Exercise Sheet № 5

Task 5.1: More on Schwarz's Theorem

Let $g\colon \mathbb{R} \to \mathbb{R}$ be differentiable in $t = 0$. Moreover suppose g is globally bounded, i.e. $\exists M \in \mathbb{R}^+ \colon \forall t \in \mathbb{R} \colon |g(t)| \leq M$. Define $F\colon \mathbb{R}^2 \to \mathbb{R}$ by setting:

$$F(x,y) = \begin{cases} x^2 g\left(\frac{y}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that $\partial_{xy}F(0,0) = \partial_{yx}F(0,0)$ iff $g'(0) = 0$.

$$\begin{aligned} \partial_x F(\mathbf{0}) &= \lim_{t\to 0} \frac{F(t,0)}{t} = \lim_{t\to 0} tg(0) \leq \lim_{t\to 0} tM = 0 \\ \partial_y F(\mathbf{0}) &= \lim_{t\to 0} \frac{F(0,t)}{t} = 0 \end{aligned}$$

If F is differentiable in $\mathbf{0}$, then there exists a function $r\colon \mathbb{R}^2 \to \mathbb{R}$ with $\lim_{\boldsymbol{x}\to\mathbf{0}} r(\boldsymbol{x}) = 0$, such that:

$$\begin{aligned} &|F(\boldsymbol{x}) - F(\mathbf{0}) - \mathbf{0}^T\boldsymbol{x}| = r(\boldsymbol{x})\|\boldsymbol{x}\| \\ &\Rightarrow r(x,y) = \frac{|x^2 g(\frac{x}{y})|}{\sqrt{x^2+y^2}} \\ &\lim_{\boldsymbol{x}\to\mathbf{0}} r(\boldsymbol{x}) = \lim_{(x,y)\to(0,0)} \frac{|x^2 g(\frac{x}{y})|}{\sqrt{x^2+y^2}} \leq \lim_{(x,y)\to(0,0)} \frac{Mx^2}{\sqrt{x^2}} = \lim_{(x,y)\to(0,0)} Mx = 0 \end{aligned}$$

Thus F is differentiable in $\mathbf{0}$. Computing the partial derivatives yields:

$$\begin{aligned} \partial_x F(x,y) &= \begin{cases} 2xg(\frac{y}{x}) - yg'\left(\frac{y}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \\ \partial_y F(x,y) &= \begin{cases} xg'\left(\frac{y}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \partial_{yx} F(\mathbf{0}) &= \lim_{h\to 0} \frac{\partial_x F(0,h)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \\ \partial_{xy} F(\mathbf{0}) &= \lim_{h\to 0} \frac{\partial_y F(h,0)}{h} = \lim_{h\to 0} \frac{hg'(0)}{h} = g'(0) \end{aligned}$$

If $g'(0)$ is 0, then $\partial_{xy}F(\mathbf{0}) = \partial_{yx}F(\mathbf{0})$.

Define $\mathbb{S}^n = \{\boldsymbol{v} \in \mathbb{R}^n : \|\boldsymbol{v}\| = 1\}$.

Consider the following theorem: Let $A \subset \mathbb{R}^n$ be open. If $F: A \to \mathbb{R}$ is differentiable in $\boldsymbol{z}_0 \in A$, then all directional derivatives of F exist in z_0 and:

$$\forall \boldsymbol{v} \in \mathbb{S}^n : \mathrm{d}_{\boldsymbol{v}} F = \langle \nabla F(\boldsymbol{z}_0), \boldsymbol{v} \rangle$$

Task 5.2

Define $F: \mathbb{R}^2 \to \mathbb{R}$ by setting

$$F(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \boldsymbol{x} \neq \boldsymbol{0} \\ 0 & \boldsymbol{x} = \boldsymbol{0} \end{cases}$$

a) Prove that $\mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{0})$ exists for all $\boldsymbol{v} \in \mathbb{S}^2$ and compute it
b) Prove $\exists \boldsymbol{v} \in \mathbb{S}^2 : \mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{0}) \neq \langle \nabla F(\boldsymbol{0}), \boldsymbol{v} \rangle$
c) Can F be differentiable in $\boldsymbol{0}$?

Subtask a:

Let $\boldsymbol{z}_0 \in \mathbb{R}^2$, then $\mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{z}_0)$ is given by:

$$\lim_{h \to 0} \frac{F(\boldsymbol{z}_0 + h\boldsymbol{v}) - F(\boldsymbol{z}_0)}{h}$$

For $\boldsymbol{z}_0 = \boldsymbol{0}$, let $\boldsymbol{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$:

$$\mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{0}) = \lim_{h \to 0} \frac{F(h\boldsymbol{v})}{h} = \lim_{h \to 0} \frac{\frac{h^3 v_1 v_2^2}{h^2 v_1^2 + h^4 v_2^4}}{h} = \lim_{h \to 0} \frac{h^3 v_1 v_2^2}{h^3(v_1^2 + h^2 v_2^4)} = \lim_{h \to 0} \frac{v_1 v_2^2}{v_1^2 + h^2 v_2^4} = \frac{v_2^2}{v_1}$$

For $\boldsymbol{v} = \boldsymbol{e}_2$, we get $\mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{z}_0) = \partial_y F(\boldsymbol{z}_0)$.

Subtask b:

$$\partial_x F(\boldsymbol{0}) = \lim_{h \to 0} \frac{F(h,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$
$$\partial_y F(\boldsymbol{0}) = \lim_{h \to 0} \frac{F(0,h)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

Thus $\nabla F(\boldsymbol{0}) = \boldsymbol{0}$ and therefore for $\boldsymbol{v} \in \mathbb{S}^2$ with $v_2 \neq 0$ we get $\langle \nabla F(\boldsymbol{0}), \boldsymbol{v} \rangle = 0 \neq \mathrm{d}_{\boldsymbol{v}} F(\boldsymbol{0})$

Subtask c: Since F is differentiable on $\mathbb{R}^2 \setminus \{\boldsymbol{0}\}$, we check differentiability in $\boldsymbol{0}$. We introduce a coordinate transform. Let $x = r\cos(\theta(r))$ and $y = r\sin(\theta(r))$, then we get:

$$\lim_{\boldsymbol{x} \to \boldsymbol{0}} \frac{F(\boldsymbol{x}) - F(\boldsymbol{0})}{\|\boldsymbol{x}\|} = \lim_{r \to 0} \frac{r^3 \cos(\theta(r)) \sin^2(\theta(r))}{r(r^2\cos^2(\theta(r)) + r^4 \sin^4(\theta(r)))}$$
$$= \lim_{r \to 0} \frac{r^3 \cos(\theta(r)) \sin^2(\theta(r))}{r^3(\cos^2(\theta(r)) + r^2 \sin^4(\theta(r)))} = \tan(\theta(r)) \sin(\theta(r))$$

For the path with $\theta(r) = \frac{\pi}{2}$, we see that F is not differentiable in $\boldsymbol{0}$.

Consider the following theorem: Let $A \subset \mathbb{R}^n$ be open. Let $\boldsymbol{F}\colon A \to \mathbb{R}^n$ with $\boldsymbol{F} \in \mathcal{C}^1(A)$ and suppose $\det(\mathrm{J}\boldsymbol{F}(\boldsymbol{z}_0)) \neq 0$ for some $\boldsymbol{z}_0 \in A$. Then $\boldsymbol{F}$ is *locally invertible* around $\boldsymbol{z}_0$, that is, there exists a neighborhood $U \subset A$ of $\boldsymbol{z}_0$, and a neighborhood V of $\boldsymbol{F}(\boldsymbol{z}_0)$ and a mapping $\boldsymbol{G}\colon V \to U$ such that $\boldsymbol{G}$ is a $\mathcal{C}^1$ diffeomorphism and:

$$\mathrm{J}\boldsymbol{G}(\boldsymbol{w}) = (\mathrm{J}\boldsymbol{G}^{-1}(\boldsymbol{w}))^{-1}$$

Task 5.3: Inverse Function Theorem

a) Consider the map $\boldsymbol{F}\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$\boldsymbol{F}(x,y,z) = \begin{bmatrix} xz \\ 2xy \\ 3yz \end{bmatrix}$$

Find all points in $\mathbb{R}^3$ where $\boldsymbol{F}$ is locally invertible.

b) Consider the map $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\boldsymbol{F}(x,y) = \begin{bmatrix} \mathrm{e}^x \cos y \\ \mathrm{e}^x \sin y \end{bmatrix}$$

Show that $\boldsymbol{F}$ is locally invertible in every point of $\mathbb{R}^2$. Is $\boldsymbol{F}$ globally invertible?

Subtask a: We see that $\boldsymbol{F}$ is differentiable on $\mathbb{R}^3$. $\boldsymbol{F}$ is not locally invertible in $\boldsymbol{z}_0$, iff $\det \mathrm{J}\boldsymbol{F}(\boldsymbol{z}_0) = 0$. Therefore we compute $\mathrm{J}\boldsymbol{F}$:

$$\mathrm{J}\boldsymbol{F}(x,y,z) = \begin{bmatrix} z & 0 & x \\ 2y & 2x & 0 \\ 0 & 3z & 3y \end{bmatrix}$$
$$\Rightarrow \det \mathrm{J}\boldsymbol{F}(x,y,z) = 6xyz + 6xyz = 12xyz$$

We see that $\forall \boldsymbol{v} \in \operatorname{span}(\boldsymbol{e}_1) \cup \operatorname{span}(\boldsymbol{e}_2) \cup \operatorname{span}(\boldsymbol{e}_3)\colon \det \mathrm{J}\boldsymbol{F}(\boldsymbol{v}) = 0$, therefore $\boldsymbol{F}$ is locally invertible in $\mathbb{R}^3 \setminus (\operatorname{span}(\boldsymbol{e}_1) \cup \operatorname{span}(\boldsymbol{e}_2) \cup \operatorname{span}(\boldsymbol{e}_3))$

Subtask b: We again compute $\mathrm{J}\boldsymbol{F}$:

$$\mathrm{J}\boldsymbol{F}(x,y) = \begin{bmatrix} \mathrm{e}^x \cos(y) & -\mathrm{e}^x \sin(y) \\ \mathrm{e}^x \sin(y) & \mathrm{e}^x \cos(y) \end{bmatrix} \Rightarrow \det \mathrm{J}\boldsymbol{F}(x,y) = \mathrm{e}^{2x}(\cos^2(y) + \sin^2(y)) = \mathrm{e}^{2x}$$

Since $\forall x \in \mathbb{R}\colon \mathrm{e}^x \neq 0$ we get that $\forall \boldsymbol{x} \in \mathbb{R}^2\colon \det \mathrm{J}\boldsymbol{F}(\boldsymbol{x}) \neq 0$. Therefore $\boldsymbol{F}$ is locally invertible on $\mathbb{R}^2$. We call $\boldsymbol{F}$ globally invertible on $\mathbb{R}^2$, if there exists a mapping $\boldsymbol{F}^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2$, such that $\boldsymbol{F} \circ \boldsymbol{F}^{-1} = \mathrm{id} = \boldsymbol{F}^{-1} \circ \boldsymbol{F}$. Assuming such a function $\boldsymbol{F}^{-1}$ exists, then $\boldsymbol{F}$ must be injective on $\mathbb{R}^2$. Note however, that for a fixed $x_0 \in \mathbb{R}$ and $y_0 \in \mathbb{R}$ we get that $\forall k \in \mathbb{Z}\colon \boldsymbol{F}(x_0, y_0) = \boldsymbol{F}(x_0, y_0 + k2\pi)$. Thus $\boldsymbol{F}$ is not injective and has no global inverse.

Task 5.4

Suppose $F \in \mathcal{C}^2(\mathbb{R}^2)$ and that there exists $\boldsymbol{x}_0 \in \mathbb{R}^2$ such that:

$$F(\boldsymbol{x}_0) = \partial_x F(\boldsymbol{x}_0) = \partial_y F(\boldsymbol{x}_0) = 0$$

Moreover, assume

$$\partial_x^2 F(\boldsymbol{x}_0)\partial_y^2 F(\boldsymbol{x}_0) > \partial_{xy} F^2(\boldsymbol{x}_0)$$

Prove there exists a neighborhood U of $\boldsymbol{x}_0$ such that

$$\forall \boldsymbol{x} \in U \setminus \{\boldsymbol{x}_0\} \colon F(\boldsymbol{x}_0) \neq 0$$

Since $F \in \mathcal{C}^2(\mathbb{R}^2)$ we can apply Schwarz's Theorem, thus:

$$\operatorname{det}\mathrm{H} F = \partial_x^2 F \partial_y^2 F - \partial_{xy} F^2$$

Since $\boldsymbol{x}_0$ is a critical point of F and $\partial_x^2 F(\boldsymbol{x}_0)\partial_y^2 F(\boldsymbol{x}_0) > \partial_{xy} F^2(\boldsymbol{x}_0)$, we see that $\mathrm{H}F(\boldsymbol{x}_0) > 0$, therefore $\boldsymbol{x}_0$ is a local extremum of F. By the definition of a local extremum there exists a neighborhood U of $\boldsymbol{x}_0$ such that

$$\forall \boldsymbol{x} \in U \setminus \{\boldsymbol{x}_0\} \colon F(\boldsymbol{x}) < F(\boldsymbol{x}_0) \vee F(\boldsymbol{x}) > F(\boldsymbol{x}_0) \Rightarrow \forall \boldsymbol{x} \in U \setminus \{\boldsymbol{x}_0\} \colon F(\boldsymbol{x}) \neq F(\boldsymbol{x}_0) = 0$$