Exercise Sheet № 5

Task 22: Determinants

a) Show that the following determinant is independent of θ:

$$\triangle = \begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta)-\cos(\theta) & \sin(\theta)+\cos(\theta) & 1 \end{vmatrix}$$

b) Show that the matrices

$$\mathrm{A} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \qquad \mathrm{B} = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute iff

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

c) Use the fact that 21375, 38798, 34162,40223 and 79154 are all divisible by 19 to show that

$$\triangle = \begin{vmatrix} 2 & 1 & 3 & 7 & 5 \\ 3 & 8 & 7 & 9 & 8 \\ 3 & 4 & 1 & 6 & 2 \\ 4 & 0 & 2 & 2 & 3 \\ 7 & 9 & 1 & 5 & 4 \end{vmatrix}$$

is divisible by 19 without evaluating the determinant.

Subtask a): We use laplacian expansion after the third column:

$$\triangle = 1 \begin{vmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{vmatrix} = \sin^2(\theta) + \cos^2(\theta) = 1$$

Subtask b):

We compute AB and BA:

$$\mathrm{AB} = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix} \qquad \mathrm{BA} = \begin{bmatrix} da & db+ec \\ 0 & cf \end{bmatrix}$$

$$\mathrm{AB} = \mathrm{BA} \iff \mathrm{AB} - \mathrm{BA} = 0$$

$$\mathrm{AB} - \mathrm{BA} = \begin{bmatrix} 0 & ae+bf-bd-ce \\ 0 & 0 \end{bmatrix} \implies e(a-c) + b(f-d) = 0$$

$$\iff b(d-f) - e(a-c) = 0$$

Now

$$\begin{vmatrix} b & a-c \\ d & d-f \end{vmatrix} = b(d-f) - e(a-c)$$

Subtask c): Notice that the rows containt exactly the digits of the given numbers. We now perform column-operations:

$$\widetilde{\boldsymbol{c}}_5 = \boldsymbol{c}_5 + 10\boldsymbol{c}_4 + 100\boldsymbol{c}_3 + 1000\boldsymbol{c}_2 + 10000\boldsymbol{c}_1$$

Resulting in:

$$\triangle = \begin{vmatrix} 2 & 1 & 3 & 7 & 21375 \\ 3 & 8 & 7 & 9 & 38798 \\ 3 & 4 & 1 & 6 & 34162 \\ 4 & 0 & 2 & 2 & 40223 \\ 7 & 9 & 1 & 5 & 79154 \end{vmatrix}$$

Using laplacian expansion after the last column yields:

$$\triangle = 2137\underbrace{\det(A_{51})}_{\in\mathbb{Z}} - 38798\underbrace{\det(A_{52})}_{\in\mathbb{Z}} + 34162\underbrace{\det(A_{53})}_{\in\mathbb{Z}} - 40223\underbrace{\det(A_{54})}_{\in\mathbb{Z}} + 79154\underbrace{\det(A_{55})}_{\in\mathbb{Z}}$$

Notice that $\triangle$ is a linear combination of multiples of 19 over $\mathbb{Z}$, hence $19|\triangle$.

Task 23: Eigenvalues and -vectors

Finde the eigenvalues and the bases for the corresponding eigenspaces of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

We first compute the characteristic polynomial:

$$\chi_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda - 2 & -1 \\ 0 & \lambda - 3 \end{vmatrix} + 2 \begin{vmatrix} -1 & \lambda - 2 \\ -1 & 0 \end{vmatrix} = \lambda(\lambda - 2)(\lambda - 3) + 2(\lambda - 2)$$

$$= (\lambda - 2)(\lambda^2 - 3\lambda + 2) \stackrel{!}{=} 0$$

$$\lambda_1 = 2$$

$$\lambda_{2,3} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2} \implies \lambda_2 = 2, \lambda_3 = 1$$

$$\implies \chi_A(\lambda) = (\lambda - 2)^2(\lambda - 1)$$

For the bases of the eigenspaces we want to solve $(\lambda_i I - A)\boldsymbol{v}_i = \mathbf{0}$. For $\lambda = 2$:

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies v_{11} = -v_{13}, v_{12} \in \mathbb{R} \implies \operatorname{Eig}(A, \lambda_1) = \operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

For $\lambda = 1$:

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies v_{21} = -2v_{23}, v_{22} = -v_{21} - v_{23} = v_{23}$$

$$\implies \operatorname{Eig}(A, \lambda_2) = \operatorname{span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right)$$

Task 24: Vandermonde Matrix

Let $\mathbb{F}$ be some field and $x_1, \dots, x_n \in \mathbb{F}$. Assume a matrix $\mathrm{A} \in \mathbb{F}^{n\times n}$ satisfies $a_{ij} = x_j^{i-1}$. Prove that

$$\det(\mathrm{A}) = \prod_{1\le j<i\le n} (x_i - x_j)$$

We first subtract the first row from all the others:

$$V_n = \det(\mathrm{A}_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-2} & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-2} & x_{n-1}^{n-1} \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^{n-1} \\ 0 & x_2 - x_1 & x_2^2 - x_1^2 & \cdots & x_2^{n-2} - x_1^{n-2} & x_2^{n-1} - x_1^{n-1} \\ 0 & x_3 - x_1 & x_3^2 - x_1^2 & \cdots & x_3^{n-2} - x_1^{n-2} & x_3^{n-1} - x_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-1} - x_1 & x_{n-1}^2 - x_1^2 & \cdots & x_{n-1}^{n-2} - x_1^{n-2} & x_{n-1}^{n-1} - x_1^{n-1} \\ 0 & x_n - x_1 & x_n^2 - x_1^2 & \cdots & x_n^{n-2} - x_1^{n-2} & x_n^{n-1} - x_1^{n-1} \end{vmatrix}$$

Now we subtract, in order, $\widetilde{\boldsymbol{c}}_k = \boldsymbol{c}_k - x_1 \boldsymbol{c}_{k-1}$, for $k = n, \dots, 2$. Notice that we now have entries of the form $a_{ij} = (x_i^{j-1} - x_1^{j-1}) - (x_1 x_i^{j-2} - x_1^{j-1}) = (x_i - x_1)x_i^{j-2}$:

$$V_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & x_2^{n-3}(x_2 - x_1) & x_2^{n-2}(x_2 - x_1) \\ 0 & x_3 - x_1 & (x_2 - x_3)x_3 & \cdots & x_3^{n-3}(x_3 - x_1) & x_3^{n-2}(x_3 - x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x_{n-1} - x_1 & (x_{n-1} - x_1)x_{n-1} & \cdots & x_{n-1}^{n-3}(x_{n-1} - x_1) & x_{n-1}^{n-2}(x_{n-1} - x_1) \\ 0 & x_n - x_1 & (x_n - x_1)x_n & \cdots & x_n^{n-3}(x_n - x_1) & x_n^{n-2}(x_n - x_1) \end{vmatrix}$$

We introduced the constant factor $(x_k - x_1)$ into every row except the first one, hence we can extract it:

$$V_n = \prod_{k=2}^{n}(x_k - x_1) \begin{vmatrix} 1 & x_2 & \cdots & x_2^{n-3} & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-3} & x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_{n-1} & \cdots & x_{n-1}^{n-3} & x_{n-1}^{n-2} \\ 1 & x_n & \cdots & x_n^{n-3} & x_n^{n-2} \end{vmatrix}$$

$$= \prod_{k=2}^{n}(x_k - x_1) V_{n-1}$$

We can now redo all these steps with x_2 and get:

$$V_{n-1} = \prod_{k=3}^{n}(x_k - x_2)V_{n-2} \implies V_n = \prod_{k_1=2}^{n}(x_{k_1} - x_1) \prod_{k_2=3}^{n}(x_{k_2} - x_2)V_{n-2} = \prod_{j=1}^{2}\prod_{i=j+1}^{n}(x_i - x_j)V_{n-2}$$

Hence we get to the following expression for V_{n-k}:

$$V_n = \prod_{j=1}^{k}\prod_{i=j+1}^{n}(x_i - x_j)V_{n-k}$$

With $V_2 = x_n - x_{n-1}$ and $V_1 = 1$ we get:

$$V_n = \prod_{j=1}^{n-1}\prod_{i=j+1}^{n}(x_i - x_j) = \prod_{1\le j<i\le n}(x_i - x_j)$$

Task 25: Characteristic Polynomial

Find $\det(\mathrm{A})$ given the characteristic polynomial $\chi_{\mathrm{A}}(\lambda)$:
a) $\chi_{\mathrm{A}}(\lambda) = \lambda^3 - 2\lambda^2 + \lambda + 5$
b) $\chi_{\mathrm{A}}(\lambda) = \lambda^4 - \lambda^3 + 7$

Recall that $\chi_{\mathrm{A}}(\lambda) = \det(\lambda \mathrm{I} - \mathrm{A})$, hence $\chi_{\mathrm{A}}(0) = \det(-\mathrm{A}) = (-1)^n \det(\mathrm{A})$ for $\mathrm{A} \in \mathbb{K}^{n \times n}$. Subtask a): Given the polynomial is of degree three, we know that $\det(\mathrm{A}) = -\chi_{\mathrm{A}}(0)$:

$$\chi_{\mathrm{A}}(0) = 5 \implies \det(\mathrm{A}) = -5$$

Subtask b): The polynomial is of degree four, hence $\det(\mathrm{A}) = \chi_{\mathrm{A}}(0) = 7$.

Task 26: Wronskian Determinant

Let $\boldsymbol{f}_1 = f_1(x), \boldsymbol{f}_2 = f_2(x), \ldots, \boldsymbol{f}_n = f_n(x)$ be functions in $\mathcal{C}^{n-1}(\mathbb{R})$. Prove that if $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ is linearly dependent in $\mathcal{C}^{n-1}(\mathbb{R})$, then

$$\forall x \in \mathbb{R} \colon W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ \frac{\mathrm{d}f_1}{\mathrm{d}x}(x) & \frac{\mathrm{d}f_2}{\mathrm{d}x}(x) & \cdots & \frac{\mathrm{d}f_n}{\mathrm{d}x}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\mathrm{d}^{n-1}f_1}{\mathrm{d}x^{n-1}}(x) & \frac{\mathrm{d}^{n-1}f_2}{\mathrm{d}x^{n-1}}(x) & \cdots & \frac{\mathrm{d}^{n-1}f_n}{\mathrm{d}x^{n-1}}(x) \end{vmatrix} = 0$$

We first introduce some notation:

$$F_k = \{f_1^{(k)}(x), \ldots, f_n^{(k)}(x)\} \qquad k \in \{0, \ldots, n-1\} \qquad f^{(0)}(x) = f(x)$$

If F_0 is linearly dependent, then wlog:

$$f_n = \sum_{i=1}^{n-1} \alpha_i f_i$$

where $\exists i \in [n-1]$ such that $\alpha_i \neq 0$. From this we get:

$$f_n^{(k)} = \frac{\mathrm{d}^k}{\mathrm{d}x^k} \sum_{i=1}^{n-1} \alpha_i f_i = \sum_{i=1}^{n-1} \alpha_i \frac{\mathrm{d}^k f_i}{\mathrm{d}x^k}$$

Hence all $\forall k \in \{0, \ldots, n-1\}$ we showed that F_k is linearly dependent. Particularly, the n-th column can be expressed as a linear combination of the first $n-1$ columns, hence we can produce a zero column, therefore $W \equiv 0$.

Task 27: Orthogonality of Eigenspaces

Let A be a square matrix, with eigenvalue λ and corresponding eigenvector $\boldsymbol{x}$. Further let $\boldsymbol{y}$ be an eigenvector of A^{t} for eigenvalue μ. Show that $\lambda \neq \mu \implies \boldsymbol{x}^{\mathsf{t}}\boldsymbol{y} = 0$.

Notice the following:

$$\mathrm{A}^{\mathsf{t}}\boldsymbol{y} = \mu\boldsymbol{y} \iff \boldsymbol{y}^{\mathsf{t}}\mathrm{A} = \mu\boldsymbol{y}^{\mathsf{t}}$$

Let wlog $\lambda, \mu \neq 0$ and assume $\boldsymbol{x}^{\mathsf{t}}\boldsymbol{y} \neq 0$:

$$\boldsymbol{x}^{\mathsf{t}}\boldsymbol{y} = \frac{1}{\lambda}\boldsymbol{x}^{\mathsf{t}}\mathrm{A}\boldsymbol{y} = \frac{\mu}{\lambda}\boldsymbol{x}^{\mathsf{t}}\boldsymbol{y}$$

This is a contradiction to our assumption that $\mu \neq \lambda$, hence $\boldsymbol{x}^{\mathsf{t}}\boldsymbol{y} = 0$. Notice that for $\lambda = 0$:

$$\boldsymbol{x}^{\mathsf{t}}\boldsymbol{y} = \frac{1}{\mu}\boldsymbol{x}^{\mathsf{t}}\mathrm{A}\boldsymbol{y} = \frac{1}{\mu}\boldsymbol{0}^{\mathsf{t}}\boldsymbol{y} = 0$$

The case $\mu = 0$ can be handled analogous.