MH4514 Financial Mathematics

- Final Revision -

Naoki Honda

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1 Brownian Motion and Stochastic Calculus

Definition 5.1

The standard Brownian motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ such that

1. $B_0 = 0$ almost surely,
2. The sample trajectories $t \longrightarrow B_t$ are continuous, with probability 1.
3. For any finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments

$$B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}$$

   are mutually independent random variables.
4. For any given times $0 \geq s < t$, $B_t - B_s$ has the Gaussian distribution $\mathcal{N}(0, t-s)$.

Lemma 5.3

The stochastic integral $\int_0^T f(t)dB_t$ has the centered Gaussian distribution

$$\int_0^T f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^T |f(t)|^2 dt\right)$$

Stochastic Calculus

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2$$

Proposition 5.12

Given the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

The solution is given by

$$S_t = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}$$

2 The Black-Scholes PDE

Proposition 6.1

A portfolio allocation $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ with price

$$V_t = \eta_t A_t + \xi_t S_t$$

is self-financing if and only if the relation

$$dV_t = \eta_t dA_t + \xi_t dS_t$$

holds.

Definition 6.4

A probability measure $\mathbb{P}^*$ on Ω is called a risk-neutral measure if it satisfies

$$\mathbb{E}^*[S_t|\mathcal{F}_u] = e^{(t-u)r}S_u, \qquad 0 \le u \le t,$$

where $\mathbb{E}^*$ denotes the expectation under $\mathbb{P}^*$.

Proposition 6.13

The arbitrage price $\pi_t(C)$ at time $t \in [0, T]$ of the option with payoff $C = h(S_T)$ is given by $\pi_t(C) = g(t, S_t)$, where the function $g(t, x)$ is solution of the following Black Scholes PDE:

$$\begin{cases} rg(t,x) = \dfrac{\partial g}{\partial t}(t,x) + rx\dfrac{\partial g}{\partial x}(t,x) + \dfrac{1}{2}\sigma^2 x^2 \dfrac{\partial^2 g}{\partial x^2}(t,x), \\ g(T,x) = h(x), \qquad x > 0. \end{cases}$$

Proposition 6.17

When $h(x) = (x - K)^+$, the solution of the Black-Scholes PDE is given by

$$f(t,x) = x\Phi(d_+(T-t)) - Ke^{-(T-t)r}\Phi(d_-(T-t)),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \qquad x \in \mathbb{R},$$

and

$$d_\pm(T-t) := \frac{\log(x/K) + (r \pm \sigma^2/2)(T-t)}{|\sigma|\sqrt{T-t}}$$

3 Martingale Approach to Pricing and Hedging

3.1 Prep

Proposition 7.1

The indefinite stochastic integral $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ of a square-integrable adapted process $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ is a martingale, *i.e.*:

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \middle| \mathcal{F}_s\right] = \int_0^s u_\tau dB_\tau, \qquad 0 \leq s \leq t$$

Theorem 7.2; Girsanov Theorem

Let $(\phi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 dt\right)\right] < \infty,$$

and let $\mathbb{Q}$ denote the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T |\psi_s|^2 ds\right)$$

Then

$$\hat{B}_t := B_t + \int_0^t \psi_s ds, \qquad 0 \leq t \leq T$$

3.2 Pricing by the Martingale Method

Recall that from the first fundamental theorem of mathematical finance, a continuous market is without arbitrate opportunities if there exist (at least) a risk-neutral probability measure $\mathbb{P}^*$ under which the discounted price process

$$\tilde{S}_t := e^{-rt} S_t, \qquad t \in \mathbb{R}_+$$

is a martingale under $\mathbb{P}^*$. In addition, when the risk-neutral probability measure is unique, the market is said to be *complete*.
In case the price process $(S_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \qquad t \in \mathbb{R}_+, \qquad S_0 > 0$$

we have

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}, \qquad t \in \mathbb{R}_+$$

and the discounted price process

$$\tilde{S}_t := e^{-rt} S_t = S_0 e^{(\mu - r - \sigma^2/2)t + \sigma B_t}, \qquad t \in \mathbb{R}_+$$

is a martingale under the probability measure $\mathbb{P}^*$ defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T \right)$$

which makes $\hat{B}_t := \frac{\mu - r}{\sigma} t + B_t$ a standard Brownian motion.
Moreover, we have

$$\begin{aligned}
d\tilde{S}_t &= (\mu - r)\tilde{S}_t dt + \sigma \tilde{S}_t dB_t \\
&= \sigma \tilde{S}_t \left(\frac{\mu - r}{\sigma} dt + dB_t \right) \\
&= \sigma \tilde{S}_t d\hat{B}_t, \qquad t \in \mathbb{R}_+
\end{aligned}$$

hence the discounted value $\tilde{V}_t$ of a self-financing portfolio can be written as

$$\begin{aligned}
\tilde{V}_t &= \tilde{V}_0 + \int_0^t \xi_u d\tilde{S}_u \\
&= \tilde{V}_0 + \sigma \int_0^t \xi_u \tilde{S}_u d\hat{B}_u
\end{aligned}$$

and by Proposition 7.1 it becomes a martingale under $\mathbb{P}^*$

Theorem 7.3

Let $(\xi_t, \eta_t)_{t\in[0,T]}$ be a portfolio strategy with price

$$V_t = \eta A_t + \xi_t S_t, \qquad t \in [0,T]$$

and let C be a contingent claim, such that

1. $(\xi_t, \eta_t)_{t\in[0,T]}$ is a self-financing portfolio
2. $(\xi_t, \eta_t)_{t\in[0,T]}$ hedges the claim C, *i.e.* we have $V_T = C$

Then the arbitrage price of the claim C is given by

$$\pi_t(C) = V_t = e^{-(T-t)r}\mathbb{E}^*[C|\mathcal{F}_t], \qquad 0 \le t \le T$$

where $\mathbb{E}^*$ denotes expectation under the risk-neutral probability measure $\mathbb{P}^*$.

Proof:

Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is self-financing, we have

$$\tilde{V}_t = \tilde{V}_0 + \sigma\int_0^t \xi_u \tilde{S}_u d\hat{B}_u = \tilde{V}_0 + \int_0^t \xi_u d\hat{S}_u, \qquad t \in \mathbb{R}_+$$

which is a martingale under $\mathbb{P}^*$ from Proposition 7.1, hence

$$\begin{aligned}\tilde{V}_t &= \mathbb{E}^*[\tilde{V}_T|\mathcal{F}_t]\\ &= e^{-rT}\mathbb{E}^*[V_T|\mathcal{F}_t]\\ &= e^{-rT}\mathbb{E}^*[C|\mathcal{F}_t]\end{aligned}$$

which implies

$$V_t = e^{rt}\tilde{V}_t = r^{-(T-t)r}\mathbb{E}^*[C|\mathcal{F}_t], \qquad 0 \le t \le T$$

Proposition 7.4

The price at time t of a European call option with strike price K and maturity T is given by

$$C(t, S_t) = S_t\Phi\big(d_+(T-t)\big) - Ke^{-(T-t)r}\Phi\big(d_-(T-t)\big), \qquad 0 \le t \le T$$

with

$$d_\pm(T-t) := \frac{\log(x/K) + (r \pm \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

Proof:

Using the relation

$$S_T = S_t e^{(r-\sigma^2/2)(T-t)+\sigma(\hat{B}_T-\hat{B}_t)}, \qquad 0 \le t \le T$$

by Theorem 7.3 the price of the portfolio hedging C is given by

$$\begin{aligned} V_t &= e^{-(T-t)r}\mathbb{E}^*[C|\mathcal{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}^*[(S_T-K)^+|\mathcal{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}^*[(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(\hat{B}_T-\hat{B}_t)}-K)^+|\mathcal{F}_t] \\ &= e^{-(T-t)r}\mathbb{E}^*[(x e^{(r-\sigma^2/2)(T-t)+\sigma(\hat{B}_T-\hat{B}_t)}-K)^+]_{x=S_t} \\ &= e^{-(T-t)r}\mathbb{E}^*[(e^{m(x)+X}-K)^+]_{x=S_t} \end{aligned}$$

where

$$m(x) := (T-t)r - \frac{\sigma^2}{2}(T-t) + \log x$$

and

$$X := \sigma(\hat{B}_T-\hat{B}_t) \simeq \mathcal{N}(0,\sigma^2(T-t))$$

is a centered Gaussian random variable with variance

$$\mathrm{Var}[X] = \mathrm{Var}[\sigma(\hat{B}_T-\hat{B}_t)] = \sigma^2\mathrm{Var}[\hat{B}_T-\hat{B}_t] = \sigma^2(T-t)$$

under $\mathbb{P}^*$. Hence we have

$$\begin{aligned} C(t,S_t) &= V_t \\ &= e^{-(T-t)r}\mathbb{E}^*[(e^{m(x)+X}-K)^+]_{x=S_t} \\ &= e^{-(T-t)r}e^{m(S_t)+\sigma^2(T-t)/2}\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) \\ &\quad - Ke^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\ &= S_t\Phi\left(v+\frac{m(S_t)-\log K}{v}\right) - Ke^{-(T-t)r}\Phi\left(\frac{m(S_t)-\log K}{v}\right) \\ &= S_t\Phi\big(d_+(T-t)\big) - Ke^{-(T-t)r}\Phi\big(d_-(T-t)\big), \qquad 0 \le t \le T \end{aligned}$$