Exercise Sheet № 8

Task 41: Gram-Schmidt orthonormalization

Find an orthonormal basis of the following subspace of $\mathbb{R}^5$:

$$\operatorname{span}\left\{ \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix}, \begin{bmatrix}2\\1\\0\\2\\3\end{bmatrix} \right\}$$

Recall that for the k-th vector $\widetilde{\boldsymbol{u}}_k$ in the corresponding orthogonal basis satisfies

$$\widetilde{\boldsymbol{u}}_k = \boldsymbol{v}_k - \sum_{j=1}^{k-1} \langle \boldsymbol{v}_k, \boldsymbol{u}_j \rangle \boldsymbol{u}_j \implies \boldsymbol{u}_k = \frac{1}{\|\boldsymbol{u}_k\|}\boldsymbol{u}_k$$

Thus $\boldsymbol{u}_1 = \boldsymbol{v}_1$, since $\|\boldsymbol{v}_1\| = 1$. Now

$$\widetilde{\boldsymbol{u}}_2 = \begin{bmatrix}1\\0\\1\\0\\0\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = \boldsymbol{u}_2$$

Next

$$\begin{aligned} &\langle \boldsymbol{v}_3, \boldsymbol{u}_1 \rangle = 1 \qquad \langle \boldsymbol{v}_3, \boldsymbol{u}_2 \rangle = 1 \\ &\implies \widetilde{\boldsymbol{u}}_3 = \begin{bmatrix}1\\1\\1\\0\\2\end{bmatrix} - \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} - \begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\0\\0\\2\end{bmatrix} \\ &\|\widetilde{\boldsymbol{u}}_3\| = \sqrt{5} \end{aligned}$$

And at last

$$\begin{aligned} &\langle \boldsymbol{v}_4, \boldsymbol{u}_1 \rangle = 2 \qquad \langle \boldsymbol{v}_4, \boldsymbol{u}_2 \rangle = 0 \qquad \langle \boldsymbol{v}_4, \boldsymbol{u}_3 \rangle = \frac{7}{\sqrt{5}} \\ &\widetilde{\boldsymbol{u}}_4 = \boldsymbol{v}_4 - 2\boldsymbol{u}_1 - \frac{7}{\sqrt{5}}\boldsymbol{u}_3 = \begin{bmatrix}2\\1\\0\\2\\3\end{bmatrix} - \begin{bmatrix}2\\0\\0\\0\\0\end{bmatrix} - \frac{7}{5}\begin{bmatrix}0\\1\\0\\0\\2\end{bmatrix} = \frac{1}{5}\left(\begin{bmatrix}0\\5\\0\\10\\15\end{bmatrix} - \begin{bmatrix}0\\7\\0\\0\\14\end{bmatrix} \right) \\ &= \frac{1}{5}\begin{bmatrix}0\\-2\\0\\10\\1\end{bmatrix} \implies \|\widetilde{\boldsymbol{u}}_4\| = \frac{1}{5}\sqrt{105} \implies \boldsymbol{u}_4 = \frac{1}{\sqrt{105}}\begin{bmatrix}0\\-2\\0\\10\\1\end{bmatrix} \end{aligned}$$

Task 42: Linear independence and inner products

Let V be a finite-dimensional vectorspace with inner product $\langle\cdot,\cdot\rangle : V^2 \to \mathbb{K}$ and $B = (\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ be an orthonormal Family in V. Proof the following statements are equivalent:

a) $\operatorname{span}(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) = V$
b) Let $\boldsymbol{v} \in V$, then $\forall i = 1, \dots, n\colon \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle = 0$ implies that $\boldsymbol{v} = \boldsymbol{0}$
c) If $\boldsymbol{v} \in V$ then $\boldsymbol{v} = \sum_{i=1}^n \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \boldsymbol{v}_i$
d) $\forall \boldsymbol{v}, \boldsymbol{w} \in V\colon \langle \boldsymbol{v}, \boldsymbol{w} \rangle = \sum_{i=1}^n \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \langle \boldsymbol{w}, \boldsymbol{v}_i \rangle$
e) $\forall \boldsymbol{v} \in V\colon \|\boldsymbol{v}\|^2 = \sum_{i=1}^n |\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle|^2$

a) $\implies$ c): Let $\boldsymbol{v} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i$, then

$$\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle = \left\langle \sum_{j=1}^n \lambda_j \boldsymbol{v}_j, \boldsymbol{v}_i \right\rangle = \sum_{j=1}^n \lambda_j \langle \boldsymbol{v}_j, \boldsymbol{v}_i \rangle = \sum_{j=1}^n \lambda_j \delta_{ij} = \lambda_i$$
$$\implies \boldsymbol{v} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i = \sum_{i=1}^n \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \boldsymbol{v}_i$$

c) $\implies$ d): Let $\boldsymbol{v} = \sum_{i=1}^n \lambda_i \boldsymbol{v}_i$ and $\boldsymbol{w} = \sum_{i=1}^n \mu_i \boldsymbol{v}_i$:

$$\langle \boldsymbol{v}, \boldsymbol{w} \rangle = \left\langle \sum_{i=1}^n \lambda_i \boldsymbol{v}_i, \sum_{j=1}^n \mu_j \boldsymbol{v}_j \right\rangle = \sum_{i=1}^n \lambda_i \left\langle \boldsymbol{v}_i, \sum_{j=1}^n \mu_j \boldsymbol{v}_j \right\rangle = \sum_{i=1}^n \lambda_i \sum_{j=1}^n \mu_j \langle \boldsymbol{v}_i, \boldsymbol{v}_j \rangle$$
$$= \sum_{i=1}^n \lambda_i \sum_{j=1}^n \mu_j \delta_{ij} = \sum_{i=1}^n \lambda_i \mu_j = \sum_{i=1}^n \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \langle \boldsymbol{w}, \boldsymbol{v}_i \rangle$$

d) $\implies$ e):

$$\|\boldsymbol{v}\|^2 = \langle \boldsymbol{v}, \boldsymbol{v} \rangle = \sum_{i=1}^n \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle \langle \boldsymbol{v}, \boldsymbol{v}_i \rangle = \sum_{i=1}^n |\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle|^2$$

d) $\implies$ b):

$$\|\boldsymbol{v}\|^2 = \sum_{i=1}^n |\langle \boldsymbol{v}, \boldsymbol{v}_i \rangle|^2 = \sum_{i=1}^n 0 = 0 \qquad \implies \|\boldsymbol{v}\| = 0$$
$$\|\boldsymbol{v}\| = 0 \iff \boldsymbol{v} = \boldsymbol{0}$$

b) $\implies$ a): We prove that $\boldsymbol{v}_i$ are linearly independent. Assume the contrary, then $\exists i \in \{1, \dots, n\}$ such that $\sum_{j=1}^n \lambda_j \boldsymbol{v}_j = \boldsymbol{0}$ and $\lambda_i \neq 0$:

$$\boldsymbol{0} = \sum_{j=1}^n \lambda_j \boldsymbol{v}_j = \lambda_i \boldsymbol{v}_i \neq \sum_{i=1}^n \langle \boldsymbol{0}, \boldsymbol{v}_i \rangle \boldsymbol{v}_i = \boldsymbol{0}$$

This is a contradiction, hence $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$ are linearly independent and thus span V, i.e. B is a basis of V.

Task 43: Solutions of differential equations I

Let $I \subseteq \mathbb{R}$, $I = (a, b)$, be an open interval. Recall that $\mathcal{C}(I)$, the set of continuous functions on I, and $\mathcal{C}^\infty(I, \mathbb{R}^n)$, the set of smooth curve-parametrizations on I, are vectorspaces over $\mathbb{R}$. We analyze the following homogenous, linear differential equation

$$\boldsymbol{y}' = \mathrm{A}\boldsymbol{y}$$

where $\mathrm{A} \in \mathcal{C}(I)^{n\times n}$ and $\boldsymbol{y} \in \mathcal{C}^\infty(I, \mathbb{R}^n)$. We define the space of solutions as $\mathcal{L}_0 = \{\boldsymbol{\varphi} \in \mathcal{C}^\infty(I, \mathbb{R}^n) : \boldsymbol{\varphi}' = \mathrm{A}\boldsymbol{\varphi}\}$.

a) Prove that $\mathcal{L}_0 \subseteq \mathcal{C}^\infty(I, \mathbb{R}^n)$ is a subspace
b) Prove that the following statements for solutions $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n \in \mathcal{L}_0$ are equivalent
   i) $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$ are linearly independent over $\mathbb{R}$
   ii) $\exists x_0 \in I$ such that $\boldsymbol{\varphi}_1(x_0), \ldots, \boldsymbol{\varphi}_n(x_0) \in \mathbb{R}^n$ linearly independent
   iii) $\det(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n) \not\equiv 0$
c) prove that $\dim \mathcal{L}_0 = n$

Subtask a): Let $\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2 \in \mathcal{L}_0$ and $\lambda, \mu \in \mathbb{R}$. Let $\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}_1 + \mu\boldsymbol{\varphi}_2$. Now $\boldsymbol{\varphi} \in \mathcal{L}_0 \iff \boldsymbol{\varphi}' = \mathrm{A}\boldsymbol{\varphi}$:

$$\boldsymbol{\varphi}' = \lambda\boldsymbol{\varphi}_1' + \mu\boldsymbol{\varphi}_2' = \lambda\mathrm{A}\boldsymbol{\varphi} + \mu\mathrm{A}\boldsymbol{\varphi}_2 = \mathrm{A}(\lambda\boldsymbol{\varphi}_1 + \mu\boldsymbol{\varphi}_2) = \mathrm{A}\boldsymbol{\varphi}$$

Thus $\boldsymbol{\varphi} \in \mathcal{L}_0$, hence $\mathcal{L}_0$ is a subspace.

Subtask b): i) $\implies$ ii): Recall that $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n \in \mathcal{L}_0$ are linearly independent iff

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{\varphi}_i \equiv \mathbf{0} \iff \lambda_1 = \cdots = \lambda_n = 0$$

And $f \equiv 0$ if $\forall x \in I\colon f(x) = 0$, thus

$$\forall x \in I\colon \sum_{i=1}^{n} \lambda_i \boldsymbol{\varphi}_i(x) = \mathbf{0} \iff \lambda_1 = \cdots = \lambda_n = 0$$

But then $\forall x \in I$ we have that $\boldsymbol{\varphi}_1(x), \ldots, \boldsymbol{\varphi}_n(x)$ are linearly independent in $\mathbb{R}^n$.

ii) $\implies$ iii): Note that $\det(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n) \not\equiv 0$ iff $\exists x_0 \in I$ such that $\boldsymbol{\varphi}_1(x_0), \ldots, \boldsymbol{\varphi}_n(x_0)$ are linearly independent, since $\det(\boldsymbol{\varphi}_1(x_0), \ldots, \boldsymbol{\varphi}_n(x_0)) \neq 0$ and thus $\exists x_0 \in I\colon \det(\boldsymbol{\varphi}_1(x_0), \ldots, \boldsymbol{\varphi}_n(x_0)) \neq 0 \iff \det(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n) \not\equiv 0$.

iii) $\implies$ i): Recall that iff $\det(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \neq 0$, then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent, thus $\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n$ are linearly independent, since $\det(\boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_n) \not\equiv 0$.

Subtask c): Let $\chi = \frac{a+b}{2}$. Consider $\boldsymbol{\varphi} \in \mathcal{L}_0$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{\varphi}' = \lambda\boldsymbol{\varphi}$. Now for $\boldsymbol{c} \in \mathbb{R}^n$:

$$\boldsymbol{\varphi}(x) = \mathrm{e}^{x-\chi}\boldsymbol{c} \implies \boldsymbol{\varphi}' = \mathrm{e}^{x-\chi}\boldsymbol{c} = \boldsymbol{\varphi} \qquad \lambda = 1$$

We can find n linearly independent vectors $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n \in \mathbb{R}^n$ and define

$$\boldsymbol{\varphi}_i(x) = \mathrm{e}^{x-\chi}\boldsymbol{c}_i$$

Notice that $\boldsymbol{\varphi}_i$ are linearly independent, since $\exists x_0 \in I$ such that $\boldsymbol{\varphi}_1(x_0), \ldots, \boldsymbol{\varphi}_n(x_0)$ are linearly, namely $x_0 = \chi \in I$, since

$$\boldsymbol{\varphi}_i(\chi) = \boldsymbol{c}_i \in \mathbb{R}^n$$

and we know that $\boldsymbol{c}_i$ are linearly independent. We found the maximum number of linearly independent vectors in $\mathcal{L}_0$ to be n, thus $\dim \mathcal{L}_0 = n$.

Task 44: Solutions of differential equations II

Let $I \subseteq \mathbb{R}$ be an open interval and $\mathrm{A} \in \mathbb{R}^{n\times n}$. We consider the following homogenous linear differential equations

$$\boldsymbol{y}' = \mathrm{A}\boldsymbol{y} \tag{1}$$

We already know that the corresponding solution-space $\mathcal{L}_0$ has dimension n. To find solutions we choose the ansatz

$$\boldsymbol{\varphi}(t) = \mathrm{e}^{\lambda t}\boldsymbol{v}$$

where $\lambda \in \mathbb{R}$ and $\boldsymbol{v} \in \mathbb{R}^n$. Prove the following:

a) $\boldsymbol{\varphi}(t) = \mathrm{e}^{\lambda t}\boldsymbol{v} \neq \mathbf{0}$ is a solution of eq. (1) iff $\mathrm{A}\boldsymbol{v} = \lambda\boldsymbol{v}$

b) solutions $\boldsymbol{\varphi}_i(t) = \mathrm{e}^{\lambda_i t}\boldsymbol{v}_i$, $i = 1,\dots,n$ are linearly independent iff $\boldsymbol{v}_i$ are linearly independent

Subtask a): $\implies$: Let $\boldsymbol{\varphi} \in \mathcal{L}_0$, i.e. $\boldsymbol{\varphi}' = \mathrm{A}\boldsymbol{\varphi}$:

$$\lambda\mathrm{e}^{\lambda t}\boldsymbol{v} = \boldsymbol{\varphi}'(t) = \mathrm{A}\boldsymbol{\varphi}(t) = \mathrm{A}\mathrm{e}^{\lambda t}\boldsymbol{v} \iff \mathrm{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$

hence $\lambda \in \mathrm{spec}(\mathrm{A})$ and $\boldsymbol{v} \in \ker(\lambda\mathrm{I} - \mathrm{A})$.

$\impliedby$: Let $\mathrm{A}\boldsymbol{v} = \lambda\boldsymbol{v}$:

$$\mathrm{A}\boldsymbol{\varphi}(t) = \mathrm{A}\mathrm{e}^{\lambda t}\boldsymbol{v} = \mathrm{e}^{\lambda t}\mathrm{A}\boldsymbol{v} = \lambda\mathrm{e}^{\lambda t}\boldsymbol{v} = \boldsymbol{\varphi}'(t)$$

Thus $\boldsymbol{\varphi} \in \mathcal{L}_0$.

Subtask b): From task 43 b), we know that $\boldsymbol{\varphi}_i$ are linearly independent, iff there exists $x_0 \in I$ such that $\boldsymbol{\varphi}_i(x_0)$ are linearly independent. We consider the shifted solutions $\widetilde{\boldsymbol{\varphi}}_i(t) = \boldsymbol{\varphi}_i(t-\chi)$, where $\chi \in I$. Notice that for $t = \chi$ we get $\widetilde{\boldsymbol{\varphi}}_i(\chi) = \boldsymbol{v}_i$.

$\implies$: If $\widetilde{\boldsymbol{\varphi}}_i$ are linearly independent, then $\widetilde{\boldsymbol{\varphi}}_i(x)$ are linearly independent for all $x \in I$. Thus for $x = \chi$ we get $\widetilde{\boldsymbol{\varphi}}_i(\chi) = \boldsymbol{v}_i$ are linearly independent.

$\impliedby$: If $\boldsymbol{v}_i$ are linearly independent, then $\exists x_0 \in I$, namely $x_0 = \chi$, such that $\boldsymbol{\varphi}_i(x_0)$ are linearly independent. From task 43 b) we can directly follow, that $\boldsymbol{\varphi}_i$ are linearly independent.

Task 45: Solutions of differential equations III

If a mass is suspended on a spring, which is elongated to position $z(t_0) = \alpha$ at time $t_0 = 0$ and velocity $\dot{z}(0) = \beta$, then the following motion of the mass is described via the following differential equation

$$\ddot{z} + 2\mu\dot{z} + \omega^2 z = 0 \tag{2}$$

where $\mu > 0$ as a coefficient of friction and $\omega > 0$ is a spring-constant. Write eq. (2) in the form of eq. (1) by defining $\boldsymbol{y} = \begin{bmatrix} z & \dot{z}\end{bmatrix}^{\mathrm{t}}$.

$$\ddot{z} = -2\mu\dot{z} - \omega^2 z$$
$$\dot{\boldsymbol{y}} = \begin{bmatrix}\dot{z}\\ \ddot{z}\end{bmatrix} = \begin{bmatrix}\dot{z}\\ -2\mu\dot{z} - \omega^2 z\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega^2 & -2\mu\end{bmatrix}\begin{bmatrix}z\\ \dot{z}\end{bmatrix} \qquad \boldsymbol{y}(0) = \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

Task 46: Solutions of differential equations IV

Let $\alpha, \beta \in \mathbb{R}$ and $\mu, \omega > 0$. We work with the system $\dot{\boldsymbol{y}} = \mathrm{A}\boldsymbol{y}$ with initial condition $\boldsymbol{y}(0) = \begin{bmatrix}\alpha & \beta\end{bmatrix}^t$ and

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\mu \end{bmatrix}$$

a) In the case $\mu > \omega$ is A diagonalizable with a real matrix. Compute a basis of $\mathbb{R}^2$ of eigenvectors of A and find a basis of the solution space. Find the solution that satisfies the initial condition.

b) For $\mu < \omega$ ist A complex diagonalizable. Compute the eigenvalues of A and find a basis of $\mathbb{C}^2$ of eigenvectors of A. Find the solution satisfying the initial condition. (You may use without proof, that for a complex eigenvector $\boldsymbol{v}$ of A corresponding to eigenvalue λ, that $\{\mathfrak{Re}(\mathrm{e}^{\lambda t}\boldsymbol{v}), \mathfrak{Im}(\mathrm{e}^{\lambda t}\boldsymbol{v})\}$ is a basis for the solution space)

Subtask a):

$$\begin{aligned}\chi_{\mathrm{A}}(\lambda) &= \lambda(\lambda + 2\mu) + \omega^2 = \lambda^2 + 2\lambda\mu + \omega^2 \\ \lambda_{1,2} &= -\mu \pm \sqrt{\mu^2 - \omega^2}\end{aligned}$$

Let $\Omega^2 = \mu^2 - \omega^2$, then for $\mu > \omega$ we have $\Omega^2 > 0$ and thus $\lambda_{1,2} = -\mu \pm \Omega$.

$$\begin{aligned}\lambda_1\mathrm{I} - \mathrm{A} &= \begin{bmatrix} -\mu + \Omega & -1 \\ \omega^2 & \mu + \Omega \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\mu + \Omega & -1 \\ \omega^2 + (\mu + \Omega)(\Omega - \mu) & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\mu + \Omega & -1 \\ \omega^2 + \Omega^2 - \mu^2 & 0 \end{bmatrix} = \begin{bmatrix} -\mu + \Omega & -1 \\ \omega^2 + \mu^2 - \omega^2 - \mu^2 & 0 \end{bmatrix} = \begin{bmatrix} -\mu + \Omega & -1 \\ 0 & 0 \end{bmatrix} \\ &\implies (\Omega - \mu)v_1 = v_2 \implies \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \Omega - \mu \end{bmatrix}\end{aligned}$$

And

$$\begin{aligned}\lambda_2\mathrm{I} - \mathrm{A} &= \begin{bmatrix} -\mu - \Omega & -1 \\ \omega^2 & \mu - \Omega \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\mu - \Omega & -1 \\ \omega^2 - (\mu + \Omega)(\mu - \Omega) & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\mu - \Omega & -1 \\ \omega^2 - (\mu^2 - \Omega^2) & 0 \end{bmatrix} = \begin{bmatrix} -\mu - \Omega & -1 \\ \omega^2 - \mu^2 + \Omega^2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\mu - \Omega & -1 \\ \omega^2 - \mu^2 + \mu^2 - \omega^2 & 0 \end{bmatrix} = \begin{bmatrix} -\mu - \Omega & -1 \\ 0 & 0 \end{bmatrix} \\ &\implies -(\mu + \Omega)v_1 = v_2 \implies \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\mu - \Omega \end{bmatrix}\end{aligned}$$

Now we have the following solutions

$$\boldsymbol{\varphi}_1(t) = \mathrm{e}^{\lambda_1 t}\boldsymbol{v}_1 \qquad \boldsymbol{\varphi}_2(t) = \mathrm{e}^{\lambda_2 t}\boldsymbol{v}_2 \implies \mathcal{L}_0 = \mathrm{span}(\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2)$$

We want to find the solution that satisfies the initial condition:

$$\begin{aligned}&\boldsymbol{y}(0) = a\boldsymbol{v}_1 + b\boldsymbol{v}_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \iff \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &\mathrm{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\mu + \Omega & -\mu - \Omega \end{bmatrix} \implies \mathrm{V}^{-1} = \frac{1}{-\mu - \Omega - \Omega + \mu}\begin{bmatrix} -\mu - \Omega & -1 \\ \mu - \Omega & 1 \end{bmatrix} = \frac{1}{2\Omega}\begin{bmatrix} \mu + \Omega & 1 \\ \Omega - \mu & -1 \end{bmatrix} \\ &\implies \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{2\Omega}\begin{bmatrix} \mu + \Omega & 1 \\ \Omega - \mu & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{2\Omega}\begin{bmatrix} \alpha(\mu + \Omega) + \beta \\ \alpha(\Omega - \mu) - \beta \end{bmatrix} \\ &\implies \boldsymbol{y}(t) = \frac{1}{2\Omega}\mathrm{e}^{-\mu t}\left(\begin{bmatrix} \alpha(\mu + \Omega) + \beta \\ (\alpha(\mu + \Omega) + \beta)(\Omega - \mu) \end{bmatrix}\mathrm{e}^{\Omega t} + \begin{bmatrix} \alpha(\Omega - \mu) - \beta \\ (\alpha(\Omega - \mu) - \beta)(-\mu - \Omega) \end{bmatrix}\mathrm{e}^{-\Omega t}\right)\end{aligned}$$

We can simplify that

$$\frac{1}{2\Omega}\mathrm{e}^{-\mu t}\begin{bmatrix} \alpha\mu(\mathrm{e}^{\Omega t} - \mathrm{e}^{-\Omega t}) + \alpha\Omega(\mathrm{e}^{\Omega t} + \mathrm{e}^{-\Omega t}) + \beta(\mathrm{e}^{\Omega t} - \mathrm{e}^{-\Omega t}) \\ (\alpha\mu\Omega + \alpha\Omega^2 + \beta\Omega - \alpha\mu^2 - \alpha\mu\Omega - \beta\mu)\mathrm{e}^{\Omega t} + (-\alpha\mu\Omega + \alpha\mu^2 + \beta\mu - \alpha\Omega^2 + \alpha\mu\Omega + \beta\Omega)\mathrm{e}^{-\Omega t} \end{bmatrix}$$

$$
\begin{aligned}
&= \frac{1}{2\Omega}\mathrm{e}^{-\mu t}\begin{bmatrix} 2\alpha\mu\sinh(\Omega t) + 2\alpha\Omega\cosh(\Omega t) + 2\beta\sinh(\Omega t) \\ (\alpha(\Omega^2-\mu^2)+\beta(\Omega-\mu))\mathrm{e}^{\Omega t} + (\alpha(\mu^2-\Omega^2)+\beta(\Omega+\mu))\mathrm{e}^{-\Omega t}\end{bmatrix} \\
&= \frac{1}{2\Omega}\mathrm{e}^{-\mu t}\begin{bmatrix} 2\alpha(\mu\sinh(\Omega t) + \Omega\cosh(\Omega t)) + 2\beta\sinh(\Omega t) \\ \alpha(\Omega^2-\mu^2)(\mathrm{e}^{\Omega t}-\mathrm{e}^{-\Omega t}) + \beta\Omega(\mathrm{e}^{\Omega t}+\mathrm{e}^{-\Omega t}) - \beta\mu(\mathrm{e}^{\Omega t}+\mathrm{e}^{-\Omega t})\end{bmatrix} \\
&= \frac{1}{2\Omega}\mathrm{e}^{-\mu t}\begin{bmatrix} 2\alpha(\mu\sinh(\Omega t) + \Omega\cosh(\Omega t)) + 2\beta\sinh(\Omega t) \\ 2\alpha(\Omega^2-\mu^2)\sinh(\Omega t) + 2\beta\cosh(\Omega t)(\Omega-\mu)\end{bmatrix} = \frac{1}{\Omega}\mathrm{e}^{-\mu t}\begin{bmatrix} \alpha(\mu\sinh(\Omega t) + \Omega\cosh(\Omega t)) + \beta\sinh(\Omega t) \\ \alpha(\Omega^2-\mu^2)\sinh(\Omega t) + \beta\cosh(\Omega t)(\Omega-\mu)\end{bmatrix}
\end{aligned}
$$

Subtask b): If $\mu < \omega$, then $\sqrt{\mu^2-\omega^2} = \sqrt{(-1)(\omega^2-\mu^2)} = \mathrm{i}\sqrt{\omega^2-\mu^2} = \mathrm{i}\Omega$ and $\Omega \in \mathbb{R}$. Now A has eigenvalues $-\mu \pm \mathrm{i}\Omega$.

$$
\begin{aligned}
\lambda_1\mathrm{I} - \mathrm{A} &= \begin{bmatrix} -\mu+\mathrm{i}\Omega & -1 \\ \omega^2 & \mu+\mathrm{i}\Omega \end{bmatrix} \rightsquigarrow \begin{bmatrix} -\mu+\mathrm{i}\Omega & -1 \\ \omega^2 + (\mathrm{i}\Omega-\mu)(\mathrm{i}\Omega+\mu) & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\mu+\mathrm{i}\Omega & -1 \\ \omega^2 + \mathrm{i}^2\Omega^2 - \mu^2 & 0 \end{bmatrix} = \begin{bmatrix} -\mu+\mathrm{i}\Omega & -1 \\ \omega^2 - \omega^2 + \mu^2 - \mu^2 & 0 \end{bmatrix} = \begin{bmatrix} -\mu+\mathrm{i}\Omega & -1 \\ 0 & 0 \end{bmatrix} \\
&\implies v_1(\mathrm{i}\Omega - \mu) = v_2 \implies \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \mathrm{i}\Omega - \mu \end{bmatrix} \implies \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\mathrm{i}\Omega - \mu \end{bmatrix}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\mathrm{e}^{\lambda t} = \mathrm{e}^{-\mu t + \mathrm{i}\Omega t} = \mathrm{e}^{-\mu t}(\cos(\Omega t) + \mathrm{i}\sin(\Omega)t) \\
&\implies \mathfrak{Re}(\boldsymbol{v}\mathrm{e}^{\lambda t}) = \begin{bmatrix} 1 \\ -\mu \end{bmatrix}\mathrm{e}^{-\mu t}\cos(\Omega t) \\
&\implies \mathfrak{Im}(\boldsymbol{v}\mathrm{e}^{\lambda t}) = \begin{bmatrix} 0 \\ \Omega \end{bmatrix}\mathrm{e}^{-\mu t}\sin(\Omega t) \\
&\implies \boldsymbol{y}(t) = \mathrm{e}^{-\mu t}\left(a\begin{bmatrix} 1 \\ -\mu \end{bmatrix}\cos(\Omega t) + b\begin{bmatrix} 0 \\ \Omega \end{bmatrix}\sin(\Omega t)\right)
\end{aligned}
$$