problem_1

October 10, 2022

1 Computational Mathematics 1: Exercise Sheet 1

1.1 Example 1.1: Taylor Series Expansion

Recall that the Taylor-series expansion of a smooth function $f\colon \mathbb{R} \to \mathbb{R}$, i.e. $f \in \mathcal{C}^\infty$, at a point $a \in \mathbb{R}$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}^n f(a)}{\mathrm{d}x^n} \cdot \frac{(x-a)^n}{n!}$$

The partial sum of order k given by

$$T_k(x) = \sum_{n=0}^{k} \frac{\mathrm{d}^n f(a)}{\mathrm{d}x^n} \cdot \frac{(x-a)^n}{n!}$$

Tasks:

1. Verify analytically that the Taylor-series expansion of the function $f(x) = \cos(x)$ at $x = 0$ is given by:

$$\cos(x) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

2. Write a python-script containing two functions:

   1. A function `taylor_cos(x, k)` that prints the approximate function values `cos_approx` of $\cos(x)$ at the point $\frac{\pi}{4}$ using its Taylor polynomial of order k

   2. A function `taylor_plot(x_int, k)` that plots the function $\cos(x)$ and the Taylor polynomials of orders k in a given interval `x_int`. Set the y-axis limits at $[-4, 4]$. Test the function using $k = 0, 2, 4, 6, 8$ in the interval $x \in [-2\pi, 2\pi]$.

3. From subtask 2, compute the 1-norm error of approximating $\cos(\pi/4)$ using the Taylor polynomials and discuss your observations as you increase the order of the polynomials.

[ ]:
```
from numpy import ndarray, array, pi, linspace, cos as npcos, zeros, abs,␣
↪arange, sqrt, column_stack, ceil, max
from typing import Callable, Optional
from scipy.special import factorial
from warnings import filterwarnings
from matplotlib.pyplot import Axes
from matplotlib.lines import Line2D
from sys import stderr # only for error messages
```

```python
import matplotlib.pyplot as plt
from pandas import DataFrame

# matplotlib setup

text_colors = {
    'jupyter': 'white',
    'regular': 'black'
}

mode = 'regular'
plot_params = {
    'axes.labelcolor': text_colors[mode],
    'xtick.color': text_colors[mode],
    'ytick.color': text_colors[mode],
    'text.usetex': True,
    'font.size': 18,
    'grid.linestyle': '--'
}

plt.rcParams.update(plot_params)

# util functions and constants

def list_to_string(l : list, delim : Optional[str] = ',') -> str:
    rstring = ''
    for i,x in enumerate(l):
        if i < len(l) - 1:
            rstring += f'{x},'
        else:
            rstring += str(x)
    return rstring

axis_resolution = 400
```

The cell above is required to run the cells below, as it loads all required components and packages. You may notice that classes from the `typing` module are loaded. While not required for functionality, they allow me to write proper type-hints to keep the code maintainable and readable.

1.1.1 Subtask 1: Analytic Verification

Given the derivative of $\cos(x)$ repeats cyclically with period four, we can sort them into four classes:

$$\frac{\mathrm{d}^n \cos}{\mathrm{d}x^n} = \begin{cases} \cos & n \equiv 0 \mod 4 \\ -\sin & n \equiv 1 \mod 4 \\ -\cos & n \equiv 2 \mod 4 \\ \sin & n \equiv 3 \mod 4 \end{cases} \implies \frac{\mathrm{d}^n \cos}{\mathrm{d}x^n}(0) = (-1)^n \delta_{n \mod 2}$$

Given that $\sin(0) = 0$ and $\cos(0) = 1$, we get the series-expansion:

$$\cos(x) = \sum_{k=0}^{\infty} \frac{\cos^{(n)}(0)}{n!} x^n = \sum_{k=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n!)}$$

1.1.2 Subtask 2: Taylor-Series implementation

We begin by writing a generic class for arbitrary power-series. This allows us a great level of abstraction and keeps the code modular and reuseable. Given a power series is uniquely defined by it's sequence a_k and the point of development a, we only need to supply these two. For good measure, we allow for non-zero starting indices by setting a value k_0. Additionally, passing a keyword-argument `precomp` allows us to pre-compute sequence-values in order to save on computation time later on. The real magic happens in the `__call__` method, which marks any instance `obj` of the class as callable, meaning we can simply use `obj(<xval>, <kval>)` in order to compute the k-th partial sum. If any sequence-values are computed in the constructor, these are of course used. For larger k than supplied in `precomp`, the sequence-function is called per iteration.

Note that the power of $(x - a)$ is given by the function `self._int_x_power()`, which allows us to use exponents like $2n + 1$ with a lambda expression. Additionally, large factorials become unwieldy, and `scipy` throws a `RuntimeWarning` instance for too large values of n, thus if the warning is caught with the `except` statement, then iteration is aborted and the result returned immediately. Note that the instance will print a warning to `stderr`.

The `plot()` method allows us to pass an x-axis and an axes-object, and optionally a k for the k-th partial sum, and plot the values of the taylor-series for each value in the passed x-axis. The method returns the line-object created by `ax.plot()`.

[ ]:
```
class PowerSeries:
    def __init__(self, sequence : Callable[[int], float],
                       devpoint : Optional[float] = 0.0,
                       start_index : Optional[int] = 0,
                       **kwargs) -> None:
        precomp = kwargs.get('precomp', 0)
        n_mod   = kwargs.get('xpower', lambda n: 1)

        self._seq = sequence
        self._devpoint = devpoint
        self._precomp = precomp
        self._int_x_power = n_mod
        if self._precomp > 0:
            self._seqvals = array([0.0] * self._precomp)
            for k in range(self._precomp):
                self._seqvals[k] = self._seq(k)

    def __call__(self, x : float, k : Optional[int] = 20) -> float:
        rval : float = 0.0
        filterwarnings('error')
        for k in range(k+1):
            try:
```

```
                if k < self._precomp:
                    rval += self._seqvals[k] * ( x - self._devpoint)**(self.
↪_int_x_power(k))
                else:
                    rval += self._seq(k) * ( x - self._devpoint)**(self.
↪_int_x_power(k))
            except RuntimeWarning:
                print(f'encountered runtime warning at index {k}, aborting␣
↪summation', file=stderr)
                return rval

        return rval

    def arrval(self, x_vals : ndarray, k : Optional[int] = 20) -> ndarray:
        return array([self(x,k) for x in x_vals])

    def plot(self, x_axis : ndarray, ax : Axes, k : Optional[int] = 20,␣
↪**kwargs) -> Line2D:
        return ax.plot(x_axis, self(x_axis, k))[0]
```

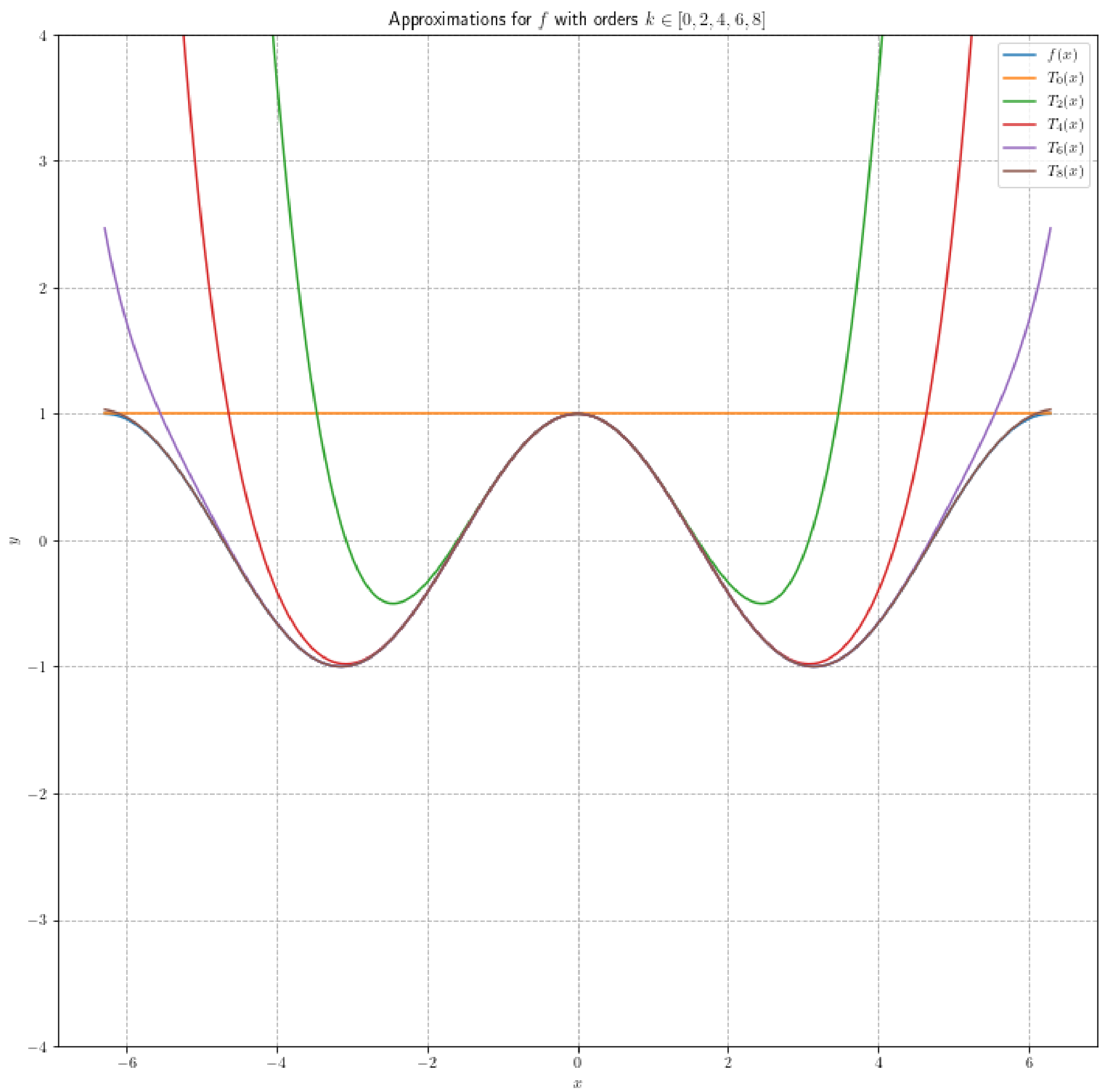
Given a generic class for handling power series, we can now simply create a child-class and change how we construct the object. We now only define a function computing the derivative of f at a and pass a lambda dividing by a, possibly changed, factorial, i.e. for $\cos(x)$ we can pass `n_multiplies = lambda n : 2*n`

[ ]:
```
class TaylorSeries(PowerSeries):

    def __init__(self, derivatives : Callable[[int, float], float],
                 devpoint : Optional[float] = 0.0,
                 n_multiplies : Optional[Callable[[int],int]] = lambda n :
↪ n,
                 **kwargs) -> None:
        sequence = lambda n : derivatives(n, devpoint) /␣
↪factorial(n_multiplies(n) )
        self._name = kwargs.get('name', 'f')
        super().__init__(sequence, devpoint, 0, xpower= n_multiplies, precomp =␣
↪kwargs.get('precomp', 0))

    def dispval(self, x : float = 0.25 * pi) -> None:
        fx = self(x)
        print(f'(f{x}) = {fx}')

    def compare_plots(self, x : ndarray, k_values : ndarray, ax : Axes,␣
↪true_values : ndarray, **kwargs) -> list:
        y_limits = kwargs.get('ylimits', (-4,4))

        if y_limits[0] >= y_limits[1]:
```

```python
            raise ValueError('supplied y-limits are invalid!')

        if kwargs.get('grid', False):
            ax.grid()

        l0 = ax.plot(x, true_values, label=fr'${self._name}(x)$')
        lines = []
        for k in k_values:
            lines.append(ax.plot(x, self.arrval(x, k), label=rf'$T_{k}(x)$'))

        ax.set_title(fr'Approximations for ${self._name}$ with orders $k\in␣
↪[{list_to_string(k_values)}]$', color=text_colors[mode])
        ax.set_ylim(y_limits[0], y_limits[1])
        return [l0].append(lines)

    def check_error(self, x : float, true : float, k_values : ndarray) ->␣
↪ndarray:
        return array([abs(self(x,k) - true) for k in k_values])

    def check_errors(self, x : ndarray, true : ndarray, k : Optional[int] = 10)␣
↪-> ndarray:
        return abs(self.arrval(x, k) - true)
```

Given this sub-class, we can now simply set `derivatives = lambda n, x : (-1)**n` and `n_multiplies = lambda n : 2*n`, to produce the taylor-polynomials of $\cos(x)$. Using the `compare_plots()` method, we can pass any x-axis, axes-object and set of valid values of k to produce a plot displaying all the taylor-polynomials and the true-function values passed via `true`. See the figure below. Note how the taylor-polynomials of order k are listed in the given order in the legend, so one can differentiate the various plots.

[ ]:
```python
fig, ax = plt.subplots()
x = linspace(-2*pi, 2*pi, axis_resolution)
taylor_cos = TaylorSeries(lambda n, x : (-1)**n, 0.0, lambda n : 2*n)
taylor_cos.compare_plots(x, [0,2,4,6,8], ax, npcos(x), grid=True)
ax.legend(bbox_to_anchor=(1, 1))
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$y$')

fig.set_size_inches(12,12)
plt.show()
```



1.1.3

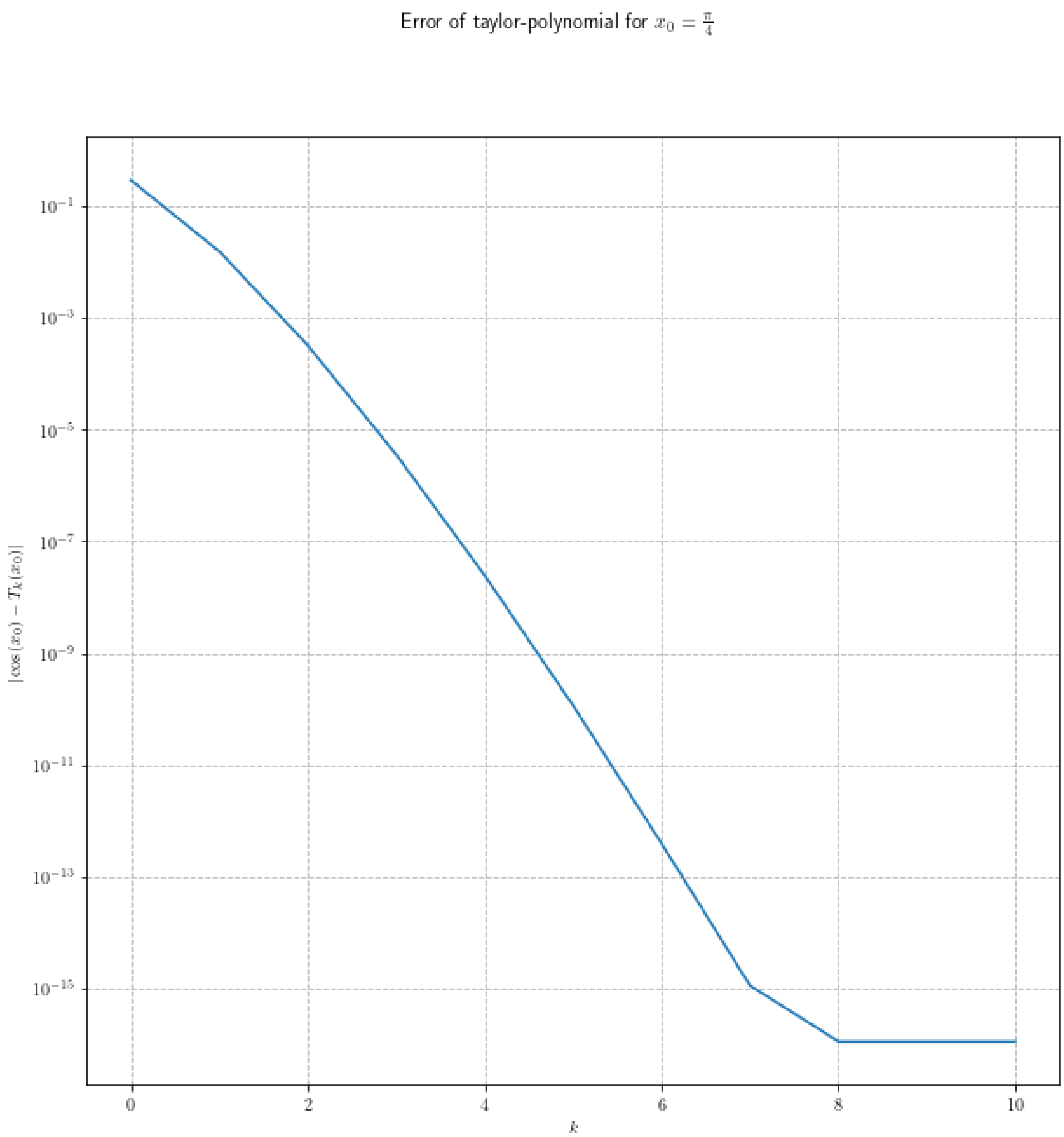
1.1.4 Subtask 3: Absolute Error

We already implemented the two methods `TaylorSeries.check_error()` and `TaylorSeries.check_errors()` which respectively compute the absolute error of the k-taylor-polynomials for a certain x or an arbitrary x-axis of values for a fixed k. Given that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, we can very easily compute the error of our cosine-approximation.

[ ]:
```
k_min = 0
k_top = 10
k_step = 1
k_values = arange(k_min, k_top + k_step, k_step, dtype=int)
true_single = sqrt(2.0) * 0.5
```

```
x_val = pi * 0.25
errors = taylor_cos.check_error(x_val, true_single, k_values)

display(DataFrame({
    'k': k_values,
    'error': errors,
}))

fig, ax = plt.subplots()
ax.semilogy(k_values, errors)
ax.grid()
ax.set_xlabel(r'$k$')
ax.set_ylabel(r'$|\cos(x_0) - T_k(x_0)|$')
fig.set_size_inches(10,10)
fig.suptitle(r'Error of taylor-polynomial for $x_0 = \frac{\pi}{4}$',
 color=text_colors[mode])
plt.show()

fig.savefig('ex_1_3.png', dpi=300)
```

	k	error
0	0	2.928932e-01
1	1	1.553192e-02
2	2	3.224255e-04
3	3	3.566364e-06
4	4	2.449675e-08
5	5	1.146229e-10
6	6	3.886891e-13
7	7	1.110223e-15
8	8	1.110223e-16
9	9	1.110223e-16
10	10	1.110223e-16



We observe that for rising k, the absolute error $|\cos - T_k|$ quickly decreases (note that the y-axis is logarithmic) and seems to reach an equilibrium point of approximately $1.11 \cdot 10^{-16}$ (see the table above).

Below we use the method `check_errors()` to compute the absolute error of the taylor-polynomial over an arbitrary axis, and plot the error.

[ ]:
```
k_min = 0
k_top = 8
k_step = 2
k_values = arange(k_min, k_top + k_step, k_step)
x_axis = linspace(-2*pi, 2*pi, axis_resolution)
```

```
true_values = npcos(x_axis)
fig, ax = plt.subplots()

for k in k_values:
    errors = taylor_cos.check_errors(x_axis, true_values, k)
    ax.plot(x_axis, errors, label=r'$|\cos - T_{' f'{k}'+ r'}|$')

ax.legend(bbox_to_anchor=(1, 1))
fig.set_size_inches(10,10)
ax.grid()
ax.set_xlabel(r'$x$')
ax.set_ylabel(r'$y$')
ax.set_ylim(0,8)
fig.suptitle(r'Absolute error of $\cos$ and the taylor-polynomials')
plt.show()
```