Exercise Sheet № 10

Task 54

a) Let $a, b \in \mathbb{R}$. Prove that the eigenvalues of

$$\mathrm{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

are $\lambda = a \pm \mathrm{i}b$

b) If $a^2 + b^2 \neq 0$, prove that C can be factored as

$$\mathrm{C} = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

where $\theta = \arg(a + \mathrm{i}b)$.

Subtask a):

$$\chi_{\mathrm{C}}(\lambda) = (\lambda - a)^2 + b^2 = \lambda^2 - 2\lambda a + a^2 + b^2$$
$$\implies \lambda_{1,2} = a \pm \sqrt{a^2 - a^2 - b^2} = a \pm \mathrm{i}b$$

Subtask b): Recall that $r\mathrm{e}^{\mathrm{i}\theta} = r\cos(\theta) + \mathrm{i}r\sin(\theta)$, hence $\mathfrak{Re}(r\mathrm{e}^{\mathrm{i}\theta}) = r\cos(\theta)$ and $\mathfrak{Im}(r\mathrm{e}^{\mathrm{i}\theta}) = r\sin(\theta)$. Further $\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\theta}$:

$$\begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} |\lambda|\cos(\theta) & -|\lambda|\sin(\theta) \\ |\lambda|\sin(\theta) & |\lambda|\cos(\theta) \end{bmatrix}$$
$$= \begin{bmatrix} \mathfrak{Re}(\lambda) & -\mathfrak{Im}(\lambda) \\ \mathfrak{Im}(\lambda) & \mathfrak{Re}(\lambda) \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \mathrm{C}$$

Task 55: Orthogonal Projection

Let $B = (\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)$ be an arbitrary basis of a subspace $W \subset V$. Show that the projection of any vector $\boldsymbol{v} \in V$ onto W can be written as

$$P_W(\boldsymbol{v}) = \sum_{j=1}^{k} \alpha_j \boldsymbol{v}_j \qquad \boldsymbol{b} = \begin{bmatrix} \langle \boldsymbol{v}_1, \boldsymbol{v} \rangle \\ \vdots \\ \langle \boldsymbol{v}_k, \boldsymbol{v} \rangle \end{bmatrix}$$

satisfying Gram$(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k)\boldsymbol{\alpha} = \boldsymbol{b}$.

Maybe maybe maybe

Task 56

A matrix $\mathrm{K} \in \mathbb{R}^{n \times n}$ is called positive definite, iff K is symmetric and $\forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\} \colon \boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x} > 0$. K is called positive semidefinite, iff $\forall \boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x} \geq 0$.

a) Prove that the following matrix is positive definite

$$\mathrm{K} = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$$

b) Find conditions for $a, b, c \in \mathbb{R}$ such that

$$\mathrm{K} = \begin{bmatrix} a & c \\ c & v \end{bmatrix}$$

is positive definite.

Subtask a): Let $\boldsymbol{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 - 2x_1 \end{bmatrix}$$
$$= 4x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1x_2 = 4x_1^2 - 4x_1x_2 + 3x_2^2$$
$$4x_1^2 - 4x_1x_2 + 3x_2^2 > 4x_1^2 - 4x_1x_2 + x_2^2 = (2x_1 - x_2)^2 \geq 0$$

Subtask b):

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathrm{K} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + 2cx_1x_2 + bx_2^2 > 0 \iff 2cx_1x_2 \leq ax_1^2 + bx_2^2$$
$$\iff cx_1^2 + 2cx_1x_2 + cx_2^2 \leq ax_1^2 + bx_2^2 + cx_1^2 + cx_2^2$$
$$\iff (\sqrt{c}x_1 + \sqrt{c}x_2)^2 \leq x_1^2(a+c) + x_2^2(b+c)$$
$$\iff c(x_1 + x_2)^2 \leq x_1^2(a+c) + x_2^2(b+c)$$

Task 57

a) Show that every diagonal entry of a positive definite matrix must be positive
b) Write down a symmetric matrix with positive diagonal entries that is not positive definite
c) Find a nonzero matrix with one or more zero diagonal entries that is positive-semidefinite
d) Give an example of two matrices that are not positive definite, but whose sum is

Subtask a): For $\mathrm{K} \in \mathbb{K}^{n \times n}$ to be positive definite, $\forall \boldsymbol{x} \in \mathbb{K}^n \setminus \{\mathbf{0}\} \colon \boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x} > 0$, hence for the j-th canonical basis vector $\boldsymbol{e}_j$, we have $\boldsymbol{e}_j^{\mathrm{t}}\mathrm{K}\boldsymbol{e}_j > 0$, but, if $\mathrm{K} = [k_{ij}]_{i,j=1}^n$, then $\boldsymbol{e}_j^{\mathrm{t}}\mathrm{K}\boldsymbol{e}_j = k_{jj}$, which exactly are the diagonal entries. Thus $\forall j = 1, \ldots, n \colon k_{jj} > 0$.

Subtask b):

$$\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

This matrix is not positive definite, as it's determinant is -8.

Subtask c):

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 \geq 0$$

Subtask d):

$$\mathrm{A} = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \qquad \implies \mathrm{A} + \mathrm{B} = \mathrm{I}$$
$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathrm{I} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + x_2^2 > 0 \qquad \boldsymbol{x} \neq \mathbf{0}$$

Task 58

a) Prove that if K is positive, then $c\mathrm{K}$ is also positive definite for $c > 0$
b) Show that if K and L are positive definite, so is $\mathrm{K} + \mathrm{L}$
c) Let $\mathrm{K} \in \mathrm{GL}(n, \mathbb{K})$ be positive definite. Prove K^{-1} is also positive definite

Subtask a): Let $\boldsymbol{x} \neq \mathbf{0}$:

$$\boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x} > 0 \implies \boldsymbol{x}^{\mathrm{t}}c\mathrm{K}\boldsymbol{x} = c\boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x} > 0$$

Subtask b): Let $\boldsymbol{x} \neq \mathbf{0}$:

$$\boldsymbol{x}^{\mathrm{t}}(\mathrm{K} + \mathrm{L})\boldsymbol{x} = (\boldsymbol{x}^{\mathrm{t}}\mathrm{K} + \boldsymbol{x}^{\mathrm{t}}\mathrm{L})\boldsymbol{x} = \underbrace{\boldsymbol{x}^{\mathrm{t}}\mathrm{K}\boldsymbol{x}}_{>0} + \underbrace{\boldsymbol{x}^{\mathrm{t}}\mathrm{L}\boldsymbol{x}}_{>0} > 0$$

Subtask c): Let $\boldsymbol{x} \neq \mathbf{0}$ and $\boldsymbol{y} = \mathrm{K}\boldsymbol{x}$:

$$\boldsymbol{y}^{\mathsf{t}}\mathrm{K}^{-1}\boldsymbol{y} = \boldsymbol{x}^{\mathsf{t}}\mathrm{K}^{\mathsf{t}}\mathrm{K}^{-1}\mathrm{K}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{t}}\mathrm{K}^{\mathsf{t}}\boldsymbol{x} > 0$$

Where $\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x} = (\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x})^{\mathsf{t}} = \boldsymbol{x}^{\mathsf{t}}\mathrm{K}^{\mathsf{t}}\boldsymbol{x}$.

Task 59

a) Prove that a positive definite matrix is regular
b) Let K and L be positive definite. Prove that $\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{t}}\mathrm{L}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^n$ iff $\mathrm{K} = \mathrm{L}$

Subtask a): Assume that K is positive definite and singular, hence $\exists \boldsymbol{x}_0 \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\mathrm{K}\boldsymbol{x} = 0$ and thus $\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x} = 0$, but $\forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ we know $\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x} > 0$. This is a contradiction, hence K must be regular.

Subtask b): Let $\boldsymbol{x} \neq \mathbf{0}$:

$$\boldsymbol{x}^{\mathsf{t}}\mathrm{K}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{t}}\mathrm{L}\boldsymbol{x} \iff \boldsymbol{x}^{\mathsf{t}}(\mathrm{K} - \mathrm{L})\boldsymbol{x} = 0$$

Thus $\mathrm{K} - \mathrm{L} = 0$ or $\mathrm{K} = \mathrm{L}$.