

Exercise Sheet № 4

Remark: Let $A \subseteq \mathbb{R}^n$ be open and $F: A \rightarrow \mathbb{R}$. Suppose that there exists $z_0 \in A$ and a neighborhood $U \subset A$ of z_0 , such that ∇F exists on U and $\nabla F \in \mathcal{C}(U)$, then F is differentiable in z_0 .

Task 4.1: Continuity and Differentiability

Define $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting

$$F(x, y) = \begin{cases} 0 & y = 0 \\ y^2 \cos\left(\frac{1}{y}\right) & y \neq 0 \end{cases}$$

- a) Prove that $\partial_y F$ is not continuous in $(x, 0)$ for $x \in \mathbb{R}$
- b) Prove that F is differentiable in $(x, 0)$ for $x \in \mathbb{R}$

Subtask a:

$$\partial_y F(x, y) = \begin{cases} 0 & y = 0 \\ \sin\left(\frac{1}{y}\right) + 2y \cos\left(\frac{1}{y}\right) & y \neq 0 \end{cases}$$

Let $f(y) = \partial_y F(x, y)$. We prove that $f(y)$ is discontinuous in $y = 0$ for $x \in \mathbb{R}$. To prove that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is discontinuous in $y_0 \in \mathbb{R}$, we use the negation of the sequence-criterion of continuity:

$$\exists (\xi_n)_{n \in \mathbb{N}}: \lim_{n \rightarrow \infty} F(\xi_n) \neq F(\lim_{n \rightarrow \infty} \xi_n)$$

Let

$$\begin{aligned} \xi_n &= \begin{bmatrix} x \\ \frac{1}{\frac{(4n+1)\pi}{2}} \end{bmatrix} \Rightarrow \lim_{n \rightarrow \infty} \xi_n = \begin{bmatrix} x \\ 0 \end{bmatrix} \\ \partial_y F(\xi_n) &= \sin\left(\frac{(4n+1)\pi}{2}\right) = 1 \Rightarrow \lim_{n \rightarrow \infty} \partial_y F(\xi_n) = 1 \\ \partial_y F(\lim_{n \rightarrow \infty} \xi_n) &= \partial_y F(x, 0) = 0 \neq 1 \end{aligned}$$

Therefore $\partial_y F$ is discontinuous on $(x, 0)$ for $x \in \mathbb{R}$.

Subtask b: Given that $\forall x \in \mathbb{R}: F(x, 0) = 0$, we get

$$\lim_{h \rightarrow 0} \frac{F(x, h) - F(x, 0)}{h} = \lim_{h \rightarrow 0} \frac{h^2 \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \rightarrow 0} h \cos \frac{1}{h} \leq \lim_{h \rightarrow 0} h = 0$$

Remark: Let $A \subset \mathbb{R}^n$ be open and $F: A \rightarrow \mathbb{R}$. Suppose that there exists $z_0 \in A$ and a neighborhood $U \subset A$ of z_0 such that $\nabla^2 F$ exists and is continuous on U . Then $\partial_{x_i x_j} F(z_0) = \partial_{x_j x_i} F(z_0)$ for all $i, j \in \{1, \dots, n\}$.

Task 4.2

Define $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ by setting

$$F(x, y) = \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{x^3 y}{x^2 + y^2} & \mathbf{x} \neq \mathbf{0} \end{cases}$$

- Prove that F is continuous on \mathbb{R}^2
- Prove that F is differentiable on \mathbb{R}^2 and compute ∇F
- Prove that $\partial_{xy} F$ and $\partial_{yx} F$ exist on \mathbb{R}^2 and

$$\partial_{xy} F(\mathbf{0}) \neq \partial_{yx} F(\mathbf{0})$$

- Prove that $\partial_{xy} F$ and $\partial_{yx} F$ are not continuous in $\mathbf{0}$

Subtask a: Since for $\mathbf{x} \neq \mathbf{0}$ we get that F is continuous in \mathbf{x} . Therefore we want to show that F is continuous in $\mathbf{0}$. Let $(\xi_n)_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} \xi_n = \mathbf{0}$. Let $\xi_n = [x_n \ y_n]^T$, then, let n be sufficiently large, such that $|x_n| < \frac{\varepsilon}{2}$ and $|y_n| < \frac{\varepsilon}{2}$ for $\varepsilon > 0$.

$$|F(\xi_n)| = \left| \frac{x_n^3 y_n}{x_n^2 + y_n^2} \right| < \left| \frac{x_n^3 y_n}{x_n^2} \right| = |x_n y_n| < \frac{\varepsilon^2}{4}$$

Therefore $\lim_{n \rightarrow \infty} |F(\xi_n)| = 0 \Leftrightarrow \lim_{n \rightarrow \infty} F(\xi_n) = 0 = F(\mathbf{0})$, it follows that F is continuous in $\mathbf{0}$, and thus on \mathbb{R}^2 .

Subtask b: F is differentiable on \mathbb{R}^2 , if $\partial_x F$ and $\partial_y F$ exist and are continuous.

$$\begin{aligned} \partial_x F(x, y) &= \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{2x^2 y(x^2 + y^2) - x^3 y(2x)}{(x^2 + y^2)^2} = \partial_x f(x, y) & \mathbf{x} \neq \mathbf{0} \end{cases} \\ \partial_y F(x, y) &= \begin{cases} 0 & \mathbf{x} = \mathbf{0} \\ \frac{x^3(x^2 + y^2) - x^3 y(2y)}{(x^2 + y^2)^2} = \partial_y f(x, y) & \mathbf{x} \neq \mathbf{0} \end{cases} \\ \partial_x f(x, y) &= \frac{2x^4 y + 2x^2 y^3 - 2x^4 y}{(x^2 + y^2)^2} = \frac{2x^2 y^3}{(x^2 + y^2)^2} \\ \partial_y f(x, y) &= \frac{x^5 + x^3 y^2 - 2x^3 y^2}{(x^2 + y^2)^2} = \frac{x^5 - x^3 y^2}{(x^2 + y^2)^2} \end{aligned}$$

We change our coordinate system. Let $x = r \cos(\theta)$ and $y = r \sin(\theta)$, then :

$$\begin{aligned} \partial_x F(x, y) &= \frac{2r^5 \cos^2(\theta) \sin^3(\theta)}{r^4} = 2r \cos^2(\theta) \sin^3(\theta) \\ \lim_{(x, y) \rightarrow \mathbf{0}} |\partial_x F(x, y)| &= \lim_{r \rightarrow 0} |2r \cos^2(\theta(r)) \sin^3(\theta(r))| \leq \lim_{r \rightarrow 0} |2r| = 0 \\ \Rightarrow \lim_{(x, y) \rightarrow \mathbf{0}} \partial_x F(x, y) &= 0 \end{aligned}$$

by the squeeze-theorem. Note we use an arbitrary angle $\theta(r)$ in order to account for all paths leading to $\mathbf{0}$.

$$\begin{aligned} \partial_y F(x, y) &= \frac{r^5 \cos^5(\theta) - r^5 \cos^3(\theta) \sin^2(\theta)}{r^4} = r(\cos^5(\theta) - \cos^3(\theta) \sin^2(\theta)) \\ \lim_{(x, y) \rightarrow \mathbf{0}} |\partial_y F(x, y)| &= \lim_{r \rightarrow 0} |r(\cos^5(\theta(r)) - \cos^3(\theta(r)) \sin^2(\theta(r)))| \end{aligned}$$

$$\begin{aligned} &\leq \lim_{r \rightarrow 0} |r \cos^5(\theta(r))| + \lim_{r \rightarrow 0} |r \cos^3(\theta(r)) \sin^2(\theta(r))| \\ &\leq \lim_{r \rightarrow 0} |r| + \lim_{r \rightarrow 0} |r| = 0 \\ &\Rightarrow \lim_{(x,y) \rightarrow \mathbf{0}} \partial_y F(x, y) = 0 \end{aligned}$$

Hence both $\partial_x F$ and $\partial_y F$ are continuous on \mathbb{R}^2 , and thus F is differentiable on \mathbb{R}^2 .

$$\nabla F = \begin{cases} \mathbf{0} & \mathbf{x} = \mathbf{0} \\ \begin{bmatrix} \frac{2x^2y^3}{(x^2+y^2)^2} \\ \frac{x^5-x^3y^2}{(x^2+y^2)^2} \end{bmatrix} & \mathbf{x} \neq \mathbf{0} \end{cases}$$

Subtask c:

$$\partial_y \partial_x F = \frac{6x^2y^2(x^2+y^2)^2 - 2x^2y^2}{(x^2+y^2)^4}$$