

## Exercise Sheet № 14

### Task 78: Matrix Exponential

Let

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Compute  $e^A$ .

We begin by diagonalizing  $A$ :

$$\begin{aligned} \chi_A(\lambda) &= (\lambda + 2)(\lambda + 3) - 2 = \lambda^2 + 5\lambda + 4 \\ \implies \lambda_1 &= -4 \quad \lambda_2 = -1 \\ -4I - A &= \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ -I - A &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ V &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \implies V^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

Given a diagonalizable matrix  $A = V \text{diag}(\lambda_1, \dots, \lambda_n) V^{-1}$  we have:

$$\begin{aligned} e^A &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} A^k = \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} (V \text{diag}(\lambda_1, \dots, \lambda_n) V^{-1})^k \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} V \text{diag}(\lambda_1^k, \dots, \lambda_n^k) V^{-1} = \lim_{m \rightarrow \infty} V \sum_{k=0}^m \frac{1}{k!} \text{diag}(\lambda_1^k, \dots, \lambda_n^k) V^{-1} \\ &= V \left( \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} \text{diag}(\lambda_1^k, \dots, \lambda_n^k) \right) V^{-1} = V \left( \lim_{m \rightarrow \infty} \sum_{k=0}^m \text{diag} \left( \frac{\lambda_1^k}{k!}, \dots, \frac{\lambda_n^k}{k!} \right) \right) V^{-1} \\ &= V \left( \lim_{m \rightarrow \infty} \text{diag} \left( \sum_{k=0}^m \frac{\lambda_1^k}{k!}, \dots, \sum_{k=0}^m \frac{\lambda_n^k}{k!} \right) \right) V^{-1} = V \text{diag} \left( \sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!}, \dots, \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \right) V^{-1} \\ &= V \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) V^{-1} \end{aligned}$$

Thus:

$$e^A = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-4} & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-4} & e^{-1} \\ -2e^{-4} & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-4} - e^{-1} & -e^{-4} + e^{-1} \\ -2e^{-4} + 2e^{-1} & 2e^{-4} + e^{-1} \end{bmatrix}$$

**Task 80: Matrix Exponential of Skew-Symmetric Matrix**

Let  $A \in \mathbb{R}^{n \times n}$  be skew-symmetric. Prove that  $e^A$  is an orthogonal matrix.

We first consider  $M \in \mathbb{R}^{n \times n}$  and compute  $\exp(M^t)$ :

$$\begin{aligned}\exp(M^t) &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} (M^t)^k = \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} (M^k)^t \\ &= \lim_{m \rightarrow \infty} \left( \sum_{k=0}^m M^k \right)^t = \left( \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} M^k \right)^t = \exp(M)^t\end{aligned}$$

If  $e^A$  is orthogonal, then  $\exp(A)(\exp(A))^t = I$ :

$$\exp(A)(\exp(A))^t = \exp(A) \exp(A^t) = \exp(A + A^t) = \exp(A - A) = \exp(0)$$

By convention we set  $0^0 = I$  and get

$$\exp(0) = \sum_{k=0}^{\infty} \frac{1}{k!} 0^k = \frac{1}{0!} 0^0 = I$$