Exercise Sheet № 2

Task 7: Transition Matrix I

Let $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$ and $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for $\mathbb{R}^3$ with

$$\boldsymbol{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$\boldsymbol{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

a) Find the transition matrix $\mathrm{M}(\mathcal{B}, \mathcal{C})$
b) Let

$$\boldsymbol{w} = \begin{bmatrix} -5 \\ 8 \\ -5 \end{bmatrix}$$

and compute the coordinate vector of $\boldsymbol{w}$ with respect to $\mathcal{B}$
c) Compute the coordinate vector of $\boldsymbol{w}$ with respect to $\mathcal{C}$

Let V, W, $\dim V = n$, $\dim W = m$, be vector spaces with bases B and C. Furthermore let $f \in \mathrm{Hom}(V, W)$. We want to find $g \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$, $\Phi_B \in \mathrm{Hom}(V, \mathbb{K}^n)$ and $\Phi_C \in \mathrm{Hom}(W, \mathbb{K}^m)$, such that

$$f = \Phi_C^{-1} \circ g \circ \Phi_B$$

We call g the matrix-representation of f with respect to B and C. We define the following isomorphism

$$\Phi_B(\boldsymbol{x}) = \Phi_B\left(\sum_{k=1}^{n} x_k \boldsymbol{b}_k\right) = \sum_{k=1}^{n} x_k \boldsymbol{e}_k$$

where $\boldsymbol{e}_k$ are the canonical basis-vectors of $\mathbb{K}^n$. Since Φ_B is linear, we get

$$\Phi_B(\boldsymbol{x}) = \sum_{k=1}^{n} x_k \Phi_B(\boldsymbol{e}_k)$$

For Φ_C^{-1} we get

$$\Phi_C^{-1}(\boldsymbol{x}) = \sum_{k=1}^{m} x_k \boldsymbol{f}_k = \sum_{k=1} x_k \Phi_C^{-1}(\boldsymbol{f}_k)$$

where $\boldsymbol{f}_k$ are the canonical basis-vectors of $\mathbb{K}^m$. For $V = W = \mathbb{K}^n$, we call $\Phi_C^{-1}\Phi_B$ the transition matrix from B to C.

Let $\mathrm{B} = \begin{bmatrix} \boldsymbol{u}_1 & \boldsymbol{u}_2 & \boldsymbol{u}_3 \end{bmatrix}$ and $\mathrm{C} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \boldsymbol{v}_3 \end{bmatrix}$. We first find C^{-1}:

$$\mathrm{C}^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 & 1 \\ -2 & 1 & -1 \\ 2 & 4 & 2 \end{bmatrix} \implies \Phi_C^{-1}\Phi_B = \begin{bmatrix} 3 & 2 & \frac{5}{2} \\ -2 & -3 & -\frac{1}{2} \\ 5 & 1 & 6 \end{bmatrix}$$

Subtask b). To find the coordinate vector of $\boldsymbol{w}$ with respect to $\mathcal{B}$, we need to find the Transition Matrix $\Phi_B^{-1}\Phi_E$, where $E = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ is the canonical basis of $\mathbb{R}^3$. Hence we compute B^{-1}:

$$\mathrm{B}^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -5 \\ -1 & -1 & 3 \\ -2 & 0 & 4 \end{bmatrix} \implies \Phi_B^{-1}\Phi_E = \mathrm{B}^{-1}$$

Hence

$$\mathrm{B}^{-1}\boldsymbol{w} = \begin{bmatrix} 9 \\ -9 \\ -5 \end{bmatrix} = \widetilde{\boldsymbol{w}}$$

Subtask c). Using our knowledge of $\Phi_C^{-1}\Phi_B$, we can directly compute the coordinate of $\widetilde{\boldsymbol{w}}$ with respect to the basis $\mathcal{C}$:

$$\Phi_C^{-1}\Phi_B\widetilde{\boldsymbol{w}} = \frac{1}{2}\begin{bmatrix} -7 \\ 23 \\ 12 \end{bmatrix}$$

Task 8: Simple Function-Space

Let V be the space spanned by $f_1 = \sin$ and $f_2 = \cos$.

a) Show that $g_1 = 2\sin + \cos$ and $g_2 = 3\cos$ form a basis of V

b) Find the transition matrix from $\mathcal{C} = \{g_1, g_2\}$ to $\mathcal{B} = \{f_1, f_2\}$

c) Find the transition matrix from $\mathcal{B}$ to $\mathcal{C}$

d) Let $h = 2\sin - 5\cos$. Find the coordinate vector of h with respect to $\mathcal{C}$

We see that $V \simeq \mathbb{R}^2$, since $\dim V = 2$. Hence if g_1 and g_2 are linearly independent, they form a basis of V. We use Φ_B, since linear independence is invariant under isomorphisms, and check wether $\Phi_B(g_1)$ and $\Phi_B(g_2)$ are linearly independent:

$$\begin{bmatrix} \Phi_B(g_1) & \Phi_B(g_2) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \overset{II-I}{\rightsquigarrow} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Hence g_1 and g_2 are linearly independent.

Subtask b): We find the transition matrix from $(\Phi_B(f_1), \Phi_B(f_2)) = \mathrm{B}$ to $(\Phi_B(g_1), \Phi_B(g_2)) = \mathrm{C}$:

$$\mathrm{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{C} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \implies \mathrm{C}^{-1} = \frac{1}{6}\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$
$$\Phi_C^{-1}\Phi_B = \mathrm{C}^{-1}$$

Subtask c): Notice $(\Phi_C^{-1}\Phi_B)^{-1} = \Phi_B^{-1}\Phi_C$, thus:

$$\Phi_B^{-1}\Phi_C = \mathrm{C}$$

Subtask d): We use the transition matrix $\Phi_C^{-1}\Phi_B$ on $\Phi_B(h)$:

$$\Phi_C^{-1}\Phi_B\Phi_B(h) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Task 9: Transition Matrix II

Let

$$\mathrm{P} = \begin{bmatrix} 5 & 8 & 6 & -13 \\ 3 & -1 & 0 & -9 \\ 0 & 1 & -1 & 0 \\ 2 & 4 & 3 & 5 \end{bmatrix} \qquad \boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 3 \\ -5 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix} 3 \\ -1 \\ 0 \\ -9 \end{bmatrix} \qquad \boldsymbol{v}_4 = \begin{bmatrix} 5 \\ 8 \\ 6 \\ -13 \end{bmatrix}$$

Find a basis $\mathcal{B} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \boldsymbol{u}_4\}$ of $\mathbb{R}^4$, for which P is the transition matrix from $\mathcal{B}$ to $\mathcal{C} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$.

We get:

$$\Phi_C^{-1}\Phi_B = \mathrm{P} \iff \mathrm{C}^{-1}\mathrm{B} = \mathrm{P} \iff \mathrm{B} = \mathrm{CP}$$

Hence

$$\mathrm{B} = \begin{bmatrix} 20 & 39 & 24 & -1 \\ 39 & 62 & 49 & -21 \\ 24 & 49 & 36 & 0 \\ -51 & -101 & -60 & 0 \end{bmatrix}$$

Task 10: Similar Matrices

We call two matrices $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{n\times n}$ similar, of there exists $\mathrm{P} \in \mathrm{GL}(n, \mathbb{R})$, such that $\mathrm{B} = \mathrm{P}^{-1}\mathrm{AP}$.

a) Consider the matrices:

$$\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Is A similar to B?

b) Show, that for $a, b \in \mathbb{R}$, the following matrices are similar:

$$\mathrm{A} = \begin{bmatrix} 1 & a \\ 0 & 2 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 1 & b \\ 0 & 2 \end{bmatrix}$$

c) Show that the following matrices are not similar:

$$\mathrm{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Subtask a). We can simply solve $\mathrm{PB} - \mathrm{AP} = 0$

$$\mathrm{AP} = \begin{bmatrix} 2p_1 + p_3 & 2p_2 + p_4 \\ 3p_3 & 3p_4 \end{bmatrix} \qquad \mathrm{PB} = \begin{bmatrix} 2p_1 & 3p_2 \\ 2p_3 & 3p_4 \end{bmatrix}$$

From $2p_3 - 3p_3 = 0$ we get $p_3 = 0$, hence:

$$\mathrm{PB} - \mathrm{AP} = \begin{bmatrix} 0 & p_2 - p_4 \\ 0 & 0 \end{bmatrix}$$

Thus $p_2 = p_4$ and $p_1, p_2 \in \mathbb{R}$.

Subtask b).

$$\mathrm{AP} = \begin{bmatrix} p_1 + ap_3 & p_2 + ap_4 \\ 2p_3 & 2p_4 \end{bmatrix} \qquad \mathrm{PB} = \begin{bmatrix} p_1 & bp_1 + 2p_2 \\ p_3 & bp_3 + 2p_4 \end{bmatrix}$$

From $2p_3 - p_3 = 0$ we again get $p_3 = 0$, hence

$$\mathrm{PB} - \mathrm{AP} = \begin{bmatrix} 0 & bp_1 + p_2 - ap_4 \\ 0 & 0 \end{bmatrix}$$

Thus for $a, b \neq 0$:

$$p_4 = \frac{bp_1 + p_2}{a}$$

Where $p_1, p_2 \in \mathbb{R}$. If $a = 0$, then $p_2 = -bp_1$. If $b = 0$, $p_2 = ap_4$. If $a = b = 0$ $p_2 = 0$.

Subtask c): Notice $\mathrm{B} = 2\mathrm{I}$, hence if we would find P, such that $\mathrm{PBP}^{-1} = \mathrm{A}$:

$$\mathrm{A} = 2\mathrm{I}$$

which is a contradiction.

Task 11: Properties of rank and trace

Let $\mathrm{A} \in \mathbb{R}^{n \times n}$ and $\mathrm{P} \in \mathrm{GL}(n, \mathbb{R})$. Prove that:

a) $\mathrm{trace}\mathrm{A} = \mathrm{trace}\mathrm{P}^{-1}\mathrm{AP}$

b) $\mathrm{rank}\mathrm{A} = \mathrm{rank}\mathrm{P}^{-1}\mathrm{AP}$

Subtask a):

$$\mathrm{trace}\mathrm{AB} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \mathrm{trace}\mathrm{BA}$$

Using this result, we get:

$$\mathrm{trace}\mathrm{P}^{-1}\mathrm{AP} = \mathrm{trace}\mathrm{PP}^{-1}\mathrm{A} = \mathrm{trace}\mathrm{A}$$

Subtask b): Since $\mathrm{rank}\mathrm{P}$ is n, we get $\mathrm{im}\mathrm{P} = \mathbb{R}^n$, thus $\mathrm{im}\mathrm{P}^{-1}\mathrm{A}|_{\mathrm{im}\mathrm{P}} = \mathrm{im}\mathrm{P}^{-1}\mathrm{A}$. Let $\mathrm{rank}\mathrm{A} = k \leq n$, then

$$\mathrm{im}\mathrm{P}^{-1}\mathrm{A} = \mathrm{im}\mathrm{P}^{-1}|_{\mathrm{im}\mathrm{A}}$$

Since $\mathrm{im}\mathrm{A}$ is a k-dimensional subspace of $\mathbb{R}^n$ and P^{-1} is bijective, $\dim\mathrm{im}\mathrm{P}^{-1}\mathrm{A} = k$.