Exercise Sheet № 7

Task 7.1: Fixed Points

a) Suppose that $f\colon [0,1] \to [0,\infty)$ is continuos and $f(1) = 0$. Show that there exists $\overline{x} \in [0,1]$ such that $f(\overline{x}) = \overline{x}$

b) Let $n \geq 1$. Suppose that $f\colon \mathbb{S}^n \to \mathbb{R}$ is continuos. Show that there exists $\overline{\boldsymbol{x}} \in \mathbb{S}^n$, such that $f(\overline{\boldsymbol{x}}) = f(-\overline{\boldsymbol{x}})$

Subtask a): Let $g(x) = f(x) - x$, then $g(1) = -1$. Note that for $x = 0$ we get that $g(0) \geq 0$, since $f(x) \geq 0$. Be the intermediate-value theorem, there exists $\overline{x} \in [0,1]$, such that $g(\overline{x}) = 0$, i.e. $f(\overline{x}) = \overline{x}$.

Subtask b): Let $g(\boldsymbol{x}) = f(\boldsymbol{x}) - f(-\boldsymbol{x})$. If there exists a $\overline{\boldsymbol{x}} \in \mathbb{S}^n$, such that $g(\boldsymbol{x}) = 0$, then $\overline{\boldsymbol{x}}$ fulfills $f(\overline{\boldsymbol{x}}) = f(-\overline{\boldsymbol{x}})$. Let $\boldsymbol{x}_0 \in \mathbb{S}^n$. If $g(\boldsymbol{x}_0) = 0$, then we found our point. Otherwise it follows that either $g(\boldsymbol{x}_0) > 0$ or $g(\boldsymbol{x}_0) < 0$. Without loss of generality, assume that $g(\boldsymbol{x}_0) > 0$. Notice that $g(-\boldsymbol{x}_0) = f(-\boldsymbol{x}_0) - f(\boldsymbol{x}_0) = -g(\boldsymbol{x}_0)$, therefore, if $g(\boldsymbol{x}_0) > 0$, it follows that $g(-\boldsymbol{x}_0) < 0$. By the intermediate value theorem, there exists $\overline{\boldsymbol{x}} \in \mathbb{S}^n$, such that $g(\overline{\boldsymbol{x}}) = 0$, i.e. $f(\overline{\boldsymbol{x}}) = f(-\overline{\boldsymbol{x}})$.

Definition: Polygonal Path

Consider the points $(\boldsymbol{z}_1, \dots, \boldsymbol{z}_m)$ with $\boldsymbol{z}_i \in \mathbb{R}^n$. Let $S_k = [\boldsymbol{z}_k, \boldsymbol{z}_{k+1}]$ and set

$$P = \bigcup_{i=1}^{m-1} S_i$$

We call P a polygonal path through $(\boldsymbol{z}_1, \dots, \boldsymbol{z}_m)$, or that P connects $\boldsymbol{z}_1$ to $\boldsymbol{z}_m$.

Definition: Polygonal-Path Connectedness A subset $A \subseteq \mathbb{R}^n$ is called *polygonally path-connected*, if $\forall \boldsymbol{x}, \boldsymbol{y} \in A$, there exists a polygonal path $P \subset A$ connecting $\boldsymbol{x}$ to $\boldsymbol{y}$.

Task 7.2: Polygonally path-connected space

Fix some integer $n \geq 2$ and let $A \subset \mathbb{R}^n$, such that $|A| \leq \aleph_0$. Prove that $\mathbb{R}^n \setminus A$ is polygonally path-connected.

Let Γ be a perpendicular bisector of $[\boldsymbol{x}, \boldsymbol{y}]$, then

$$\mathfrak{L}(\boldsymbol{x}) = \{\mathrm{span}(\boldsymbol{x} - \boldsymbol{a}), \boldsymbol{a} \in A\}$$
$$\mathfrak{I} = \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{z} \in \Gamma \cap \ell | \ell \in \mathfrak{L}(\boldsymbol{x}) \vee \ell \in \mathfrak{L}(\boldsymbol{y})\}$$

It follows that $|\mathfrak{I}| \leq \aleph_0$. Therefore we can choose $\boldsymbol{p} \in \Gamma$, such that $\boldsymbol{p} \notin \mathfrak{I}$. Therefore $[\boldsymbol{x}, \boldsymbol{p}] \cap A = \emptyset$ and $[\boldsymbol{p}, \boldsymbol{y}] \cap A = \emptyset$, thus the polygonal path $[\boldsymbol{x}, \boldsymbol{p}] \cup [\boldsymbol{p}, \boldsymbol{y}]$ does no intersect A, therefore $\mathbb{R}^n$ is polygonally path connected.

Task 7.3: Path-Connectedness under set-difference

Fix some integer $n \geq 2$ and let $A \subset \mathbb{R}^n$ be convex and bounded. Prove that $\mathbb{R}^n \setminus A$ is path-connected.

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus A$. If $[\boldsymbol{x}, \boldsymbol{y}] \cap A = \emptyset$, then we found a continuos path connecting $\boldsymbol{x}$ and $\boldsymbol{y}$. If $[\boldsymbol{x}, \boldsymbol{y}] \cap A \neq \emptyset$, we select $\boldsymbol{z}_0 \in A$ and $r > 0$, such that $A \subseteq \mathcal{B}_r(\boldsymbol{z}_0)$. Let Γ be a perpendicular bisector, such that $\boldsymbol{z}_0 \in \Gamma$, where $\Gamma = \mathrm{span}(\boldsymbol{p})$ where $\boldsymbol{z}_0 = \lambda \boldsymbol{p}$ for some $\lambda \in \mathbb{R}^-$. Now set $\boldsymbol{r} = 2r\boldsymbol{p} \in \mathbb{R}^n \setminus A$, then $[\boldsymbol{x}, \boldsymbol{r}] \cap A = \emptyset$ and $[\boldsymbol{r}, \boldsymbol{y}] \cap A = \emptyset$. Therefore $([\boldsymbol{x}, \boldsymbol{r}] \cup [\boldsymbol{r}, \boldsymbol{y}]) \cap A = \emptyset$ and $[\boldsymbol{x}, \boldsymbol{r}] \cup [\boldsymbol{r}, \boldsymbol{y}]$ is a polygonal path connecting $\boldsymbol{x}$ and $\boldsymbol{y}$.

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be a curve-parametrization.

Definition: Piecewise regular We say γ is *regular*, if $\gamma \in \mathcal{C}^1([a,b])$ and $\|\gamma\|_2 > 0$. We say that γ is *piecewise regular*, if there exist $a = t_0 < t_1 < \cdots t_k = b$, such that $\gamma|_{[t_{i-1},t_i]}$ is regular for $i = 1, \ldots, n$.

Theorem: Length of a curve Let $\gamma\colon I \to \mathbb{R}^n$, where $I = [a,b]$, be a piecewise regular curve-parametrization. Then the length of $[\gamma]$ is given by:

$$\ell(\gamma) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \|\gamma'(t)\|_2 \,\mathrm{d}t$$

Definition: Scalar Line Integral Let $\gamma\colon I \to \mathbb{R}^n$ be a piecewise regular curve-parametrization and $F\colon \gamma[I] \to \mathbb{R}$ be continuos. The integral of F along γ is defined by

$$\int_\gamma F \,\mathrm{d}s = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} F(\gamma(t))\|\gamma'(t)\|_2 \,\mathrm{d}t$$

Task 7.5: Line Integral

Let $\gamma\colon [0,\pi] \to \mathbb{R}^2$ be defined by

$$\gamma(t) = \begin{bmatrix} \cos^3(t) \\ \sin^3(t) \end{bmatrix}$$

a) Prove that $\gamma(t)$ is piecewise regular
b) Compute $\ell(\gamma)$
c) Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $F(x,y) = \sqrt[3]{|xy|}$ and compute

$$\int_\gamma F \,\mathrm{d}s$$

Subtask a):

$$\begin{aligned}
&\gamma'(t) = \begin{bmatrix} -3\sin(t)\cos^2(t) \\ 3\sin^2(t)\cos(t) \end{bmatrix} \\
&\Rightarrow \|\gamma'(t)\|_2 = 3\sqrt{\sin^2\cos^4(t) + \sin^4(t)\cos^2(t)} = 3\sqrt{\sin^2(t)\cos^2(t)(\cos^2(t)+\sin^2(t))} \\
&= 3\sqrt{\sin^2(t)\cos^2(t)} = 3\sqrt{\frac{1}{4}\sin^2(2t)} = \frac{3}{2}\sqrt{\sin^2(2t)} = \frac{3}{2}|\sin(2t)|
\end{aligned}$$

$\sin(2t) = 0$ for $2t = k\pi$ for $k \in \mathbb{Z}$. Thus we get $t = \frac{k}{2}\pi$ for $k \in \mathbb{Z}$. For the given problem, the relevant values for t are:

$$t_0 = 0 \qquad t_1 = \frac{\pi}{2} \qquad t_2 = \pi$$

For $t \in \left(0, \frac{\pi}{2}\right)$ we get $\|\gamma'(t)\|_2 > 0$ and for $t \in \left(\frac{\pi}{2}, \pi\right)$ as well. Therefore γ is piecewise regular.

Subtask b):

$$\begin{aligned}
\ell(\gamma) &= \frac{3}{2}\left(\int_0^{\frac{\pi}{2}} \sin(2t)\,\mathrm{d}t - \int_{\frac{\pi}{2}}^{\pi} \sin(2t)\,\mathrm{d}t\right) = \frac{3}{2}\left(\frac{1}{2}(-\cos(2t))\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\cos(2t)\Big|_{\frac{\pi}{2}}^{\pi}\right) \\
&= \frac{3}{4}(2+2) = 3
\end{aligned}$$

Subtask c):

$$
\begin{aligned}
I &= \int_\gamma F \,\mathrm{d}s = \frac{3}{2}\int_0^{\frac{\pi}{2}} \sqrt[3]{|\sin^3(t)\cos^3(t)|}\cdot\sin(2t)\,\mathrm{d}t - \frac{3}{2}\int_{\frac{\pi}{2}}^{\pi} \sqrt[3]{|\sin^3(t)\cos^3(t)|}\cdot\sin(2t)\,\mathrm{d}t \\
&= \frac{3}{2}\left(\int_0^{\frac{\pi}{2}} \sin(t)\cos(t)\sin(2t)\,\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} \sin(t)\cos(t)\sin(2t)\,\mathrm{d}t\right) \\
&\sin(t)\cos(t) = \frac{1}{2}\sin(2t) \\
&\Rightarrow I = \frac{3}{4}\left(\int_0^{\frac{\pi}{2}} \sin^2(2t)\,\mathrm{d}t + \int_{\frac{\pi}{2}}^{\pi} \sin^2(2t)\,\mathrm{d}t\right) = \frac{3}{32}\left(4t - \sin(4t)\Big|_0^{\frac{\pi}{2}} + 4t - \sin(4t)\Big|_{\frac{\pi}{2}}^{\pi}\right) \\
&= \frac{3}{32}\left(2\pi - 2\pi + 4\pi\right) = \frac{3\pi}{8}
\end{aligned}
$$