

Exercise Sheet № 4

Task 4.1

Let X be a real vectorspace and $p: X \rightarrow [0, \infty)$ be a function such that

$$\begin{aligned}\forall \mathbf{x} \in X: p(\mathbf{x}) = 0 &\implies \mathbf{x} = \mathbf{0} \\ \forall \xi \in \mathbb{R}: \forall \mathbf{x} \in X: p(\xi \mathbf{x}) &= |\xi| p(\mathbf{x})\end{aligned}$$

Prove that p is a norm on X , iff the unit sphere $B = \{\mathbf{x} \in X: p(\mathbf{x}) \leq 1\}$ is convex.

\implies : Let p be a norm on X , then the triangle inequality holds and thus for $\alpha \in [0, 1]$ and $\mathbf{y}, \mathbf{x} \in B$:

$$p(\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}) \leq p(\alpha \mathbf{x}) + p((1 - \alpha) \mathbf{y}) = |\alpha| p(\mathbf{x}) + (1 - \alpha) p(\mathbf{y}) \leq \alpha + (1 - \alpha) = 1$$

Thus the convex-combination $\alpha \mathbf{x} + (1 - \alpha) \mathbf{y}$ lies in B , hence B is convex.

\Leftarrow : Let B be convex. We only have to show that p satisfies the triangle inequality. For $\mathbf{x}, \mathbf{y} \in X \setminus \{\mathbf{0}\}$, then

$$p\left(\frac{1}{p(\mathbf{x})}\mathbf{x}\right) = \frac{1}{p(\mathbf{x})}p(\mathbf{x}) = 1$$

Hence $\frac{1}{p(\mathbf{x})}\mathbf{x}, \frac{1}{p(\mathbf{y})}\mathbf{y} \in B$. Further notice that $\frac{p(\mathbf{x})}{p(\mathbf{x})+p(\mathbf{y})} \leq 1$, hence

$$\begin{aligned}\frac{p(\mathbf{x})}{p(\mathbf{x})+p(\mathbf{y})} \frac{1}{p(\mathbf{x})}\mathbf{x} + \left(1 - \frac{p(\mathbf{x})}{p(\mathbf{x})+p(\mathbf{y})}\right) \frac{1}{p(\mathbf{y})}\mathbf{y} &\in B \\ \frac{p(\mathbf{x})}{p(\mathbf{x})+p(\mathbf{y})} \frac{1}{p(\mathbf{x})}\mathbf{x} + \left(1 - \frac{p(\mathbf{x})}{p(\mathbf{x})+p(\mathbf{y})}\right) \frac{1}{p(\mathbf{y})}\mathbf{y} &= \frac{1}{p(\mathbf{x})+p(\mathbf{y})}\mathbf{x} + \frac{1}{p(\mathbf{y})+p(\mathbf{x})}\mathbf{y} = \frac{1}{p(\mathbf{x})+p(\mathbf{y})}(\mathbf{x} + \mathbf{y}) \\ p\left(\frac{1}{p(\mathbf{x})+p(\mathbf{y})}(\mathbf{x} + \mathbf{y})\right) \leq 1 &\quad p\left(\frac{1}{p(\mathbf{x})+p(\mathbf{y})}(\mathbf{x} + \mathbf{y})\right) = \frac{p(\mathbf{x} + \mathbf{y})}{p(\mathbf{x})+p(\mathbf{y})} \leq 1 \iff p(\mathbf{x} + \mathbf{y}) \leq p(\mathbf{x}) + p(\mathbf{y})\end{aligned}$$

Thus p satisfies the triangle inequality and is hence a norm on X .

Task 4.2

Let $U \subseteq X$ be a proper subspace of a normed real vectorspace X . Prove that $\text{int } U = \emptyset$.

Notice that $\text{int } U$ is all points $\mathbf{u} \in U$, such that there exists $r > 0$ with $\mathcal{B}_r(\mathbf{u}) \subseteq U$. Given a proper subspace U , we can always find a linear independent vector \mathbf{v} . Notice that $\mathbf{v} \neq \mathbf{0}$, hence $\mathbf{u} + \frac{r}{\|\mathbf{v}\|}\mathbf{v} \in \mathcal{B}_\varepsilon(\mathbf{u})$ for $\varepsilon > r$. Now $\mathbf{v} \in \mathcal{B}_\varepsilon(\mathbf{u})$ but $\mathbf{v} \notin U$ hence $\mathcal{B}_\varepsilon(\mathbf{u}) \not\subseteq U$.

Task 4.3

Prove that every finite-dimensional subspace of a normed real vectorspace X is closed.

Since U is finite-dimensional with $\dim U = n$, there exists a finite basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ of U , hence $\forall \mathbf{x} \in U: \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{b}_i$. Let $(\mathbf{x}_m)_{m \in \mathbb{N}}$ be a convergent sequence with $\lim_{m \rightarrow \infty} \mathbf{x}_m = \mathbf{x}$, then $\mathbf{x}_m = \sum_{i=1}^n \lambda_{i,m} \mathbf{b}_i$. Since \mathbf{x}_m is convergent with limit \mathbf{x} , we know that $\lim_{m \rightarrow \infty} \sum_{i=1}^n \lambda_{i,m} \mathbf{b}_i = \sum_{i=1}^n \lambda_i \mathbf{b}_i$. Hence \mathbf{x} is a linear combination of vectors in U , therefore $\mathbf{x} \in U$.