Exercise Sheet № 6

Task 6.1: Taylor Expansion of multidimensional functions

Define $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\boldsymbol{F}(x,y) = \begin{bmatrix} \sin(xy) + x\cos(y) \\ \mathrm{e}^{x+y} - \frac{1}{1+x^2+y^2} \end{bmatrix} = \begin{bmatrix} F_1(x,y) \\ F_2(x,y) \end{bmatrix}$$

a) Show that $\boldsymbol{F}$ is locally invertible in $\mathbf{0}$

b) Denote $\boldsymbol{G} = \left[G_1, G_2\right]^T$ as the local inverse of $\boldsymbol{F}$ around $\mathbf{0}$. Compute the first order Taylor-Approximation of G_i around $\mathbf{0}$, which is given by:

$$G_i(\boldsymbol{x}) = G_i(\mathbf{0}) + \partial_x G_i(\mathbf{0})x + \partial_y G_i(\mathbf{0})y + o\left(\|\boldsymbol{x}\|_2\right)$$

Subtask a)

Remember that $\boldsymbol{F}$ is locally invertible in $\boldsymbol{x}_0 \in \mathbb{R}^2$, iff $\det \mathrm{J}\boldsymbol{F}(\boldsymbol{x}_0) \neq 0$:

$$\mathrm{J}\boldsymbol{F}(\boldsymbol{x}) = \begin{bmatrix} y\cos(xy) + \cos(y) & x\cos(xy) - x\sin(y) \\ \mathrm{e}^{x+y} + \frac{2x}{(1+x^2+y^2)^2} & \mathrm{e}^{x+y} + \frac{2y}{(1+x^2+y^2)} \end{bmatrix} \Rightarrow \mathrm{J}\boldsymbol{F}(\mathbf{0}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$\Rightarrow \det \mathrm{J}\boldsymbol{F}(\mathbf{0}) = 1 \neq 0$$

Therefore $\boldsymbol{F}$ is locally invertible in $\mathbf{0}$.

Subtask b)

Since $\boldsymbol{F}$ is locally invertible in $\mathbf{0}$, there exists $\boldsymbol{G}$ on a neighborhood U of $\mathbf{0}$, such that $\boldsymbol{F}^{(-1)} = \boldsymbol{G}$ on $\boldsymbol{F}[U]$. Therefore we know the following:

$$\mathrm{J}\boldsymbol{G} = (\mathrm{J}\boldsymbol{F})^{-1}$$

Thus

$$\mathrm{J}\boldsymbol{G} = \begin{bmatrix} \partial_x G_1 & \partial_y G_1 \\ \partial_x G_2 & \partial_y G_2 \end{bmatrix} = \frac{1}{\det \mathrm{J}\boldsymbol{F}} \begin{bmatrix} \partial_y F_2 & -\partial_y F_1 \\ -\partial_x F_2 & \partial_x F_1 \end{bmatrix}$$
$$\Rightarrow \mathrm{J}\boldsymbol{G}(\mathbf{0}) = \begin{bmatrix} \partial_x G_1(\mathbf{0}) & \partial_y G_1(\mathbf{0}) \\ \partial_x G_2(\mathbf{0}) & \partial_y G_2(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} \partial_y F_2(\mathbf{0}) & -\partial_y F_1(\mathbf{0}) \\ -\partial_x F_2(\mathbf{0}) & \partial_x F_1(\mathbf{0}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Thus we only need $\boldsymbol{x}_0 \in \mathbb{R}^2$, such that $\boldsymbol{F}(\boldsymbol{x}_0) = \mathbf{0}$. Notice that $\boldsymbol{F}(\mathbf{0}) = \mathbf{0}$ and therefore $G(\mathbf{0}) = \mathbf{0}$, given $\boldsymbol{G}$ is bijective around $\mathbf{0}$. Thus:

$$G_1(x,y) = x + o(\|\boldsymbol{x}\|_2)$$
$$G_2(x,y) = y - x + o(\|\boldsymbol{x}\|_2)$$

Let X be a non-empty set and $(X, \mathcal{O})$ be a topological space. We call X *connected*, if the only open subsets that are both open and closed are $\emptyset$ and X.

Task 6.2: Connectedness

Let $(X, \mathcal{O})$ be a topological space. Prove the equivalence of the following statements:

a) X is disconnected

b) $X = A_1 \cup A_2$, where $A_1, A_2 \subset X$, $A_1 \cap A_2 = \emptyset$ and $A_1, A_2 \in \mathcal{O}$

c) $X = C_1 \cup C_2$, where C_1, C_2 are both closed, disjoint and proper

a) $\Rightarrow$ b) Assuming X is disconnected, then there exists a proper subset A_1 of X such that A_1 is both open and closed. Since $A_1 \subset X$ and $A_1 \neq \emptyset$, it follows that $A_1^c = A_2$ is proper, non empty and both open and closed. Note that of course $A_1 \cap A_2 = A_1 \cap A_1^c = \emptyset$ and $A_1 \cup A_2 = X$.

b) $\Rightarrow$ c) Take A_1 and A_2, since they are both closed, open, disjoint and proper we found $C_1 = A_1$ and $C_2 = A_2$.

c) $\Rightarrow$ a) Given that $C_1 \cup C_2 = X$ and $C_1 \cap C_2 = \emptyset$, then $C_1^c = C_2$ and $C_2^c = C_1$. Thus both are open and closed and proper disjoint subsets of X, thusly X is disconnected.

Definition: The euclidean topology τ on $\mathbb{R}^n$ is defined as the collection of sets

$$\tau = \{A \subset \mathbb{R}^n \colon \forall \boldsymbol{x} \in A \colon \exists \varepsilon > 0 \colon \mathcal{B}_\varepsilon(\boldsymbol{x}) \subset A\}$$

For the following exercises, $\mathbb{R}^n$ is always equipped with the euclidean topology. Any subset $A \subset \mathbb{R}^n$ is itself a topological space with the topology induced by the euclidean one.

Definition: A topological space $(X, \mathcal{O})$ is call *path-connected*, if

$$\forall x, y \in X \colon \exists \alpha \colon [0,1] \to X \colon \alpha \in \mathcal{C}([0,1]) \wedge \alpha(0) = x \wedge \alpha(1) = y \wedge \operatorname{im}\alpha \subset X$$

We call α a path between x and y.

Task 6.3: Connectedness under continuity

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be topological spaces

a) Assume $f \colon X \to Y$ is continuous. Show that if X is connected, then $f[X]$ is connected
b) Show that if X is path-connected, then it is connected
c) Suppose $A, B \subset X$ are path connected and $A \cap B \neq \emptyset$. Show that $A \cup B$ is path connected
d) Assume that $f \colon X \to Y$ is continuous. Show that if X is path-connected, then $f[X]$ is path-connected

Subtask a): f is continuous, if $\forall O \in \mathcal{O}_X \colon f[O] \in \mathcal{O}_Y$. Assume $f[X]$ is disconnected, then there exist two open and closed, disjoint, proper subsets $A_1, A_2 \in \mathcal{O}_Y$ such that $f[X] = A_1 \cup A_2$. Since f is continuous, we get that $f^{-1}[A_1], f^{-1}[A_2] \in \mathcal{O}_X$. Since $A_1 \cap f[X] \neq \emptyset$, we get $\exists x \in X \colon f(x) \in A_1$. Thus $x \in f^{-1}[A_1] \subset X$. Similarly for A_2, therefore $X \cap f^{-1}[A_1] \neq \emptyset$ and $X \cap f^{-1}[A_2] \neq \emptyset$. Suppose $\exists x \in X \colon x \in X \cap f^{-1}[A_1] \cap f^{-1}[A_2]$, then $f(x) \in f[X] \cap A_1 \cap A_2 = \emptyset$, since $A_1 \cap A_2 = \emptyset$. Therefore X is disconnected, which is a contradiction to our assumption, that X is connected.

Subtask b): Given X is path-connected, there always exists a continuous map $\alpha \colon [0,1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$ for any $x, y \in X$. Assume that X is not connected, then there exist two open, disjoint proper subsets U, V of X with $U \cup V = X$. Take $x \in U$ and $y \in V$. Since X is path connected, there exists a continuous path α between x and y, with $\operatorname{im}\alpha \subset X$. Given α is continuous and $U \cap V = \emptyset$, we get that $\alpha^{-1}[U] \cap \alpha^{-1}[V] = \emptyset$, which means that $[0,1]$ is not connected, which is a contradiction. Thus X is connected.

Subtask c): Let $C = A \cap B$, then $C \subseteq A$ and $C \subseteq B$. For $x \in A$, $y \in C$ and $z \in B$ we define the paths:

$$\alpha_1 \colon [0,1] \to A, \alpha_1(0) = x, \alpha_1(1) = y$$
$$\alpha_2 \colon [0,1] \to B, \alpha_2(1) = z, \alpha_2(0) = y$$
$$\alpha(t) = \begin{cases} \alpha_1(2t) & t \in [0, \frac{1}{2}) \\ y & t = \frac{1}{2} \\ \alpha_2(2t-1) & t \in (\frac{1}{2}, 1] \end{cases}$$

Where α_1, α_2 are paths between x and y and y and z respectively. If α is continuous, then $A \cup B$ is path-connected. Note that by construction, we get that $\lim_{t \to \frac{1}{2}^+} \alpha(t) = y = \lim_{t \to \frac{1}{2}^-} \alpha(t)$, thus α is continuous in $\frac{1}{2}$ and therefore on $[0,1]$.

Subtask d): Let $x, y \in X$ and α a path between x and y. Since both f and α are continuous, their composition $f \circ \alpha$ is continuous. Thus there exists a continuous map $f \circ \alpha$ between $f(x)$ and $f(y)$.

Task 6.4: Path-Connectedness

a) Let $X = \mathbb{R} \setminus \{0\}$. Prove that X is disconnected
b) Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ and define the line segment

$$[\boldsymbol{a}, \boldsymbol{b}] = \{t\boldsymbol{a} + (1-t)\boldsymbol{b}, t \in [0,1]\}$$

Prove that $[\boldsymbol{a}, \boldsymbol{b}]$ is path-connected
c) Let $C \subset \mathbb{R}^n$ be convex. Show that C is path-connected

Subtask a): $X = (-\infty, 0) \cup (0, \infty)$, i.e. we found to open, proper and disjoint subsets of X, such that $X = C_1 \cup C_2$, such that $C_1 \neq X$ and $C_2 \neq \emptyset$. Thus X is not connected.

Subtask c): Given C is convex, we know that $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in C\colon [\boldsymbol{x}_1, \boldsymbol{x}_2] \subseteq C$. We found a continuous map $\boldsymbol{\gamma}\colon [0,1] \to C$ such that $\boldsymbol{\gamma}(1) = \boldsymbol{x}_1$ and $\boldsymbol{\gamma}(0) = \boldsymbol{x}_2$ with $\operatorname{im}\boldsymbol{\gamma} \subseteq C$, i.e. C is path-connected.

Subtask b): By definition $[\boldsymbol{a}, \boldsymbol{b}]$ is convex, therefore it is path-connected by Subtask c).

Task 6.5: Homeomorphism

a) Let $\mathbb{S}^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} \colon \|\boldsymbol{x}\| = 1\}$. Show that $\mathbb{S}^n$ is path-connected
b) Show that $\mathbb{S}^1$ is not homeomorphic to $(0,1)$
c) Prove that the intervals $(0,1)$ and $(0,1]$ are not homeomorphic

Subtask a): Let $X = \mathbb{R}^{n+1} \setminus \{\boldsymbol{0}\}$. Define $f\colon X \to \mathbb{S}^n$ by setting $f(\boldsymbol{x}) = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. Note that $\|\boldsymbol{x}\|$ is continuous, thus $f \in \mathcal{C}(X)$. Given X is path-connected and f is continuous, it follows that $f[X] = \mathbb{S}^n$ is path-connected.

Subtask b): Two topological spaces are homeomorphic, if there exists a continuous bijection between them. Since $\mathbb{S}^1$ is compact, if there would exist such a mapping $f\colon \mathbb{S}^1 \to (0,1)$, this would imply that $\operatorname{im} f = (0,1)$ is compact, which is a contradiction.

Subtask c): Assume there exists $f\colon (0,1) \to (0,1]$ such that f is continuous and bijective. Let $A = (0,1) \setminus \{f^{-1}(1)\}$. Given f is bijective and continuous, it's strictly monotone, thus $A = (0, f^{-1}(1)) \cup (f^{-1}(1), 1)$ is disconnected, which is a contradiction to $f^{-1}[(0,1]] = (0,1)$.