Exercise Sheet № 12

Task 68

Let $L \in \text{End}(V)$. A subspace $W \subseteq V$ is said to be invariant with respect to L, if $L[W] \subseteq W$. Prove the following statements:

a) A one-dimensional subspace is invariant under $L(\boldsymbol{x}) = \mathrm{A}\boldsymbol{x}$ iff it is an eigenspace of A
b) If W is an invariant subspace of A, then it is also invariant under A^2
c) Let $\mathrm{A} \in \mathbb{R}^{n\times n}$ be symmetric. If $\boldsymbol{v}$ is an eigenvector of A prove that $\boldsymbol{v}^\perp$ is an invariant subspace under A
d) If $W^\perp$ is an invariant subspace under A, then so is it's orthogonal complement $W^\perp$

Subtask a): Let $W \subseteq V$ be subspace with basis $\{\boldsymbol{b}\}$. Notice that $\forall \boldsymbol{w} \in W \colon \forall \lambda \in \mathbb{R} \colon \lambda\boldsymbol{w} \in W$, since W is a subspace.

$\implies$: Let W be invariant under A, i.e. $\forall \boldsymbol{w} \in W \colon \mathrm{A}\boldsymbol{w} \in W$. Hence $\exists \mu \in \mathbb{R} \colon \mathrm{A}\boldsymbol{w} = \mu\boldsymbol{b}$, since W is one-dimensional. Notice however, that $\mathrm{A}\boldsymbol{b} = \lambda\boldsymbol{b}$ implies, that $\boldsymbol{b}$ is an eigenvector of A with eigenvalue λ. Hence W is an eigenspace of A.

$\impliedby$: Let $\mathrm{A}\boldsymbol{b} = \lambda\boldsymbol{b}$ be an eigenvector of A with eigenvalue λ. Notice that for any $\boldsymbol{w} \in W$, there exists $\mu \in \mathbb{R}$ such that $\boldsymbol{w} = \mu\boldsymbol{b}$, since $\dim(W) = 1$. Hence $\mathrm{A}\boldsymbol{w} = \mu\mathrm{A}\boldsymbol{b} = \mu\lambda\boldsymbol{b} \in W$, therefore $\mathrm{A}W \subseteq W$.

Subtask b): This follows trivially:

$$\mathrm{A}^2\boldsymbol{w} = \mathrm{A}\underbrace{\mathrm{A}\boldsymbol{w}}_{=\boldsymbol{v}\in W} = \mathrm{A}\boldsymbol{v} \in W$$

Subtask c): Let $\boldsymbol{w} \in \boldsymbol{v}^\perp$:

$$\langle \mathrm{A}\boldsymbol{w}, \boldsymbol{v}\rangle = \langle \boldsymbol{w}, \mathrm{A}\boldsymbol{v}\rangle = \overline{\lambda}\,\langle \boldsymbol{w}, \boldsymbol{v}\rangle = 0 \implies \mathrm{A}\boldsymbol{w} \in \boldsymbol{v}^\perp$$

Subtask d): Let $\boldsymbol{w} \in W$, $\widetilde{\boldsymbol{w}} = \mathrm{A}\boldsymbol{w} \in W$ and $\boldsymbol{v} \in W^\perp$:

$$\langle \mathrm{A}\boldsymbol{v}, \boldsymbol{w}\rangle = \langle \boldsymbol{v}, \widetilde{\boldsymbol{w}}\rangle = 0 \implies \mathrm{A}\boldsymbol{w} \in W^\perp$$

Task 69: Nice

Prove that a symmetric matrix $\mathrm{K} \in \mathbb{R}^{n\times n}$ is positive definite iff $\forall \lambda \in \text{spec}(\mathrm{K}) \colon \lambda > 0$.

$\implies$: By the spectral theorem, there exists an orthogonal matrix V such that $\mathrm{A} = \mathrm{V}^{\mathsf{t}}\text{diag}(\lambda_1, \dots, \lambda_n)\mathrm{V}$. Let $\boldsymbol{x} \in R^n \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{y} = \mathrm{V}\boldsymbol{x} \neq \boldsymbol{0}$, since V is regular. Hence

$$\boldsymbol{x}^{\mathsf{t}}\mathrm{A}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{t}}\mathrm{V}^{\mathsf{t}}\text{diag}(\lambda_1, \dots, \lambda_n)\mathrm{V}\boldsymbol{x} = \boldsymbol{y}^{\mathsf{t}}\text{diag}(\lambda_1, \dots, \lambda_n)\boldsymbol{y} = \sum_{i=1}^{n} \lambda_i y_i^2$$

Since $\lambda_i > 0$ for $i = 1, \dots, n$, $\boldsymbol{x}^{\mathsf{t}}\mathrm{A}\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$.

$\impliedby$: Let $\mathrm{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ with $\boldsymbol{x} \neq \boldsymbol{0}$. Let wlog $\boldsymbol{x}^{\mathsf{t}}\boldsymbol{x} = 1$, hence

$$0 < \boldsymbol{x}^{\mathsf{t}}\mathrm{A}\boldsymbol{x} = \lambda\boldsymbol{x}^{\mathsf{t}}\boldsymbol{x} = \lambda$$

Hence for any $\lambda \in \text{spec}(\mathrm{A})$ we have $\lambda > 0$.

Task 70: Square Roots of matrices

a) Prove that every positive definite matrix $\mathrm{K} \in \mathbb{R}^{n\times n}$ has a unique positive definite square-root, i.e. there exists $\mathrm{B} > 0$ such that $\mathrm{B}^2 = \mathrm{K}$

b) Find the square root of

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Subtask a): If $\mathrm{K} > 0$, then all eigenvalues $\lambda_1, \dots, \lambda_n$ are strictly positive, thus $\mathrm{B} = \mathrm{Q}^t \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})\mathrm{Q}$, where $\mathrm{K} = \mathrm{Q}^t \mathrm{diag}(\lambda_1, \dots, \lambda_n)\mathrm{Q}$ is the orthogonal diagonalization of K. Now, notice that K and B are similar, thus $\mathrm{spec}(\mathrm{K}) = \mathrm{spec}(\mathrm{B})$ and further

$$\mathrm{B}^2 = \mathrm{Q}^t \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})\mathrm{Q}\mathrm{Q}^t \mathrm{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n})\mathrm{Q} = \mathrm{Q}^t \mathrm{diag}(\lambda_1, \dots, \lambda_n)\mathrm{Q} = \mathrm{K}$$

It remains to show that B is unique. Let $\mathrm{B} > 0$ with $\mathrm{B}^2 = \mathrm{A}$. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be an orthonormal-basis of eigenvectors of A:

$$\mathrm{B}^2\boldsymbol{u}_i = \mathrm{A}\boldsymbol{u}_i = \lambda_i\boldsymbol{u}_i \implies \mathrm{spec}(\mathrm{B}^2) = \{\mu_i^2 | \mu_i \in \mathrm{spec}(\mathrm{B})\}$$

For every $\lambda_i \in \mathrm{spec}(\mathrm{A})$ there exists $\mu_i \in \mathrm{spec}(\mathrm{B})$ such that $\lambda_i = \mu_i^2$. Since $\mathrm{B} > 0$ we know $\mu_i > 0$. Since $\mathrm{AB} = \mathrm{B}^2\mathrm{B} = \mathrm{BA}$, we know that $\boldsymbol{u}_i$ are eigenvectors of B, hence $\mathrm{B}\boldsymbol{u}_i = \mu_i\boldsymbol{u}_i$, thus B is unique.

Subtask b):

$$\begin{aligned} &\chi_{\mathrm{A}}(\lambda) = (\lambda - 2)^2 - 1 = \lambda^2 - 4\lambda + 3 \implies \lambda_{1,2} = 2 \pm 1 \\ &\lambda_1 = 1 \qquad \lambda_2 = 3 \end{aligned}$$

Now:

$$\begin{aligned} &\mathrm{I} - \mathrm{A} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \implies \boldsymbol{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &3\mathrm{I} - \mathrm{A} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \implies \boldsymbol{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &\implies \mathrm{Q} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \implies \mathrm{Q}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \mathrm{Q}^t \\ &\mathrm{B} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \end{aligned}$$