Exercise Sheet № 1

Task 1.1: Sequences in metric spaces I

Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence X and $x \in X$.

i) Prove that $\lim_{n\to\infty} x_n = x$, iff for every partial sequence $(x_{k_1})_{k_1\in J_1}$, there exists a partial sequence $(x_{k_2})_{k_2\in J_2}$ which converges to x

ii) Assume that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence in (X, d) and x is an accumulation point of $(x_n)_{n\in\mathbb{N}}$. Prove that $\lim_{n\to\infty} x_n = x$

Subtask i):

$\Longrightarrow$: We use a contradiction and assume $(x_n)_{n\in\mathbb{N}}$ does not converge, i.e:

$$\exists\varepsilon > 0\colon \forall N \in \mathbb{N}\colon \exists n \geq N\colon d(x_n, x) \geq \varepsilon$$

Thus $\exists J \subseteq \mathbb{N}$ with $|J| = \aleph_0$, such that $\forall n \in J\colon d(x_n, x) \geq \varepsilon$. Let $(x_k)_{k\in J}$ be a partial sequence. By our assumption $(x_k)_{k\in J}$ has the accumulation point x, however $\forall k \in J\colon d(x_k, x) \geq \varepsilon$, thus x cannot be an accumulation point of $(x_k)_{k\in J}$, which is a contradiction to our assumption. Thus $(x_n)_{n\in\mathbb{N}}$ converges to x.

$\Longleftarrow$: $(x_n)_{n\in\mathbb{N}}$ converges in (X, d) to x, iff:

$$\forall\varepsilon > 0\colon \exists N \in \mathbb{N}\colon n \geq N \implies d(x_n, x) < \varepsilon$$

Let $(x_{k_2})_{k_2\in J_2}$ be a partial sequence of $(x_{k_1})_{k_1\in J_1}$ with $J_2 \subseteq J_1 \subseteq \mathbb{N}$ and $|J_2| = |J_1| = \aleph_0$. The convergence of $(x_n)_{n\in\mathbb{N}}$ to x implies for $k_2 \in J_2$ with $k \geq N$:

$$k_2 \geq N \implies d(x_{k_2}, x) < \varepsilon$$

Hence $\lim_{k_2\to\infty} x_{k_2} = x$.

Subtask ii):

If $(x_n)_{n\in\mathbb{N}}$ has a convergent partial sequence $(x_{n_k})_{k\in\mathbb{N}}$, then

$$\forall\varepsilon > 0\colon \exists K \in \mathbb{N}\colon k \geq K \implies d(x_{n_k}, x) < \frac{\varepsilon}{2}$$

Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence:

$$\exists N \in \mathbb{N}\colon n, m \geq N \implies d(x_n, d_m) < \frac{\varepsilon}{2}$$

Let $\widetilde{N} = \max(N, K)$, then

$$\forall n, k \geq N\colon d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon$$

Task 1.2: Sequences in metric spaces II

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a complete metric space (X, d):

i) assume $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < \infty$. Prove that $(x_n)_{n\in\mathbb{N}}$ converges

ii) Does $(x_n)_{n\in\mathbb{N}}$ converge, if we only assume $\sum_{n\in\mathbb{N}} d^2(x_n, x_{n+1}) < \infty$?

Subtask i): Let $y_n = d(x_n, x_{n+1})$ be the corresponding sequence of real numbers. Note that $\forall n \in \mathbb{N}\colon y_n \geq 0$. Since

$$\sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \sum_{n\in\mathbb{N}} y_n$$

converges, we know that $\lim_{n\to\infty} y_n = 0$, hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, i.e:

$$\forall\varepsilon > 0\colon \exists N \in \mathbb{N}\colon n \geq N\colon d(x_n, x_{n+1}) < \frac{\varepsilon}{m}$$

for some $m \in \mathbb{N}$.

Since d is a metric on X, we get:

$$d(x_n, x_{n+2}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) < \frac{2\varepsilon}{m}$$

Hence:

$$d(x_n, x_{n+m}) \leq \sum_{i=0}^{m-1} d(x_{n+i}, x_{n+i+1}) < \sum_{i=0}^{m-1} \frac{\varepsilon}{m} = \varepsilon$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy-sequence. Since (X, d) is complete, $(x_n)_{n\in\mathbb{N}}$ converges.

Subtask ii): No, since we are not guaranteed, that $\sum_{n\in\mathbb{N}} d(x_n, x_{n+1})$ converges.

Task 1.3: Closure and Interior

i) Let (X, d) be a metric space and $C, D \subseteq X$. Prove $\operatorname{cls}(C \cup D) = \operatorname{cls} C \cup \operatorname{cls} D$ and $\operatorname{int}(C \cap D) = \operatorname{int} C \cap \operatorname{int} D$

ii) Find a metric space (X, d) and subsets $C, D \subseteq X$, such that

$$\operatorname{cls}(C \cap D) \neq \operatorname{cls} C \cap \operatorname{cls} D \qquad \wedge \qquad \operatorname{int}(C \cup D) \neq \operatorname{int} C \cup \operatorname{int} D$$

We denote $\mathcal{T}_d = \{A \subseteq X | \forall x \in A\colon \exists r \in \mathbb{R}\colon \mathcal{B}_r(x) \subseteq A\}$ the topology generated by d on X. Recall the definition of the closure and interior:

$$\operatorname{cls} A = \bigcap \{B \subseteq X\colon A \subseteq B \wedge B^{\mathsf{c}} \in \mathcal{T}_d\}$$
$$\operatorname{int} A = \bigcup \{C \in \mathcal{T}_d\colon C \subseteq A\}$$

Let $\mathcal{C}_d = \{A \subseteq X\colon A^{\mathsf{c}} \in \mathcal{T}_d\}$ be the set of closed sets in (X, d).

Subtask i): Using the distributive properties of $\cap$ and $\cup$, we get:

$$\begin{aligned}
&\operatorname{cls}(C \cup D) = \bigcap \{A \in \mathcal{C}_d\colon C \cup D \subseteq A\} = \bigcap \{A \in \mathcal{C}_d\colon C \subseteq A \vee D \subseteq A\} \\
&= \bigcap (\{A \in \mathcal{C}_d\colon C \subseteq A\} \cup \{B \in \mathcal{C}_d\colon D \subseteq B\}) \\
&= \left(\bigcap \{A \in \mathcal{C}_d\colon C \subseteq A\}\right) \cup \left(\bigcap \{B \in \mathcal{C}_d\colon D \subseteq B\}\right) = \operatorname{cls}(C) \cup \operatorname{cls}(D)
\end{aligned}$$

Similarly for the interior:

$$\begin{aligned}
&\operatorname{int}(C \cap D) = \bigcup \{A \in \mathcal{T}_d\colon A \subseteq C \cap D\} = \bigcup \{A \in \mathcal{T}_d\colon A \subseteq C \wedge A \subseteq D\} \\
&= \bigcup (\{A \in \mathcal{T}_d\colon A \subseteq C\} \cap \{B \in \mathcal{T}_d\colon B \subseteq D\}) \\
&= \left(\bigcup \{A \in \mathcal{T}_d\colon A \subseteq C\}\right) \cap \left(\bigcup \{B \in \mathcal{T}_d\colon B \subseteq D\}\right) = \operatorname{int}(C) \cap \operatorname{int}(D)
\end{aligned}$$

Subtask ii): Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Further let $C = (0, 1]$ and $D = (1, 2)$. Then

$$\begin{aligned}
\operatorname{cls} C = [0, 1] \qquad \operatorname{cls} D = [1, 2] &\implies \operatorname{cls} C \cap \operatorname{cls} D = \{1\} \\
\operatorname{int} C = (0, 1) \qquad \operatorname{int} D = (1, 2) &\implies \operatorname{int} C \cup \operatorname{int} D = (0, 1) \cup (1, 2)
\end{aligned}$$

Further $C \cap D = \emptyset$. Since $\emptyset \in \mathcal{T}_d$ and $\emptyset \subseteq \emptyset$, we get $\operatorname{cls} \emptyset = \emptyset$, hence

$$\begin{aligned}
&\operatorname{cls}(C \cap D) = \emptyset \neq \{1\} = \operatorname{cls} C \cap \operatorname{cls} D \\
&\operatorname{int}(C \cup D) = \operatorname{int}((0, 2)) = (0, 2) \neq (0, 1) \cup (1, 2) = \operatorname{int} C \cup \operatorname{int} D
\end{aligned}$$