Assignment 1 - MH4514 Financial Mathematics

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Question 1.

Solution. In the presence of a daily dividend, we need to multiply the price by the fraction of non-dividend amount, i.e. $(1-\alpha)$. Therefore we have

$$S_k^{(1)} = \left\{ \begin{array}{ll} (1-\alpha)(1+b)S_{k-1}^{(1)} & \text{if } R_k = b \\ \\ (1-\alpha)(1+a)S_{k-1}^{(1)} & \text{if } R_k = a \end{array} \right\} = (1-\alpha)(1+R_k)S_{k-1}^{(1)}$$

Question 2.

Solution. Dividend amount at time k can be expressed as follows.

$$(\text{Dividend amount}) = \alpha(1+R_k)S_{k-1}^{(1)}$$

Thus, we can express the dividend amount as a percentage of the ex-dividend price $S_k^{(1)}$ as

$$\begin{aligned} \frac{(\text{Dividend amount})}{(\text{Ex-dividend price})} &= \alpha(1+R_k)S_{k-1}^{(1)} \div S_k^{(1)} \\ &= \alpha(1+R_k)S_{k-1}^{(1)} \div (1-\alpha)(1+R_k)S_{k-1}^{(1)} \\ &= \frac{\alpha}{(1-\alpha)} \end{aligned}$$

Next, the return of the risky asset under the risk-neutral probability measure satisfies

$$
\begin{aligned}
\mathbb{E}^*\left[\frac{S_{k+1}^{(1)}}{1-\alpha}\middle|\mathcal{F}_k\right] &= \mathbb{E}^*\left[\frac{(1-\alpha)(1+R_{k+1})S_k^{(1)}}{(1-\alpha)}\middle|\mathcal{F}_k\right] \\
&= \mathbb{E}^*\left[(1+R_{k+1})S_k^{(1)}\middle|\mathcal{F}_k\right] \\
&= S_k^{(1)}\left(1+\mathbb{E}^*\left[R_{k+1}\middle|\mathcal{F}_k\right]\right) \\
&= S_k^{(1)}\left\{1+b\mathbb{P}^*(R_{k+1}=b|\mathcal{F}_k)+a\mathbb{P}^*(R_{k+1}=a|\mathcal{F}_k)\right\} \\
&= S_k^{(1)}\left(1+b\frac{r-a}{b-a}+a\frac{b-r}{b-a}\right) \\
&= S_k^{(1)}\left(1+\frac{1}{b-a}(br-ab+ab-ar)\right) \\
&= S_k^{(1)}\left(1+\frac{1}{b-a}r(b-a)\right) \\
&= (1+r)S_k^{(1)}
\end{aligned}
$$

Question 3.

Solution. From question 1 and 2, at time k, the price of portfolio with dividend can be express as

$$
\begin{aligned}
\xi_k\left\{S_k^{(1)}+(\text{dividend}_k)\right\}+\eta_k S_k^{(0)} &= \xi_k\left\{S_k^{(1)}+\frac{\alpha}{1-\alpha}S_k^{(1)}\right\}+\eta_k S_k^{(0)} \\
&= \xi_k\frac{S_k^{(1)}}{1-\alpha}+\eta_k S_k^{(0)}
\end{aligned}
$$

Consider rebalansing this portfolio for the next time step, we change the portfolio strategy from (ξ_k,η_k) to (ξ_{k+1},η_{k+1}) at time k with the price of portfolio.
In order to condition this as a self-financing strategy, we have to satisfy the following condition.

$$
\xi_k\frac{S_k^{(1)}}{1-\alpha}+\eta_k S_k^{(0)} = \xi_{k+1}S_k^{(1)}+\eta_{k+1}S_k^{(0)}
$$

In addition, V_N can be express as follows

$$
V_N = \xi_N\frac{S_N^{(1)}}{1-\alpha}+\eta_N S_N^{(0)}
$$

Question 4.

Solution. As V_N is discussed in Question 3, $\tilde{V}_k$ can be rewritten as below.

$$\begin{aligned}
\tilde{V}_k &:= \frac{V_k}{S_k^{(0)}} \\
&= \frac{\xi_k S_k^{(1)}/(1-\alpha) + \eta_k S_k^{(0)}}{S_k^{(0)}} = \xi_k \frac{S_k^{(1)}}{(1-\alpha)S_k^{(0)}} + \eta_k \\
&= \frac{\xi_{k+1} S_k^{(1)} + \eta_{k+1} S_k^{(0)}}{S_k^{(0)}} = \frac{\xi_{k+1} S_k^{(1)}}{S_k^{(0)}} + \eta_{k+1}
\end{aligned}$$

Now, under the risk-neutral probability measure $\mathbb{P}^*$, $\tilde{V}_k$ is a martingale which satisfies $\mathbb{E}^*[\tilde{V}_{k+1}|\mathcal{F}_k] = \tilde{V}_k$.
Proof is following, with using the result of Question 2 and predictability of portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\dots N}$.

$$\begin{aligned}
\mathbb{E}^*\left[\tilde{V}_{k+1}|\mathcal{F}_k\right] &= \mathbb{E}^*\left[\xi_{k+1}\frac{S_{k+1}^{(1)}}{(1-\alpha)S_{k+1}^{(0)}} + \eta_{k+1}\middle|\mathcal{F}_k\right] \\
&= \frac{\xi_{k+1}}{S_{k+1}^{(0)}}\mathbb{E}^*\left[\frac{S_{k+1}^{(1)}}{(1-\alpha)}\middle|\mathcal{F}_k\right] + \eta_{k+1} \\
&= \frac{\xi_{k+1}}{S_{k+1}^{(0)}}(1+r)S_k^{(1)} + \eta_{k+1} \\
&= \frac{\xi_{k+1}}{(1+r)S_k^{(0)}}(1+r)S_k^{(1)} + \eta_{k+1} \\
&= \frac{\xi_{k+1}S_k^{(1)}}{S_k^{(0)}} + \eta_{k+1} \\
&= \tilde{V}_k
\end{aligned}$$

Question 5.

Solution. Since the claim C is attainable, there exists a self-financing portfolio strategy $(\xi_k, \eta_k)_{k=1,2,\dots N}$ such that $C = V_N$, or equivalently $\tilde{C} = \tilde{V}_N$.
In addition, from the question 4, the process $(\tilde{V}_t)_{t=1,2,\dots,N}$ is a martingale, hence we have

$$\begin{aligned}
\tilde{V}_k &= \mathbb{E}^*[\tilde{V}_N|\mathcal{F}_k] = \mathbb{E}^*[\tilde{C}|\mathcal{F}_k] \\
\Leftrightarrow V_k &= \frac{1}{(1+r)^{N-k}}\mathbb{E}^*[C|\mathcal{F}_k]
\end{aligned}$$

Question 6.

Solution. From the definition of the pricing function C_0, we know that we need to specify the variables of k and x.
For k, it is easy to see substituting t into k will do the job.
For x, compare the payoff function and treat l as the number of event $R_i = b$ happened from time k to N, we have

$$
\begin{aligned}
h(S_N^{(1)}) &= h\left(x(1+b)^l(1+a)^{N-k-l}\right) \\
&= h\left(x\prod_{i=k+1}^{N}(1+R_i)\right) \\
S_N^{(1)} &= x\prod_{i=k+1}^{N}(1+R_i) \\
x &= \frac{S_N^{(1)}}{\prod_{i=k+1}^{N}(1+R_i)} = S_k^{(1)}
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
V_t &= C_0(t, S_t^{(1)}, N, a, b, r) \\
&= \frac{1}{(1+r)^{N-t}}\sum_{l=0}^{N-t}\binom{N-t}{l}(p^*)^l(q^*)^{N-t-l}h\left(S_t^{(1)}(1+b)^l(1+a)^{N-t-l}\right)
\end{aligned}
$$

Question 7.

Solution. From the question 2, with tower property we have

$$
\begin{aligned}
(1+r)S_k^{(1)} &= \mathbb{E}^*\left[\frac{S_{k+1}^{(1)}}{1-\alpha}\middle|\mathcal{F}_k\right] \\
S_k^{(1)} &= \left(\frac{1}{(1-\alpha)(1+r)}\right)\mathbb{E}^*\left[S_{k+1}^{(1)}\middle|\mathcal{F}_k\right] \\
S_k^{(1)} &= \left(\frac{1}{(1-\alpha)(1+r)}\right)^2\mathbb{E}^*\left[\mathbb{E}^*[S_{k+2}^{(1)}|\mathcal{F}_{k+1}]\middle|\mathcal{F}_k\right] \\
S_k^{(1)} &= \left(\frac{1}{(1-\alpha)(1+r)}\right)^2\mathbb{E}^*\left[S_{k+2}^{(1)}\middle|\mathcal{F}_k\right] \\
&\vdots \\
S_k^{(1)} &= \left(\frac{1}{(1-\alpha)(1+r)}\right)^{N-k}\mathbb{E}^*\left[S_N^{(1)}\middle|\mathcal{F}_k\right]
\end{aligned}
$$

Whereas for the risky asset without dividend, we have

$$S_k^{(1)} = \frac{1}{(1+r)^{N-k}} \mathbb{E}^* \left[S_N^{(1)} \middle| \mathcal{F}_k \right]$$

By comparing the two, we know that we need to adjust $C_0(k, S_k^{(1)}, N, a, b, r)$ by multiplying $(1-\alpha)^{N-k}$, therefore we have

$$C_\alpha(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) = (1-\alpha)^{N-k} C_0(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)$$

Question 8.

Solution. From the question 4, we know that the discounted portfolio price process $(\tilde{V}_k)_{k=0,1,\ldots,N}$ is a martingale, and C_α is a Markov process so that we have

$$\begin{aligned}
\tilde{V}_k &= \mathbb{E}^* \left[\tilde{V}_{k+1} \middle| \mathcal{F}_k \right] \\
\frac{V_k}{S_k^{(0)}} &= \mathbb{E}^* \left[\frac{V_{k+1}}{S_{k+1}^{(0)}} \middle| \mathcal{F}_k \right] \\
(1+r)V_k &= \mathbb{E}^* \left[V_{k+1} \middle| \mathcal{F}_k \right] \\
(1+r)C_\alpha(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) &= \mathbb{E}^* \left[C_\alpha(k+1, S_{k+1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) \middle| \mathcal{F}_k \right] = \mathbb{E}^* \left[\cdot | S_k \right] \\
(1+r)C_\alpha(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) &= q^* C_\alpha(k+1, S_k^{(1)}(1+a), N, a_\alpha, b_\alpha, r_\alpha) \\
&\quad + p^* C_\alpha(k+1, S_k^{(1)}(1+b), N, a_\alpha, b_\alpha, r_\alpha)
\end{aligned}$$

Question 9.

Solution. As discussed in question 4 and 7, we have

$$\begin{aligned}
\tilde{V}_k &= \xi_k \frac{S_k^{(1)}}{(1-\alpha)S_k^{(0)}} + \eta_k \\
&= \frac{(1-\alpha)^{N-k}}{S_k^{(0)}} C_0(k, S_k^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)
\end{aligned}$$

From which we deduce the two equations

$$\begin{cases}
\xi_k(1+a)S_{k-1}^{(1)} + \eta_k = (1-\alpha)^{N-k} C_0(k, (1-\alpha)(1+a)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha) \\
\xi_k(1+b)S_{k-1}^{(1)} + \eta_k = (1-\alpha)^{N-k} C_0(k, (1-\alpha)(1+b)S_{k-1}^{(1)}, N, a_\alpha, b_\alpha, r_\alpha)
\end{cases}$$

Now, think of ξ_k as a function of $S_{k-1}^{(1)}$, and solve previous equations for ξ_k, we have [1]

$$\xi_k\left(S_{k-1}^{(1)}\right)=\frac{(1-\alpha)^{N-k}\{C_0(k,(1-\alpha)(1+b)S_{k-1}^{(1)})-C_0(k,(1-\alpha)(1+a)S_{k-1}^{(1)})\}}{(b-a)S_{k-1}^{(1)}}$$

Question 10.

Solution. Consider the quantity of risky asset $(\xi_k)_{k=1,2,\ldots,N}$ from the Question 9, the dividend rate α works as a shrinking term which decreases ξ_k since $0<(1-\alpha)^{N-k}\leq 1$ for $k=1,2,\ldots,N$.

Thus, in the presence of a daily dividend rate $\alpha>0$, the rate investing on the risky assets with dividend tends to decrease, compared with a risky asset without dividend.
In other words, the dividend is more likely to be reinvested more into the riskless asset than the risky asset.

In addition, as time proceeds (i.e. k increases), the shrinking term gets closer to 1 ($\lim_{k\to N}(1-\alpha)^{N-k}=1$), meaning the rate at which the shrinking term adjusts the dividend reinvested in risky asset in each time steps increases every time.

[1] Here, following variables $N, a_\alpha, b_\alpha, r_\alpha$ of C_0 are omitted for the sake of simplicity