Exercise Sheet № 9

Task 9.2

Let $(X, \|\cdot\|)$ be a normed vectorspace and $f\colon X \to \mathbb{R}$ be an unbound linear functional. Prove that $\ker f$ is dense in X, but $\ker f \neq X$.

Since f is unbounded, there exists a sequence $(x_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} x_k = 0$ and $\forall k \in \mathbb{N}\colon f(x_k) = 1$. Choose $y \in X$ arbitrarily. Notice that

$$f(y - f(y)x_n) = f(y) - f(y) = 0$$

Hence $y_k = y - f(y)x_k$ is convergent with limit y and since $x_k \notin \ker f \forall k \in \mathbb{N}$, we now that $\ker f \neq X$.

This holds for any $y \in X$. Let $\widetilde{y}, y \in X$

Task 9.3

Subtask i): Let $N \in \mathbb{N}$:

$$\sum_{k=1}^{N} |\xi_k| \cdot |\eta_k| \leq \|y\|_\infty \sum_{k=1}^{N} |\xi_k| \leq \|y\|_\infty \cdot \|x\|_1$$

Therefore $Tx(y)$ converges absolutely and

$$|Tx(y)| \leq \sum_{k=1}^{\infty} |\xi_k \eta_k| \leq \|x\|_1 \cdot \|y\|_\infty$$

Therefore Tx is bounded. Since $\mathfrak{c}_0(\mathbb{R})$ is a subspace of $\mathfrak{c}(\mathbb{R})$, we know that $\alpha_1 y_1 + \alpha_2 y_2 \in \mathfrak{c}_0(\mathbb{R})$. By the absolute convergence of $Tx(y)$ we get:

$$Tx(\alpha_1 y_1 + \alpha_2 y_2) = \alpha_1 \sum_{k=1}^{\infty} \xi_k \eta_{1,k} + \alpha_2 \sum_{k=1}^{\infty} \xi_k \eta_{2,k} = \alpha_1 Tx(y_1) + \alpha_2 Tx(y_2)$$

Hence Tx is linear and thus continuous. Furthermore, we showed that $Tx \in \mathfrak{c}_0(\mathbb{R})^*$. Further let $y \in \mathfrak{c}_0(\mathbb{R})$ with $\|y\|_\infty = 1$, then

$$\|Tx(y)\| \leq \|x\|_1 \implies \|Tx\| \leq \|x\|_1$$

Subtask ii): We define $e_j \in \mathfrak{c}_0$ with $e_{j,k} = \delta_{jk}$. Notice

$$\forall j \in \mathbb{N}\colon Tx(e_j) = \xi_j = 0 \iff \forall k \in \mathbb{N}\colon x_k = 0 \iff x = 0$$

Thus T is injective. For surjectivity, we consider the unit-sequences $e_j \in \mathfrak{c}_0(\mathbb{R})$, where $e_{j,k} = \delta_{jk}$. Let $S \in \mathfrak{c}_0(\mathbb{R})^*$. We want to find $x \in \ell^1(\mathbb{R})$ such that $Tx = S$. Notice that for $y \in \mathfrak{c}_0(\mathbb{R})$:

$$y = \sum_{j\in\mathbb{N}} \eta_j e_j \implies S(y) = \sum_{j\in\mathbb{N}} \eta_j S(e_j)$$

Let $g_n = S(e_n)$, then $Tg(y) = S(y)$ for $y \in \mathfrak{c}_0(\mathbb{R})$, i.e. $\xi_k = g_k = S(e_k)$ defines x. We now define for $N \in \mathbb{N}$

$$\eta_k = \begin{cases} \operatorname{sign}(g_k) & g_k \neq 0 \wedge k \leq N \\ 0 & g_k = 0 \vee k > N \end{cases}$$

Notice that $\|y\|_\infty \leq 1$, therefore

$$\sum_{k=1}^{N} |\eta_k| = \sum_{k=1}^{N} \eta_k g_k = |S(y)| \leq \|S\| \cdot \|y\|_\infty \leq \|S\|$$

$$\implies \sum_{k=1}^{\infty} |g_k| = \sup_{N \in \mathbb{N}} \sum_{k=1}^{N} |g_k| \leq \|S\| < \infty$$

since S is a bounded linear functional, hence $x \in \ell^1(\mathbb{R})$. Thus T is an isomorphism. It remains to show, that T is an isometry. Since Tx is a bounded linear functional, we only need to find $x \in \ell^1(\mathbb{R})$ such that $\|Tx\| \geq \|x\|_1$. The case $Tx = 0$ is trivial. Let $x \in \ell^1(\mathbb{R})$ with $x \neq 0$, we define

$$f_k = \begin{cases} \operatorname{sign}(\xi_k) & k \leq N \\ 0 & k > N \end{cases} \implies f \in \mathfrak{c}_0(\mathbb{R})$$

Further $\|f\|_\infty = 1$ and $Tx(f) = \sum_{k \in \mathbb{N}} f_k \xi_k = \sum_{k \in \mathbb{N}} |\xi_k| = \|x\|_1$:

$$\|x\|_1 = |Tx(f)| \leq \|Tx\| \cdot \|f\| = \|Tx\|$$