

Exercise Sheet № 14

Task 78: Matrix Exponential

Let

$$A = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Compute e^A .

We begin by diagonalizing A:

$$\begin{aligned} \chi_A(\lambda) &= (\lambda + 2)(\lambda + 3) - 2 = \lambda^2 + 5\lambda + 4 \\ \implies \lambda_1 &= -4 \quad \lambda_2 = -1 \\ -4I - A &= \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \implies v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ -I - A &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \implies v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ V &= \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \implies V^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

Given a diagonalizable matrix $A = V \text{diag}(\lambda_1, \dots, \lambda_n) V^{-1}$ we have:

$$\begin{aligned} e^A &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} A^k = \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} (V \text{diag}(\lambda_1, \dots, \lambda_n) V^{-1})^k \\ &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} V \text{diag}(\lambda_1^k, \dots, \lambda_n^k) V^{-1} = \lim_{m \rightarrow \infty} V \sum_{k=0}^m \frac{1}{k!} \text{diag}(\lambda_1^k, \dots, \lambda_n^k) V^{-1} \\ &= V \left(\lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} \text{diag}(\lambda_1^k, \dots, \lambda_n^k) \right) V^{-1} = V \left(\lim_{m \rightarrow \infty} \sum_{k=0}^m \text{diag} \left(\frac{\lambda_1^k}{k!}, \dots, \frac{\lambda_n^k}{k!} \right) \right) V^{-1} \\ &= V \left(\lim_{m \rightarrow \infty} \text{diag} \left(\sum_{k=0}^m \frac{\lambda_1^k}{k!}, \dots, \sum_{k=0}^m \frac{\lambda_n^k}{k!} \right) \right) V^{-1} = V \text{diag} \left(\sum_{k=0}^{\infty} \frac{\lambda_1^k}{k!}, \dots, \sum_{k=0}^{\infty} \frac{\lambda_n^k}{k!} \right) V^{-1} \\ &= V \text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n}) V^{-1} \end{aligned}$$

Thus:

$$e^A = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{-4} & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-4} & e^{-1} \\ -2e^{-4} & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^{-4} - e^{-1} & -e^{-4} + e^{-1} \\ -2e^{-4} + 2e^{-1} & 2e^{-4} + e^{-1} \end{bmatrix}$$

Task 80: Matrix Exponential of Skew-Symmetric Matrix

Let $A \in \mathbb{R}^{n \times n}$ be skew-symmetric. Prove that e^A is an orthogonal matrix.

We first consider $M \in \mathbb{R}^{n \times n}$ and compute $\exp(M^t)$:

$$\begin{aligned}\exp(M^t) &= \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} (M^t)^k = \lim_{m \rightarrow \infty} \sum_{k=0}^{\infty} \frac{1}{k!} (M^k)^t \\ &= \lim_{m \rightarrow \infty} \left(\sum_{k=0}^{\infty} M^k \right)^t = \left(\lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{1}{k!} M^k \right)^t = \exp(M)^t\end{aligned}$$

If e^A is orthogonal, then $\exp(A)(\exp(A))^t = I$:

$$\exp(A)(\exp(A))^t = \exp(A) \exp(A^t) = \exp(A + A^t) = \exp(A - A) = \exp(0)$$

By convention we set $0^0 = I$ and get

$$\exp(0) = \sum_{k=0}^{\infty} \frac{1}{k!} 0^k = \frac{1}{0!} 0^0 = I$$