Exercise Sheet № 10

Note: The german abbreviation *erklpf* denotes a locally parametrized, embedded regular k-dimensional surface in $\mathbb{R}^n$, which we abbreviate erklps.

Task 10.1

Let $M = f^{-1}(0)$ be erklps in $\mathbb{R}^2$ with dimension 1, where $f\colon \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable, such that $\mathrm{D}f$ has full rank for all $\boldsymbol{x}$, where $f(\boldsymbol{x}) = 0$, and $\mathbf{0} \notin M$. Show that

$$R = \left\{(x, y, z) \in \mathbb{R}^3 | f\left(\sqrt{x^2 + y^2}, z\right) = 0\right\}$$

is a 2-dimensional erklps of $\mathbb{R}^3$. Visualize this result with a torus.

Remember that an implicitly defined set $S = \{\boldsymbol{x} \in D\colon \boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{p}\}$ is erklps, iff $\boldsymbol{h}$ is continuously differentiable on D and $\mathrm{D}\boldsymbol{h}$ has full rank on D.

Let $\boldsymbol{h}\colon \mathbb{R}^3 \to \mathbb{R}^2$ with:

$$\boldsymbol{h}(x, y, z) = \begin{bmatrix} \sqrt{x^2 + y^2} \\ z \end{bmatrix} \Rightarrow \mathrm{D}\boldsymbol{h}(x, y, z) = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that $\operatorname{rank} \mathrm{D}\boldsymbol{h}(\boldsymbol{x}) = 2$ for $\boldsymbol{x} \notin \operatorname{span}(\boldsymbol{e}_3)$. Let $D = \mathbb{R}^3 \setminus \operatorname{span}(\boldsymbol{e}_3)$