Exercise Sheet № 5

Task 5.1: Norm on ℓ^1

Recall

$$\ell^1(\mathbb{R}) = \left\{ x = (\xi_k)_{k\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0} : \sum_{k\in\mathbb{N}_0} |\xi_k| < \infty \right\} \qquad \|x\|_1 = \sum_{k\in\mathbb{N}_0} |\xi_k|$$

We define

$$\|x\| = \sup_{p\in\mathbb{N}_0} \left| \sum_{k=0}^{p} \xi_k \right|$$

Prove or disprove the following statements:

i) $\|\cdot\|$ is a norm on $\ell^1(\mathbb{R})$

ii) $(\ell^1(\mathbb{R}), \|\cdot\|)$ is a Banach-space

iii) $\|\cdot\|$ is equivalent to $\|\cdot\|_1$

Subtask i): Recall that a norm must satisfy the following properties:

- $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{0}$
- $\|\lambda\boldsymbol{x}\| = |\lambda|\|\boldsymbol{x}\|$
- $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

We first check that $\|\cdot\|$ is not degenerate. Assume $\|\boldsymbol{x}\| = 0$, then:

$$\forall p \in \mathbb{N}_0 : \left| \sum_{k=0}^{p} \xi_k \right| = 0$$

Assume now $\boldsymbol{x} \neq \boldsymbol{0}$ then $\exists m \in \mathbb{N} : \xi_m \neq 0$, but then:

$$\left| \sum_{k=0}^{m} \xi_k \right| = \left| \xi_m + \sum_{k=0}^{m-1} \xi_k \right| = |\xi_m| \neq 0$$

Hence either $\boldsymbol{x} = \boldsymbol{0}$ or $\|\boldsymbol{x}\| \neq 0$. For homogeneity. Let $\lambda \in \mathbb{R}$:

$$\|\lambda\boldsymbol{x}\| = \sup_{p\in\mathbb{N}_0} \left| \sum_{k=0}^{p} \lambda\xi_k \right| = \sup_{p\in\mathbb{N}_0} \left| p \sum_{k=0}^{p} \xi_k \right| = \sup_{p\in\mathbb{N}_0} |\lambda| \cdot \left| \sum_{k=0}^{p} \xi_k \right|$$

Since both $|\lambda| \geq 0$ and $|\sum_{k=0}^{p} \xi_k| \geq 0$, we can factor $|\lambda|$:

$$\|\lambda\boldsymbol{x}\| = |\lambda| \cdot \sup_{p\in\mathbb{N}_0} \left| \sum_{k=0}^{p} \xi_k \right|$$

At last, the triangle inequality:

$$\left| \sum_{k=0}^{p} \xi_k + \mu_k \right| = \left| \sum_{k=0}^{p} \xi_k + \sum_{k=0}^{p} \mu_k \right| \leq \left| \sum_{k=0}^{p} \xi_k \right| + \left| \sum_{k=0}^{p} \mu_k \right|$$

Let

$$N_p = \left| \sum_{k=0}^{p} \xi_k + \mu_k \right| \qquad S_p = \left| \sum_{k=0}^{p} \xi_k \right| + \left| \sum_{k=0}^{p} \mu_k \right|$$

then $\forall p \in \mathbb{N}_0 : N_p \leq S_p$. Let N_{p_0} such that $\forall p \in \mathbb{N}_0 : N_p \leq N_{p_0}$, then $\forall p \in \mathbb{N}_0 : N_p \leq N_{p_0} \leq S_{p_0}$ and thus $\sup_{p\in\mathbb{N}_0} N_p \leq \sup_{p\in\mathbb{N}_0} S_p$.

Subtask ii): We consider the following sequence in $\ell^1(\mathbb{R})$

$$(x_n)_{n\in\mathbb{N}_0} \qquad \xi_{k,n} = \begin{cases} \frac{(-1)^k}{k+1} & k \leq n \\ 0 & k > n \end{cases}$$

Notice that $\forall n \in \mathbb{N}\colon x_n \in \ell^1(\mathbb{R})$. Let $\mu_k = \frac{(-1)^k}{k+1}$, then

$$\sum_{k=0}^{n} |\mu_k| = \sum_{k=0}^{n} \frac{1}{k+1} \implies \sum_{k\in\mathbb{N}_0} |\mu_k| = \infty \implies (\mu_k)_{k\in\mathbb{N}_0} \notin \ell^1(\mathbb{R})$$

Let $m, n \in \mathbb{N}$ and wlog n be odd:

$$\boldsymbol{x}_{m+n} - \boldsymbol{x}_n = \left(\underbrace{0,\dots,0}_{n\text{ times}}, \frac{(-1)^{n+1}}{n+1}, \dots, \frac{(-1)^{n+m}}{n+m}, 0, \dots\right)$$

Now let $s_l = \sum_{k=n+1}^{n+1+l} \frac{(-1)^k}{k}$. Notice that $s_0 = \frac{1}{n+1}$ as $n+1$ is even. We prove that for $n \in \mathbb{N}\colon s_n \leq s_0$ by induction:

$$s_1 \leq s_0 \iff \frac{1}{n+1} - \frac{1}{n+2} \leq \frac{1}{n+1} \iff -\frac{1}{n+2} \leq 0$$

Thus:

$$s_k = \frac{(-1)^{n+1+k}}{n+k+1} + s_{k-1} \leq \frac{(-1)^{n+1+k}}{n+1+k} + s_0 \iff 0 \leq s_0$$

Therefore:

$$\|\boldsymbol{x}_{n+m} - \boldsymbol{x}_n\| = \frac{1}{n+1}$$

Hence given any $\varepsilon > 0$, we just choose $n > \frac{1}{\varepsilon} - 1$ to satisfy that $\|\boldsymbol{x}_{n+m} - \boldsymbol{x}_n\| < \varepsilon$. Thus $(\boldsymbol{x}_n)_{n\in\mathbb{N}}$ is Cauchy-sequence whose limit is not in $\ell^1(\mathbb{R})$

Task 5.2: Subspaces of $\ell^p(\mathbb{R})$

Let $p \in [1, \infty)$ and set

$$U_p = \left\{ (\xi_k)_{k\in\mathbb{N}_0} \in \ell^p(\mathbb{R})\colon \sum_{k=0}^{\infty} \xi_k = 0 \right\}$$

Prove the following statements

i) U_p is a subspace of $\ell^p(\mathbb{R})$

ii) If $p > 1$ then U_p is not closed in $\ell^p(\mathbb{R})$

iii) U_1 is closed in $\ell^1(\mathbb{R})$

Subtask i: Obviously $U_p \subseteq \ell^p(\mathbb{R})$. Now let $\lambda \in \mathbb{R}$ and $\boldsymbol{x} \in U_p$. We define $\Xi_n = \sum_{k=0}^{n} \xi_k$:

$$\lambda\boldsymbol{x} \in U_p \iff \lim_{k\to\infty} \lambda\Xi_k = 0$$

Using properties of limits we get $\lim_{k\to\infty} \lambda\Xi_k = \lambda \lim_{k\to\infty} \Xi_k = 0$ hence $\lambda\boldsymbol{x} \in U_p$. Similarly let $\boldsymbol{y} \in U_p$ with $\boldsymbol{y} = (\mu_k)_{k\in\mathbb{N}_0}$ and $M_n = \sum_{k=0}^{n} \mu_k$ then

$$\boldsymbol{x} + \boldsymbol{y} \in U_p \iff \lim_{k\to\infty} \Xi_k + M_k = 0$$

Again, using properties of limits we know that $\lim_{k\to\infty} \Xi_k + M_k = \lim_{k\to\infty} \Xi_k + \lim_{k\to\infty} M_k = 0$ hence $\boldsymbol{x}+\boldsymbol{y} \in U_p$ and thus U_p is a subspace.