Exercise Sheet № 13

Task 73

Given is a skew-symmetric matrix $\mathrm{A} \in \mathbb{R}^{2\times 2}$ with eigenvalue $\lambda = \mathrm{e}^{\mathrm{i}\alpha}$ for $\alpha \in [0, 2\pi)$. Let $\boldsymbol{v}$ be an eigenvector corresponding to λ with $\|\boldsymbol{v}\| = 1$.

a) Prove

$$\|\mathfrak{Re}(\boldsymbol{v})\| = \|\mathfrak{Im}(\boldsymbol{v})\| = \frac{1}{\sqrt{2}}$$

b) Verify the following

$$\begin{aligned}\mathrm{A}\mathfrak{Re}(\boldsymbol{v}) &= \cos(\alpha)\mathfrak{Re}(\boldsymbol{v}) - \sin(\alpha)\mathfrak{Im}(\boldsymbol{v})\\ \mathrm{A}\mathfrak{Im}(\boldsymbol{v}) &= \sin(\alpha)\mathfrak{Re}(\boldsymbol{v}) + \cos(\alpha)\mathfrak{Im}(\boldsymbol{v})\end{aligned}$$

c) Let

$$\mathrm{S} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathfrak{Re}(\boldsymbol{v}) & \mathfrak{Im}(\boldsymbol{v})\end{bmatrix}$$

Compute $\mathrm{S}^{\mathrm{t}}\mathrm{AS}$.

Let A be skew-symmetric. Based on $\det(\mathrm{A}) = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{-\mathrm{i}\alpha} = 1$ and $\mathrm{trace}(\mathrm{A}) = \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha} = 2\cos(\alpha)$:

$$\begin{aligned}&\chi_{\mathrm{A}}(\lambda) = \lambda^2 - \mathrm{trace}(\mathrm{A})\lambda + \det(\mathrm{A}) = \lambda^2 - 2\lambda\cos(\alpha) + 1\\ &= \lambda^2 - 2\lambda\cos(\alpha) + \cos^2(\alpha) + \sin^2(\alpha) = (\lambda - \cos(\alpha))^2 + \sin^2(\alpha)\\ &\implies \mathrm{A} = \begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\end{aligned}$$

Subtask a): Using this knowledge:

$$\begin{aligned}&\mathrm{e}^{\mathrm{i}\alpha}\mathrm{I} - \mathrm{A} = \begin{bmatrix}\mathrm{i}\sin(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \mathrm{i}\sin(\alpha)\end{bmatrix} \xrightarrow{\mathrm{i}I - II} \begin{bmatrix}\mathrm{i}\sin(\alpha) & \sin(\alpha)\\ 0 & 0\end{bmatrix} \implies \mathrm{i}\sin(\alpha)v_1 = -\sin(\alpha)v_2\\ &\iff v_1 = -\frac{1}{\mathrm{i}}v_2 = \mathrm{i}v_2 \implies \boldsymbol{v} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathrm{i}\\ 1\end{bmatrix} \implies \mathfrak{Re}(\boldsymbol{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 1\end{bmatrix} \quad \mathfrak{Im}(\boldsymbol{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ 0\end{bmatrix}\\ &\implies \|\mathfrak{Re}(\boldsymbol{v})\| = \frac{1}{\sqrt{2}} = \|\mathfrak{Im}(\boldsymbol{v})\|\end{aligned}$$

Subtask b):

$$\begin{aligned}&\mathrm{A}\mathfrak{Re}(\boldsymbol{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = -\frac{\sin(\alpha)}{\sqrt{2}}\begin{bmatrix}1\\ 0\end{bmatrix} + \frac{\cos(\alpha)}{\sqrt{2}}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \cos(\alpha)\mathfrak{Re}(\boldsymbol{v}) - \sin(\alpha)\mathfrak{Im}(\boldsymbol{v})\\ &\mathrm{A}\mathfrak{Im}(\boldsymbol{v}) = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \frac{\cos(\alpha)}{\sqrt{2}}\begin{bmatrix}1\\ 0\end{bmatrix} + \frac{\sin(\alpha)}{\sqrt{2}}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \cos(\alpha)\mathfrak{Im}(\boldsymbol{v}) + \sin(\alpha)\mathfrak{Re}(\boldsymbol{v})\end{aligned}$$

Subtask c):

$$\begin{aligned}&\mathrm{S}^{\mathrm{t}}\mathrm{AS} = 2\begin{bmatrix}\mathfrak{Re}(\boldsymbol{v}) & \mathfrak{Im}(\boldsymbol{v})\end{bmatrix}^{\mathrm{t}}\begin{bmatrix}\mathrm{A}\mathfrak{Re}(\boldsymbol{v}) & \mathrm{A}\mathfrak{Im}(\boldsymbol{v})\end{bmatrix}\\ &= 2\begin{bmatrix}\mathfrak{Re}(\boldsymbol{v})^{\mathrm{t}}\\ \mathfrak{Im}(\boldsymbol{v})^{\mathrm{t}}\end{bmatrix}\begin{bmatrix}\cos(\alpha)\mathfrak{Re}(\boldsymbol{v}) - \sin(\alpha)\mathfrak{Im}(\boldsymbol{v}) & \cos(\alpha)\mathfrak{Im}(\boldsymbol{v}) + \sin(\alpha)\mathfrak{Re}(\boldsymbol{v})\end{bmatrix}\\ &= 2\begin{bmatrix}\cos(\alpha)\|\mathfrak{Re}(\boldsymbol{v})\|^2 - \sin(\alpha)\mathfrak{Re}(\boldsymbol{v})^{\mathrm{t}}\mathfrak{Im}(\boldsymbol{v}) & \sin(\alpha)\|\mathfrak{Re}(\boldsymbol{v})\|^2 + \cos(\alpha)\mathfrak{Re}(\boldsymbol{v})^{\mathrm{t}}\mathfrak{Im}(\boldsymbol{v})\\ \cos(\alpha)\mathfrak{Im}(\boldsymbol{v})^{\mathrm{t}}\mathfrak{Re}(\boldsymbol{v}) - \sin(\alpha)\|\mathfrak{Im}(\boldsymbol{v})\|^2 & \cos(\alpha)\|\mathfrak{Im}(\boldsymbol{v})\|^2 + \sin(\alpha)\mathfrak{Im}(\boldsymbol{v})^{\mathrm{t}}\mathfrak{Re}(\boldsymbol{v})\end{bmatrix}\\ &= \begin{bmatrix}\cos(\alpha) & \sin(\alpha)\\ -\sin(\alpha) & \cos(\alpha)\end{bmatrix} = \mathrm{A}^{\mathrm{t}}\end{aligned}$$

Task 74: Lemma of Fitting

Let $G \in \mathrm{End}(V)$ and $G^0 = \mathrm{id}$. Prove the following statements without using the lemma of Fitting.

a) $V \supseteq \mathrm{im}G \supseteq \mathrm{im}G^2 \supseteq \mathrm{im}G^3 \supseteq \dots$

We want to show, that $\forall k \in \mathbb{N}_0\colon \mathrm{im}(G^{k+1}) \subseteq \mathrm{im}(G^k)$. We can achieve this via a simple induction. For $k = 0$ we get $\mathrm{im}(G) \subseteq \mathrm{im}(\mathrm{id}) = V$ which is trivially fulfilled, since $G \in \mathrm{End}(V)$.