

## Exercise Sheet № 4

Remark: Let  $A \subseteq \mathbb{R}^n$  be open and  $F: A \rightarrow \mathbb{R}$ . Suppose that there exists  $z_0 \in A$  and a neighborhood  $U \subset A$  of  $z_0$ , such that  $\nabla F$  exists on  $U$  and  $\nabla F \in \mathcal{C}(U)$ , then  $F$  is differentiable in  $z_0$ .

### Task 4.1: Continuity and Differentiability

Define  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  by setting

$$F(x, y) = \begin{cases} 0 & y = 0 \\ y^2 \cos\left(\frac{1}{y}\right) & y \neq 0 \end{cases}$$

- a) Prove that  $\partial_y F$  is not continuous in  $(x, 0)$  for  $x \in \mathbb{R}$
- b) Prove that  $F$  is differentiable in  $(x, 0)$  for  $x \in \mathbb{R}$

Subtask a:

$$\partial_y F(x, y) = \begin{cases} 0 & y = 0 \\ \sin\left(\frac{1}{y}\right) + 2y \cos\left(\frac{1}{y}\right) & y \neq 0 \end{cases}$$

Let  $f(y) = \partial_y F(x, y)$ . We prove that  $f(y)$  is discontinuous in  $y = 0$  for  $x \in \mathbb{R}$ . To prove that a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is discontinuous in  $y_0 \in \mathbb{R}$ , we use the negation of the sequence-criterion of continuity:

$$\exists (\xi_n)_{n \in \mathbb{N}}: \lim_{n \rightarrow \infty} F(\xi_n) \neq F(\lim_{n \rightarrow \infty} \xi_n)$$

Let

$$\begin{aligned} \xi_n &= \begin{bmatrix} x \\ \frac{1}{(4n+1)\pi} \end{bmatrix} \Rightarrow \lim_{n \rightarrow \infty} \xi_n = \begin{bmatrix} x \\ 0 \end{bmatrix} \\ \partial_y F(\xi_n) &= \sin\left(\frac{(4n+1)\pi}{2}\right) = 1 \Rightarrow \lim_{n \rightarrow \infty} \partial_y F(\xi_n) = 1 \\ \partial_y F(\lim_{n \rightarrow \infty} \xi_n) &= \partial_y F(x, 0) = 0 \neq 1 \end{aligned}$$

Therefore  $\partial_y F$  is discontinuous on  $(x, 0)$  for  $x \in \mathbb{R}$ .

Subtask b: Given that  $\forall x \in \mathbb{R}: F(x, 0) = 0$ , we get

$$\lim_{h \rightarrow 0} \frac{F(x, h) - F(x, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \rightarrow 0} h \cos \frac{1}{h} \leq \lim_{h \rightarrow 0} h = 0$$

Remark: Let  $A \subset \mathbb{R}^n$  be open and  $F: A \rightarrow \mathbb{R}$ . Suppose that there exists  $z_0 \in A$  and a neighborhood  $U \subset A$  of  $z_0$  such that  $\nabla^2 F$  exists and is continuous on  $U$ . Then  $\partial_{x_i x_j} F(z_0) = \partial_{x_j x_i} F(z_0)$  for all  $i, j \in \{1, \dots, n\}$ .

**Task 4.2**

Define  $F: \mathbb{R}^2 \rightarrow \mathbb{R}$  by setting

$$F(x, y) = \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{x^3 y}{x^2 + y^2} & \mathbf{x} \neq \mathbf{0} \end{cases}$$

- a) Prove that  $F$  is continuous on  $\mathbb{R}^2$
- b) Prove that  $F$  is differentiable on  $\mathbb{R}^2$  and compute  $\nabla F$
- c) Prove that  $\partial_{xy} F$  and  $\partial_{yx} F$  exist on  $\mathbb{R}^2$  and

$$\partial_{xy} F(\mathbf{0}) \neq \partial_{yx} F(\mathbf{0})$$

- d) Prove that  $\partial_{xy} F$  and  $\partial_{yx} F$  are not continuous in  $\mathbf{0}$

Subtask a: Since for  $\mathbf{x} \neq \mathbf{0}$  we get that  $F$  is continuous in  $\mathbf{x}$ . Therefore we want to show that  $F$  is continuous in  $\mathbf{0}$ . Let  $(\xi_n)_{n \in \mathbb{N}}$  with  $\lim_{n \rightarrow \infty} \xi_n = \mathbf{0}$ . Let  $\xi_n = [x_n \ y_n]^T$ , then, let  $n$  be sufficiently large, such that  $|x_n| < \frac{\varepsilon}{2}$  and  $|y_n| < \frac{\varepsilon}{2}$  for  $\varepsilon > 0$ .

$$|F(\xi_n)| = \left| \frac{x_n^3 y_n}{x_n^2 + y_n^2} \right| < \left| \frac{x_n^3 y_n}{x_n^2} \right| = |x_n y_n| < \frac{\varepsilon^2}{4}$$

Therefore  $\lim_{n \rightarrow \infty} |F(\xi_n)| = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(\xi_n) = 0 = F(\mathbf{0})$ , it follows that  $F$  is continuous in  $\mathbf{0}$ , and thus on  $\mathbb{R}^2$ .

Subtask b:  $F$  is differentiable on  $\mathbb{R}^2$ , if  $\partial_x F$  and  $\partial_y F$  exists and are continuous.

$$\begin{aligned} \partial_x F(x, y) &= \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{2x^2 y(x^2 + y^2) - x^3 y(2x)}{(x^2 + y^2)^2} = \partial_x f(x, y) & \mathbf{x} \neq \mathbf{0} \end{cases} \\ \partial_y F(x, y) &= \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{x^3(x^2 + y^2) - x^3 y(2y)}{(x^2 + y^2)^2} = \partial_y f(x, y) & \mathbf{x} \neq \mathbf{0} \end{cases} \\ \partial_x f(x, y) &= \frac{2x^4 y + 2x^2 y^3 - 2x^4 y}{(x^2 + y^2)^2} = \frac{2x^2 y^3}{(x^2 + y^2)^2} \\ \partial_y f(x, y) &= \frac{x^5 + x^3 y^2 - 2x^3 y^2}{(x^2 + y^2)^2} = \frac{x^5 - x^3 y^2}{(x^2 + y^2)^2} \end{aligned}$$

We change our coordinate system. Let  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ , then :

$$\begin{aligned} \partial_x F(x, y) &= \frac{2r^5 \cos^2(\theta) \sin^3(\theta)}{r^4} = 2r \cos^2(\theta) \sin^3(\theta) \\ \lim_{(x,y) \rightarrow \mathbf{0}} |\partial_x F(x, y)| &= \lim_{r \rightarrow 0} |2r \cos^2(\theta(r)) \sin^3(\theta(r))| \leq \lim_{r \rightarrow 0} |2r| = 0 \\ \Rightarrow \lim_{(x,y) \rightarrow \mathbf{0}} \partial_x F(x, y) &= 0 \end{aligned}$$

by the squeeze-theorem. Note we use an arbitrary angle  $\theta(r)$  in order to account for all paths leading to  $\mathbf{0}$ .

$$\begin{aligned} \partial_y F(x, y) &= \frac{r^5 \cos^5(\theta) - r^5 \cos^3(\theta) \sin^2(\theta)}{r^4} = r(\cos^5(\theta) - \cos^3(\theta) \sin^2(\theta)) \\ \lim_{(x,y) \rightarrow \mathbf{0}} |\partial_y F(x, y)| &= \lim_{r \rightarrow 0} |r(\cos^5(\theta(r)) - \cos^3(\theta(r)) \sin^2(\theta(r)))| \end{aligned}$$

$$\begin{aligned} &\leq \lim_{r \rightarrow 0} |r \cos^5(\theta(r))| + \lim_{r \rightarrow 0} |r \cos^3(\theta(r)) \sin^2(\theta(r))| \\ &\leq \lim_{r \rightarrow 0} |r| + \lim_{r \rightarrow 0} |r| = 0 \\ \Rightarrow \quad &\lim_{(x,y) \rightarrow \mathbf{0}} \partial_y F(x, y) = 0 \end{aligned}$$

Hence both  $\partial_x F$  and  $\partial_y F$  are continuous on  $\mathbb{R}^2$ , and thus  $F$  is differentiable on  $\mathbb{R}^2$ .

$$\nabla F = \begin{cases} \mathbf{0} & \mathbf{x} = \mathbf{0} \\ \begin{bmatrix} \frac{2x^2y^3}{(x^2+y^2)^2} \\ \frac{x^5-x^3y^2}{(x^2+y^2)^2} \end{bmatrix} & \mathbf{x} \neq \mathbf{0} \end{cases}$$

Subtask c:

$$\partial_y \partial_x F = \frac{6x^2y^2(x^2+y^2)^2 - 2x^2y^2}{(x^2+y^2)^2}$$