Exercise Sheet № 3

Task 2.1: Analyzing Critical Points on Lines

a) Suppose that $\boldsymbol{z} = \mathbf{0}$ is a local minimum for a given function $F\colon \mathbb{R}^n \to \mathbb{R}$. Let $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and consider the restriction of F along the line of direction $\boldsymbol{v}$, that is, the function $g_{\boldsymbol{v}}(t) = F(t\boldsymbol{v})$ for $t \in \mathbb{R}$. Prove that $t = 0$ is a local minimum for $g_{\boldsymbol{v}}$.

b) We now show that the converse of point a) does not hold, even if $F \in \mathcal{C}^\infty$. To this end, consider the function $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$F(x, y) = (y - x^2)(y - 2x^2)$$

Prove the following statements:

i $\mathbf{0}$ is the only critical point of F

ii $\det(\mathrm{H}F(\mathbf{0})) = 0$

iii consider the restriction of F along the lines through origin

$$g_m(x) = \begin{cases} F(x, mx) & m \geq 0 \\ F(0, x) & m = \infty \end{cases}$$

Show that for all $m \in [0, \infty]$ the point $x = 0$ local minimum for g_m

iv Show that $\mathbf{0}$ is a saddle point for F

Subtask a): Let $\boldsymbol{l}_{\boldsymbol{v}}(t) = t\boldsymbol{v}$ for $\boldsymbol{v} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Further let $V = \mathrm{span}(\boldsymbol{v})$, then $F|_V = F \circ \boldsymbol{l}_{\boldsymbol{v}} = g_{\boldsymbol{v}}$. Furthermore let $\varepsilon > 0$ be sufficiently be small such that $\forall \boldsymbol{x} \in \mathcal{B}_\varepsilon(\mathbf{0})\colon F(\mathbf{0}) \geq F(\boldsymbol{x})$. Notice $V \cap \mathcal{B}_\varepsilon(\mathbf{0}) = \boldsymbol{l}_{\boldsymbol{v}}(\mathcal{B}_\varepsilon(0))$. It follows now that $\forall t \in \mathcal{B}_\varepsilon(0)\colon g_{\boldsymbol{v}}(t) \geq g_{\boldsymbol{v}}(0)$. We found an open neighborhood U of $t = 0$ such that $\forall t \in U\colon g_{\boldsymbol{v}}(t) \geq g_{\boldsymbol{v}}(0)$, therefore $t = 0$ is a local minimum of $g_{\boldsymbol{v}}$.

Subtask b): We begin by computing ∇F:

$$\begin{aligned} F(x, y) &= y^2 - 3x^2y + 2x^4 \\ \Rightarrow \nabla F(x, y) &= \begin{bmatrix} -6xy + 8x^3 \\ 2y - 3x^2 \end{bmatrix} \Rightarrow \nabla F(\mathbf{0}) = \mathbf{0} \end{aligned}$$

Subtask i: Let $x \neq 0$:

$$\begin{aligned} 8x^3 - 6xy = 0 \Leftrightarrow 8x^2 - 6y = 0 \Leftrightarrow \frac{4}{3}x^2 = y \\ \rightsquigarrow \frac{8}{3}x^2 - 3x^2 = -\frac{1}{3}x^2 \overset{!}{=} 0 \Leftrightarrow x = 0 \end{aligned}$$

Analogous let $y \neq 0$:

$$x^2 = \frac{3}{4}y \rightsquigarrow 2y - \frac{9}{4}y = -\frac{1}{4}y \overset{!}{=} 0 \Leftrightarrow y = 0$$

Subtask ii: We first compute $\mathrm{H}F$:

$$\mathrm{H}F(x, y) = \begin{bmatrix} -6y + 24x^2 & -6x \\ -6x & 2 \end{bmatrix} \Rightarrow \mathrm{H}F(\mathbf{0}) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Therefore $\det(\mathrm{H}F(\mathbf{0})) = 0$.

Subtask iii: We compute $F(x, mx)$ for $m \in [0, \infty)$:

$$\begin{aligned} F(x, mx) &= (mx - x^2)(mx - 2x^2) = m^2x^2 - 2mx^3 - mx^3 + 2x^4 = m^2x^2 - 3mx^3 + 2x^4 \\ &= x^2(m^2 - 3mx + 2x^2) \\ \frac{\mathrm{d}g_m}{\mathrm{d}x} &= 2m^2x - 9mx^2 + 8x^3 \qquad \frac{\mathrm{d}^2g_m}{\mathrm{d}x^2} = 2m^2 - 18mx + 24x^2 \end{aligned}$$

It follows that $g_m''(0) = 2m^2 \geq 0 \forall m \in \mathbb{R}$ and therefore $\forall m \in [0, \infty)\colon g_m''(0) > 0$. As $g_m'(0) = 0$, $x = 0$ is a local minimum for g_m. Secondly we set $m = \infty$ and analyze $g_\infty(x)$:

$$g_\infty(x) = F(0, x) = x^2 \Rightarrow \frac{\mathrm{d}g_\infty}{\mathrm{d}x} = 2x \Rightarrow \frac{\mathrm{d}^2 g_\infty}{\mathrm{d}x^2} = 2$$

As $g_\infty'' \geq 0 \forall x \in \mathbb{R}$, we get that $x = 0$ is local minimum for g_∞. Therefore $g_m(x)$ has a local minimum at $x = 0$ for $m \in [0, \infty]$.

Subtask iv: As $\det(\mathrm{H}F(\mathbf{0})) = 0$, we cannot use the hessian to classify the critical point $\mathbf{0}$. We have already seen, that $\forall m \in [0, \infty]$, that g_m has a local minimum at $x = 0$. Let $y \geq 0$, we find X_1 such that $\forall x \in X_1\colon F(x, y) < 0$ and X_2 such that $\forall x \in X_2\colon F(x, y) > 0$. As we require $y > 0$, we can focus on x. If $y < 2x^2$ then, $F > 0$. If $y < x^2$ then $y < 2x^2$ therefore $F > 0$. At last, if $x^2 < y < 2x^2$, then $F < 0$. Let $y = \frac{3}{2}x^2$, therefore $\forall x \in \mathbb{R}\colon x^2 \leq y \leq 2x^2$. We define $f_m(x) = F\left(x, mx^2\right)$ for $m \in (1, 2)$:

$$\begin{aligned}
f(x) &= (mx^2 - x^2)(mx^2 - 2x^2) = (m-1)(m-2)x^4 \\
\frac{\mathrm{d}f_m}{\mathrm{d}x} &= 4(m-1)(m-2)x^3 \Rightarrow \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 12(m-1)(m-2)x^2 \\
\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} &= 24(m-1)(m-2)x \Rightarrow \frac{\mathrm{d}^4 f}{\mathrm{d}x^4} = 24(m-1)(m-2)
\end{aligned}$$

As $m \in (1, 2)$, we get that $m - 1 > 0$ and $m - 2 < 0$, therefore $24(m-1)(m-2) < 0$. As $f^{(4)}(0) \neq 0$ and $f^{(4)}(0) < 0$, we now know that $x = 0$ is a local maximum for f_m. Hence $\mathbf{0}$ is a saddle point of F, as approaching $\mathbf{0}$ on $L_m = \{(x, f_m(x)), x \in \mathbb{R}\}$, we get that F has a local maximum. However, for $m \in [0, \infty]$, approaching $\mathbf{0}$ on $L_m = \{(x, g_m(x)), x \in \mathbb{R}\}$, we find that F has a local minimum.

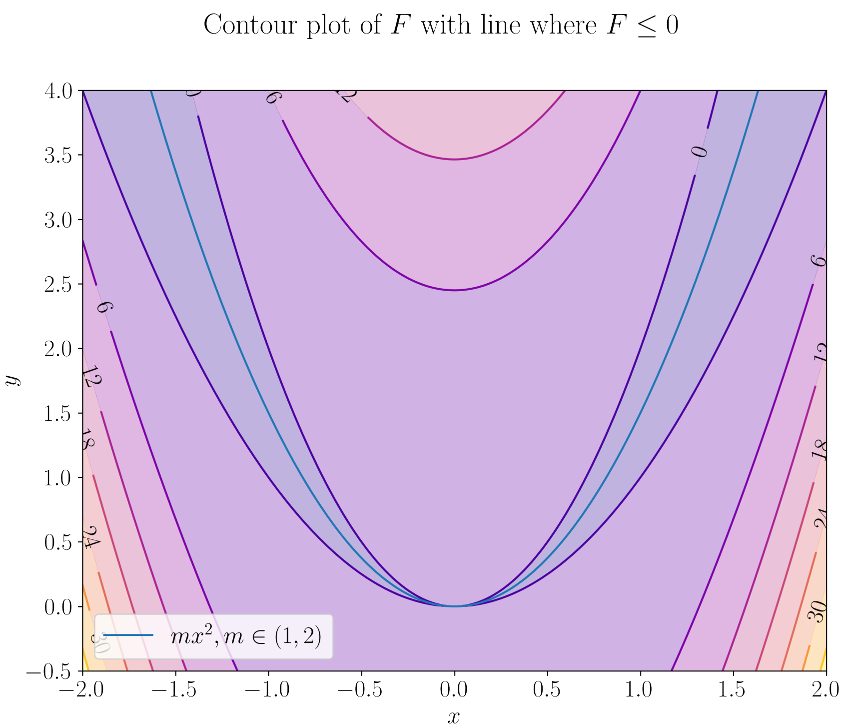

Figure 1: Contour plot of F on $[-2, 2] \times [-0.5, 4]$ and a line L where $F|_L \leq 0$

Recall on the ***Implicit Function Theorem***. Let $A \subset \mathbb{R}^2$ be open and $F\colon A \to \mathbb{R}$ with $F \in \mathcal{C}^1(A)$. Assume there exists a point $\boldsymbol{x}_0 = \begin{bmatrix} x_0 & y_0 \end{bmatrix}^T \in A$ sucht that

$$F(\boldsymbol{x}_0) = 0 \qquad \partial_y F(\boldsymbol{x}_0) \neq 0$$

Then there exist neighborhoods U of x_0 and V of y_0 and $f\colon U \to V$ such that

$$F(x, f(x)) = 0 \forall x \in U$$

The function f is called the *implicit function* defined by the equation $F = 0$. Moreover $f \in \mathcal{C}^1(U)$ and

$$f'(x) = -\frac{\partial_x F(x, f(x))}{\partial_y F(x, f(x))}$$

Task 2.2: Applying the Implicit Function Theorem

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 3xy$$

Find all the points $x_0 \in \mathbb{R}$ sucht that $F = 0$ implicitly defines a map $y = f(x)$ in a neighborhood of x_0.

Contour plot of F

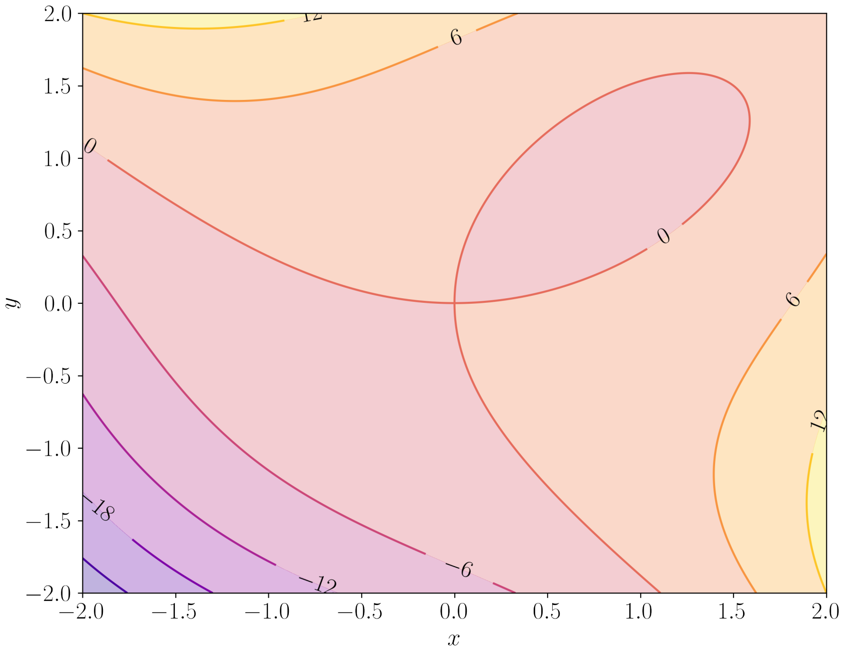

Figure 2: Contour plot of F on $[-2, 2]^2$

$$\partial_x F(x, y) = 3x^2 - 3y$$
$$\partial_y F(x, y) = 3y^2 - 3x$$

We immediately find $F(\mathbf{0}) = 0$. Note however, that $\partial_y F(\mathbf{0}) = 0$, therefore $F = 0$ does not define $y = f(x)$ in a neighborhood around 0. Let $y = tx$, then we set $f_t(x) = F(x, tx) = x^3(1 + t^3) - 3x^2 t$. For constant $x = x_0 \neq 0$, $f_t(x_0)$ is a polynomial of odd degree in t, therefore $\exists t_0 \in \mathbb{R} \colon f_{t_0}(x_0) = 0$.

$$\partial_y F(x, tx) = 3t^2 x^2 - 3x^2 t$$

As only for $x = 0 \Rightarrow f_0(x) = 0$, we know $t_0 \neq 0$, therefore $\partial_y F(x_0, t_0 x_0) \neq 0$, thus

$F = 0$ implicitly defines a function $f \colon \mathcal{B}_\varepsilon(x_0) \to \mathcal{B}_\delta(t_0 x_0)$ where $y = f(x)$, for ε and δ sufficiently small.

Task 2.3: Applying the Implicit Function Theorem

Let $F \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = 2y^3 + 4x^2 y - 3x^4 + x + 6y$$

Prove that the equation $F = 0$ defines an implicit function $y = f(x)$ for all $x \in \mathbb{R}$.

$$\begin{aligned} \partial_x F(x, y) &= 8xy - 12x^3 + 1 \\ \partial_y F(x, y) &= 6y^2 + 4x^2 + 6 \end{aligned}$$

Note that $\forall x \in \mathbb{R} \colon x^2 \geq 0$, therefore $\forall \boldsymbol{x} \in \mathbb{R}^2 \colon \partial_y F(\boldsymbol{x}) \geq 6$. Let $x = x_0 \in \mathbb{R}$ be constant, then $f(y) = F(x_0, y)$ is a polynomial of degree 3 in y, which has at least one real root $y_0 \in \mathbb{R}$. Therefore, by solving $f(y) = 0$ we get a point $\boldsymbol{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ such that $F(\boldsymbol{x}_0) = 0$ and $\partial_y F(\boldsymbol{x}_0) \neq 0$. Thus, for ε and δ sufficiently small:

$$\forall x_0 \in \mathbb{R} \colon \exists y_0 \in \mathbb{R} \colon F(x_0, y_0) = 0 \wedge \partial_y F(x_0, y_0) \neq 0 \Rightarrow \exists f \colon \mathcal{B}_\varepsilon(x_0) \to \mathcal{B}_\delta(y_0) \colon f(x) = y$$

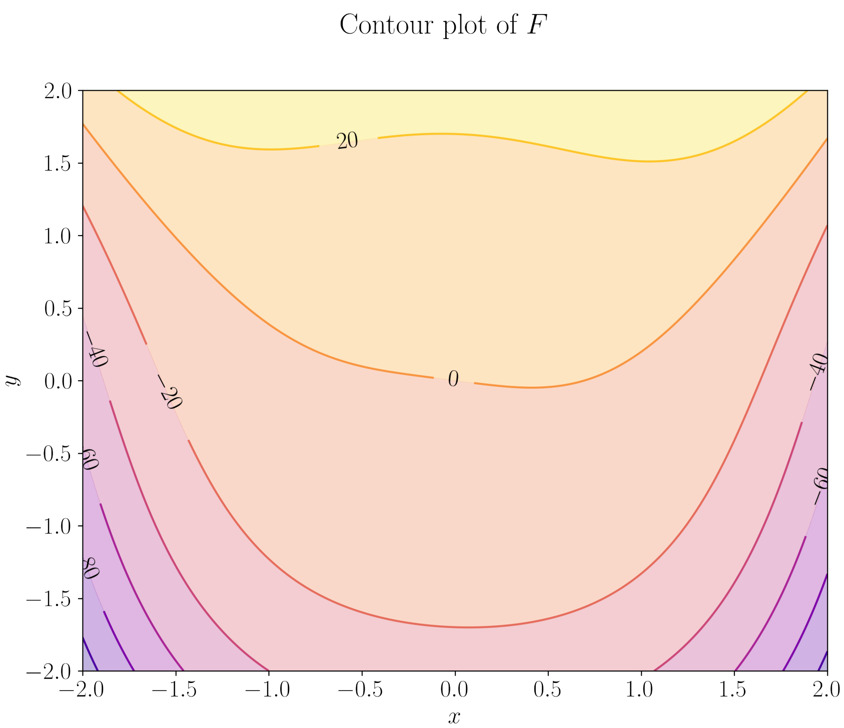

Figure 3: Contour plot of F on $[-2, 2]^2$

Remark on the ***Tangent line to a set***: Let $A \subset \mathbb{R}^2$ be open and $F: A \to \mathbb{R}$ with $F \in \mathcal{C}^1(A)$. Define the set

$$Z = \{\boldsymbol{x} \in A : F(\boldsymbol{x}) = 0\}$$

Suppose the point $\boldsymbol{x}_0 = (x_0, y_0) \in Z$ is such that either $\partial_x F(\boldsymbol{x}_0) \neq 0$ or $F_y(\boldsymbol{x}_0) \neq 0$. Then the equation of the tangent line to Z at $\boldsymbol{x}_0$ is given by:

$$\partial_x F(\boldsymbol{x}_0)(x - x_0) + \partial_y F(\boldsymbol{x}_0)(y - y_0) = 0$$

Task 2.4: Tangent Line

Let $F: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F(x, y) = x^3 + y^3 - 4x^2y + 2$$

a) Show that the equation $F = 0$ defines an implicit function $y = f(x)$ around the point $x_0 = 1$
b) Compute $f'(1)$
c) Compute the equation of the tangent-line to the set

$$Z = \{\boldsymbol{x} \in \mathbb{R}^2 : F(\boldsymbol{x}) = 0\}$$

at the point $\boldsymbol{x} = (1, 1)$

Subtask a)

$$\begin{aligned}
\partial_x F(x, y) &= 3x^2 - 8xy \\
\partial_y F(x, y) &= 3y^2 - 4x^2 \\
F(1, y) &= 1 + y^3 - 4y + 2 = y^3 - 4y + 3
\end{aligned}$$

One solution of $F(1, y) = 0$ is $y_1 = 1$, so we can factor out $y - 1$:

$$\frac{y^3 - 4y + 3}{y - 1} = y^2 + y - 3 \Rightarrow y_{2,3} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 3} = \frac{-1 \pm \sqrt{13}}{2}$$

Using $\partial_y F(x, y) = 3y^2 - 4x^2$ we get:

$$\begin{aligned}
\partial_y F(1, y) &= 3y^2 - 4 \\
\partial_y F(1, y_1) &= -1 \neq 0 \\
\partial_y F(1, y_2) &= \frac{3}{4}(\sqrt{13} - 1)^2 - 4 \neq 0 \\
\partial_y F(1, y_3) &= \frac{3}{4}(\sqrt{13} + 1)^2 - 4 \neq 0
\end{aligned}$$

Subtask b): Given we have three points on $(1, y)$ where $F = 0$, we can compute $f'(1)$ thrice:

$$\begin{aligned}
f'(1) &= -\frac{\partial_x F(1, y_1)}{\partial_y F(1, y_1)} = -5 \\
f'(1) &= -\frac{\partial_x F(1, y_2)}{\partial_y F(1, y_2)} = 4\frac{4\sqrt{13} - 7}{3(\sqrt{13} - 1)^2 - 16} \\
f'(1) &= -\frac{\partial_x F(1, y_3)}{\partial_y F(1, y_2)} = -4\frac{4\sqrt{13} + 7}{3(\sqrt{13} + 1)^2 - 16}
\end{aligned}$$

Subtask c):

$$\partial_x F(1,1)(x-1) + \partial_y F(1,1)(y-1) = 0 \Leftrightarrow -5(x-1) - (y-1) = 0 \Leftrightarrow y = 1 - 5(x-1) = 6 - 5x$$

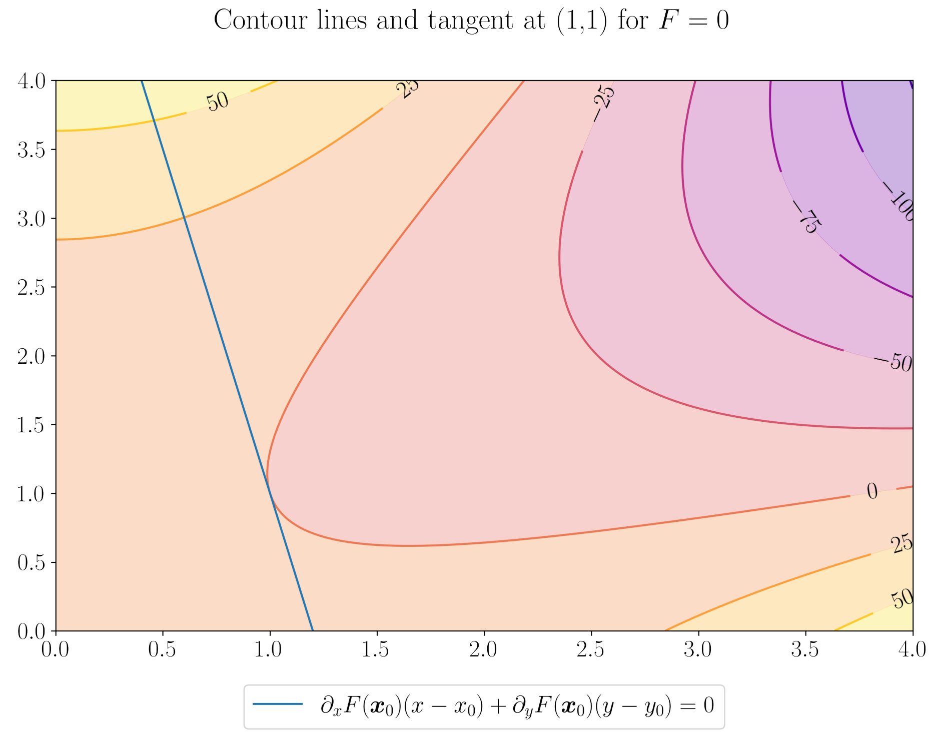

Figure 4: Contour plot of F on $[0,4]^2$ with tangent to contour-line for $F = 0$ at $(1,1)$