Large Scale Modes Reconstruction with Short Wavelength Modes

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I. INTRODUCTION

[1]

II. STANDARD PERTURBATION THEORY

In this section we review the SPT[1] approach of describing small scale nonlinearity. In the case of a perfect pressurelesee fluid, the nonrelativistic cosmological fluid equations are continuity, Euler and Poisson equations:

$$\frac{\partial\delta(\vec{x},\tau)}{\partial\tau}+\vec{\nabla}\times[(1+\delta(\vec{x},\tau))\vec{v}(\vec{x},\tau)]=0 \quad (1)$$

$$(\frac{\partial}{\partial\tau}+\vec{v}(\vec{x},\tau)\times\vec{\nabla})\vec{v}(\vec{x},\tau)=-\frac{da}{d\tau}\frac{\vec{v}(\vec{x},\tau)}{a}-\vec{\nabla}\Phi \quad (2)$$

$$\nabla^2\Phi=4\pi Ga^2\bar{\rho}_{\rm m}\delta(\vec{x},\tau) \quad (3)$$

these equations fully determine the time evolution of the local density contrast δ and the peculiar velocity field $\vec{v}=d\vec{x}/d\tau$. Taking the divergence of Eq. (2) we can eliminate the gravitational potential Φ in Fourier space and get:

$$\frac{\partial\delta}{\partial\tau}+\theta=-\int\frac{d^3\vec{k}_1}{(2\pi)^3}\int\frac{d^3\vec{k}_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\vec{k}-\vec{k}_{12})\times\frac{\vec{k}\cdot\vec{k}_1}{k_1^2}\theta(\vec{k}_1,\tau)\delta(\vec{k}_2,\tau) \quad (4)$$

$$\frac{\partial\delta}{\partial\tau}+\frac{da}{d\tau}\frac{\theta}{a}+\frac{6}{\tau^2}\delta=-\int\frac{d^3\vec{k}_1}{(2\pi)^3}\int\frac{d^3\vec{k}_2}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\vec{k}-\vec{k}_{12})\times\frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{2k_1^2k_2^2}\theta(\vec{k}_1,\tau)\theta(\vec{k}_2,\tau) \quad (5)$$

here $\vec{k}_{12}=\vec{k}_1+\vec{k}_2$ and more generally $\vec{k}_{1\cdots n}=\vec{k}_1+\cdots+\vec{k}_n$; $\theta\equiv\vec{\nabla}\cdot\vec{v}$ is the divergence of velocity field.
In Einstein-de Sitter space, linear growth function $D_1(a)=a$ and we can solve these equations perturbatively using expansion [?]

$$\delta(\vec{k},\tau)=\sum_{n=1}^{\infty}\delta^{(n)}(\vec{k},\tau)=\sum_{n=1}^{\infty}a^n(\tau)\delta_n(\vec{k}) \quad (6)$$

$$\theta(\vec{k},\tau)=\sum_{n=1}^{\infty}\theta^{(n)}(\vec{k},\tau)=-H(\tau)\sum_{n=1}^{\infty}a^{n+1}(\tau)\theta_n(\vec{k}) \quad (7)$$

where the superscript (n) means the order of perturbation theory, and first order term $\delta^{(1)}$ corresponds to linear evolution. Linear power spectrum is given by this term via:

$$\langle\delta(\vec{k})\delta(\vec{k}')\rangle=(2\pi)^3\delta_{\rm D}(\vec{k}+\vec{k}')P_{\rm lin}(k) \quad (8)$$

Substituting the two perturbative series into Eq. (4) and Eq. (5) we get recursion relations for $\delta_n(\vec{k})$ and $\theta_n(\vec{k})$ [?] with solution:

$$\delta_n(\vec{k})=\int\frac{d^3\vec{k}_1}{(2\pi)^3}\cdots\int\frac{d^3\vec{k}_n}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\vec{k}-\vec{k}_{1\cdots n})\times F_n(\vec{k}_1,\cdots,\vec{k}_n)\delta_1(\vec{k}_1)\cdots\delta_1(\vec{k}_n) \quad (9)$$

$$\theta_n(\vec{k})=\int\frac{d^3\vec{k}_1}{(2\pi)^3}\cdots\int\frac{d^3\vec{k}_n}{(2\pi)^3}(2\pi)^3\delta_{\rm D}(\vec{k}-\vec{k}_{1\cdots n})\times G_n(\vec{k}_1,\cdots,\vec{k}_n)\delta_1(\vec{k}_1)\cdots\delta_1(\vec{k}_n) \quad (10)$$

and recursion relations are encoded in kernels F_n and G_n. Since we only consider up to second-order, the second-order symmetrized kernels are given by:

$$F_2(\vec{k}_1,\vec{k}_2)=\frac{5}{7}+\frac{2}{7}\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{k_1^2k_2^2}+\frac{\vec{k}_1\cdot\vec{k}_2}{2k_1k_2}(\frac{k_1}{k_2}+\frac{k_2}{k_1}) \quad (11)$$

$$G_2(\vec{k}_1,\vec{k}_2)=\frac{3}{7}+\frac{4}{7}\frac{(\vec{k}_1\cdot\vec{k}_2)^2}{k_1^2k_2^2}+\frac{\vec{k}_1\cdot\vec{k}_2}{2k_1k_2}(\frac{k_1}{k_2}+\frac{k_2}{k_1}) \quad (12)$$

This approach gives a nearly accurate description of ΛCDM universe [?] with a slight generalization:

$$\delta(\vec{k},\tau)=\sum_{n=1}^{\infty}\delta^{(n)}(\vec{k},\tau)=\sum_{n=1}^{\infty}D_1^n(\tau)\delta_n(\vec{k}) \quad (13)$$

$$\theta(\vec{k},\tau)=\sum_{n=1}^{\infty}\theta^{(n)}(\vec{k},\tau)=-\frac{d\ln D_1(\tau)}{d\tau}\sum_{n=1}^{\infty}D_1^n(\tau)\theta_n(\vec{k}) \quad (14)$$

Time-dependent second-order density contrast can be expressed as convolution of two linear density fields with kernel F_2:

$$\delta^{(2)}(\vec{k},\tau)=\int\frac{d^3\vec{k}_1}{(2\pi)^3}F_2(\vec{k}_1,\vec{k}-\vec{k}_1)\delta^{(1)}(\vec{k}_1,\tau)\delta^{(1)}(\vec{k}-\vec{k}_1,\tau) \quad (15)$$

Discussion (Accuracy?...)

[1] SPT stands for standard perturbation theory.

III. QUADRATIC ESTIMATOR

Comparison with CMB lensing [?]
Compute the correlation function of two short-wavelength modes $\vec{k}_s$ and $\vec{k}'_s$, in the squeezed limit $|\vec{k}_s+\vec{k}'_s| \ll |\vec{k}_s|, |\vec{k}'_s|$ up two second-order (τ is fixed):

$$\begin{aligned} &\langle\delta(\vec{k}_s)\delta(\vec{k}'_s)\rangle|_{(\vec{k}_s+\vec{k}'_s)\neq 0} \\ &= \langle\delta^{(1)}(\vec{k}_s)\delta^{(2)}(\vec{k}'_s)\rangle + \langle\delta^{(2)}(\vec{k}_s)\delta^{(1)}(\vec{k}'_s)\rangle \end{aligned} \tag{16}$$

Substituting Eq. (15) into the first bracket we get:

$$\begin{aligned} \langle\delta^{(1)}(\vec{k}_s)\delta^{(2)}(\vec{k}'_s)\rangle &= \int \frac{d^3\vec{k}}{(2\pi)^3} F_2(\vec{k}, \vec{k}'_s - \vec{k}) \\ &\times \langle\delta^{(1)}(\vec{k}_s)\delta^{(1)}(\vec{k}'_s - \vec{k})\delta^{(1)}(\vec{k})\rangle \end{aligned} \tag{17}$$

This correlation function with three Gaussian fields is nonzero under following consideration: if a three-point correlation function consists of two short-wavelength modes and one long-wavelength mode, we can do the following contraction:

$$\langle\delta(\vec{k}_s)\delta(\vec{k}'_s)\delta(\vec{k}_l)\rangle = \langle\delta(\vec{k}_s)\delta(\vec{k}'_s)\rangle\delta(\vec{k}_l) \tag{18}$$

Since in a real life survey, we can only measure this long-wavelength mode once. Thus it won't have any statistical property and we can safely extract this term out of the bracket.
We can do this contraction twice in this integral. One occurs when $|\vec{k}| \ll |\vec{k}_s|, |\vec{k}'_s - \vec{k}|$, we can extract $\delta^{(1)}(\vec{k})$ out; the other one occurs when $|\vec{k}'_s - \vec{k}| \ll |\vec{k}|, |\vec{k}_s|$.
Use Eq. (8) the integral in Eq. (17) can be evaluated as:

$$\begin{aligned} &\int \frac{d^3\vec{k}}{(2\pi)^3} F_2(\vec{k}, \vec{k}'_s - \vec{k}) \langle\delta^{(1)}(\vec{k}_s)\delta^{(1)}(\vec{k}'_s - \vec{k})\delta^{(1)}(\vec{k})\rangle \\ &= \int d^3\vec{k} F_2(\vec{k}, \vec{k}'_s - \vec{k})\delta_{\rm D}(\vec{k}_s + \vec{k}'_s - \vec{k}) P_{\rm lin}(k_s)\delta^{(1)}(\vec{k}) \\ &+ \int d^3\vec{k} F_2(\vec{k}, \vec{k}'_s - \vec{k})\delta_{\rm D}(\vec{k}_s + \vec{k}) P_{\rm lin}(k_s)\delta^{(1)}(\vec{k}'_s - \vec{k}) \\ &= 2F_2(-\vec{k}_s, \vec{k}_s + \vec{k}'_s) P_{\rm lin}(k_s)\delta^{(1)}(\vec{k}_s + \vec{k}'_s) \end{aligned} \tag{19}$$

Finally we have:

$$\langle\delta(\vec{k}_s)\delta(\vec{k}'_s)\rangle = f(\vec{k}_s, \vec{k}'_s)\delta^{(1)}(\vec{k}_s + \vec{k}'_s) \tag{20}$$

with

$$\begin{aligned} f(\vec{k}_s, \vec{k}'_s) &= 2F_2(-\vec{k}_s, \vec{k}_s + \vec{k}'_s) P_{\rm lin}(k_s) \\ &+ 2F_2(-\vec{k}'_s, \vec{k}_s + \vec{k}'_s) P_{\rm lin}(k'_s) \end{aligned} \tag{21}$$

Eq. (20) suggests that we can estimate long-wavelength modes with appropriate average over pairs of shart-wavelength modes. General form of the quadratic estimator can be written as:

$$\hat{\delta}^{(1)}(\vec{k}_l) = A(\vec{k}_l) \int \frac{d^3\vec{k}_s}{(2\pi)^3} g(\vec{k}_s, \vec{k}'_s)\delta(\vec{k}_s)\delta(\vec{k}'_s) \tag{22}$$

with g being weighting function, $\vec{k}'_s = \vec{k}_l - \vec{k}_s$ and A is defined via $\langle\hat{\delta}^{(1)}(\vec{k}_l)\rangle = \delta^{(1)}(\vec{k}_l)$:

$$A(\vec{k}_l) = \left[\int \frac{d^3\vec{k}_s}{(2\pi)^3} g(\vec{k}_s, \vec{k}'_s) f(\vec{k}_s, \vec{k}'_s)\right]^{-1} \tag{23}$$

The Gaussian noise is given by:

$$\langle\hat{\delta}^{(1)}(\vec{k}_l)\hat{\delta}^{(1)}(\vec{k}'_l)\rangle = (2\pi)^3\delta_{\rm D}[\vec{k}_l - \vec{k}'_l)(P_{\rm lin}(k_l) + N(\vec{k}_l)] \tag{24}$$

with

$$\begin{aligned} N(\vec{k}_l) &= 2A^2(k_l) \\ &\times \int \frac{d^3\vec{k}_s}{(2\pi)^3} g^2(\vec{k}_s, \vec{k}_l - \vec{k}_s) P_{\rm nl}(k_s) P_{\rm nl}(|\vec{k}_l - \vec{k}_s|) \end{aligned} \tag{25}$$

where $P_{\rm nl}$ is the nonlinear power spectrum. Minimizing the noise term we can fix the form of g to be:

$$g(\vec{k}_s, \vec{k}'_s) = \frac{f(\vec{k}_s, \vec{k}'_s)}{2P_{\rm nl}(k_s)P_{\rm nl}(k'_s)} \tag{26}$$

Noise term reduces simply to $N(\vec{k}_l) = A(\vec{k}_l)$.

IV. DISCUSSION

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[1] W. Hu and T. Okamoto, Astrophys. J. **574**, 566 (2002), arXiv:astro-ph/0111606 [astro-ph].