Exercise Sheet № 11

Task 61: Conic Sections

Given $f\colon \mathbb{R}^2 \to \mathbb{R}$, we call the set $\{\boldsymbol{x} \in \mathbb{R}^2 \colon f(\boldsymbol{x}) = 0\}$ a conic section. Determine the type and find the principal axis transform of the conic sections generated by the following functions.

a) $f(x_1, x_2) = 2x_1^2 - 2\sqrt{6}x_1x_2 + x_2^2 + 3x_1 + 3x_2 - 1$

b) $f(x_1, x_2) = 4x_1^2 - 2x_1x_2 + 2x_2^2 + 2$

Conic sections are always generated by quadratic forms, which can be understood as a quadratic function with a vector argument. Let $\mathrm{Q} \in \mathbb{R}^{n\times n}$, $\boldsymbol{c} \in \mathbb{R}^n$ and $c_0 \in \mathbb{R}$, then the most general quadratic form is given as

$$q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{t}}\mathrm{Q}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{t}}\boldsymbol{x} + c_0$$

Using the principal axis theorem (PAT), we can transform $q(\boldsymbol{x}) = 0$ into one of two forms. Let $u, v\colon \mathbb{R}^2 \to \mathbb{R}$:

$$u^2(x_1, x_2) + v^2(x_1, x_2) = 1 \qquad \text{(Ellipse)}$$
$$u^2(x_1, x_2) - v^2(x_1, x_2) = 1 \qquad \text{(Hyperbola)}$$

To apply the PAT, we first diagonalize Q. Given the corresponding change of basis matrix V, we transform our coordinate system by applying V to $\boldsymbol{x}$. This eliminates any factors $q_{ij}x_ix_j$ that my be present in q. This allows us to complete the square and find one of the given forms from above. Notice that normalizing the eigenvectors of Q has the beneficial property that V becomes a simple rotation matrix. Given the diagonalization $\mathrm{Q} = \mathrm{V}\operatorname{diag}(\lambda_1, \dots, \lambda_n)\mathrm{V}^{-1}$, recall that quadratic forms are generated by symmetric matrices, thus V is orthogonal, hence:

$$q(\boldsymbol{x}) = \boldsymbol{x}^{\mathrm{t}}\mathrm{V}^{\mathrm{t}}\operatorname{diag}(\lambda_1, \dots, \lambda_n)\mathrm{V}\boldsymbol{x} + \boldsymbol{c}^{\mathrm{t}}\boldsymbol{x} + c_0$$

Let $\boldsymbol{y} = \mathrm{V}\boldsymbol{x}$, then

$$q(\boldsymbol{x}) = \boldsymbol{y}^{\mathrm{t}}\operatorname{diag}(\lambda_1, \dots, \lambda_n)\boldsymbol{y} + \boldsymbol{c}^{\mathrm{t}}\boldsymbol{x} + c_0$$

Recall completing the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Subtask a): The coefficients of our quadratic form are

$$\mathrm{Q} = \begin{bmatrix} 2 & -\sqrt{6} \\ -\sqrt{6} & 1 \end{bmatrix} \qquad \boldsymbol{c} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad c_0 = -1$$

Q has the characteristic equation $(\lambda - 2)(\lambda - 1) - 6 = \lambda^2 - 3\lambda - 4 = 0$, thus $\lambda_1 = -1$ and $\lambda_2 = 4$. Thus the eigenvectors of Q are

$$\boldsymbol{v}_1 = \begin{bmatrix} \sqrt{6} \\ 3 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} -\sqrt{6} \\ 2 \end{bmatrix} \implies \mathrm{V} = \begin{bmatrix} \frac{\sqrt{6}}{\sqrt{15}} & -\frac{\sqrt{6}}{\sqrt{10}} \\ \frac{3}{\sqrt{15}} & \frac{2}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{2} & -\sqrt{3} \\ \sqrt{3} & \sqrt{2} \end{bmatrix}$$

Now let $\boldsymbol{x} = \mathrm{V}\boldsymbol{y}$, hence

$$\boldsymbol{x} = \mathrm{V}\begin{bmatrix} \xi \\ \mu \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \sqrt{2}\xi - \sqrt{3}\mu \\ \sqrt{3}\xi + \sqrt{2}\mu \end{bmatrix}$$

Now:

$$\begin{aligned} f(\xi, \mu) =& \frac{2}{5}(\sqrt{2}\xi - \sqrt{3}\mu)^2 + \frac{1}{5}(\sqrt{3}\xi + \sqrt{2}\mu)^2 - 2\frac{\sqrt{6}}{5}(\sqrt{2}\xi - \sqrt{3}\mu)(\sqrt{3}\xi + \sqrt{2}\mu) \\ &+ \frac{3}{\sqrt{5}}(\sqrt{2}\xi - \sqrt{3}\mu + \sqrt{3}\xi + \sqrt{2}\mu) - 1 \end{aligned}$$

$$= \frac{7}{5}\xi^2 - \frac{2}{5}\sqrt{6}\xi\mu + \frac{8}{5}\mu^2 - \frac{2}{5}\sqrt{6}(\sqrt{6}\xi^2 - \xi\mu - \sqrt{6}\mu^2) + \frac{3}{\sqrt{5}}(\xi(\sqrt{2}+\sqrt{3}) + \mu(\sqrt{2}-\sqrt{3})) - 1$$
$$= \frac{7}{5}\xi^2 + \frac{8}{5}\mu^2 - \frac{12}{5}\xi^2 + \frac{12}{5}\mu^2 + \frac{3(\sqrt{2}+\sqrt{3})}{\sqrt{5}}\xi + \frac{3(\sqrt{2}-\sqrt{3})}{\sqrt{5}}\mu - 1$$
$$= -\xi^2 + 4\mu^2 + \frac{3(\sqrt{2}+\sqrt{3})}{\sqrt{5}}\xi + \frac{3(\sqrt{2}-\sqrt{3})}{\sqrt{5}}\mu - 1$$

Completing the square for ξ and μ separately yields:

$$f(\xi,\mu) = -\left(\xi - \frac{3(\sqrt{2}+\sqrt{3})}{2\sqrt{5}}\right)^2 + \frac{9(\sqrt{2}+\sqrt{3})^2}{20} + 4\left(\mu + \frac{3(\sqrt{2}-\sqrt{3})}{8\sqrt{5}}\right)^2 - \frac{9(\sqrt{2}-\sqrt{3})^2}{80} - 1$$
$$= 4\left(\mu + \frac{3(\sqrt{2}-\sqrt{3})}{8\sqrt{5}}\right)^2 - \left(\xi - \frac{3(\sqrt{2}+\sqrt{3})^2}{2\sqrt{5}}\right)^2 + \frac{36(2+2\sqrt{6}+3)}{80} - \frac{9(2-2\sqrt{6}+3)}{80} - 1$$
$$= 4\left(\mu + \frac{3(\sqrt{2}-\sqrt{3})}{8\sqrt{5}}\right)^2 - \left(\xi - \frac{3(\sqrt{2}+\sqrt{3})^2}{2\sqrt{5}}\right)^2 + \frac{135+90\sqrt{6}}{80} - 1$$
$$= 4\left(\mu + \frac{3(\sqrt{2}-\sqrt{3})}{8\sqrt{5}}\right)^2 - \left(\xi - \frac{3(\sqrt{2}+\sqrt{3})^2}{2\sqrt{5}}\right)^2 + \frac{11}{16} + \frac{9}{8}\sqrt{6}$$

We see that f generates a hyperbola.

Subtask b): The coefficients for our quadratic form are

$$\mathrm{Q} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \qquad \boldsymbol{c} = \mathbf{0} \qquad c_0 = 2$$

Q has the characteristic equation $(\lambda-4)(\lambda-2) - 1 = \lambda^2 - 6\lambda + 7 = 0$, thus $\lambda_1 = 3+\sqrt{2}$ and $\lambda_2 = 3-\sqrt{2}$. Hence the eigenvectors of Q are

$$\boldsymbol{v}_1 = \begin{bmatrix} -1-\sqrt{2} \\ 1 \end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1+\sqrt{2} \\ 1 \end{bmatrix} \implies \mathrm{V} = \begin{bmatrix} \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}} & \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} & \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}$$

Now let $\boldsymbol{x} = \mathrm{V}\boldsymbol{y}$, hence

$$\boldsymbol{x} = \mathrm{V}\begin{bmatrix} \xi \\ \mu \end{bmatrix} = \begin{bmatrix} \frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\xi + \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\mu \\ \frac{1}{\sqrt{4+2\sqrt{2}}}\xi + \frac{1}{\sqrt{4-2\sqrt{2}}}\mu \end{bmatrix}$$

Now:

$$f(\xi,\mu) = 4\left(\frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\xi + \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\mu\right)^2 - 2\left(\frac{-1-\sqrt{2}}{\sqrt{4+2\sqrt{2}}}\xi + \frac{-1+\sqrt{2}}{\sqrt{4-2\sqrt{2}}}\mu\right)\left(\frac{1}{\sqrt{4+2\sqrt{2}}} + \frac{1}{\sqrt{4-2\sqrt{2}}}\mu\right)$$
$$+ 2\left(\frac{1}{\sqrt{4+2\sqrt{2}}}\xi + \frac{1}{\sqrt{4-2\sqrt{2}}}\mu\right)^2 + 2$$

F this

Task 62: Shift-Operator

Let $V = \mathcal{C}_0(\mathbb{R},\mathbb{C}) = \{f \in \mathcal{C}(\mathbb{R},\mathbb{C}) \colon \exists M \in \mathbb{R}^+ \colon |x| > M \implies f(x) = 0\}$. We consider the linear operator $A \in \mathrm{End}(V)$ with $Af(x) = f(x-1)$. Prove that A has no eigenvalues.

Assume there exists $\lambda \in \mathbb{R}\setminus\{0\}$ and $f \in V\setminus\{0\}$, such that $f(x-1) = \lambda f(x)$. Further notice $A^k f(x) = f(x-k)$ by induction.

Task 63: Nilpotent

Let V be a vectorspace over $\mathbb{F}$. We call $F \in \mathrm{End}(V)$ nilpotent, if $\exists k \in \mathbb{N}\colon F^k = 0$.

a) Prove that a nilpotent endomorphism only has zero as eigenvalue

b) Let $F \in \mathrm{End}(\mathbb{C}^n)$ be self-adjoint and nilpotent. Prove $F = 0$.

Subtask a): Let $\lambda \in \mathrm{spec}(F)$ and $v \in V$ such that $Fv = \lambda v$. Then we know by induction

$$F^n v = \lambda^n v$$

Let $k \in \mathbb{N}$ such that $F^k = 0$ and $v \neq 0$, then $0 = F^k v = \lambda^k v$. Since $v \neq 0$, it follows that $\lambda = 0$, since for any $x \in \mathbb{R} \setminus \{0\}$ we know $x^k \neq 0$ for $k \in \mathbb{N}$.

Subtask b): Assume $F \neq 0$. Let $G = F^{k-1}$, where $F^k = 0$. Notice that $G^2 = F^{2k-2} = 0$. Further

$$(G^* G\boldsymbol{x})^*(G^* G\boldsymbol{x}) = \boldsymbol{x}^* G^* G G^* G\boldsymbol{x} = \boldsymbol{x}^* G^* G^* G G x \boldsymbol{x} = 0$$

Notice that this implies $\forall \boldsymbol{x} \in \mathbb{C}^n\colon \|G\boldsymbol{x}\|^2 = \boldsymbol{x}^* G^* G\boldsymbol{x} = 0$, hence $G\boldsymbol{x} = \mathbf{0}$, but that implies that $F^{k-1}\boldsymbol{x} = \mathbf{0}$, which is a contradiction to our assumption that $F \neq 0$.

Task 65: Composition of self-adjoint operators

Let V be a finite-dimensional unitary vectorspace and $F, G \in \mathrm{End}(V)$ be self-adjoint. Prove that FG is self-adjoint iff $FG = GF$.

$\implies$:

$$GF = G^* F^* = (FG)^* = FG$$

$\impliedby$:

$$(FG)^* = G^* F^* = GF = FG$$