Exercise Sheet № 3

Task 2.2: Matrices

1. Prove the following: Let $\|\cdot\|_M$ be a matrix-norm induced by a vector-norm $\|\cdot\|_V$. Show the following:
   a) $\|\mathrm{A}\boldsymbol{x}\|_V \leq \|\mathrm{A}\|_M \cdot \|\boldsymbol{x}\|_V$
   b) $\|\mathrm{AB}\|_M \leq \|\mathrm{A}\|_M \cdot \|\mathrm{B}\|_M$
   c) $\|\mathrm{I}\|_M = 1$
2. Prove the following. Let $\mathrm{A} = (a_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix. Prove the following:
   a) $a_{ii} > 0 \forall 1 \leq i \leq n$
   b) $a_{ij}^2 < a_{ii}a_{jj}$ with $i \neq j$ and $1 \leq i, j \leq n$
   c) $\exists k \in \{1, \ldots, n\}\colon \max_{1\leq i,j\leq n} |a_{ij}| = a_{kk}$
3. Consider the $n \times n$ matrix used to approximate a second derivative via centered difference quotient:

$$\mathrm{D} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

   Let $n = 3$:
   a Find the eigenvalues of D
   b Compute the spectral radius $\varrho(\mathrm{D} + 2\mathrm{I})$
   c Determine the condition number $\kappa_2(\mathrm{D})$

   Now perform the computations when D is an $n \times n$ matrix. Discuss the behavior of the condition number $\kappa_2(\mathrm{D})$ as $n \to \infty$.

Subtask 1: Remember the induced matrix-norm:

$$\|\mathrm{A}\|_M = \sup_{\boldsymbol{x}\neq\mathbf{0}} \frac{\|\mathrm{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \Leftrightarrow \|\mathrm{A}\|_M = \inf\left\{r \geq 0 \colon \mathrm{A}\mathbb{S}^{n-1} \subseteq \mathcal{B}_r(\mathbf{0})\right\}$$

Where for any set $M \subseteq V$, we define $\mathrm{A}M = \{\mathrm{A}\boldsymbol{x}, \boldsymbol{x} \in M\}$. From the definition of $\|\mathrm{A}\|_M$ we get, that $\|\mathrm{A}\boldsymbol{x}\| \leq r\|\boldsymbol{x}\| = \|\mathrm{A}\|_M \cdot \|\boldsymbol{x}\|$. Since $\|\mathrm{A}\boldsymbol{x}\| \leq \|\mathrm{A}\|_M \cdot \|\boldsymbol{x}\|$, we further get

$$\|\mathrm{AB}\|_M = \sup_{\boldsymbol{x}\in\mathbb{S}^{n-1}} \|\mathrm{AB}\boldsymbol{x}\| \leq \sup_{\boldsymbol{x}\in\mathbb{S}^{n-1}} \|\mathrm{A}\|_M \cdot \|\mathrm{B}\boldsymbol{x}\| = \|\mathrm{A}\|_M \cdot \|\mathrm{B}\|_M$$

For $\|\mathrm{I}\|_M$:

$$\|\mathrm{I}\|_M = \sup_{\boldsymbol{x}\in\mathbb{S}^{n-1}} \|\mathrm{I}\boldsymbol{x}\| = \sup_{\boldsymbol{x}\in\mathbb{S}^{n-1}} \|\boldsymbol{x}\| = 1$$

Subtask 2: Since A is positive definite, $\forall \boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{x}^T\mathrm{A}\boldsymbol{x} > 0$, i.e. for $\boldsymbol{e}_k$, we get:

$$\boldsymbol{e}_k^T\mathrm{A}\boldsymbol{e}_k = a_{kk} > 0$$

by the definiteness of A.

Let $\alpha \subseteq \{1, \ldots, n\}$ be an index family and $\mathrm{A}[\alpha]$ be the submatrix with columns and rows labelled by the indices in α. Let $\boldsymbol{x} \in \mathbb{R}^n$ with $\boldsymbol{x}[\alpha] \neq \mathbf{0}_{|\alpha|}$ and $\boldsymbol{x}[\alpha^C] = \mathbf{0}_{n-|\alpha|}$, thus $\boldsymbol{x}[\alpha]^T\mathrm{A}[\alpha]\boldsymbol{x}[\alpha] = \boldsymbol{x}^T\mathrm{A}\boldsymbol{x} > 0$. Hence $\mathrm{A}[\alpha] > 0$. For $\alpha = \{i, j\}$, we get

$$\det(\mathrm{A}[\alpha]) = a_{ii}a_{jj} - a_{ij}^2 > 0 \Leftrightarrow a_{ii}a_{jj} > a_{ij}^2$$

Let $\boldsymbol{x} = s\boldsymbol{e}_i - \boldsymbol{e}_j$, where $s \in \mathbb{R}$, then:

$$\boldsymbol{x}^T \mathrm{A} \boldsymbol{x} = s^2 a_{ii} + 2 s a_{ij} + a_{jj}$$

If we assume that $a_{ji} = a_{ij}$ is the largest entry of A for $i \neq j$, we can set $s = 1$ and receive:

$$\boldsymbol{x}^T \mathrm{A} \boldsymbol{x} = a_{ii} - 2a_{ij} + a_{jj} = (a_{ii} - a_{ij}) + (a_{jj} - a_{ij}) < 0$$

which is a contradiction.

Subtask 3:

Subtask a: The eigenvalues of D solve the equation $\det(\lambda \mathrm{I} - \mathrm{D}) = 0$:

$$\begin{aligned} \chi_{\mathrm{D}}(\lambda) &= \lambda^3 + 6\lambda^2 + 10\lambda + 4 \\ \lambda_1 &= -2 \\ \lambda_{2,3} &= -2 \pm \sqrt{2} \end{aligned}$$

Subtask b: The spectral radius of a matrix $\mathrm{M} \in \mathbb{K}^{n\times n}$ is given by $\max_{i=1,\ldots,n} |\lambda_i|$. Computing the eigenvalues of $\mathrm{D} + 2\mathrm{I}$ yields $\lambda_1 = 0$, $\lambda_2 = -\sqrt{2}$ and $\lambda_3 = \sqrt{2}$, therefore $\varrho(\mathrm{D} + 2\mathrm{I}) = \sqrt{2}$.

Subtask c: The condition number κ_2 of a given definite[1] matrix $\mathrm{A} \in \mathbb{R}^{n\times n}$ is given by:

$$\kappa_2(\mathrm{A}) = \frac{\max_{i=1,\ldots,n} \lambda_i}{\min_{i=1,\ldots,n} \lambda_i}$$

Therefore we get:

$$\begin{aligned} \kappa_2(\mathrm{D}) &= \frac{-2 - \sqrt{2}}{-2 + \sqrt{2}} = \frac{2 + \sqrt{2}}{2 - \sqrt{2}} = \frac{(2 + \sqrt{2})^2}{(2 - \sqrt{2})(2 + \sqrt{2})} \\ &= \frac{4(1 + \sqrt{2}) + 2}{2} = 3 + 2\sqrt{2} \end{aligned}$$

Notice that D is a tridiagonal Töplitz-matrix. The eigenvalues are thus given by:

$$\lambda_k = -2 + 2\cos\left(\frac{\pi k}{n+1}\right)$$

The spectral radius of $\varrho(\mathrm{D} + 2\mathrm{I})$ is thus given by:

$$\varrho(\mathrm{D} + 2\mathrm{I}) = \cos\left(\frac{\pi}{n+1}\right)$$

The condition number becomes:

$$\kappa_2(\mathrm{D}) = \frac{\cos\left(\frac{\pi}{n+1}\right) - 1}{\cos\left(\frac{n\pi}{n+1}\right) - 1}$$

Since $\lim_{n\to\infty} \frac{n\pi}{n+1} = \pi \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \pi$ and $\lim_{n\to\infty} \frac{\pi}{n+1} = 0$ we get:

$$\lim_{n\to\infty} \kappa_2(\mathrm{D}) = \frac{\cos(0) - 1}{\cos(\pi) - 1} = 0$$

[1] either negative or positive, ***not*** indefinite or semidefinite