# Assignment 3 - MH4514 Financial Mathematics

Name: Honda Naoki
Matriculation: N1804369J

March 29, 2019

**Question 1.**

**Solution.** Using the properties of covariance and Ito multiplication table, we can calculate the fraction as

$$\begin{aligned}\frac{Cov\left(\frac{dS_t}{S_t}, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]} &= \frac{Cov\left(rdt + \beta\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]} \\ &= \frac{Cov\left(\beta\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\left[\frac{dM_t}{M_t}\right]} = \beta\end{aligned}$$

**Question 2.**

**Solution.** By (1) and (2), observe that the evolution of $(S_t)_{t\in\mathbb{R}_+}$ is given by

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \beta\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t \\ &= (r + \beta(\mu - r))dt + \beta\sigma_M dB_t + \sigma_S dW_t\end{aligned}$$

Now, consider its volatility, by squaring both side we have

$$\begin{aligned}\left(\frac{dS_t}{S_t}\right)^2 &= ((r + \beta(\mu - r))dt + \beta\sigma_M dB_t + \sigma_S dW_t)^2 \\ &= \left(\beta^2\sigma_M^2 + \sigma_S^2\right) dt\end{aligned}$$

Thus, we can think of $(S_t)_{t\in\mathbb{R}_+}$ as a geometric Brownian motion with volatility $\sqrt{\beta^2\sigma_M^2 + \sigma_S^2}$. Finally with the excess return $\alpha$, we can express its evolution as

$$\begin{aligned}\frac{dS_t}{S_t} &= (r + \alpha + \beta(\mu - r))dt + \sqrt{\beta^2\sigma_M^2 + \sigma_S^2}dW_t \\ dS_t &= (r + \alpha + \beta(\mu - r))S_t dt + \sqrt{\beta^2\sigma_M^2 + \sigma_S^2}S_t dW_t\end{aligned}$$

**Question 3.**

**Solution.** Consider the equations (1), we have

$$\begin{aligned}\frac{dM_t}{M_t} &= \mu dt + \sigma_M dB_t \\ &= rdt + \sigma_M\left(dB_t + \frac{\mu - r}{\sigma_M}dt\right) \\ &= rdt + \sigma_M dB_t^* \qquad\qquad \left(\text{where} \quad dB_t^* = \left(dB_t + \frac{\mu - r}{\sigma_M}dt\right)\right)\end{aligned}$$

Next, we can rewrite (2) as

$$\begin{aligned}\frac{dS_t}{S_t} &= rdt + \beta(M_t)\left(\frac{dM_t}{M_t} - rdt\right) + \sigma_S dW_t \\ &= rdt + \beta(M_t)(rdt + \sigma_M dB_t^* - rdt) + \sigma_S dW_t \\ &= rdt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^* \qquad\qquad (\text{where} \quad dW_t^* = dW_t)\end{aligned}$$

Hence, by setting $B_t^* = \left(B_t + \frac{\mu - r}{\sigma_M}t\right)$ and $W_t^* = W_t$ we have

$$\begin{cases} \dfrac{dM_t}{M_t} & = rdt + \sigma_M dB_t^* \\ \dfrac{dS_t}{S_t} & = rdt + \sigma_M \beta(M_t) dB_t^* + \sigma_S dW_t^* \end{cases}$$

**Question 4.**

**Solution.** Consider the new probability measure $\mathbb{P}^*$ defined on $\mathcal{F}_T$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left\{ -\frac{\mu - r}{\sigma_M}B_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma_M}\right)^2 T\right\}$$

Setting

$$B_t^* = B_t + \frac{\mu - r}{\sigma_M}t, \qquad W_t^* = W_t$$

then under $\mathbb{P}^*$, the processes $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ are two independent standard Brownian motions.

**Question 5.**

**Solution.** From Question 4, we know under $\mathbb{P}^*$, the processes $(B_t^*)_{t\in\mathbb{R}_+}$ and $(W_t^*)_{t\in\mathbb{R}_+}$ are two independent standard Brownian motions. Using this, the discounted asset price processes $(\tilde{S}_t, \tilde{M}_t)_{t\in\mathbb{R}_+}$ can be shown as martingales.

$$
\begin{aligned}
\mathbb{E}^*\left[\tilde{M}_t \middle| \mathcal{F}_k\right] &= e^{-rt}\mathbb{E}^*\left[\int_0^t rM_u du + \sigma_M \int_0^t M_u dB_u^* \middle| \mathcal{F}_k\right] \\
&= e^{-rt}\mathbb{E}^*\left[\int_0^t rM_u du \middle| \mathcal{F}_k\right] = e^{-rt}e^{(t-k)r}M_k \\
&= \tilde{M}_k
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}^*\left[\tilde{S}_t \middle| \mathcal{F}_k\right] &= e^{-rt}\mathbb{E}^*\left[\int_0^t rS_u du + \sigma_M \int_0^t \beta(M_u)S_u dB_u^* + \sigma_S \int_0^t S_u dW_u^* \middle| \mathcal{F}_k\right] \\
&= e^{-rt}\mathbb{E}^*\left[\int_0^t rS_u du \middle| \mathcal{F}_k\right] = e^{-rt}e^{(t-k)r}S_k \\
&= \tilde{S}_k
\end{aligned}
$$

From Proposition 6.6, the probability measure $\mathbb{P}^*$ is risk-neutral since the discounted risky asset price processes $(\tilde{S}_t, \tilde{M}_t)_{t\in\mathbb{R}_+}$ are multingales under $\mathbb{P}^*$.

Hence, by Theorem 6.8, the market based on the assets $S_t$ and $M_t$ is without arbitrage opportunities since it admits at least one equivalent risk-neutral probability measure $\mathbb{P}^*$.

**Question 6.**

**Solution.** By Ito's calculus we have

$$
\begin{aligned}
dV_t = \xi_t dS_t &+ S_t d\xi_t + d\xi_t dS_t \\
&+ \zeta_t dM_t + M_t d\zeta_t + d\zeta_t dM_t \\
&\quad + \eta_t dA_t + A_t d\eta_t + d\eta_t dA_t
\end{aligned}
$$

Here, since we know $S_t d\xi_t + M_t d\zeta_t + A_t d\eta_t = 0$ being true under the self-financing condition, and $d\xi_t dS_t = d\zeta_t dM_t = d\eta_t dA_t = 0$, we can rewrite $dV_t$ as

$$
dV_t = \xi_t dS_t + \zeta_t dM_t + \eta_t dA_t
$$

**Question 7.**

**Solution.** From Question 2 and the same derivation of Question 3 we have

$$\begin{cases} dM_t &= rM_t dt + \sigma_M M_t dB_t^* \\ \\ dS_t &= rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^* \end{cases}$$

Using the above system and Ito's formula[1], we have[2]

$$\begin{aligned} df(t, S_t, M_t) = &\left\{ \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) S_t^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2} + \beta(M_t)\sigma_M^2 M_t S_t \frac{\partial^2 f}{\partial x \partial y} \right\} dt \\ &+ \frac{\partial f}{\partial x} dS_t + \frac{\partial f}{\partial y} dM_t \end{aligned}$$

Further expand $dS_t$ and $dM_t$ we have

$$\begin{aligned} df(t, S_t, M_t) = &\left\{ \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) S_t^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 M_t^2 \frac{\partial^2 f}{\partial y^2} \right. \\ &\left. + \beta(M_t)\sigma_M^2 M_t S_t \frac{\partial^2 f}{\partial x \partial y} + r\left( S_t \frac{\partial f}{\partial x} + M_t \frac{\partial f}{\partial y} \right) \right\} dt \\ &+ \left( \sigma_M \beta(M_t) S_t \frac{\partial f}{\partial x} + \sigma_M M_t \frac{\partial f}{\partial y} \right) dB_t^* + \sigma_S S_t \frac{\partial f}{\partial x} dW_t^* \end{aligned} \tag{7.1}$$

Using the self-financing condition of Question 6, we also can express $df(t, S_t, M_t)$ as follows

$$\begin{aligned} df(t, S_t, M_t) &= dV_t = \xi_t dS_t + \zeta_t dM_t + \eta_t dA_t \\ &= \xi_t(rS_t dt + \sigma_M \beta(M_t) S_t dB_t^* + \sigma_S S_t dW_t^*) + \zeta_t(rM_t dt + \sigma_M M_t dB_t^*) + r\eta_t A_t dt \\ &= rf(t, S_t, M_t)dt + (\sigma_M \beta(M_t)\xi_t S_t + \zeta_t \sigma_M M_t) dB_t^* + \sigma_S \xi_t S_t dW_t^* \end{aligned} \tag{7.2}$$

By identification of the term in $dt$ in (7.1) and (7.2) we get the PDE with the terminal condition:

$$\begin{cases} rf(t, x, y) &= \frac{\partial f}{\partial t} + \frac{1}{2}\left(\sigma_M^2 \beta^2(M_t) + \sigma_S^2\right) x^2 \frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_M^2 y^2 \frac{\partial^2 f}{\partial y^2} \\ &\quad + \beta(M_t)\sigma_M^2 xy \frac{\partial^2 f}{\partial x \partial y} + r\left( x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} \right) \\ \\ f(T, x, y) &= h(x, y) \end{cases}$$

[1] In the text book, P191, (5.5.7)

[2] Here we omit the argument bracket of function $f(t, S_t, M_t)$ for the sake of simplicity

**Question 8.**

**Solution.** Similar to the way we derived the PDE, by identifying the coefficient on $dB_t^*$ and $dB_t^*$ in (7.1)-(7.2) we have system of equation

$$\begin{cases} \sigma_M \beta(M_t) S_t \dfrac{\partial f}{\partial x} + \sigma_M M_t \dfrac{\partial f}{\partial y} = \sigma_M \beta(M_t) \xi_t S_t + \zeta_t \sigma_M M_t \\ \\ \sigma_S S_t \dfrac{\partial f}{\partial x} = \sigma_S \xi_t S_t \end{cases}$$

which gives a solution

$$\xi_t = \frac{\partial f}{\partial x}(t,x,y), \qquad \zeta_t = \frac{\partial f}{\partial y}(t,x,y).$$

For $\eta_t$, from previous result and the definition of the portfolio price $V_t$ we have

$$\begin{aligned} \eta_t A_t &= V_t - \xi_t S_t - \zeta_t M_t \\ &= f(t,x,y) - S_t \frac{\partial f}{\partial x}(t,x,y) - M_t \frac{\partial f}{\partial y}(t,x,y) \\ \eta_t &= \frac{1}{A_0 e^{rt}} \left\{ f(t,x,y) - S_t \frac{\partial f}{\partial x}(t,x,y) - M_t \frac{\partial f}{\partial y}(t,x,y) \right\} \end{aligned}$$

**Question 9.**

**Solution.** From Question 7, with constant $\beta$ and terminal condition of a call option payoff, we have PDE of

$$\begin{cases} rf(t,x,y) &= \dfrac{\partial f}{\partial t} + \dfrac{1}{2}\left(\sigma_M^2 \beta^2 + \sigma_S^2\right) x^2 \dfrac{\partial^2 f}{\partial x^2} + \dfrac{1}{2}\sigma_M^2 y^2 \dfrac{\partial^2 f}{\partial y^2} \\ & \quad + \beta \sigma_M^2 xy \dfrac{\partial^2 f}{\partial x \partial y} + r\left(x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y}\right) \\ \\ f(T,x,y) &= (x-K)^+ \end{cases}$$

Noticing by the terminal condition, we think of $f(t,x,y)$ as a function of $t, x$, which has a solution of

$$f(t,x;K) = \begin{cases} x\Phi(d_+(T-t,x;K)) - e^{-r(T-t)} K \Phi(d_-(T-t,x;K)) & 0 \le t < T \\ (x-K)^+ & t = T \end{cases}$$

where

$$\begin{aligned} d_\pm(s,x;K) &:= \frac{1}{\sqrt{s(\sigma_M^2\beta^2 + \sigma_S^2)}} \left[ log\left(\frac{x}{K}\right) + s\left(r \pm \frac{\sigma_M^2 \beta^2 + \sigma_S^2}{2}\right)\right] \\ \Phi(z) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2} du \end{aligned}$$

The hedging portfolio is

$$\xi_t = \frac{\partial f}{\partial x}(t, x; K) = \Phi(d_+(T-t, x; K)), \qquad \zeta_t = \frac{\partial f}{\partial y}(t, x; K) = 0$$

$$\begin{aligned}\eta_t &= \frac{1}{A_0 e^{rt}}\big(f(t, x; K) - S_t\xi_t - M_t\zeta_t\big) \\ &= -\frac{K}{A_0 e^{rT}}\Phi(d_-(T-t, x; K))\end{aligned}$$

Thus, the replicating portfolio consists of holding a quantity $\xi_t = \Phi(d_+(T-t, x; K)) \geq 0$ of risky asset $S_t$, and borrowing a quantity $-\eta_t = \frac{K}{A_0 e^{rT}}\Phi(d_-(T-t, x; K)) \geq 0$ of the riskless asset $A_t$.

**Question 10.**

**Solution.** The payoff function of put option can be expressed as

$$h(S_T) = (K - S_T)^+ = -(S_T - K) + (S_T - K)^+$$

Hence, with the result of Question 9, we can compute the pricing function at time $t$ $(\leq T)$ as

$$\begin{aligned}f(t, x; K) &= -(x - e^{-r(T-t)}K) + x\Phi(d_+(T-t, x; K)) - e^{-r(T-t)}K\Phi(d_-(T-t, x; K)) \\ &= e^{-r(T-t)}K\big(1 - \Phi(d_-(T-t, x; K))\big) - x\big(1 - \Phi(d_+(T-t, x; K))\big) \\ &= e^{-r(T-t)}K\Phi(-d_-(T-t, x; K)) - x\Phi(-d_+(T-t, x; K))\end{aligned}$$

(Here we used the relation $1 - \Phi(a) = \Phi(-a)$)

The hedging portfolio is

$$\xi_t = \frac{\partial f}{\partial x}(t, x; K) = -\Phi(-d_+(T-t, x; K)), \qquad \zeta_t = \frac{\partial f}{\partial y}(t, x; K) = 0$$

$$\begin{aligned}\eta_t &= \frac{1}{A_0 e^{rt}}\big(f(t, x; K) - S_t\xi_t - M_t\zeta_t\big) \\ &= \frac{K}{A_0 e^{rT}}\Phi(-d_-(T-t, x; K))\end{aligned}$$

Thus, the replicating portfolio consists of short selling a quantity $-\xi_t = \Phi(-d_+(T-t, x; K)) \geq 0$ of risky asset $S_t$, and investing (saving) a quantity $\eta_t = \frac{K}{A_0 e^{rT}}\Phi(-d_-(T-t, x; K)) \geq 0$ of the riskless asset $A_t$.