Exercise Sheet № 8

Definition: Submanifold A k-dimensional submanifold of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that for every $\boldsymbol{x} \in M$ there exists an open set $U \subset \mathbb{R}^n$ with $\boldsymbol{x} \in U$ such that $M \cap U$ is a regular k-dimensional surface, that is, locally M can be parametrized by a regular k-dimensional surface parametrization, as seen in the lecture. Let $\boldsymbol{f}: D \to M$ be a local parametrization at $\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{u}) \in M$, then the tangent space of M at $\boldsymbol{x}$, denote $T_{\boldsymbol{x}}M$ can be computed as $\mathrm{im} \mathrm{J}\boldsymbol{f}$.

Theorem: Submanifold Characterisation Let $\boldsymbol{F}: \mathbb{R}^n \to \mathbb{R}^{n-k}$ be differentiable, $\boldsymbol{c} \in \mathbb{R}^{n-k}$ and denote $M = \{\boldsymbol{x} \in \mathbb{R}^n : F(\boldsymbol{x}) = \boldsymbol{c}\}$. If $\mathrm{D}\boldsymbol{F}(\boldsymbol{x})$ is surjective on M, then M is a k dimensional submanifold. In this case the tangent space at $\boldsymbol{x}$ can be computed as $\ker(\mathrm{D}F(\boldsymbol{x}))$.

Task 8.1: Real Unitary Matrices

Consider the set of real unitary matrices $\mathcal{O}(n) = \{\mathrm{A} \in \mathbb{R}^{n\times n} : \mathrm{A}^{-1} = \mathrm{A}^T\} \subset \mathbb{R}^{n\times n}$.

a) Show that $\mathcal{O}(n)$ is not connected

b) Show that $\mathcal{O}(n)$ is a $\frac{n(n-1)}{2}$ dimensional submanifold and that the tangent space of $\mathcal{O}(n)$ in I is the set of skew-symmetric matrices.
Hint: Characterize $\mathcal{O}(n)$ as the zero-set of a mapping $F: \mathbb{R}^{n\times n} \to \mathbb{R}_s^{n\times n}$, where $\mathbb{R}_s^{n\times n}$ ist the space of all symmetric $n \times n$ matrices

Subtask a): Notice that for $\mathrm{A} \in \mathcal{O}(n)$ we get $\det(\mathrm{A}) = 1$ or $\det(\mathrm{A}) = -1$, since

$$\det(\mathrm{A}\mathrm{A}^{-1}) = \det(\mathrm{A})\det(\mathrm{A}^T) = \det^2(\mathrm{A}) = 1$$

Let $P = \{\mathrm{A} \in \mathbb{R}^{n\times n} : \det(\mathrm{A}) = 1\} \cap \mathcal{O}(n)$ and $N = \{\mathrm{A} \in \mathbb{R}^{n\times n} : \det(\mathrm{A}) = -1\} \cap \mathcal{O}(n)$. Note that $\forall \mathrm{A} \in \mathcal{O}(n) : \det(\mathrm{A}) \neq 0$, i.e. $\mathcal{O} = P \cup N$. Further notice that $P \cap N = \emptyset$. If P and M are both open or closed, then $\mathcal{O}$ is not connected. Let $f : \mathcal{O}(n) \to \{-1, 1\}$ where $f(\mathrm{A}) = \det(\mathrm{A})$. Since $\det$ is continuous on $\mathbb{R}^{n\times n}$, we use that the pre-image of a closed set under a continuous map is closed. Note that $f^{-1}[\{1\}] = P$ and $f^{-1}[\{-1\}] = N$, i.e. P and N are closed.

Subtask b):

Let $\mathrm{f}: \mathbb{R}^{n\times n} \to \mathbb{R}_s^{n\times n}$ where $\mathrm{f}(\mathrm{A}) = \mathrm{A}\mathrm{A}^t - \mathrm{I}$. Then $\mathrm{f}|_{\mathcal{O}(n)} = 0$. Let $\mathrm{V} \in \mathbb{R}^{n\times n}$ and $\mathrm{O} \in \mathcal{O}(n)$, then

$$\begin{aligned}
&\mathrm{Df}(\mathrm{O})\mathrm{V} = \lim_{t\to 0} \frac{1}{t}(\mathrm{f}(\mathrm{O} + t\mathrm{V}) - \mathrm{f}(\mathrm{O})) \\
&= \lim_{t\to 0} \frac{1}{t}((\mathrm{O} + t\mathrm{V})(\mathrm{O} + t\mathrm{V})^T - \mathrm{I} - \mathrm{O}\mathrm{O}^T + \mathrm{I}) = \lim_{t\to 0} \frac{1}{t}\left(\mathrm{O}\mathrm{O}^T + t\mathrm{O}\mathrm{V}^T + t\mathrm{V}\mathrm{O}^T + t^2\mathrm{V}\mathrm{V}^T - \mathrm{O}\mathrm{O}^T\right) \\
&= \lim_{t\to 0} \mathrm{O}\mathrm{V}^T + \mathrm{V}\mathrm{O}^T + t\mathrm{V}\mathrm{V}^T = \mathrm{O}\mathrm{V}^T + \mathrm{V}\mathrm{O}^T = \mathrm{Df}(\mathrm{O})\mathrm{V}
\end{aligned}$$

Therefore f is differentiable, since $\mathrm{O}\mathrm{V}^T + \mathrm{O}^T\mathrm{V}$ is continuous. Next we want to show that Df is surjective. Let $\mathrm{A} \in \mathbb{R}_s^{n\times n}$. We want to find $\mathrm{V} \in \mathbb{R}^{n\times n}$, such that $\mathrm{Df}(\mathrm{O})\mathrm{V} = \mathrm{A}$:

$$\mathrm{V} = \frac{1}{2}\mathrm{A}\mathrm{O} \Rightarrow \mathrm{Df}(\mathrm{O})\mathrm{V} = \frac{1}{2}\mathrm{O}\mathrm{O}^T\mathrm{A}^T + \frac{1}{2}\mathrm{A}\mathrm{O}\mathrm{O}^T = \frac{1}{2}(\mathrm{A} + \mathrm{A}^T) = \mathrm{A}$$

The last equality only holds for symmetric matrices A. Therefore $\mathcal{O}(n)$ is a $\frac{n(n-1)}{2}$ dimensional submanifold. Note that $\mathrm{Df}(\mathrm{I})\mathrm{V} = \mathrm{V} + \mathrm{V}^T$. Hence $\mathrm{Df}(\mathrm{I})\mathrm{V} = 0 \Leftrightarrow \mathrm{V} = -\mathrm{V}^T$, which is the definition of a skew-symmetric matrix.

Task 8.2: Connected Dense Subset

Let $(X, \mathcal{T})$ be a topological space. Show that if there exists a connected dense subset $A \subset X$ of X, then X is connected.

We show that X is connected by contradiction. Since A is dense in X, then $\forall \omega \in \mathcal{T} : A \cap \omega \neq \emptyset$. Furthermore, assume there exist $U, V \in \mathcal{T}$, such that $U, V \neq \emptyset$, $U \cap V = \emptyset$ and $U \cup V = X$, i.e. X is disconnected. Since A is dense, we get that $A \cap U = C \neq \emptyset$ and $A \cap V = D \neq \emptyset$, thus $A = C \cup D$. Since A is assumed to be connected, it is open, thus C and D are open, since they are finite intersections of open sets, which leads to the contradiction that A is the disjoint union of two open sets, equivalent to the fact that A is disconnected.

Task 8.3

Let A and B be subspaces of a topological space such that $A \cup B$ and $A \cap B$ are connected. Prove if A and B are either both open or both closed, then A and B are connected.

Let without loss of generality A and B be open. Assume A is disconnected, then there exist $C, D \subseteq A$, such that C, D are open and $A = C \uplus D$. Thus:

$$(B \cap C) \uplus (B \cap D) = A \cap B$$

Since $A \cap B$ is connected, it follows that either $B \cap C = \emptyset$ or $B \cap D = \emptyset$. If $B \cap C = \emptyset$, then $A \cup B = (D \cup B) \uplus C$, i.e. $A \cup B$ is disconnected, which is a contradiction. Analogous for $B \cap D = \emptyset$. The same argument holds true for the case that A and B are closed.

Task 8.4: Tangent Space

Determine the tangent space of the surface defined by $x^2 + y^2 - z^2 = 25$ in all points where $z = 0$.

We want to characterize the set $x^2 + y^2 = 25$ by the kernel of $F(x, y, z) = x^2 + y^2 - 25$. Note that $\mathrm{D}F(x, y, z) = \begin{bmatrix} 2x & 2y & 0 \end{bmatrix}$, where $\mathrm{D}F\colon \mathbb{R}^3 \to \mathbb{R}$. If $\mathrm{D}F$ is surjective on $\ker F$, then $T_{\boldsymbol{x}} \ker F = \ker \mathrm{D}F(\boldsymbol{x})$. Let $a \in \mathbb{R}$ and $\boldsymbol{w} \in \mathbb{R}^3$, then

$$\begin{aligned}
&y \neq 0, x \neq 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2xw_1 + 2yw_2 \overset{!}{=} a \Leftrightarrow 2xw_1 - a = -2yw_2 \Leftrightarrow w_2 = \frac{1}{2y}(a - 2xw_1) \\
&y = 0, x \neq 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2xw_1 \overset{!}{=} a \Leftrightarrow w_1 = \frac{a}{2x} \wedge w_2 \in \mathbb{R} \\
&y \neq 0, x = 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2yw_2 \overset{!}{=} a \Leftrightarrow w_2 = \frac{a}{2y} \wedge w_1 \in \mathbb{R}
\end{aligned}$$

Therefore $\mathrm{D}F(\boldsymbol{x})$ is surjective for $\boldsymbol{x} \in \ker F$. We now want to find $\boldsymbol{w} \in \mathbb{R}^3$, such that $\mathrm{D}F(\boldsymbol{w})\boldsymbol{w} = 0$:

$$\begin{aligned}
&x \neq 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2xw_1 + 2yw_2 \overset{!}{=} 0 \Leftrightarrow xw_1 = -yw_2 \Leftrightarrow w_1 = -\frac{y}{x}w_2 \\
&x = 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2yw_2 \overset{!}{=} 0 \Rightarrow w_1 \in \mathbb{R} \wedge w_2 = 0 \\
&y = 0 \qquad \mathrm{D}F(\boldsymbol{x})\boldsymbol{w} = 2xw_1 \overset{!}{=} 0 \Rightarrow w_1 = 0 \wedge w_2 \in \mathbb{R}
\end{aligned}$$

Addendum: The surjectiveness of $\mathrm{D}F$ directly follows by the rank of $\mathrm{D}F$, which is 1, thus $\dim(\operatorname{im} \mathrm{D}F) = 1$, i.e, $\operatorname{im} \mathrm{D}F \simeq \mathbb{R}$.

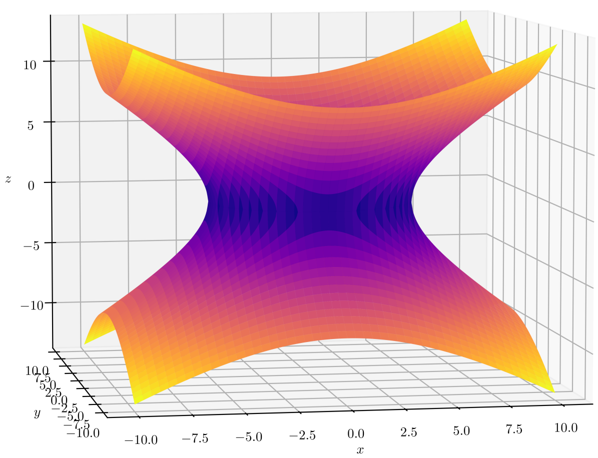

Figure 1: Described Surface

Alternative Method for Tangent Space: Note that $\ker F = \operatorname{im}\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}\colon [0, 2\pi] \to \mathbb{R}^3$ with

$$\boldsymbol{\gamma}(t) = \begin{bmatrix} 5\cos(t) \\ 5\sin(t) \\ 0 \end{bmatrix}$$

Note that $\boldsymbol{\gamma}$ is a regular $\mathcal{C}^\infty$ curve parametrization by arc-length, thus

$$\boldsymbol{T}(t) = \frac{1}{5}\boldsymbol{\gamma}'(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{bmatrix}$$
$$\boldsymbol{N}(t) = \boldsymbol{T}'(t) = \begin{bmatrix} -\cos(t) \\ -\sin(t) \\ 0 \end{bmatrix}$$
$$\boldsymbol{B}(t) = \boldsymbol{T}(t) \times \boldsymbol{N}(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We see that $T_{\boldsymbol{x}}\operatorname{im}\boldsymbol{\gamma} = \boldsymbol{x} + \boldsymbol{N}(t)^\perp = \boldsymbol{x} + \operatorname{span}(\boldsymbol{T}(t), \boldsymbol{B}(t))$, where $\boldsymbol{x} = \boldsymbol{\gamma}(t)$.