Task 12: Permutations I

a) Let $\mathfrak{A}_n = \{\pi \in \mathfrak{S}_n : \text{sign}(\pi) = 1\}$ be the set of even permutations. Prove that $(\mathfrak{A}_n, \circ)$ is a group, where $\circ$ denotes function composition.
b) Write down all elements of A_n for $n = 1, 2, 3, 4$ and show, that $(A_n, \circ)$ is abelian for $n \leq 3$
c) Is the set of odd permutations $\mathfrak{S}_n \setminus \mathfrak{A}_n$ a group with $\circ$?

Subtask a): Recall that every permutation $\sigma \in \mathfrak{S}_n$ can be decomposed into a minimal factorization of transpositions τ_j

$$\sigma = \bigcirc_{j=1}^{k} \tau_j$$

Using the facts that $\text{sign}(\sigma)\text{sign}(\pi) = \text{sign}(\sigma \circ \pi)$ and $\text{sign}(\tau) = -1$ for any transposition, we get:

$$\text{sign}(\sigma) = \text{sign}\left(\bigcirc_{j=1}^{k} \tau_j\right) = \prod_{j=1}^{k} \text{sign}(\tau_j) = (-1)^k$$

If k is even, then $\text{sign}(\sigma) = 1$. To show that $U \subseteq G$, where G is finite, is a subgroup, we only have to show that $\forall a, b \in U : ab \in U$ and $1 \in U$. Let us show, that $\mathfrak{A}_n$ is closed under function composition. Let $\pi, \sigma \in \mathfrak{A}_n$, then

$$\text{sign}(\pi \circ \sigma) = \text{sign}(\pi)\text{sign}(\sigma) = 1$$

Recall the definition of the signature of a composition:

$$\text{sign}(\sigma) = (-1)^{f_\sigma} \qquad f_\sigma = |\{(i, j) : i < j \land \sigma(i) > \sigma(j) | i, j \in [n]\}|$$

Since forall $i, j \in [n]$ with $i < j$, we have $\text{id}(i) = i < j = \text{id}(j)$, we see that $\text{sign}(\text{id}) = 1$, hence $\text{id} \in \mathfrak{A}_n$, thus $(\mathfrak{A}_n, \circ)$ is a subgroup of $(\mathfrak{S}_n, \circ)$.

Subtask b):

$$\begin{aligned}
&\mathfrak{S}_1 = \{\text{id}\} \implies \mathfrak{A}_1 = \{\text{id}\} \qquad \text{id} \circ \text{id} = \text{id} \\
&\mathfrak{S}_2 = \left\{\text{id}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\right\} \implies \mathfrak{A}_2 = \{\text{id}\} \qquad \text{id} \circ \text{id} = \text{id} \\
&\mathfrak{S}_3 = \left\{\text{id}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right\} \\
&\quad \implies \mathfrak{A}_3 = \left\{\text{id}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}\right\}
\end{aligned}$$

For $n = 4$ we introduce the concept of cycle types. Each permutation $\sigma \in \mathfrak{S}_4$ has a unique decomposition into a product of disjoint cycles. The cycle type is a tuple consisting of natural numbers $i_1, \ldots, i_k$, where each i_j describes the number of cycles of length i_j in the product. For $\mathfrak{S}_4$ we get the following set of cycle-types

$$\mathcal{T} = \{(1, 1, 1, 1), (2, 1, 1), (2, 2), (3, 1), (4)\}$$

As an explanation, a cycle of type $(2, 2)$ is a product of two 2-cycles, i.e. transpositions. Notice, that the only permutation whose cycle-type is $(1, 1, 1, 1)$ is the identity map. Further notice, that cycles of type $(2, 2)$ are a product of 2 transpositions, thus they are even. Cycles of type $(2, 1, 1)$ are of the form $\tau\,(i_3)\,(i_4)$, where τ is a transposition, hence they are odd. Cycles of type (4) permute all 4 elements, thus they are odd. At last, cycles of type $(3, 1)$ can be represented as a product of two non-disjoint transpositions, i.e. they are even.

We now can easily list all even permutations:

cycle-type	permutations
$(1, 1, 1, 1)$	id
$(2, 2)$	$(1\ 2)(3\ 4)$, $(1\ 3)(2\ 4)$, $(1\ 4)(2\ 3)$
$(3, 1, 1)$	$(1\ 2\ 3)$, $(1\ 2\ 4)$, $(1\ 3\ 4)$, $(2\ 3\ 4)$, $(2\ 4\ 3)$, $(1\ 3\ 2)$, $(1\ 4\ 3)$, $(1\ 4\ 2)$

Table 1: All even permutations grouped by their cycle-types

We see, that function composition is not commutative on $\mathfrak{A}_4$:

$$\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}$$
$$\begin{pmatrix}2 & 3 & 4\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}$$

Subtask c): No, since we need an identity element in $\mathfrak{S}_n \setminus \mathfrak{A}_n$, but $\mathrm{id} \in \mathfrak{A}_n$.

Task 13: Permutations II

a) Write the following permutation of 7 elements as a product of transpositions

$$\pi = \begin{pmatrix}2 & 3 & 6 & 7 & 4 & 1 & 5\end{pmatrix}$$

b) For $\pi \in \mathfrak{S}_n$, we define the permutation matrix

$$\mathrm{P}_\pi = \begin{bmatrix}\boldsymbol{e}_{\pi(1)} \\ \vdots \\ \boldsymbol{e}_{\pi(n)}\end{bmatrix} \in \mathbb{K}^{n \times n}$$

where $\boldsymbol{e}_i$ is the i-th canonical row-basis-vector of $\mathbb{K}^n$. Let

$$\mathrm{A} = \begin{bmatrix}\boldsymbol{z}_1 \\ \vdots \\ \boldsymbol{z}_n\end{bmatrix} \in \mathbb{K}^{n \times n}$$

Prove the following

$$\mathrm{P}_\pi \mathrm{A} = \begin{bmatrix}\boldsymbol{z}_{\pi(1)} \\ \vdots \\ \boldsymbol{z}_{\pi(n)}\end{bmatrix}$$

Subtask a):

$$\pi = \begin{pmatrix}2 & 5\end{pmatrix}\begin{pmatrix}2 & 1\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}\begin{pmatrix}2 & 7\end{pmatrix}\begin{pmatrix}2 & 6\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}$$

Subtask b): Recall the following fact:

$$\boldsymbol{e}_{\mathrm{t}}\mathrm{A} = \boldsymbol{z}_k$$

Now:

$$\boldsymbol{e}_k \mathrm{P}\mathrm{A} = \boldsymbol{e}_{\pi(k)}\mathrm{A} = \boldsymbol{z}_{\pi(k)}$$

Task 14: Alternating Multilinear Maps

We define the following map $d\colon \mathbb{K}^3 \to \mathbb{K}$:

$$d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \langle \boldsymbol{w}, \boldsymbol{u} \times \boldsymbol{v} \rangle$$

Prove the following

a) $d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -d(\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v})$
b) $d(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -d(\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u})$

Recall the following fact for alternating multilinear maps $\triangle\colon V^n \to \mathbb{K}$:

$$\triangle(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \mathrm{sign}(\pi)\triangle(\boldsymbol{a}_{\pi(1)}, \ldots, \boldsymbol{a}_{\pi(n)})$$

for any $\pi \in \mathfrak{S}_n$. Thus we first show that d is multilinear and alternating. We first check wether or not d is alternating:

$$d(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{w}) = \langle \boldsymbol{w}, \boldsymbol{u} \times \boldsymbol{u} \rangle = \langle \boldsymbol{w}, \boldsymbol{0} \rangle = 0$$

$$d(\boldsymbol{u},\boldsymbol{v},\boldsymbol{v}) = \langle \boldsymbol{v}, \boldsymbol{u}\times\boldsymbol{v}\rangle = 0$$

The second equality follows[1] directly from the fact, that $\boldsymbol{u}\times\boldsymbol{v}\perp\boldsymbol{v}$. For multilinearity, we only have to check that the cross-product is bilinear, since the dot-product is by definition bilinear[2].

$$(\boldsymbol{a}+\boldsymbol{b})\times\boldsymbol{c} = \begin{bmatrix}(a_2+b_2)c_3-(a_3+b_3)c_2\\(a_3+b_3)c_1-(a_1+b_1)c_3\\(a_1+b_1)c_2-(a_2+b_2)c_1\end{bmatrix} = \begin{bmatrix}a_2c_3+b_2c_3-a_3c_2-b_3c_2\\a_3c_1+b_3c_1-a_1c_3-b_1c_3\\a_1c_2+b_1c_2-a_2c_1-b_2c_1\end{bmatrix}$$
$$= \begin{bmatrix}a_2c_3-a_3c_2\\a_3c_1-a_1c_3\\a_1c_2-a_2c_1\end{bmatrix} + \begin{bmatrix}b_2c_3-b_3c_2\\b_3c_1-b_1c_3\\b_1c_2-b_2c_1\end{bmatrix} = \boldsymbol{a}\times\boldsymbol{c}+\boldsymbol{b}\times\boldsymbol{c}$$
$$\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c}) = \begin{bmatrix}a_2(b_3+c_3)-a_3(b_2+c_3)\\a_3(b_1+c_1)-a_1(b_3+c_3)\\a_1(b_2+c_2)-a_2(b_1+c_1)\end{bmatrix} = \begin{bmatrix}a_2b_3+a_2c_3-a_3b_2-a_3c_2\\a_3b_1+a_3c_1-a_1b_3-a_1c_3\\a_1b_2+a_1c_2-a_2b_1-a_2c_1\end{bmatrix}$$
$$= \begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix} + \begin{bmatrix}a_2c_3-a_3c_2\\a_3c_1-a_1c_3\\a_1c_2-a_2c_1\end{bmatrix} = \boldsymbol{a}\times\boldsymbol{b}+\boldsymbol{a}\times\boldsymbol{c}$$
$$(\lambda\boldsymbol{a})\times\boldsymbol{b} = \begin{bmatrix}\lambda a_2b_3-\lambda a_3b_2\\\lambda a_3b_1-\lambda a_1b_3\\\lambda a_1b_2-\lambda a_2b_1\end{bmatrix} = \lambda\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix} = \lambda(\boldsymbol{a}\times\boldsymbol{b})$$
$$\boldsymbol{a}\times(\lambda\boldsymbol{b}) = \begin{bmatrix}\lambda a_2b_3-\lambda a_3b_2\\\lambda a_3b_1-\lambda a_1b_3\\\lambda a_1b_2-\lambda a_2b_1\end{bmatrix} = \lambda\begin{bmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{bmatrix} = \lambda(\boldsymbol{a}\times\boldsymbol{b})$$

Thus d is an alternating multilinear map.

Subtask a): We have to show, that $d(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3) = -d(\boldsymbol{u}_1,\boldsymbol{u}_3,\boldsymbol{u}_2)$. Notice that we permute the arguments with $\pi = \begin{pmatrix}2 & 3\end{pmatrix}$, which is a transposition, thus $\operatorname{sign}(\pi) = -1$ and

$$d(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3) = \operatorname{sign}(\pi)d(\boldsymbol{u}_{\pi(1)},\boldsymbol{u}_{\pi(2)},\boldsymbol{u}_{\pi(3)}) = -d(\boldsymbol{u}_1,\boldsymbol{u}_3,\boldsymbol{u}_2)$$

Subtask b): We pursue the same methodology from a). We have to show that $d(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3) = -d(\boldsymbol{u}_3,\boldsymbol{u}_2,\boldsymbol{u}_1)$. Our corresponding permutation is thus $\pi = \begin{pmatrix}1 & 3\end{pmatrix}$ hence

$$d(\boldsymbol{u}_1,\boldsymbol{u}_2,\boldsymbol{u}_3) = \operatorname{sign}(\pi)d(\boldsymbol{u}_{\pi(1)},\boldsymbol{u}_{\pi(2)},\boldsymbol{u}_{\pi(3)}) = -d(\boldsymbol{u}_3,\boldsymbol{u}_2,\boldsymbol{u}_1)$$

[1] ambitious readers may verify it through computation
[2] this only holds for $\mathbb{K} = \mathbb{R}$

Task 15: Determinants I

We already learned the rule of Sarrus for $A \in \mathbb{K}^{3\times 3}$:

$$\det(A) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

a) Find an analogous formula for $B \in \mathbb{K}^{4\times 4}$

b) Compute the determinant of the following matrix:

$$M = \begin{bmatrix} 2 & 3 & 0 & 4 \\ 0 & 1 & 1 & 0 \\ 4 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix}$$

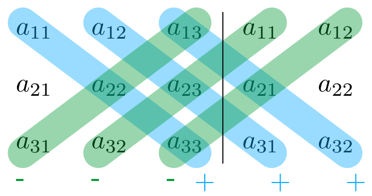

Figure 1: Mnemonic for the rule of Sarrus

Subtask a):

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

We use Laplacian expansion after the first column to compute $\det(B)$:

$$\begin{aligned}
&a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{41}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\
=&a_{11}(a_{22}a_{33}a_{44} + a_{23}a_{34}a_{42} + a_{24}a_{32}a_{43} - a_{24}a_{33}a_{42} - a_{22}a_{34}a_{42} - a_{23}a_{32}a_{44}) \\
-&a_{21}(a_{12}a_{33}a_{44} + a_{13}a_{34}a_{42} + a_{14}a_{32}a_{43} - a_{14}a_{33}a_{42} - a_{12}a_{34}a_{43} - a_{13}a_{32}a_{44}) \\
+&a_{31}(a_{12}a_{23}a_{44} + a_{13}a_{24}a_{42} + a_{12}a_{22}a_{43} - a_{12}a_{24}a_{43} - a_{12}a_{22}a_{44} - a_{14}a_{23}a_{42}) \\
-&a_{41}(a_{12}a_{23}a_{34} + a_{12}a_{24}a_{32} + a_{14}a_{22}a_{33} - a_{12}a_{24}a_{33} - a_{13}a_{22}a_{34} - a_{14}a_{23}a_{32})
\end{aligned}$$

Subtask b):

$$\det(M) = 9$$

Task 16: Determinants II

Prove or disprove the following statements:

a) $\forall \mathrm{A} \in \mathbb{R}^{3\times 3}\colon \det(-\mathrm{A}) = \det(\mathrm{A})$
b) $\forall \mathrm{A}, \mathrm{B} \in \mathbb{R}^{3\times 3}\colon \det(\mathrm{A}+\mathrm{B}) = \det(\mathrm{A}) + \det(\mathrm{B})$
c) $\forall \mathrm{A} \in \mathbb{R}^{2\times 2}\colon \det(\mathrm{A}^t\mathrm{A}) \geq 0$

Subtask a): We know the following fact from the lecture $\det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B})$, thus

$$-\mathrm{A} = (-\mathrm{I})\mathrm{A} \implies \det(-\mathrm{A}) = \det(-\mathrm{I})\det(\mathrm{A}) = (-1)^3\det(\mathrm{A}) = (-1)\det(\mathrm{A})$$

Subtask b): Let $\mathrm{A} = \mathrm{diag}(\frac{1}{2}, 1, 1)$, then $\det(\mathrm{A}) = \frac{1}{2}$, therefore $2\det(\mathrm{A}) = 1$ and $\det(2\mathrm{A}) = 4$

Subtask c): We know from the lecture, that $\det(\mathrm{A}^t) = \det(\mathrm{A})$. Using $\det(\mathrm{AB}) = \det(\mathrm{A})\det(\mathrm{B})$ we get

$$\det(\mathrm{A}^t\mathrm{A}) = \det(\mathrm{A}^t)\det(\mathrm{A}) = (\det(\mathrm{A}))^2 \geq 0$$