Assignment 3 - MH4514 Financial Mathematics

Name: Honda Naoki
Matriculation: N1804369J

March 26, 2019

Question 1.

Solution. Using the properties of covariance and Ito multiplication table, we can calculate the fraction as

$$\frac{Cov\left(\frac{dS_t}{S_t}, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]} = \frac{Cov\left(rdt + \beta\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]}$$
$$= \frac{Cov\left(\beta\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]} = \beta$$

Question 2.

Solution. By (1) and (2), observe that the evolution of $(S_t)_{t\in\mathbb{R}_+}$ is given by

$$\frac{dS_t}{S_t} = rdt + \beta\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t$$
$$= (r + \beta(\mu - r))dt + \beta\sigma_M dB_t + \sigma_S dW_t$$

Now, consider its volatility, by squaring both side we have

$$\left(\frac{dS_t}{S_t}\right)^2 = ((r + \beta(\mu - r))dt + \beta\sigma_M dB_t + \sigma_S dW_t)^2$$
$$= \left(\beta^2\sigma_M^2 + \sigma_S^2\right) dt$$

Thus, we can think of $(S_t)_{t\in\mathbb{R}_+}$ as a geometric Brownian motion with volatility $\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}$. Finally with the excess return α, we can express its evolution as

$$\frac{dS_t}{S_t} = (r + \alpha + \beta(\mu - r))dt + \sqrt{\beta^2\sigma_M^2 + \sigma_S^2} dW_t$$
$$dS_t = (r + \alpha + \beta(\mu - r))S_t dt + \sqrt{\beta^2\sigma_M^2 + \sigma_S^2} S_t dW_t$$

Question 3.

Solution. Consider the equations (1), we have

$$\begin{aligned}\frac{dM_t}{M_t} &= \mu dt + \sigma_M dB_t \\ &= rdt + \sigma_M\left(dB_t + \frac{\mu - r}{\sigma_M}dt\right) \\ &= rdt + \sigma_M dB_t^* \qquad\qquad \left(\text{where} \quad dB_t^* = \left(dB_t + \frac{\mu - r}{\sigma_M}dt\right)\right)\end{aligned}$$

Next, we can rewrite (2) as

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \beta(M_t)\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t \\ &= rdt + \beta(M_t)(rdt + \sigma_M dB_t^* - rdt) + \sigma_S dW_t \\ &= rdt + \sigma_M\beta(M_t)dB_t^* + \sigma_S dW_t^* \qquad\qquad (\text{where} \quad dW_t^* = dW_t)\end{aligned}$$

Hence, by setting $B_t^* = \left(B_t + \frac{\mu - r}{\sigma_M}t\right)$ and $W_t^* = W_t$ we have

$$\begin{cases}\dfrac{dM_t}{M_t} &= rdt + \sigma_M dB_t^* \\[2ex] \dfrac{dS_t}{S_t} &= rdt + \sigma_M\beta(M_t)dB_t^* + \sigma_S dW_t^*\end{cases}$$

Question 4.

Solution. Consider the new probability measure $\mathbb{P}^*$ defined on $\mathcal{F}_T$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{-\frac{\mu - r}{\sigma_M}B_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma_M}\right)^2 T\right\}$$

Setting

$$B_t^* = B_t + \frac{\mu - r}{\sigma_M}t, \qquad W_t^* = W_t$$

then under $\mathbb{P}^*$, the processes $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ are two independent standard Brownian motions.

Question 5.

Solution. From Question 4, we know under $\mathbb{P}^*$, the processes $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ are two independent standard Brownian motions. Using this, the discounted asset price processes $(\tilde{S}_t, \tilde{M}_t)_{t\in\mathbb{R}_+}$ can be shown as martingales.

$$\begin{aligned}
\mathbb{E}^*\left[M_t \middle| \mathcal{F}_k\right] &= e^{-rt}\mathbb{E}^*\left[\int_0^t rM_u du + \sigma_M \int_0^t M_u dB_u^* \middle| \mathcal{F}_k\right] \\
&= e^{-rt}\mathbb{E}^*\left[\int_0^t rM_u du \middle| \mathcal{F}_k\right] = e^{-rt}e^{(t-k)r}M_k \\
&= \tilde{M}_k
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}^*\left[\tilde{S}_t \middle| \mathcal{F}_k\right] &= e^{-rt}\mathbb{E}^*\left[\int_0^t rS_u du + \sigma_M \int_0^t \beta(M_u) S_u dB_u^* + \sigma_S \int_0^t S_u dW_u^* \middle| \mathcal{F}_k\right] \\
&= e^{-rt}\mathbb{E}^*\left[\int_0^t rS_u du \middle| \mathcal{F}_k\right] = e^{-rt}e^{(t-k)r}S_k \\
&= \tilde{S}_k
\end{aligned}$$

From Proposition 6.6, the probability measure $\mathbb{P}^*$ is risk-neutral since the discounted risky asset price processes $(\tilde{S}_t, \tilde{M}_t)_{t\in\mathbb{R}_+}$ are multingales under $\mathbb{P}^*$.

Hence, by Theorem 6.8, the market based on the assets S_t and M_t is without arbitrage opportunities since it admits at least one equivalent risk-neutral probability measure $\mathbb{P}^*$.

Question 6.

Solution. By Ito's calculus we have

$$\begin{aligned}
dV_t = \xi_t dS_t &+ S_t d\xi_t + d\xi_t dS_t \\
&+ \zeta_t dM_t + M_t d\zeta_t + d\zeta_t dM_t \\
&\quad + \eta_t dA_t + A_t d\eta_t + d\eta_t dA_t
\end{aligned}$$

Here, since we know $S_t d\xi_t + M_t d\zeta_t + A_t d\eta_t = 0$ being true under the self-financing condition, and $d\xi_t dS_t = d\zeta_t dM_t = d\eta_t dA_t = 0$, we can rewrite dV_t as

$$dV_t = \xi_t dS_t + \zeta_t dM_t + \eta_t dA_t$$

Question 7.

Solution. From Question 2 and the same derivation of Question 3 we have

$$\begin{cases} dM_t &= rM_t dt + \sigma_M M_t dB_t^* \\ \\ dS_t &= rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^* \end{cases}$$

Using the above system and Ito's formula[1], we have[2]

$$\begin{aligned} df(t, S_t, M_t) = &\left\{ \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) S_t^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2} + \beta(M_t)\sigma_M^2 M_t S_t \frac{\partial^2 f}{\partial x \partial y} \right\} dt \\ &+ \frac{\partial f}{\partial x} dS_t + \frac{\partial f}{\partial y} dM_t \end{aligned}$$

Further expand dS_t and dM_t we have

$$\begin{aligned} df(t, S_t, M_t) = &\left\{ \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) S_t^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2} \right. \\ &\left. + \beta(M_t)\sigma_M^2 M_t S_t \frac{\partial^2 f}{\partial x \partial y} + r\left(S_t \frac{\partial f}{\partial x} + M_t \frac{\partial f}{\partial y}\right) \right\} dt \\ &+ \left(\sigma_M \beta(M_t) S_t \frac{\partial f}{\partial x} + \sigma_M M_t \frac{\partial f}{\partial y}\right) dB_t^* + \sigma_S S_t \frac{\partial f}{\partial x} dW_t^* \end{aligned} \tag{7.1}$$

Using the self-financing condition of Question 6, we also can express $df(t, S_t, M_t)$ as follows

$$\begin{aligned} df(t, S_t, M_t) &= dV_t = \xi_t dS_t + \zeta_t dM_t + \eta_t dA_t \\ &= \xi_t(rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^*) + \zeta_t(rM_t dt + \sigma_M M_t dB_t^*) + r\eta_t A_t dt \\ &= rf(t, S_t, M_t)dt + (\sigma_M \beta(M_t)\xi_t S_t + \zeta_t \sigma_M M_t) dB_t^* + \sigma_S \xi_t S_t dW_t^* \end{aligned} \tag{7.2}$$

By identification of the term in dt in (7.1) and (7.2) we get the PDE with the terminal condition:

$$\begin{cases} rf(t, x, y) &= \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) x^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 y^2 \frac{\partial^2 f}{\partial y^2} \\ &\quad + \beta(M_t)\sigma_M^2 xy \frac{\partial^2 f}{\partial x \partial y} + r\left(x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}\right) \\ \\ f(T, x, y) &= h(x, y) \end{cases}$$

[1] In the text book, P191, (5.5.7)

[2] Here we omit the argument bracket of function $f(t, S_t, M_t)$ for the sake of simplicity

Question 8.

Solution. Similar to the way we derived the PDE, by identifying the coefficient on dB_t^* and dB_t^* in (7.1)-(7.2) we have system of equation

$$\begin{cases} \sigma_M \beta(M_t) S_t \dfrac{\partial f}{\partial x} + \sigma_M M_t \dfrac{\partial f}{\partial y} = \sigma_M \beta(M_t) \xi_t S_t + \zeta_t \sigma_M M_t \\ \\ \sigma_S S_t \dfrac{\partial f}{\partial x} = \sigma_S \xi_t S_t \end{cases}$$

which gives a solution

$$\xi_t = \frac{\partial f}{\partial x}(t, x, y), \qquad \zeta_t = \frac{\partial f}{\partial y}(t, x, y).$$

For η_t, from previous result and the definition of the portfolio price V_t we have

$$\begin{aligned} \eta_t A_t &= V_t - \xi_t S_t - \zeta_t M_t \\ &= f(t, x, y) - S_t \frac{\partial f}{\partial x}(t, x, y) - M_t \frac{\partial f}{\partial y}(t, x, y) \\ \eta_t &= \frac{1}{A_0 e^{rt}} \left\{ f(t, x, y) - S_t \frac{\partial f}{\partial x}(t, x, y) - M_t \frac{\partial f}{\partial y}(t, x, y) \right\} \end{aligned}$$

Question 9.

Solution. From Question 7, with constant β and terminal condition of a call option payoff, we have PDE of

$$\begin{cases} rf(t,x,y) &= \dfrac{\partial f}{\partial t} + \dfrac{1}{2}\left(\sigma_M^2\beta^2 + \sigma_S^2\right)x^2\dfrac{\partial^2 f}{\partial x^2} + \dfrac{1}{2}\sigma_M^2 y^2\dfrac{\partial^2 f}{\partial y^2} \\ &\quad + \beta\sigma_M^2 xy\dfrac{\partial^2 f}{\partial x \partial y} + r\left(x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}\right) \\ f(T,x,y) &= (x-K)^+ \end{cases}$$

By change of variables with $u(\tau,a,b) = e^{-rt}f\left(T-t, e^{x+\left(r-(\sigma_M^2\beta^2+\sigma_S^2)/2\right)\tau}, e^{y+(r-\sigma_M^2/2)\tau}\right)$, we have

$$\frac{\partial u}{\partial t} = \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}\frac{\partial^2 u}{\partial a^2} + \frac{\sigma_M^2}{2}\frac{\partial^2 u}{\partial b^2} + \beta\sigma_M^2\frac{\partial^2 u}{\partial a \partial b}$$

Step 1

By change of variables with $v(t,x,y) = e^{-rt}f(t,x,y)$, such that $\frac{\partial v}{\partial t} = e^{-rt}\frac{\partial f}{\partial t} - re^{-rt}f$, we have

$$\frac{\partial v}{\partial t} + \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}x^2\frac{\partial^2 v}{\partial x^2} + \frac{\sigma_M^2}{2}y^2\frac{\partial^2 v}{\partial y^2} + \beta\sigma_M^2 xy\frac{\partial^2 v}{\partial x\partial y} + r\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 0$$

Step 2

By change of variables with $w(\tau, e^a, e^b) = v(T-t, x, y)$, such that

$$\frac{\partial v}{\partial x} \to \frac{1}{x}\frac{\partial w}{\partial a}, \quad \frac{\partial^2 v}{\partial x^2} \to \frac{1}{x^2}\frac{\partial^2 w}{\partial a^2} - \frac{1}{x^2}\frac{\partial w}{\partial a}, \quad \frac{\partial v}{\partial t} \to -\frac{\partial w}{\partial \tau}$$

we have

$$\begin{aligned}\frac{\partial w}{\partial \tau} - \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}\frac{\partial^2 w}{\partial a^2} - \frac{\sigma_M^2}{2}\frac{\partial^2 w}{\partial b^2} - \beta\sigma_M^2\frac{\partial^2 w}{\partial a\partial b}& \\ - \left(r - \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}\right)\frac{\partial w}{\partial a} - \left(r - \frac{\sigma_M^2}{2}\right)\frac{\partial w}{\partial b} &= 0\end{aligned}$$

Step 3

By change of variables with $u(\tau, c, d) = w\left(\tau, x + \left(r - \frac{\sigma_M^2\beta^2+\sigma_S^2}{2}\right)\tau, y + \left(r - \frac{\sigma_M^2}{2}\right)\tau\right)$, such that

$$\frac{\partial w}{\partial \tau} \to \frac{\partial u}{\partial \tau} + \left(r - \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}\right)\frac{\partial u}{\partial c} + \left(r - \frac{\sigma_M^2}{2}\right)\frac{\partial u}{\partial d}, \qquad \frac{\partial w}{\partial a} \to \frac{\partial w}{\partial c}$$

we have

$$\frac{\partial u}{\partial \tau} - \frac{\sigma_M^2\beta^2 + \sigma_S^2}{2}\frac{\partial^2 u}{\partial c^2} - \frac{\sigma_M^2}{2}\frac{\partial^2 u}{\partial d^2} - \beta\sigma_M^2\frac{\partial^2 u}{\partial c\partial d} = 0$$

Link 1
Link 2
Link 3
Link 4

Define function $g(t,x,y)$ as the 2 dimensional Gaussian probability density function

$$g(t,x,y) = \frac{1}{2\pi t}exp\left(-\frac{x^2+y^2}{2t}\right)$$

which solves the heat equation with initial condition

$$g(0,x,y) = h(exp(\sigma x + \sigma y))$$

i.e.

$$\frac{\partial g}{\partial t} = \left(-\frac{1}{t} + \frac{x^2+y^2}{2t^2}\right)g(t,x,y) = \frac{1}{2}\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)$$

Question 10.

Solution.