# Algebraic Attacks: Theoretical Aspects

Aurélien Bœuf,[1] Léo Perrin[1]

[1]Inria, France

March 13, 2025

# In this Presentation

**Anemoi**
Crypto23

**Griffin**
Crypto23

**ArionHash**
arXiv

## In this Presentation

**Anemoi**
Crypto23

**Griffin**
Crypto23

**ArionHash**
arXiv

**Full-round break**
of some instances

# In this Presentation

**Anemoi**
Crypto23


**Full-round break**
of some instances


**Full-round break**
of some instances

# In this Presentation


**Maybe** full-round break?


**Full-round break**
of some instances


**Full-round break**
of some instances

# In this Presentation


**Maybe** full-round break?


**Full-round break**
of some instances


**Full-round break**
of some instances

How did this happen?

# Outline

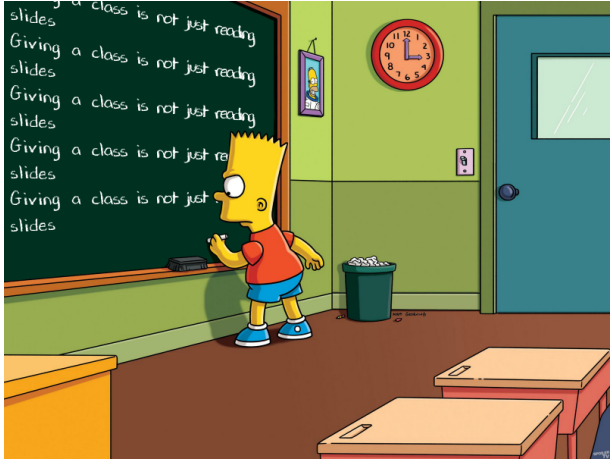
# Plan of this Section

# Some Notations

$$F(x_0, x_1, \ldots, x_{n-1}) = \sum_{d_0 \geq 0} \cdots \sum_{d_{n-1} \geq 0} \alpha_{d_0,\ldots,d_{n-1}} \underbrace{\prod_{j=0}^{n-1} x_j^{d_j}}_{\text{monomial}}$$

- $F$ is a multivariate polynomial
- Each monomial is of degree $\sum_j d_j$
- Each coefficient $\alpha_{d_0,\ldots,d_{n-1}}$ is a field element.

# Algebraic Attack: a first look at MiMC



# Pipeline of a Root-Finding Attack

1. Design an attack

# Pipeline of a Root-Finding Attack

1. Design an attack
2. Write a (system of) equation(s)

# Pipeline of a Root-Finding Attack

1. Design an attack
2. Write a (system of) equation(s)
3. ???

# Pipeline of a Root-Finding Attack

1. Design an attack
2. Write a (system of) equation(s)
3. ???
4. Deduce a preimage/CICO solution/master key...

# Pipeline of a Root-Finding Attack

1. Design an attack
2. Write a (system of) equation(s)
3. ???
4. Deduce a preimage/CICO solution/master key...

The topic of this class: the "???" part!

# Plan of this Section

# CICO Problem

**C**onstrained **I**nput **C**onstrained **O**utput

The CICO problem of size $c$ (capacity of the sponge) for a permutation $P$ consists in finding an input/output pair such that

$$P(*, \ldots, *, \underbrace{0, \ldots, 0}_{c \text{ elements}}) = (*', \ldots, *', \underbrace{0, \ldots, 0}_{c \text{ elements}})$$

# CICO Problem

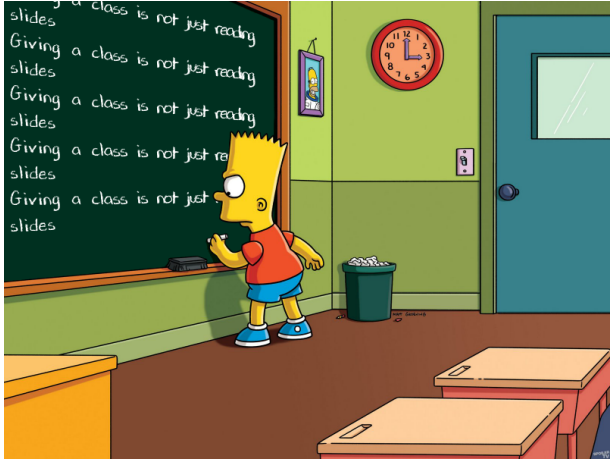
**C**onstrained **I**nput **C**onstrained **O**utput

The CICO problem of size $c$ (capacity of the sponge) for a permutation $P$ consists in finding an input/output pair such that

$$P(*, \ldots, *, \underbrace{0, \ldots, 0}_{c \text{ elements}}) = (*', \ldots, *', \underbrace{0, \ldots, 0}_{c \text{ elements}})$$

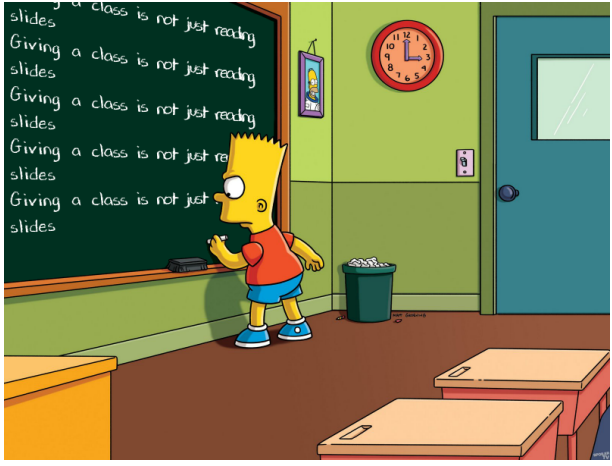
Solving CICO of size $c$ *should* take about $q^c$ operations.

# Let's Look at Feistel-MiMC



# Plan of this Section

# Deriving a Multi-Variate System: CICO-2



# Plan of this Section

# Plan of this Section

# Root Finding: Simple Cases

Consider a multivariate polynomial ring $\mathbb{F}[x_1, x_2, \ldots, x_N]$.
We want to solve:

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

# Root Finding: Simple Cases

$$\begin{cases} m_{1,1}x_1 + \cdots + m_{1,N}x_N + a_1 = 0 \\ m_{2,1}x_1 + \cdots + m_{2,N}x_N + a_2 = 0 \\ \vdots \\ m_{k,1}x_1 + \cdots + m_{k,N}x_N + a_k = 0 \end{cases}$$

Polynomials of **degree 1**: Linear system $\Rightarrow$ **Linear algebra**.

# Root Finding: Simple Cases

$$\begin{cases} p_1(x_1) = 0 \\ p_2(x_1) = 0 \\ \quad \vdots \\ p_k(x_1) = 0 \end{cases}$$

**One variable**: Univariate root finding $\Rightarrow$ **Euclidian division** (for Berlekamp-Rabin algorithm).

# Root Finding: The Less Simple Case

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \qquad\qquad\vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

Several variables,
not linear!

# Root Finding: The Less Simple Case

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

Several variables,
not linear!

## General Approach

If $p_i(x_1, \ldots, x_N) = 0$ for all $i$, then

$$\sum_i q_i(x_1, \ldots, x_N) p_i(x_1, \ldots, x_N) = 0,$$

for any set of polynomials $\{q_i\}_i$.

# Root Finding: The Less Simple Case

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

Several variables,
not linear!

## General Approach

If $p_i(x_1, \ldots, x_N) = 0$ for all $i$, then

$$\sum_i q_i(x_1, \ldots, x_N) p_i(x_1, \ldots, x_N) = 0,$$

for any set of polynomials $\{q_i\}_i$.

The set of all such linear combinations is
the ideal generated by the $\{p_i\}_{i=1}^k$

We denote it $I(p_1, \ldots, p_k)$.

# Root Finding: The Less Simple Case

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ p_2(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

Several variables,
not linear!

## General Approach

If $p_i(x_1, \ldots, x_N) = 0$ for all $i$, then

$$\sum_i q_i(x_1, \ldots, x_N) p_i(x_1, \ldots, x_N) = 0,$$

for any set of polynomials $\{q_i\}_i$.

The set of all such linear combinations is
the ideal generated by the $\{p_i\}_{i=1}^k$

We denote it $I(p_1, \ldots, p_k)$.

**Goal:** somehow, find a polynomial $r(x_1) = 0$ in this ideal!

# Plan of this Section

# Using the Ideal Structure

## Ideal

An ideal $I$ is a subgroup of a ring $R$ such that:

$$\forall i \in I, \forall r \in R, i \times r \in R\,.$$

# Using the Ideal Structure

## Ideal

An ideal $I$ is a subgroup of a ring $R$ such that:

$$\forall i \in I, \forall r \in R, i \times r \in R\,.$$

### Over the integer

There is an ideal you all know: $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$.

# Using the Ideal Structure

## Ideal

An ideal $I$ is a subgroup of a ring $R$ such that:

$$\forall i \in I, \forall r \in R, i \times r \in R\,.$$

### Over the integer

There is an ideal you all know: $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$.
Every $x \in \mathbb{Z}$ can be written **uniquely** as $a + b$, where $a \in n\mathbb{Z}$ and $b \in$

# Using the Ideal Structure

## Ideal

An ideal $I$ is a subgroup of a ring $R$ such that:

$$\forall i \in I, \forall r \in R, i \times r \in R\,.$$

### Over the integer

There is an ideal you all know: $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$.
Every $x \in \mathbb{Z}$ can be written **uniquely** as $a + b$, where $a \in n\mathbb{Z}$ and $b \in \mathbb{Z}/n\mathbb{Z}$.

# Using the Ideal Structure

## Ideal

An ideal $I$ is a subgroup of a ring $R$ such that:

$$\forall i \in I, \forall r \in R, i \times r \in R\,.$$

### Over the integer

There is an ideal you all know: $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$.
Every $x \in \mathbb{Z}$ can be written **uniquely** as $a + b$, where $a \in n\mathbb{Z}$ and $b \in \mathbb{Z}/n\mathbb{Z}$.

We want to simplify our lives and work in

$$\mathbb{F}[x_1, x_2, \ldots, x_N]/I(p_1, \ldots, p_k)\,.$$

# A Problem with Euclidian Division

- Euclidian division on **integers**:

$$a = bq + r,\ 0 \leq r < b.$$

Division of 13 by 3:

$$13 = 4 \times 3 + 1.$$

# A Problem with Euclidian Division

- Euclidian division on **integers**:

$$a = bq + r,\ 0 \leq r < b.$$

Division of 13 by 3:

$$13 = 4 \times 3 + 1.$$

- Euclidian division on **univariate polynomials** ($\mathbb{F}[X]$):

$$A = BQ + R,\ \deg(R) < \deg(B).$$

Division of $X^3 + X + 1$ by $X$:

$$X^3 + X + 1 = (X^2 + 1)X + 1.$$

# The Problem with Multivariate

- Euclidian division on **multivariate polynomials**:

$$A = BQ + R \text{... condition on } R?$$

# The Problem with Multivariate

- Euclidian division on **multivariate polynomials**:

$$A = BQ + R \text{... condition on } R?$$

Division of $x$ by $x + y$ in $\mathbb{F}[x, y]$:

$$x = 0 \cdot (x+y) + x$$

*or*

$$x = 1 \cdot (x+y) - y\,?$$

# The Problem with Multivariate

- Euclidian division on **multivariate polynomials**:

$$A = BQ + R\text{... condition on } R?$$

Division of $x$ by $x + y$ in $\mathbb{F}[x, y]$:

$$x = 0 \cdot (x+y) + x \quad \Leftarrow x < y$$

*or*

$$x = 1 \cdot (x+y) - y \quad \Leftarrow y < x$$

Need to define a **monomial ordering**.

$\Longrightarrow$ Division steps determined by **leading monomials (LM)**.

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} \;?\; y$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} <_{\text{lex}} y\,, \quad x^6yz \;?\; y^2z$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} <_{\text{lex}} y\,, \quad x^6yz <_{\text{lex}} y^2z\,.$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} <_{\text{lex}} y\,, \quad x^6yz <_{\text{lex}} y^2z\,.$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad y \quad ? \quad x^2$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} <_{\text{lex}} y\,, \quad x^6yz <_{\text{lex}} y^2z\,.$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad y <_{\text{glex}} x^2\,, \quad z^2 \quad ? \quad xyz$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad x^{1000} <_{\text{lex}} y\,, \quad x^6yz <_{\text{lex}} y^2z\,.$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\,, \quad y <_{\text{glex}} x^2\,, \quad z^2 <_{\text{glex}} xyz\,, \quad xy <_{\text{glex}} xz <_{\text{glex}} yz\,.$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\quad x^{1000} <_{\text{lex}} y\ ,\quad x^6yz <_{\text{lex}} y^2z\ .$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\quad y <_{\text{glex}} x^2\ ,\quad z^2 <_{\text{glex}} xyz\ ,\quad xy <_{\text{glex}} xz <_{\text{glex}} yz\ .$$

- **Weighted Graded LEX:** Compare the **weighted sum** of degrees, then lex if equality.
  Examples for $x <_{\text{lex}} y <_{\text{lex}} z$ and $\mathbf{wt}(x) = 6, \mathbf{wt}(y) = 1, \mathbf{wt}(z) = 2$:

$$x \quad ? \quad yz^2$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\quad x^{1000} <_{\text{lex}} y\ ,\quad x^6yz <_{\text{lex}} y^2z\ .$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\quad y <_{\text{glex}} x^2\ ,\quad z^2 <_{\text{glex}} xyz\ ,\quad xy <_{\text{glex}} xz <_{\text{glex}} yz\ .$$

- **Weighted Graded LEX:** Compare the **weighted sum** of degrees, then lex if equality.
  Examples for $x <_{\text{lex}} y <_{\text{lex}} z$ and $\mathbf{wt}(x) = 6, \mathbf{wt}(y) = 1, \mathbf{wt}(z) = 2$:

$$x >_{\text{wglex}} yz^2 \text{ because } \mathbf{wt}(x) = 6 \text{ and } \mathbf{wt}(yz) = \mathbf{wt}(y) + 2\mathbf{wt}(z) = 5\ .$$

$$x^2 \quad ? \quad z^6$$

# Monomial orderings

- **LEXicographical:** Compare degree of highest variable, then second-highest, etc.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\ x^{1000} <_{\text{lex}} y\ ,\ x^6yz <_{\text{lex}} y^2z\ .$$

- **Graded LEX:** Compare **total degree** first, then switch to lex if equality.

$$x <_{\text{lex}} y <_{\text{lex}} z\ ,\ y <_{\text{glex}} x^2\ ,\ z^2 <_{\text{glex}} xyz\ ,\ xy <_{\text{glex}} xz <_{\text{glex}} yz\ .$$

- **Weighted Graded LEX:** Compare the **weighted sum** of degrees, then lex if equality.
  Examples for $x <_{\text{lex}} y <_{\text{lex}} z$ and $\mathbf{wt}(x) = 6, \mathbf{wt}(y) = 1, \mathbf{wt}(z) = 2$:

$$x >_{\text{wglex}} yz^2 \text{ because } \mathbf{wt}(x) = 6 \text{ and } \mathbf{wt}(yz) = \mathbf{wt}(y) + 2\mathbf{wt}(z) = 5\ .$$

$$x^2 <_{\text{wglex}} z^6 \text{ because } \mathbf{wt}(x^2) = \mathbf{wt}(z^6) = 12 \text{ and } x^2 <_{\text{lex}} z^6\ .$$

# The Problem... Still.

Consider a system $\{p_1, \ldots, p_k\}$.

$\Longrightarrow$ Division of a polynomial $p$ by $\{p_1, \ldots, p_k\}$ for some ordering: **final remainder can depend on the choice of divisors!**

# The Problem... Still.

Consider a system $\{p_1, \ldots, p_k\}$.

$\Longrightarrow$ Division of a polynomial $p$ by $\{p_1, \ldots, p_k\}$ for some ordering: **final remainder can depend on the choice of divisors!**

Example: in $\mathbb{F}[x, y]$ with **lex** ordering ($x <_{lex} y$), divide $y^2$ by $\{y^2 - 1, y - x\}$.

$$\begin{array}{l} y^2 \\ \Downarrow \quad \text{red. by } y^2 - 1 \\ 1 \\ \Downarrow \quad \text{no further red.} \\ 1 \end{array} \qquad \begin{array}{l} y^2 \\ \Downarrow \quad \text{red. by } y - x \\ xy \\ \Downarrow \quad \text{red. by } y - x \\ x^2 \end{array}$$

# The Problem... Still.

Consider a system $\{p_1, \ldots, p_k\}$.

$\Longrightarrow$ Division of a polynomial $p$ by $\{p_1, \ldots, p_k\}$ for some ordering: **final remainder can depend on the choice of divisors!**

Example: in $\mathbb{F}[x, y]$ with **lex** ordering ($x <_{lex} y$), divide $y^2$ by $\{y^2 - 1, y - x\}$.

$$\begin{array}{l} y^2 \\ \Downarrow \quad \text{red. by } y^2 - 1 \\ 1 \\ \Downarrow \quad \text{no further red.} \\ 1 \end{array} \qquad \begin{array}{l} y^2 \\ \Downarrow \quad \text{red. by } y - x \\ xy \\ \Downarrow \quad \text{red. by } y - x \\ x^2 \end{array}$$

**The solution: Gröbner Bases.**

# What is a Gröbner Basis?

Let $G = \{p_1, \ldots, p_k\}$ and $<$ a monomial ordering.

## Definition

*G* is a *Gröbner basis* if and only if the reduction defined by $<$ of any polynomial *P* does not depend on the order chosen for the reductors.

# What is a Gröbner Basis?

Let $G = \{p_1, \ldots, p_k\}$ and $<$ a monomial ordering.

## Definition

$G$ is a *Gröbner basis* if and only if the reduction defined by $<$ of any polynomial $P$ does not depend on the order chosen for the reductors.

## Useful Proposition

*If $LM_<(p_1), \ldots, LM_<(p_k)$ are pairwise **coprime** (e.g. $x^2$ and $y$), then $G$ is a Gröbner basis.*

# Gröbner Basis - Examples

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).

# Gröbner Basis - Examples

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).
- However, it is a Gröbner basis for **lex** order with $x > y$. Proof: $\mathrm{LM}(y^2 - 1) = y^2$ and $\mathrm{LM}(y - x) = x$ are **coprime**.

# Gröbner Basis - Examples

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).
- However, it is a Gröbner basis for **lex** order with $x > y$. Proof: $\mathrm{LM}(y^2 - 1) = y^2$ and $\mathrm{LM}(y - x) = x$ are **coprime**.
- $\{y^3 + x, y^3 + x^2\}$ is not a Gröbner basis for any **lex** or **deglex** order.

# Gröbner Basis - Examples

In $\mathbb{F}[x, y]$:

- $\{y^2 - 1, y - x\}$ is not a Gröbner basis for **lex** order with $x < y$ (previous example).
- However, it is a Gröbner basis for **lex** order with $x > y$. Proof: $\mathrm{LM}(y^2 - 1) = y^2$ and $\mathrm{LM}(y - x) = x$ are **coprime**.
- $\{y^3 + x, y^3 + x^2\}$ is not a Gröbner basis for any **lex** or **deglex** order.
- However, it is a Gröbner basis for **weighted degree** orders with $\mathbf{wt}(x) = 2$ and $\mathbf{wt}(y) = 1$, as then $\mathrm{LM}(y^3 + x) = y^3$ and $\mathrm{LM}(y^3 + x^2) = x^2$ are **coprime**.

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

1. Define system

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \qquad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)
3. Change order to **lex** (FGLM)

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)
3. Change order to **lex** (FGLM)
4. Find the roots in $\mathbb{F}_q$ of $g_N^*$ with univariate methods, etc.

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)
3. Change order to **lex** (FGLM)

4. Find the roots in $\mathbb{F}_q$ of $g_N^*$ with univariate methods, etc.

Steps 2 and 3 are both computationally costly, but not for the same reasons. For most AOPs, step 2 dominates in practice...

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)
3. Change order to **lex** (FGLM)

4. Find the roots in $\mathbb{F}_q$ of $g_N^*$ with univariate methods, etc.

Steps 2 and 3 are both computationally costly, but not for the same reasons. For most AOPs, step 2 dominates in practice...

... or does it?

# Plan of this Section

# Plan of this Section

# Quick Overview of Griffin, Arion, Anemoi

**Our targets:**

| **Anemoi** | **Griffin** | **ArionHash** |
|---|---|---|
| Crypto23 | Crypto23 | arXiv |

- Griffin, ArionHash and AnemoiSponge are Arithmetization-Oriented families of hash functions.
- Based on Griffin-$\pi$, Arion-$\pi$ and Anemoi families of permutations (all fixed-key block ciphers).

# Quick Overview of Griffin, Arion, Anemoi

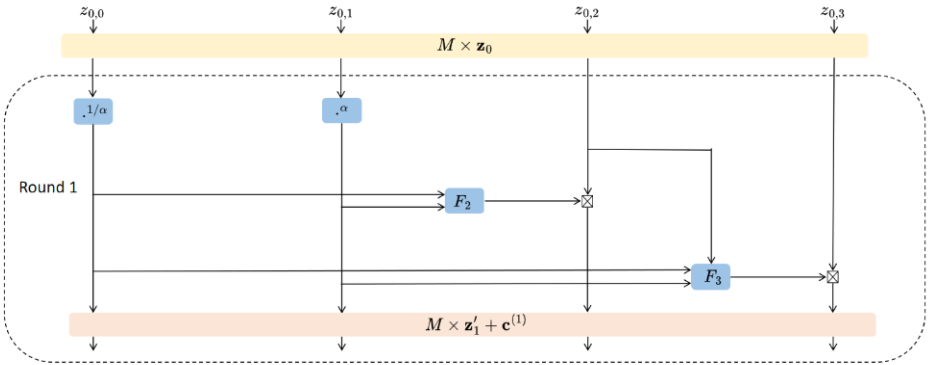
**Our targets:**

| **Anemoi** | **Griffin** | **ArionHash** |
|---|---|---|
| Crypto23 | Crypto23 | arXiv |

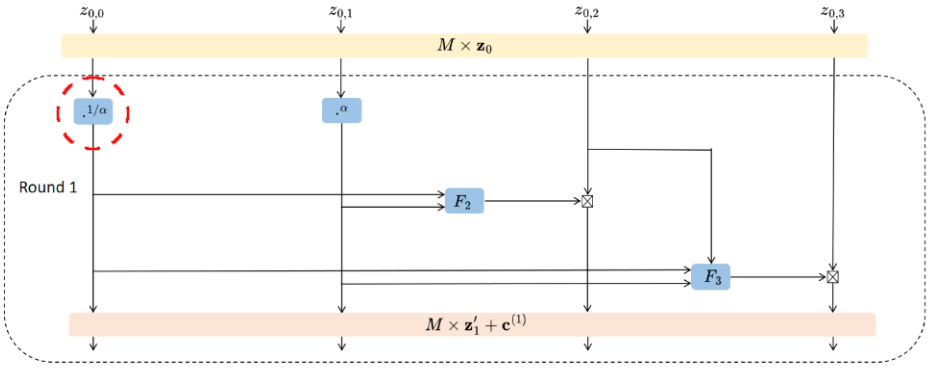
- Griffin, ArionHash and AnemoiSponge are Arithmetization-Oriented families of hash functions.
- Based on Griffin-$\pi$, Arion-$\pi$ and Anemoi families of permutations (all fixed-key block ciphers).
- **Many** instances are defined: variable $\mathbb{F}_p$, number of branches, exponents for monomial permutations...

$\Longrightarrow$ We attack some instance better than others.

# Griffin-$\pi$ - Round Function (4 branches)



# Griffin-$\pi$ - Round Function (4 branches)




$\cdot^{1/\alpha}$ is the only high-degree operation $\Longrightarrow$ add one variable per $\cdot^{1/\alpha}$.

# Griffin-$\pi$ - Model

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable $x_0$ in the input. One equation for the output (last branch at 0).
- $N_{rounds}$ equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ ($\cdot^{1/\alpha}$ S-boxes).

# Griffin-$\pi$ - Model

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable $x_0$ in the input. One equation for the output (last branch at 0).
- $N_{rounds}$ equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ ($\cdot^{1/\alpha}$ S-boxes).

**Example ($\alpha = 3$, one round)**

$$x_1^3 = ax_0 + b$$
$$x_0^7 + cx_0^4x_1 + dx_0x_1^2 + \cdots = 0$$

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases} \quad \begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

1. Define system
2. Find a GB (F4/F5)
3. Change order to **lex** (FGLM)

4. Find the roots in $\mathbb{F}_q$ of $g_N^*$ with univariate methods, etc.

Designers of Anemoi and Griffin base their security on the hardness of **Step 2**.

# Generic System Solving

$$\begin{cases} p_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ p_{k-1}(x_1, \ldots, x_N) = 0 \\ p_k(x_1, \ldots, x_N) = 0 \end{cases}$$

1. Define system

$$\begin{cases} g_1(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{\kappa-1}(x_1, \ldots, x_N) = 0 \\ g_\kappa(x_1, \ldots, x_N) = 0 \end{cases}$$

2. Find a GB (F4/F5)

$$\begin{cases} g_1^*(x_1, \ldots, x_N) = 0 \\ \vdots \\ g_{N-1}^*(x_{N-1}, x_N) = 0 \\ g_N^*(x_N) = 0 \end{cases}$$

3. Change order to **lex** (FGLM)

4. Find the roots in $\mathbb{F}_q$ of $g_N^*$ with univariate methods, etc.

Designers of Anemoi and Griffin base their security on the hardness of **Step 2**.

But we can skip it!

# Plan of this Section

# Griffin-$\pi$ - Model

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable $x_0$ in the input. One equation for the output (last branch at 0).
- $N_{rounds}$ equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ ($\cdot^{1/\alpha}$ S-boxes).

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0 + b$$
$$x_0^7 + cx_0^4x_1 + dx_0x_1^2 + \cdots = 0$$

# Griffin-$\pi$ - Model

- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable $x_0$ in the input. One equation for the output (last branch at 0).
- $N_{rounds}$ equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ ($\cdot^{1/\alpha}$ S-boxes).

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0 + b$$
$$x_0^7 + cx_0^4x_1 + dx_0x_1^2 + \cdots = 0$$

***Observation***: *$x_1$ has a lower degree than $x_0$ in the last equation.*
$\Longrightarrow$ *In **grevlex**, the leading monomials are $x_0^7$ and $x_1^3$.*
$\Longrightarrow$ ***It's a Gröbner basis** ! (coprime leading monomials)*

# Griffin-$\pi$ - Model

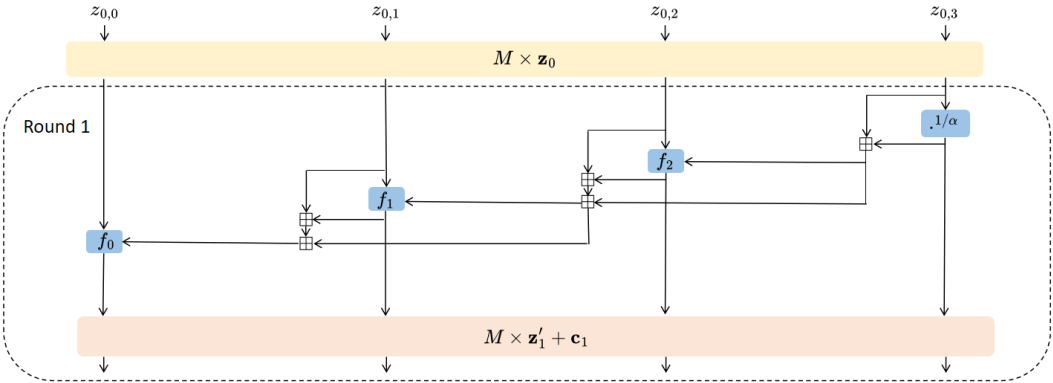
- **CICO problem**: $\mathcal{G}_\pi(\cdots||0) = (\cdots||0)$.
  $\Longrightarrow$ One variable $x_0$ in the input. One equation for the output (last branch at 0).
- $N_{rounds}$ equations of the form $x_i^\alpha = P_i(x_0, x_1, \ldots x_{i-1})$ ($\cdot^{1/\alpha}$ S-boxes).

## Example ($\alpha = 3$, one round)

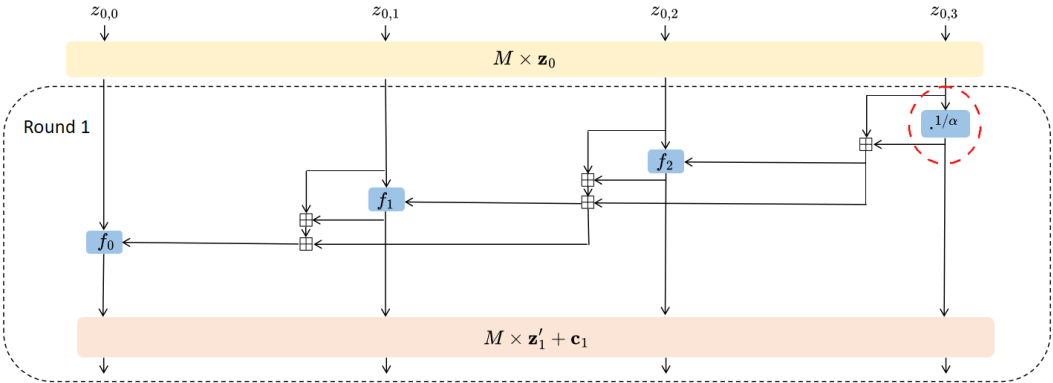
$$x_1^3 = ax_0 + b$$
$$x_0^7 + cx_0^4x_1 + dx_0x_1^2 + \cdots = 0$$

***Observation***: *$x_1$ has a lower degree than $x_0$ in the last equation.*
$\Longrightarrow$ *In **grevlex**, the leading monomials are $x_0^7$ and $x_1^3$.*
$\Longrightarrow$ ***It's a Gröbner basis** ! (coprime leading monomials)*
*For more rounds, **grevlex** doesn't work. We need **weighted degree** orders, with $\mathbf{wt}(x_0) = 1$ and $\mathbf{wt}(x_i) = 7^{i-1}$.*

# Arion-$\pi$ - Round Function (4 branches)



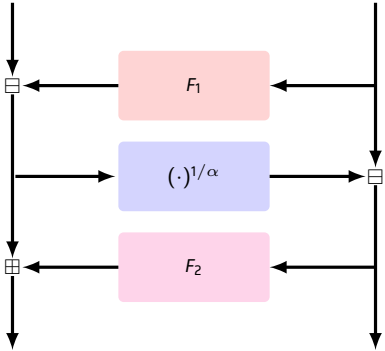
# Arion-$\pi$ - Round Function (4 branches)



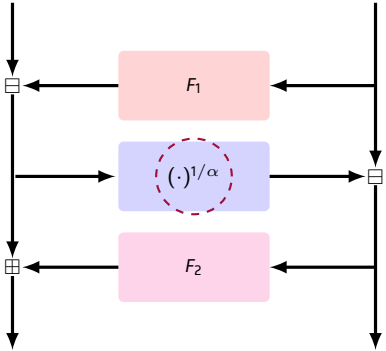

$\cdot^{1/\alpha}$ is the only high-degree operation $\Longrightarrow$ add one variable per $\cdot^{1/\alpha}$.

# Plan of this Section

# Anemoi - Nonlinear layer (2 branches)



# Anemoi - Nonlinear layer (2 branches)




$(\cdot)^{1/\alpha}$ is the only high-degree operation $\implies$ add one variable per $(\cdot)^{1/\alpha}$.

# Anemoi - Model

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$
$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

# Anemoi - Model

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$
$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

$x_0^2$ cancels out: this isn't a Gröbner basis for any order!

# Anemoi - Model

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$
$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

$x_0^2$ cancels out: this isn't a Gröbner basis for any order!

## Saving the Freelunch

**Solution:** multiply last equation by $x_1^2$ and reduce it by the first equation. We get:

$$p^*(x_0, x_1) = ax_0^3 + bdx_0^2x_1 + \cdots$$

# Anemoi - Model

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$
$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

$x_0^2$ cancels out: this isn't a Gröbner basis for any order!

## Saving the Freelunch

**Solution:** multiply last equation by $x_1^2$ and reduce it by the first equation. We get:

$$p^*(x_0, x_1) = ax_0^3 + bdx_0^2x_1 + \cdots$$

$\Longrightarrow$ The first equation and $p^*$ are a Gröbner basis for some weighted order.

# Anemoi - Model

## Example ($\alpha = 3$, one round)

$$x_1^3 = ax_0^2 + bx_0 + c$$
$$x_0x_1 + dx_1^2 + ex_0 + fx_1 + g = 0$$

$x_0^2$ cancels out: this isn't a Gröbner basis for any order!

## Saving the Freelunch

**Solution:** multiply last equation by $x_1^2$ and reduce it by the first equation. We get:

$$p^*(x_0, x_1) = ax_0^3 + bdx_0^2x_1 + \cdots$$

$\implies$ The first equation and $p^*$ are a Gröbner basis for some weighted order.
This adds a few parasitic solutions (corresponding to $x_1 = 0$), but not many.

# Plan of this Section

# FGLM in a Nutshell

- Given a zero-dimensional ideal $I$, a Gröbner basis $G_1$ for $I$ some ordering $<_1$, and an ordering $<_2$, FGLM computes a Gröbner basis $G_2$ for $<_2$ in $O(n_{var} D_I^3)$.
- $D_I$ is the degree of the ideal, a.k.a. the number of **solutions of the system** in the algebraic closure.

# FGLM in a Nutshell

- Given a zero-dimensional ideal $I$, a Gröbner basis $G_1$ for $I$ some ordering $<_1$, and an ordering $<_2$, FGLM computes a Gröbner basis $G_2$ for $<_2$ in $O(n_{var} D_I^3)$.
- $D_I$ is the degree of the ideal, a.k.a. the number of **solutions of the system** in the algebraic closure.
- **This is interesting** because a GB in **lex** order **must have** a univariate polynomial in the smallest variable, which we can solve. (This corresponds to eliminating the other variables.)

# FGLM in a Nutshell

- Given a zero-dimensional ideal $I$, a Gröbner basis $G_1$ for $I$ some ordering $<_1$, and an ordering $<_2$, FGLM computes a Gröbner basis $G_2$ for $<_2$ in $O(n_{var} D_I^3)$.
- $D_I$ is the degree of the ideal, a.k.a. the number of **solutions of the system** in the algebraic closure.
- **This is interesting** because a GB in **lex** order **must have** a univariate polynomial in the smallest variable, which we can solve. (This corresponds to eliminating the other variables.)
- Free Gröbner basis, FGLM and symmetric techniques to bypass the first rounds is already enough to break some instances of Griffin and Arion.

# Faster Change of Order Strategy

- Idea from a 2022 paper by Jérémy Berthomieu, Vincent Neiger, Mohab Safey El Din.
- Strategy: for the smallest variable $x$, compute the characteristic polynomial $\chi$ of the linear operation $P \mapsto \mathrm{Red}_<(x \cdot P, G)$.
- $\chi(x) = 0$. Generically, this is **exactly** the univariate polynomial in $x$ in the reduced GB of $I$ in **lex** order.
- **Issue:** our systems **do not** verify an important property of the original paper.

# Computing the Multiplication Matrix

**Step 1:** Compute the matrix $T$ of the linear operation in $\mathbb{F}[x_0, x_1, \ldots, x_N]$ that maps $P$ to $x_0 \cdot P$.

- Need to reduce monomials of the form $x_0^{k_0+1} x_1^{k_1} \cdots x_N^{k_N}$. **We have no tight complexity estimate for this step**.

# Computing the Multiplication Matrix

**Step 1:** Compute the matrix $T$ of the linear operation in $\mathbb{F}[x_0, x_1, \ldots, x_N]$ that maps $P$ to $x_0 \cdot P$.

- Need to reduce monomials of the form $x_0^{k_0+1} x_1^{k_1} \cdots x_N^{k_N}$. **We have no tight complexity estimate for this step.**
- The matrix is sparse. If leading monomials are $x_0^{d_0}, \ldots, x_N^{d_N}$:

$$T_0 = \underbrace{\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & * \\ \hline 1 & 0 & \cdots & 0 & * \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & * \end{array}\right)}_{(d_0-1)d_1 \cdots d_N \qquad d_1 \cdots d_N}$$

# Computing the Characteristic Polynomial

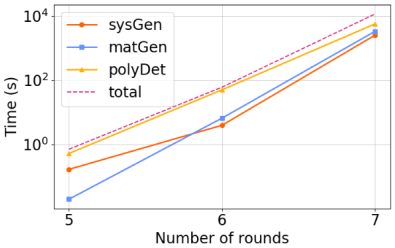
**Step 2:** Given $T$, compute $\det(XI - M)$.

- $T$ is sparse. With block matrix reasoning, this reduces to computing the determinant of a polynomial matrix of size $D_1 = d_1 \cdots d_N$.
- In Griffin and Arion, $d_0$ is by far the highest degree, so this reduces complexity by a lot.
- This can be computed with fast linear algebra, in $\mathcal{O}(d_0 \log(d_0)^2 d_1^\omega \cdots d_N^\omega)$.

# Our Full Algorithm

1. `sysGen`: Compute the Freelunch system and the order for a free Gröbner basis.
2. `matGen`: Compute the multiplication matrix $T$. **Complexity hard to evaluate**.
3. `polyDet`: Compute the characteristic polynomial $\chi$ of $T$.
   **Longest step aside from** `matGen`.
4. `uniSol`: Find roots of $\chi$ with Berlekamp-Rabin in $\mathcal{O}(D_I \log(D_I) \log(pD_I))$.

# Experimental Results



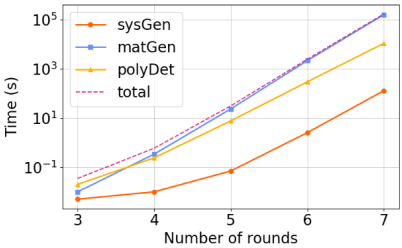
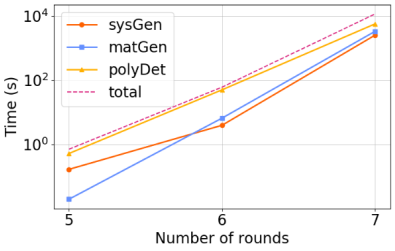




Figure: Complexity of Griffin (7 out of 10 rounds, $\alpha$=3)

Complexity of Anemoi (7 out of 21 rounds, $\alpha = 3$)

# Experimental Results



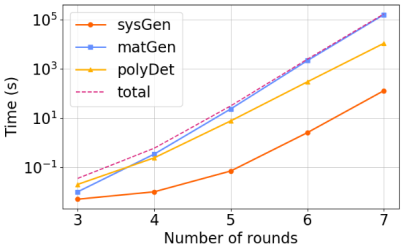




Figure: Complexity of Griffin (7 out of 10 rounds, $\alpha$=3)

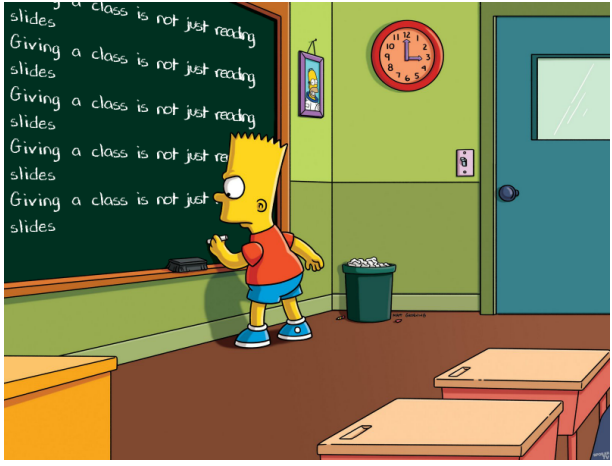
Complexity of Anemoi (7 out of 21 rounds, $\alpha = 3$)

$\Longrightarrow$ For Griffin, `polyDet` upper-bounds the others up to 7 rounds.

$\Longrightarrow$ For Anemoi, `matGen` is the bottleneck.

# Plan of this Section

# Skipping Rounds



# Conclusion

- AO primitives should not base their security on the complexity of finding a Gröbner basis (F4/F5).
- Instead, focus on the growth of $D_I$ with the number of rounds (impacts the complexity of solving algorithms).
- Anemoi, Griffin and Arion need to recompute their numbers of rounds!

# Conclusion

- AO primitives should not base their security on the complexity of finding a Gröbner basis (F4/F5).
- Instead, focus on the growth of $D_I$ with the number of rounds (impacts the complexity of solving algorithms).
- Anemoi, Griffin and Arion need to recompute their numbers of rounds!

`MatGen` and `PolyDet` are already getting obsolete!

Improved Resultant Attack against Arithmetization-Oriented Primitives

`https://eprint.iacr.org/2025/259`

*Augustin Bariant, Aurélien Boeuf, Pierre Briaud, Maël Hostettler, Morten Øygarden, Håvard Raddum*

# Conclusion

- AO primitives should not base their security on the complexity of finding a Gröbner basis (F4/F5).
- Instead, focus on the growth of $D_I$ with the number of rounds (impacts the complexity of solving algorithms).
- Anemoi, Griffin and Arion need to recompute their numbers of rounds!

`MatGen` and `PolyDet` are already getting obsolete!

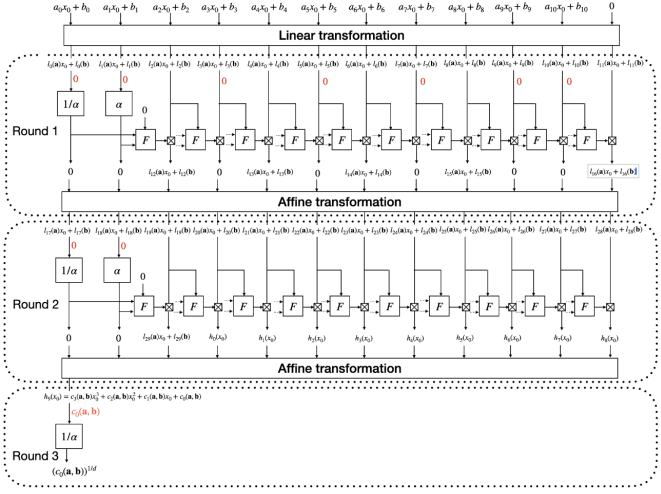
Improved Resultant Attack against Arithmetization-Oriented Primitives

`https://eprint.iacr.org/2025/259`

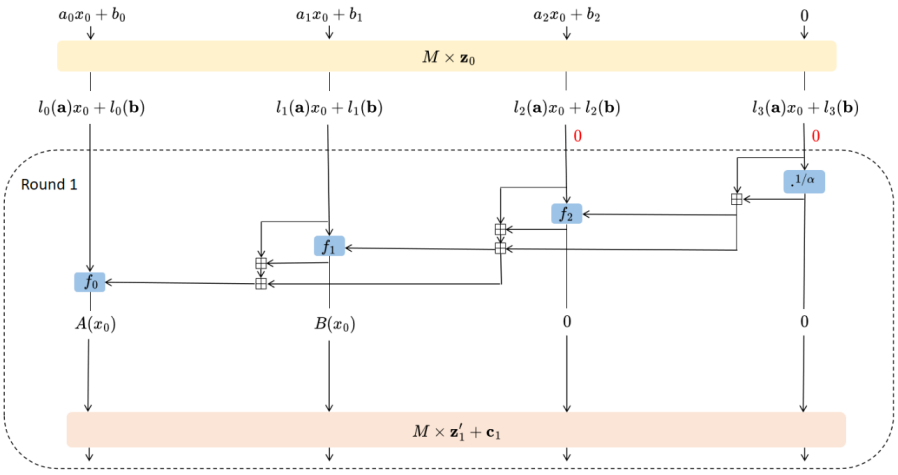
*Augustin Bariant, Aurélien Boeuf, Pierre Briaud, Maël Hostettler, Morten Øygarden, Håvard Raddum*

**Thank you!**

# Griffin Trick



# Arion Trick


$a_0x_0 + b_0$
$a_1x_0 + b_1$
$a_2x_0 + b_2$
0
$M \times \mathbf{z}_0$
$l_0(\mathbf{a})x_0 + l_0(\mathbf{b})$
$l_1(\mathbf{a})x_0 + l_1(\mathbf{b})$
$l_2(\mathbf{a})x_0 + l_2(\mathbf{b})$
$l_3(\mathbf{a})x_0 + l_3(\mathbf{b})$
0
0
Round 1
$.^{1/\alpha}$
$f_2$
$f_1$
$f_0$
$A(x_0)$
$B(x_0)$
0
0
$M \times \mathbf{z}_1' + \mathbf{c}_1$