Exercise Sheet № 3

Task 3.1

Let (X, d) be a metric space and $C \subseteq X$ not empty. We define the distance to C as

$$d_C : X \to \mathbb{R} \qquad x \mapsto \inf_{y \in C} d(x, y)$$

i) Assume C is compact. Let $x \in X$. Prove $\exists \overline{y} \in C$ such that $d(x, \overline{y}) = d_C(x)$. Is this $\overline{y}$ unique?
ii) Is $\overline{y}$ unique, if $X = \mathbb{R}^N$ and C is closed?

Subtask i): We prove a metric is continuous. Let $x \in X$ be fixed, $\varepsilon = \delta > 0$ and $y_0 \in X$. Let $y \in \mathcal{B}_\delta(y_0)$. Further let wlog $d(x, y_0) \geq d(x, y)$ now:

$$|d(x, y_0) - d(x, y)| \leq |d(x, y) + d(y, y_0) - d(x, y)| = |d(y, y_0)| < \delta$$

Hence $d(x, \cdot)$ is continuous. Since C is compact, the $f_x[C]$ is also compact, where $f_x(y) = d(x, y)$, hence $\exists a, b \in \mathbb{R}_0^+$ with $a < b$, such that $f_x[C] = [a, b]$. Therefore $\inf f_x[C] = \inf_{y \in C} d(x, y) = \inf[a, b] = a$.

$\overline{y}$ is not unique. Let $X = \mathbb{R}$ and $C = [0, 1] \cup [2, 3]$. Notice C is is still compact. For $x = \frac{3}{2}$, both $d(x, 1) = \frac{1}{2}$ and $d(x, 2) = \frac{1}{2}$.

Task 3.2

Let (X, d) be a metric space, and $(K_n)_{n \in \mathbb{N}}$ be family of non empty, closed sets, where $\forall n \in \mathbb{N} \colon K_{n+1} \subset K_n$. Assume (X, d) is compact:
i) Use 2.1 to prove that $\bigcap_{n \in \mathbb{N}} K_n \neq \emptyset$
ii) Additionally assume $\operatorname{diam} K_n \overset{n \to \infty}{\longrightarrow} 0$. Prove that $\bigcap_{n \in \mathbb{N}} K_n$ contains exactly on element of X.

Subtask ii): Assume there exists $y \in X$ such that $y \neq x$ and

$$K = \bigcap_{n \in \mathbb{N}} K_n = \{x, y\}$$

Since $x \neq y$ we know $d(x, y) > 0$, i.e. $\operatorname{diam} K = d(x, y) > 0$, which is a contradiction to our assumption.

Task 3.3

Provide an example of a complete metric space (X, d), a relatively compact subset $C \subseteq X$ and a sequence $(x_n)_{n \in \mathbb{N}} \in C^{\mathbb{N}}$, whose limit does not lie in C.

Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Further let $C = (0, 1)$. Notice that $\operatorname{cls} C = [0, 1]$ is compact, i.e. C is relatively compact. We define $x_n = \frac{1}{n+1}$. Notice that $\lim_{n \to \infty} x_n = 0$, and $\forall n \in \mathbb{N} \colon x_n \in C$, but $0 \notin C$.