Exercise Sheet № 12

Task 12.1

Let $a = (\alpha_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{R})$ and consider the sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \ell^\infty(\mathbb{R})$ defined via

$$\xi_{n,k} = \begin{cases} \alpha_k & k \geq n+1 \\ 0 & k \leq n \end{cases}$$

where $\xi_{n,k}$ is the k-th component of x_n. Prove $x_n \rightharpoonup^* 0$.

Recall that a sequence $(x_n^*)_{n\in\mathbb{N}}$ converges *weakly to $x^* \in X^*$ if

$$\forall x \in X\colon \lim_{n\to\infty} \langle x_n^*, x\rangle = \langle x^*, x\rangle$$

Since $\ell^\infty(\mathbb{R})$ is isomorphic to the dual space of $\ell^1(\mathbb{R})$, let $y \in \ell^1(\mathbb{R})$ with $y = (\eta_k)_{k\in\mathbb{N}}$. Since $y \in \ell^1(\mathbb{R})$ we know that y converges absolutely, hence for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$ we get

$$\sum_{k=n}^{\infty} \eta_k \leq \left|\sum_{k=n}^{\infty} \eta_k\right| \leq \sum_{k=n}^{\infty} |\eta_k| < \varepsilon$$

Now let $\varepsilon > 0$ and n sufficiently large such that the condition above is fulfilled

$$\langle x_n^*, y\rangle = \sum_{k=1}^{\infty} \xi_{n,k}\eta_k = \sum_{k=n+1}^{\infty} \alpha_k\eta_k \leq \|a\|_\infty \sum_{k=n+1}^{\infty} \eta_k < \|a\|_\infty \varepsilon$$

Notice that this holds for all $\varepsilon > 0$, thus

$$\lim_{n\to\infty} \langle x_n^*, y\rangle = 0 = \langle 0, y\rangle$$

and therefore $x_k \rightharpoonup^* 0$.

Task 12.2

We label the unit vectors in $\ell^2(\mathbb{R})$ with $(e_n)_{n\in\mathbb{N}}$. Determine if the following sequences converge weakly in $\ell^2(\mathbb{R})$, and if they do, find their limit.

i) Let $a \in \ell^2(\mathbb{R})$ and let $x_n = a + e_n$

ii) $x_n = ne_n$

Subtask i): Let $y^* \in \ell^2(\mathbb{R})^*$:

$$\langle y^*, x_n\rangle = \langle y^*, a + e_n\rangle = \langle y^*, a\rangle + \langle y^*, e_n\rangle$$

We propose that $x \rightharpoonup a$, i.e $\forall y^* \in \ell^2(\mathbb{R})^*\colon \lim_{n\to\infty} \langle y^*, x_n\rangle = \langle y^*, a\rangle$. Recall for $(\eta_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{R})$ we know $\lim_{k\to\infty} \eta_k = 0$:

$$\begin{aligned} \lim_{n\to\infty} \langle y^*, x_n\rangle &= \lim_{n\to\infty} \langle y^*, a\rangle + \langle y^*, e_n\rangle \\ &= \langle y^*, a\rangle + \lim_{n\to\infty} \langle y^*, e_n\rangle = \langle y^*, a\rangle + \underbrace{\lim_{n\to\infty} \eta_n}_{=0} = \langle y^*, a\rangle \end{aligned}$$

Subtask ii): Let $y^* \in \ell^2(\mathbb{R})^*$:

$$\langle y^*, x_n\rangle = n\,\langle y^*, e_n\rangle = n\eta_n \overset{n\to\infty}{\longrightarrow} \infty$$

Task 12.5

We equip $X = \mathcal{C}^1([-1,1])$ with the norm $\|x\|_X = \|x\|_\infty + \|x'\|_\infty$. We further define

$$f_0 \colon X \to \mathbb{R} \qquad x \mapsto x'(0) \qquad\qquad \varepsilon \in (0,\infty) \quad f_\varepsilon \colon X \to \mathbb{R} \qquad x \mapsto \frac{1}{2\varepsilon}(x(\varepsilon) - x(-\varepsilon))$$

i) Prove that $f_0, f_\varepsilon \in X^*$

ii) Let $(\varepsilon_k)_{k\in\mathbb{N}}$ be in $\mathfrak{c}_0((0,\infty))$. Prove that $f_{\varepsilon_n} \rightharpoonup^* f_0$

Subtask i): Let $x, y \in X$ and $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned}
f_0(\lambda x + \mu y) &= (\lambda x + \mu y)'(0) = (\lambda x' + \mu y')(0) = \lambda x'(0) + \mu y'(0) = \lambda f_0(x) + \mu f_0(y) \\
f_\varepsilon(\lambda x + \mu y) &= \frac{1}{2\varepsilon}(\lambda x(\varepsilon) + \mu y(\varepsilon) - \lambda x(-\varepsilon) - \mu y(-\varepsilon)) \\
&= \frac{\lambda}{2\varepsilon}(x(\varepsilon) - x(-\varepsilon)) + \frac{\mu}{2\varepsilon}(y(\varepsilon) - y(-\varepsilon)) = \lambda f_\varepsilon(x) + \mu f_\varepsilon(y)
\end{aligned}$$

To prove f_0 and f_ε are continuous, we show they are bounded.

$$\begin{aligned}
|f_0(x)| &= |x'(0)| \leq \|x'\|_\infty \leq \|x\|_\infty + \|x'\|_\infty = \|x\|_X \\
|f_\varepsilon(x)| &= \frac{1}{2\varepsilon}|(x(\varepsilon) - x(-\varepsilon))| \leq \frac{1}{2\varepsilon}(\underbrace{|x(\varepsilon)|}_{\leq \|x\|_\infty \leq \|x\|_X} + \underbrace{|x(-\varepsilon)|}_{\leq \|x_\infty\| \leq \|x\|_X}) \leq \frac{1}{\varepsilon}\|x\|_X
\end{aligned}$$