MH4514 Financial Mathematics

- Midterm Revision -

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1 (10.6) Properties of the Conditional Expectation

Properties of CE

- If G depends only on the information contained in $\mathcal{G}$,

$$\mathbb{E}[FG|\mathcal{G}] = G\mathbb{E}[F|\mathcal{G}]$$

- If G depends only on the information contained in $\mathcal{G}$,

$$\mathbb{E}[G|\mathcal{G}] = G$$

- **Tower Property**

$$\mathbb{E}[\mathbb{E}[F|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[F|\mathcal{H}] \quad (\text{if } \mathcal{H} \subset \mathcal{G})$$

- When F "does not depend" on the information contained in $\mathcal{G}$ (more stated, when the random variable F is *independent* of the σ-algebra $\mathcal{G}$),

$$\mathbb{E}[F|\mathcal{G}] = \mathbb{E}[F]$$

- If G depends only on $\mathcal{G}$ and F is independent of $\mathcal{G}$, then

$$\mathbb{E}[h(F,G)|\mathcal{G}] = \mathbb{E}[h(F,x)]_{x=G}$$

2 (1) Discrete-Time Martingales

2.1 Definition

Definition 1.1

An integrable[a], discrete-time process $(Z_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$ if $(Z_n)_{n\in\mathbb{N}}$ is $\mathcal{F}_n$-adapted and satisfies the property

$$\mathbb{E}[Z_{n+1}|\mathcal{F}_n] = Z_n, \qquad n \in \mathbb{N}$$

[a]Integrable means $\mathbb{E}[|Z_n|] < \infty$ for all $n \in \mathbb{N}$

2.2 Properties

A particular property of martingales is that their expectation is constant over time.

Proposition 1.2

Let $(Z_n)_{n\in\mathbb{N}}$ be martingale, we have

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \qquad n \in \mathbb{N}$$

The following Theorem 1.6 is called the Doob's stopping time theorem.

Theorem 1.6

Assume that $(M_n)_{n\in\mathbb{N}}$ is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. Then the *stopped process* $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.

2.3 Applications: Ruin Probability

Consider the standard random walk (or gambling process) $(S_n)_{n\in\mathbb{N}}$ on $\{0, 1, ..., B\}$ with independent $\{-1, 1\}$-valued increments, and

$$\mathbb{P}(S_{n+1} - S_n = +1) = p \qquad \text{and} \qquad \mathbb{P}(S_{n+1} - S_n = -1) = q, \qquad n \in \mathbb{N}$$

Let

$$T_{0,B} : \Omega \longrightarrow \mathbb{N}$$

be the first hitting time of the boundary $\{0, B\}$, define stopping time τ by

$$\tau := T_{0,B} := \inf\{n \geq 0 : S_n = B \text{ or } S_n = 0\}$$

We will recover the ruin probabilities

$$\mathbb{P}(s_\tau = 0 | S_0 = k), \qquad\qquad k = 0, 1, ..., B$$

First in the unbiased case $p = q = 1/2$

2.3.1 Unbiased Ruin Probability with Martingale

Step 1. The process $(S_n)_{n\in\mathbb{N}}$ is a martingale.

We note that the process $(S_n)_{n\in\mathbb{N}}$ has independent increments, and in the unbiased case $p = q = 1/2$ those increments are centered:

$$\mathbb{E}[S_{n+1} - S_n] = 1 \times p + (-1) \times q = 1 \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0,$$

hence $(S_n)_{n\in\mathbb{N}}$ is a *martingale.*

Step 2. The stopped process $(S_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale, as a consequence of Theorem 1.6 (Doob's stopping time theorem)

Step 3. Since the stopped process $(S_{\tau\wedge n})_{n\in\mathbb{N}}$ is a martingale, we find that its expectation $\mathbb{E}[S_{\tau\wedge n}|S_0 = k]$ is constant in $n \in \mathbb{N}$ by proposition 1.2, which gives

$$k = \mathbb{E}[S_0|S_0 = k] = \mathbb{E}[S_{\tau\wedge n}|S_0 = k], \qquad\qquad k = 0, 1, ..., B$$

Letting n go to infinity we get

$$\begin{aligned}\mathbb{E}[S_\tau|S_0 = k] &= \mathbb{E}\left[\lim_{n\to\infty} S_{\tau\wedge n}|S_0 = k\right] \\ &= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge n}|S_0 = k] = k\end{aligned}$$

where the exchange between limit and expectation is justified by the boundedness $|S_{\tau\wedge n}| \leq B$ *a.s.*, $n \in \mathbb{N}$.
Hence we have

$$\begin{cases} 0 \times \mathbb{P}(S_\tau = 0|S_0 = k) + B \times \mathbb{P}(S_\tau = B|S_0 = k) = \mathbb{E}[S_\tau|S_0 = k] = k \\ \mathbb{P}(S_\tau = 0|S_0 = k) + \mathbb{P}(S_\tau = B|S_0 = k) = 1 \end{cases}$$

which shows that

$$\mathbb{P}(S_\tau = B|S_0 = k) = \frac{k}{B} \qquad \text{and} \qquad \mathbb{P}(S_\tau = 0|S_0 = k) = 1 - \frac{k}{B}$$

2.3.2 Biased Ruin Probability with Martingale

Next, for the unbiased case where $p \neq q$. Here we note that the processs

$$M_n := \left(\frac{q}{p}\right)^{S_n}, \qquad n \in \mathbb{N}$$

is a martingale with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
Step 1. The process $(M_n)_{n\in\mathbb{N}}$ is a martingale.

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &= \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}} \middle| \mathcal{F}_n\right] = \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n}\left(\frac{q}{p}\right)^{S_n} \middle| \mathcal{F}_n\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n} \middle| \mathcal{F}_n\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \mathbb{E}\left[\left(\frac{q}{p}\right)^{S_{n+1}-S_n}\right] \\
&= \left(\frac{q}{p}\right)^{S_n} \left(\frac{q}{p}\mathbb{P}(S_{n+1}-S_n = 1) + \left(\frac{q}{p}\right)^{-1}\mathbb{P}(S_{n+1}-S_n = -1)\right) \\
&= \left(\frac{q}{p}\right)^{S_n} \left(p\frac{q}{p} + q\left(\frac{q}{p}\right)^{-1}\right) \\
&= \left(\frac{q}{p}\right)^{S_n} (q+p) = \left(\frac{q}{p}\right)^{S_n} = M_n
\end{aligned}$$

$n \in \mathbb{N}$. In particular, the expectation of $(M_n)_{n\in\mathbb{N}}$ is constant over time by Proposition 1.2 since it is a martingale, *i.e.* we have

$$\left(\frac{q}{p}\right)^k = \mathbb{E}[M_0|S_0 = k] = \mathbb{E}[M_n|S_0 = k], \qquad k = 0, 1, ..., B, \qquad n \in \mathbb{N}$$

Step 2. The stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ is also a martingale, as a consequence of Theorem 1.6

Step 3. Since the stopped process $(M_{\tau\wedge n})_{n\in\mathbb{N}}$ remains a martingale, its expected value $\mathbb{E}[M_{\tau\wedge n}|S_0 = k]$ is constant in $n \in \mathbb{N}$ by Proposition 1.2, this gives

$$\left(\frac{q}{p}\right)^k = \mathbb{E}[M_0|S_0 = k] = \mathbb{E}[M_{\tau\wedge n}|S_0 = k]$$

Next, letting n go to infinity we find

$$\begin{aligned}
\left(\frac{q}{p}\right)^k &= \mathbb{E}[M_0|S_0 = k] = \lim_{n\to\infty} \mathbb{E}[M_{\tau\wedge n}|S_0 = k] \\
&= \mathbb{E}[\lim_{n\to\infty} M_{\tau\wedge n}|S_0 = k] \\
&= \mathbb{E}[M_\tau|S_0 = k]
\end{aligned}$$

hence

$$
\begin{aligned}
\left(\frac{q}{p}\right)^k &= \mathbb{E}[M_\tau | S_0 = k] \\
&= \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \middle| S_0 = k\right) + \left(\frac{q}{p}\right)^0 \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^0 \middle| S_0 = k\right) \\
&= \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \middle| S_0 = k\right) + \mathbb{P}(M_\tau = 1 | S_0 = k)
\end{aligned}
$$

Solving the system of equations

$$
\begin{cases}
\left(\frac{q}{p}\right)^k = \left(\frac{q}{p}\right)^B \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \middle| S_0 = k\right) + \mathbb{P}(M_\tau = 1 | S_0 = k) \\
\mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \middle| S_0 = k\right) + \mathbb{P}(M_\tau = 1 | S_0 = k) = 1
\end{cases}
$$

gives

$$
\begin{aligned}
\mathbb{P}(S_\tau = B | S_0 = k) &= \mathbb{P}\left(M_\tau = \left(\frac{q}{p}\right)^B \middle| S_0 = k\right) \\
&= \frac{(q/p)^k - 1}{(q/p)^B - 1}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{P}(S_\tau = 0 | S_0 = k) &= \mathbb{P}(M_\tau = 1 | S_0 = k) \\
&= 1 - \frac{(q/p)^k - 1}{(q/p)^B - 1} \\
&= \frac{(q/p)^B - (q/p)^k}{(q/p)^B - 1}
\end{aligned}
$$

for $k = 0, 1, ..., B$.

2.4 Applications: Mean Game Duration

In this section we show how we can recover the mean game duration $\mathbb{E}[\tau|S_0 = k]$.

2.4.1 Unbiased Mean Game Duration with Martingale

In the case of a fair game $p = q = 1/2$,
Step 1. The process $(S_n^2 - n)_{n\in\mathbb{N}}$ is a martingale.

$$\begin{aligned}
\mathbb{E}[S_{n+1}^2 - (n+1)|\mathcal{F}_n] &= \mathbb{E}[(S_n + S_{n+1} - S_n)^2 - (n+1)|\mathcal{F}_n] \\
&= \mathbb{E}[S_n^2 + (S_{n+1} - S_n)^2 + 2S_n(S_{n+1} - S_n) - (n+1)|\mathcal{F}_n] \\
&= \mathbb{E}[S_n^2 - n - 1|\mathcal{F}_n] + \mathbb{E}[(S_{n+1} - S_n)^2|\mathcal{F}_n] + 2\mathbb{E}[S_n(S_{n+1} - S_n)|\mathcal{F}_n] \\
&= S_n^2 - n - 1 + \mathbb{E}[(S_{n+1} - S_n)^2] + 2S_n\mathbb{E}[S_{n+1} - S_n] \\
&= S_n^2 - n - 1 + \mathbb{E}[(S_{n+1} - S_n)^2] \\
&= S_n^2 - n
\end{aligned}$$

Step 2. The stopped process $(S_{\tau\wedge n}^2 - \tau \wedge n)_{n\in\mathbb{N}}$ is also a martingale, as a consequence of Theorem 1.6

Step 3. Since the stopped process $(S_{\tau\wedge n}^2 - \tau \wedge n)_{n\in\mathbb{N}}$ is also a martingale, its expectation $\mathbb{E}[S_{\tau\wedge n}^2 - \tau \wedge n|S_0 = k]$ is constant in $n \in \mathbb{N}$ by Proposition 1.2, hence we have

$$k^2 = \mathbb{E}[S_0^2 - 0|S_0 = k] = \mathbb{E}[S_{\tau\wedge n}^2 - \tau \wedge n|S_0 = k]$$

and after taking the limit as n tends to infinity,

$$\begin{aligned}
k^2 &= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge n}^2 - \tau \wedge n|S_0 = k] \\
&= \lim_{n\to\infty} \mathbb{E}[S_{\tau\wedge n}^2|S_0 = k] - \lim_{n\to\infty} \mathbb{E}[\tau \wedge n|S_0 = k] \\
&= \mathbb{E}\left[\lim_{n\to\infty} S_{\tau\wedge n}^2|S_0 = k\right] - \mathbb{E}\left[\lim_{n\to\infty} \tau \wedge n|S_0 = k\right] \\
&= \mathbb{E}\left[\lim_{n\to\infty} S_{\tau\wedge n}^2 - \lim_{n\to\infty} \tau \wedge n|S_0 = k\right] \\
&= \mathbb{E}[S_\tau^2 - \tau|S_0 = k]
\end{aligned}$$

since $S_{\tau\wedge n}^2 \in [0, B^2]$ for all $n \in \mathbb{N}$ and $n \mapsto \tau \wedge n$ is nondecreasing, and this gives

$$\begin{aligned}
k^2 &= \mathbb{E}[S_\tau^2 - \tau|S_0 = k] \\
&= \mathbb{E}[S_\tau^2|S_0 = k] - \mathbb{E}[\tau|S_0 = k] \\
&= B^2\mathbb{P}(S_\tau = B|S_0 = k) + 0^2\mathbb{P}(S_\tau = 0|S_0 = k) - \mathbb{E}[\tau|S_0 = k]
\end{aligned}$$

i.e.

$$\begin{aligned}
\mathbb{E}[\tau|S_0 = k] &= B^2(S_\tau = B|S_0 = k) - k^2 \\
&= B^2\frac{k}{B} - k^2 \\
&= k(B - k), \qquad\qquad k = 0, 1, ..., B
\end{aligned}$$

2.4.2 Biased Mean Game Duration with Martingale

In the case of non-symmetric case where $p \neq q$
Step 1. The process $S_n - (p-q)n$ is a martingale.

$$\mathbb{E}[S_n - S_{n-1} - (p-q)] = \mathbb{E}[S_n - S_{n-1}] - (p-q) = 0$$

Step 2. The stopped process $(S_{\tau \wedge n} - (p-q)(\tau \wedge n))_{n \in \mathbb{N}}$ is also a martingale, as a consequence of Theorem 1.6

Step 3. The expectation $\mathbb{E}[S_{\tau \wedge n} - (p-q)(\tau \wedge n) | S_0 = k]$ is constant in $n \in \mathbb{N}$

Step 4. Since the stopped process $(S_{\tau \wedge n} - (p-q)(\tau \wedge n))_{n \in \mathbb{N}}$ is a martingale, we have

$$k = \mathbb{E}[S_0 - 0 | S_0 = k] = \mathbb{E}[S_{\tau \wedge n} - (p-q)(\tau \wedge n) | S_0 = k]$$

and after taking the limit as n tends to infinity,

$$\begin{aligned} k &= \lim_{n \to \infty} \mathbb{E}[S_{\tau \wedge n} - (p-q)(\tau \wedge n) | S_0 = k] \\ &= \mathbb{E}[\lim_{n \to \infty} S_{\tau \wedge n} - (p-q) \lim_{n \to \infty} \tau \wedge n | S_0 = k] \\ &= \mathbb{E}[S_\tau - (p-q)\tau | S_0 = k] \end{aligned}$$

which gives

$$\begin{aligned} k &= \mathbb{E}[S_\tau - (p-q)\tau | S_0 = k] \\ &= \mathbb{E}[S_\tau | S_0 = k] - (p-q)\mathbb{E}[\tau | S_0 = k] \\ &= B \times \mathbb{P}(S_\tau = B | S_0 = k) + 0 \times \mathbb{P}(S_\tau = 0 | S_0 = k) - (p-q)\mathbb{E}[\tau | S_0 = k] \end{aligned}$$

i.e.

$$\begin{aligned} (p-q)\mathbb{E}[\tau | S_0 = k] &= B \times \mathbb{P}(S_\tau = B | S_0 = k) - k \\ &= B \frac{(q/p)^k - 1}{(q/p)^B - 1} - k, \end{aligned}$$

hence

$$\mathbb{E}[\tau | S_0 = k] = \frac{1}{p-q} \left(B \frac{(q/p)^k - 1}{(q/p)^B - 1} - k \right)$$

for $k = 0, 1, ..., B$

2.5 Applications: Summary

Here's the summarized table of the family of martingales.

Table 1.1: List of martingales.

	Unbiased	Biased
Ruin probability	S_n	$\left(\frac{q}{p}\right)^{S_n}$
Mean game duration	$S_n^2 - n$	$S_n - n(p-q)$

3 (2) Assets, Portfolios, and Arbitrage

3.1 (0) Back to the Introduction

Definition 0.1: Put option

A (European) *put* option is a contract that gives its holder the right to *sell* a quantity of assets at a predefined price K called the strike price and at a predefined date T called maturity

In general, the payoff of a put option takes the form

$$\phi(S_T) = (K - S_T)^+ := \begin{cases} K - S_T & S_T \leq K \\ 0 & S_T \geq K \end{cases}$$

Definition 0.2: Call option

A (European) *call* option is a contract that gives its holder the right to *buy* a quantity of assets at a predefined price K called the strike price and at a predefined date T called maturity

In general, the payoff of a put option takes the form

$$\phi(S_T) = (S_T - K)^+ := \begin{cases} 0 & S_T \leq K \\ S_T - K & S_T \geq K \end{cases}$$

Definition 0.3: Pricing and hedging in a binary model

The *arbitrage price* of the option is interpreted as the initial cost $\alpha S_0 + \$\beta$ of the portfolio hedging the claim $\mathcal{C}$

3.2 (2.4) Risk-Neutral Probability Measures

Bunch of theorem below...

Theorem 2.5

A market is *without* arbitrage opportunity $\iff$ It admits at least one (equivalent[a]) risk-neutral probability measure $\mathbb{P}^*$

[a]Referring to Definition 2.4; A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F})$ is said to be *equivalent* to another probability measure $\mathbb{P}$ when $\mathbb{P}^*(A) = 0 \iff \mathbb{P}(A) = 0$ for all $A \in \mathcal{F}$

3.3 (2.6) Market Completeness

Definition 2.10: Pricing and hedging in a binary model

A market model is said to be *complete* if every contingent claim $\mathcal{C}$ is attainable.

Definition 2.11: Pricing and hedging in a binary model

A market model without arbitrage opportunities is complete $\iff$ It admits only one (equivalent) risk-neutral probability measure $\mathbb{P}^*$.

4 (3) Discrete-Time Model

4.1 (3.5) Multingales and Conditional Expectation

Definition 3.4

A stochastic process $(M_t)_{t=0,1,...,N}$ is called a discrete-time martingale with respect to the filtration $(\mathcal{F}_t)_{t=0,1,...,N}$ if $(M_t)_{t=0,1,...,N}$ is $(\mathcal{F}_t)_{t=0,1,...,N}$-adapted and satisfies the property

$$\mathbb{E}[M_{t+1}|\mathcal{F}_t] = M_t, \qquad t = 0, 1, ..., N-1.$$

Proposition 3.5

Let $(Z_n)_{n\in\mathbb{N}}$ be martingale, we have

$$\mathbb{E}[Z_n] = \mathbb{E}[Z_0], \qquad n \in \mathbb{N}$$

4.2 (3.6) Market Completeness and Risk-Neutral Measures

Definition 3.8

A probability measure $\mathbb{P}^*$ on Ω is called a rick-neutral measure if

$$\mathbb{E}^*\left[S_{t+1}^{(i)}\middle|\mathcal{F}_t\right] = (1+r)S_t^{(i)}, \qquad i = 1, 2, ..., d$$

Definition 3.9

A probability measure $\mathbb{P}^*$ on Ω is called a rick-neutral measure if

$$\tilde{S}_t^{(i)} := \frac{S_t^{(i)}}{(1+r)^t}, \qquad t = 0, 1, ..., n$$

is a martingale under $\mathbb{P}^*$, *i.e.*

$$\mathbb{E}^*\left[\tilde{S}_{t+1}^{(i)}\middle|\mathcal{F}_t\right] = \tilde{S}_t^{(i)}, \qquad t = 0, 1, ..., N-1$$

$i = 0, 1, ..., d$.

4.3 (3.7) The Cox-Ross-Rubinstein (CRR) Market Model

Consider the portfolio consists of

- Risk-free asset priced as

$$S_t^{(0)} = S_0^{(0)}(1+r)^t, \qquad\qquad t = 0, 1, ..., N$$

- Risky asset priced as

$$S_t^{(1)} = S_0^{(1)} \prod_{k=1}^{t}(1+R_k), \qquad\qquad t = 0, 1, ..., N$$

  where $R_t \in \{a, b\}$, $t = 1, 2, ..., N$

Theorem 3.15

The CRR model is without arbitrage opportunities if and only if $a < r < b$. In this case the market is complete and the (equivalent) risk-neutral probability measure $\mathbb{P}^*$ is given by

$$\mathbb{P}^*(R_{t+1} = b|\mathcal{F}_t) = \frac{r-a}{b-a} \quad \text{and} \quad \mathbb{P}^*(R_{t+1} = a|\mathcal{F}_t) = \frac{b-r}{b-a}$$

$t = 0, 1, ..., N-1$. In particular, $(R_1, R_2, ...R_N)$ forms a sequence of $i.i.d$ random variables under $\mathbb{P}^*$, with

$$p^* := \mathbb{P}^*(R_t = b) = \frac{r-a}{b-a} \quad \text{and} \quad q^* := \mathbb{P}^*(R_t = a) = \frac{b-r}{b-a}$$

5 (4) Pricing and Hedging in Discrete Time

5.1 (4.1) Pricing of Contingent Claims

Lemma 4.2

The following statements are equivalent:

- (i) The portfolio strategy $(\bar{\xi}_t)_{t=1,2,\ldots,N}$ is self-financing

- (ii) $\bar{\xi}_t \cdot \bar{X}_t = \bar{\xi}_{t+1} \cdot \bar{X}_t$ for all $t = 1, 2, ..., N-1$.

- (iii) The discounted portfolio value $\tilde{V}_t$ can be written as the stochastic summation

$$\tilde{V}_t = \tilde{V}_0 + \underbrace{\sum_{k=1}^{t} \bar{\xi}_k \cdot (\bar{X}_k - \bar{X}_{k-1})}_{\text{sum of profits and losses}}, \qquad t = 0, 1, ..., N,$$

of discounted profits and losses.

6 (5.4) Ito Stochastic Integral: Ito Isometry

Proposition 5.9; Ito Isometry

$$Var\left[\int_0^T f(t)dB_t\right] = \mathbb{E}\left[\left(\int_0^T f(t)dB_t\right)^2\right] = \int_0^T |f(t)|^2 dt$$

Application of Ito Isometry

- i.e.1)

$$\mathbb{E}\left[\left(\int_0^T B_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T |B_t|^2 dt\right] = \int_0^T \mathbb{E}\left[|B_t|^2\right] dt = \int_0^T t dt = \frac{T^2}{2}$$

- i.e.2) For all square-integrable adapted processes $(u_t)_{t\in\mathbb{R}_+}$, $(v_t)_{t\in\mathbb{R}_+}$

$$\mathbb{E}\left[\int_0^T u_t dB_t \int_0^T v_t dB_t\right] = \mathbb{E}\left[\int_0^T u_t v_t dt\right]$$