$

Prove the following:

$$\det(\mathrm{A}_n) = (x-1)^{n-1}(x+n-1)$$

We pursue a more general approach. Let $a, b \in \mathbb{F}$ and $1_n = [1]_{i,j=1,\ldots,n} \in \mathbb{F}^n n \times n$.

$$D_n(a,b) = \det(b1_n - \mathrm{I}_n(b-a)) = \det(\mathrm{B}_n)$$

Since adding rows does not change the determinant, we can bring $b1_n - \mathrm{I}_n(b-a)$ into the following form:

$$\begin{bmatrix} a+(n-1)b & a+(n-1)b\mathbf{1}_{n-1}^{\mathrm{t}} \\ \mathbf{1}_{n-1} & \mathrm{B}_{n-1} \end{bmatrix}$$

Using Laplacian expansion after the first row yields:

$$D_n(a,b) = (a+(n-1)b)\begin{vmatrix} 1 & \mathbf{1}_{n-1}^{\mathrm{t}} \\ \mathbf{1}_{n-1} & \mathrm{B}_{n-1} \end{vmatrix}$$

Adding rows yields:

$$\begin{bmatrix} a+(n-1)b & a+(n-1)b\mathbf{1}_{n-1}^{\mathrm{t}} \\ \mathbf{1}_{n-1} & \mathrm{B}_{n-1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \mathbf{1}_{n-1}^{\mathrm{t}} \\ \mathbf{0}_{n-1} & (a-b)\mathrm{I}_{n-1} \end{bmatrix}$$

This is a triangular matrix, hence:

$$D_n(a,b) = (a+(n-1)b)\prod_{i=1}^{n-1}(a-b) = (a+(n-1)b)(a-b)^{n-1}$$

With $a = x$ and $b = 1$, we get $D_n(x,1) = (x+n-1)(x-1)^{n-1}$.

Task 18: Cofactor Matrix I

a) Compute the inverse of the following matrix using only the determinant

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$$

b) Let $(\mathbb{F}, +, \cdot)$ be a field with $a, b, c, d \in \mathbb{F}$ and $c \neq 0$. Compute the inverse of

$$B = \begin{bmatrix} 1 & a & b \\ 0 & c & d \\ 0 & 0 & 1 \end{bmatrix}$$

Recall the following fact:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = \frac{1}{\det(A)} \operatorname{Cof}(A)^t$$

Where $\operatorname{Cof}(A)$ is the cofactor matrix given by

$$\operatorname{Cof}(A) = [\widetilde{a}_{ij}]_{i,j=1,\dots,n} \qquad \widetilde{a}_{ij} = (-1)^{i+j}\det(A_{i,j}) \qquad A_{i,j} = [a_{kl}]_{\substack{k,l=1,\dots,n \\ k\neq i, l\neq l}}$$

Subtask a): We first compute $\det(A)$ using Laplacian expansion:

$$\det A = 2\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + 3\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} = 2 \cdot 8 + 3 \cdot (-4) = 4$$

Now we compute the minors of A:

$$\begin{array}{lll} \det(A_{1,1}) = 8 & \det(A_{1,2}) = 0 & \det(A_{1,3}) = -6 \\ \det(A_{2,1}) = 0 & \det(A_{2,2}) = 2 & \det(A_{2,3}) = 0 \\ \det(A_{3,1}) = -4 & \det(A_{3,2}) = 0 & \det(A_{3,3}) = 4 \end{array}$$

Hence

$$\operatorname{Cof}(A) = \begin{bmatrix} 8 & 0 & -6 \\ 0 & 2 & 0 \\ -4 & 0 & 4 \end{bmatrix} \implies A^{-1} = \frac{1}{4}\begin{bmatrix} 8 & 0 & -4 \\ 0 & 2 & 0 \\ -6 & 0 & 4 \end{bmatrix}$$

Subtask b): B is a triangular matrix, thus $\det(B) = c$. The minors are:

$$\begin{array}{lll} \det(B_{1,1}) = c & \det(B_{1,2}) = 0 & \det(B_{1,3}) = 0 \\ \det(B_{2,1}) = a & \det(B_{2,2}) = 1 & \det(B_{2,3}) = 0 \\ \det(B_{3,1}) = ad - cb & \det(B_{3,2}) = d & \det(B_{3,3}) = c \end{array}$$

Therefore

$$\operatorname{Cof}(B) = \begin{bmatrix} c & 0 & 0 \\ -a & 1 & 0 \\ ad - cb & -d & c \end{bmatrix} \implies B^{-1} = \frac{1}{c}\begin{bmatrix} c & -a & ad - cb \\ 0 & 1 & -d \\ 0 & 0 & c \end{bmatrix}$$

Task 19: Determinants II

Compute the determinants of the following matrices:

a)

$$\mathrm{A} = \begin{bmatrix} 1 & 0 & 4 \\ 2 & -1 & 3 \\ 5 & 2 & 1 \end{bmatrix}$$

b)

$$\mathrm{B} = \begin{bmatrix} -4 & 3 & 64 & 124 & 32 \\ 2 & 5 & 25 & 45 & 3 \\ 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 1 & -2 \\ 0 & 0 & -1 & 10 & 4 \end{bmatrix}$$

c) Let $(\mathbb{F}, +, \cdot)$ be a field and $a, b \in \mathbb{F}$, let

$$\mathrm{A}_n = a\mathrm{I}_n + b(\mathrm{T}_n + \mathrm{T}_n^{\mathrm{t}}) \qquad \mathrm{T}_n = \begin{bmatrix} \mathbf{0}_{n-1} & \mathrm{I}_{n-1} \\ 0 & \mathbf{0}_{n-1}^{\mathrm{t}} \end{bmatrix}$$

Find a recursion formula for computing $\det(\mathrm{A}_n)$. Which sequence arises for $a = 1$ and $b = \mathrm{i}$, if you choose $\mathbb{F} = \mathbb{C}$?

Subtask a): We use Laplacian expansion after the first row:

$$\det(\mathrm{A}) = \begin{vmatrix} -1 & 3 \\ 2 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = -1 - 6 + 4(4+5) = 30$$

Subtask b): We use Laplacian expansion after the first column:

$$\det(\mathrm{B}) = -4 \begin{vmatrix} 5 & 25 & 45 & 3 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 1 & -2 \\ 0 & -1 & 10 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 64 & 124 & 32 \\ 0 & 2 & 1 & 4 \\ 0 & 3 & 1 & -2 \\ 0 & -1 & 10 & 4 \end{vmatrix}$$

We see that the two 4×4 determinants have the same sub-matrix. If we expand after the first column, we can use the same determinant twice:

$$\begin{vmatrix} 2 & 1 & 4 \\ 3 & 1 & -2 \\ -1 & 10 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 \\ 10 & 4 \end{vmatrix} - \begin{vmatrix} 3 & -2 \\ -1 & 4 \end{vmatrix} + 4 \begin{vmatrix} 3 & 1 \\ -1 & 10 \end{vmatrix}$$
$$= 2(4+20) - (12-2) + 4(30+1) = 48 - 10 + 124 = 162$$

Hence:

$$\det(\mathrm{B}) = (-4) \cdot 5 \cdot 162 - 2 \cdot 3 \cdot 162 = -4212$$

Subtask c): We denote $\Delta_n = \det(\mathrm{A}_n)$. Notice that the first column of A_n, except for the first two rows, is the zero vector $\mathbf{0}_{n-2}$. Thus we want to use Laplacian expansion after the first column, resulting in two non-zero summands for Δ_n. Forming $\mathrm{A}_{n,2,1}$, which can be constructed by removing the first column and second row, we get:

$$\mathrm{A}_{n,2,1} = \begin{bmatrix} b & \mathbf{0}_{n-2}^{\mathrm{t}} \\ b & \\ \mathbf{0}_{n-3} & \mathrm{A}_{n-2} \end{bmatrix} \implies \det(\mathrm{A}_{n,2,1}) = b\Delta_{n-2}$$

Hence

$$\Delta_n = a\Delta_{n-1} - b^2\Delta_{n-2}$$

Using $a = 1$ and $b = \mathrm{i}$, we get:

$$\Delta_n = \Delta_{n-1} + \Delta_{n-2}$$

which produces the Fibonacci sequence, if $\Delta_0 = 0$.

Addendum (not required): Let $\mathbb{F} = \mathbb{C}$. We want to find an explicit formula for Δ_n. Notice that we can write the recurrence relation as a first-order system

$$\begin{aligned}
\begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} &= \begin{bmatrix} a\Delta_n - b^2\Delta_{n-2} \\ \Delta_{n-1} \end{bmatrix} = \begin{bmatrix} a & -b^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n-1} \\ \Delta_{n-2} \end{bmatrix} \\
\implies \begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} &= \underbrace{\begin{bmatrix} a & -b^2 \\ 1 & 0 \end{bmatrix}^{n-1}}_{=\mathrm{S}^{n-1}} \begin{bmatrix} \Delta_1 \\ \Delta_0 \end{bmatrix}
\end{aligned}$$

Where $\Delta_0 = 1$ and $\Delta_1 = a$. If we want to find arbitrary powers of S easily, we diagonalize it:

$$\begin{aligned}
&\chi_{\mathrm{S}}(\lambda) = \lambda(\lambda - a) + b^2 = \lambda^2 - \lambda a + b^2 \\
&\lambda_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2 - 4b^2}{4}} \\
&\mathrm{V} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \qquad \mathrm{S}^n = \mathrm{V}\operatorname{diag}(\lambda_1^n, \lambda_2^n)\mathrm{V}^{-1} \\
&\implies \begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} = \mathrm{V}\operatorname{diag}(\lambda_1^{n-1}, \lambda_2^{n-1})\mathrm{V}^{-1} \begin{bmatrix} \Delta_1 \\ \Delta_0 \end{bmatrix}
\end{aligned}$$

Now:

$$\Delta_n = a\frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$$

We can compare this result with the recurrence relation by computing the explicit values for $a = 1$ and $b = \mathrm{i}$:

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \sqrt{\frac{5}{4}} = \frac{1 + \sqrt{5}}{2} = \varphi \\
\lambda_2 &= 1 - \lambda_1 = -\frac{1}{\varphi} \\
\lambda_1 - \lambda_2 &= \sqrt{5}
\end{aligned}$$

Where φ is the golden ratio. Now:

$$\Delta_n = \frac{\varphi^{n+1} - (-1)^{n+1}\varphi^{-(n+1)}}{\sqrt{5}}$$

Which is exactly Binet's Formula for the Fibonacci sequence.

Task 20: Cofactor Matrix II

Let $\mathrm{A}, \mathrm{B} \in \mathrm{GL}(n, \mathbb{K})$, prove the following statements:

a) $\det(\operatorname{adj} \mathrm{A}) = \operatorname{adj}(\mathrm{A})^{n-1}$

b) $\operatorname{adj}(\operatorname{adj} \mathrm{A}) = \operatorname{adj}(\mathrm{A})^{n-2}\mathrm{A}$

c) $\operatorname{adj}(\mathrm{AB}) = \operatorname{adj}(\mathrm{B}) \operatorname{Cof}(\mathrm{A})$

Recall the following

$$\mathrm{A}^{-1} = \frac{1}{\det(\mathrm{A})} \operatorname{Cof}(\mathrm{A})^{\mathrm{t}} = \frac{1}{\det(\mathrm{A})} \operatorname{adj}(\mathrm{A})$$
$$\det(c\mathrm{A}) = c^n \det(\mathrm{A})$$

Subtask a):

$$\det(\operatorname{adj}(\mathrm{A})) = \det(\det(\mathrm{A})\mathrm{A}^{-1}) = \det(\mathrm{A})^n \det(\mathrm{A}^{-1}) = \det(\mathrm{A})^{n-1}$$

Subtask b):

$$\operatorname{adj}(\mathrm{A}) = \det(\mathrm{A})\mathrm{A}^{-1} = \mathrm{C}$$
$$\operatorname{adj}(\mathrm{C}) = \det(\operatorname{adj}(\mathrm{A})) \operatorname{adj}(\mathrm{A})^{-1} = \det(\mathrm{A})^{n-1} \frac{1}{\det(\mathrm{A})} \mathrm{A} = \det(\mathrm{A})^{n-2}\mathrm{A}$$

We may call $\operatorname{Cof}(\operatorname{Cof}(\mathrm{A}))$ the cocomatrix of A.

Subtask c):

$$\operatorname{adj}(\mathrm{AB}) = \det(\mathrm{AB})(\mathrm{AB})^{-1} = \det(\mathrm{B})\mathrm{B}^{-1}\det(\mathrm{A})\mathrm{A}^{-1} = \operatorname{adj}(\mathrm{B}) \operatorname{adj}(\mathrm{A})$$

Task 21: Block Matrices

Let $(\mathbb{F}, +, \cdot)$ be a field. Let $\mathrm{A}, \mathrm{A}' \in \mathbb{K}^{m\times m}$, $\mathrm{B}, \mathrm{B}' \in \mathbb{K}^{m\times n}$, $\mathrm{C}, \mathrm{C}' \in \mathbb{K}^{n\times m}$ and $\mathrm{D}, \mathrm{D}' \in \mathbb{K}^{n\times n}$. Verify the following:

$$\begin{bmatrix}\mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix}\begin{bmatrix}\mathrm{A}' & \mathrm{B}'\\ \mathrm{C}' & \mathrm{D}'\end{bmatrix} = \begin{bmatrix}\mathrm{AA}' + \mathrm{BC}' & \mathrm{AB}' + \mathrm{BD}'\\ \mathrm{CA}' + \mathrm{DC}' & \mathrm{CB}' + \mathrm{DD}'\end{bmatrix}$$

Also prove

$$\begin{vmatrix}\mathrm{A} & \mathrm{B}\\ 0_{n\times m} & \mathrm{D}\end{vmatrix} = \det(\mathrm{A})\det(\mathrm{D})$$

Let

$$\mathcal{A} = \begin{bmatrix}\mathrm{A} & \mathrm{B}\\ \mathrm{C} & \mathrm{D}\end{bmatrix} \qquad \mathcal{B} = \begin{bmatrix}\mathrm{A}' & \mathrm{B}'\\ \mathrm{C}' & \mathrm{D}'\end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix}\boldsymbol{x}_1\\ \boldsymbol{x}_2\end{bmatrix} \qquad \boldsymbol{x}_1 \in \mathbb{R}^m, \boldsymbol{x}_2 \in \mathbb{R}^n$$

The multiplication of $\mathcal{A}$ corresponds to the following linear map $f \in \mathrm{End}(\mathbb{R}^{m+n})$:

$$f(\boldsymbol{x}) = \begin{bmatrix}\mathrm{A}\boldsymbol{x}_1\\ \mathrm{C}\boldsymbol{x}_1\end{bmatrix} + \begin{bmatrix}\mathrm{B}\boldsymbol{x}_2\\ \mathrm{D}\boldsymbol{x}_2\end{bmatrix}$$

Now let $g \in \mathrm{End}(\mathbb{R}^{m+n})$ with $g(\boldsymbol{x}) = \mathcal{B}\boldsymbol{x}$, then:

$$\begin{aligned}
(f\circ g)(\boldsymbol{x}) &= f(g(\boldsymbol{x})) = f\left(\begin{bmatrix}\mathrm{A}'\boldsymbol{x}_1\\ \mathrm{C}'\boldsymbol{x}_1\end{bmatrix} + \begin{bmatrix}\mathrm{B}'\boldsymbol{x}_2\\ \mathrm{D}'\boldsymbol{x}_2\end{bmatrix}\right)\\
&= f\left(\begin{bmatrix}\mathrm{A}'\boldsymbol{x}_1\\ \mathrm{C}'\boldsymbol{x}_1\end{bmatrix}\right) + f\left(\begin{bmatrix}\mathrm{B}'\boldsymbol{x}_2\\ \mathrm{D}'\boldsymbol{x}_2\end{bmatrix}\right)\\
&= \begin{bmatrix}\mathrm{AA}'\boldsymbol{x}_1\\ \mathrm{CA}'\boldsymbol{x}_1\end{bmatrix} + \begin{bmatrix}\mathrm{BC}'\boldsymbol{x}_1\\ \mathrm{DC}'\boldsymbol{x}_1\end{bmatrix} + \begin{bmatrix}\mathrm{AB}'\boldsymbol{x}_2\\ \mathrm{CB}'\boldsymbol{x}_2\end{bmatrix} + \begin{bmatrix}\mathrm{BD}'\boldsymbol{x}_2\\ \mathrm{DD}'\boldsymbol{x}_2\end{bmatrix}\\
&= \begin{bmatrix}(\mathrm{AA}' + \mathrm{BC}')\boldsymbol{x}_1\\ (\mathrm{CA}' + \mathrm{DC}')\boldsymbol{x}_1\end{bmatrix} + \begin{bmatrix}(\mathrm{AB}' + \mathrm{BD}')\boldsymbol{x}_2\\ (\mathrm{CB}' + \mathrm{DD}')\boldsymbol{x}_2\end{bmatrix}
\end{aligned}$$

We know that given two linear maps $f, g \in \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ with matrix representations E, F, their composition $f \circ g$ has EF as corresponding matrix representation.

For the determinant. Assume either A or D is singular, then the corresponding block-matrix has a zero on the diagonal of it's row-echelon form. If both A and D are singular, we can bring the block matrix into upper triangular form. Let a_{ii} be the diagonal entries of A in REF, and analogous for D d_{ii}, then:

$$\begin{vmatrix}\mathrm{A} & \mathrm{B}\\ 0_{n\times m} & \mathrm{D}\end{vmatrix} = \left(\prod_{i=1}^{m} a_{ii}\right)\left(\prod_{i=1}^{n} d_{ii}\right) = \det(\mathrm{A})\det(\mathrm{D})$$