MAE5032 Final Project

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Problem

We consider the transient heat equation in a one-dimensional (1D) domain $\Omega := (0, 1)$. The boundary of the domain is $\Gamma = \{0, 1\}$. Let f be the heat supply per unit volume, u be the temperature, ρ be the density, c be the heat capacity, u_0 be the initial temperature, κ be the conductivity, n_x be the Cartesian components of the unit outward normal vector . The boundary data involves the prescribed temperature g on Γ_g and heat flux h on Γ_h. The boundary Γ admits a non-overlapping decomposition: $\Gamma = \overline{\Gamma_g \cup \Gamma_h}$ and $\emptyset = \Gamma_g \cap \Gamma_h$. The transient heat equation may be stated as follows.

$$
\begin{aligned}
\rho c \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} &= f \qquad \text{on } \Omega \times (0, T) \\
u &= g \qquad \text{on } \Gamma_g \times (0, T) \\
\kappa \frac{\partial u}{\partial x} n_x &= h \qquad \text{on } \Gamma_h \times (0, T) \\
u|_{t=0} &= u_0 \qquad \text{in } \Omega.
\end{aligned}
$$

Theoretical analysis

Explicit Euler method

$$
\frac{\partial u}{\partial t} = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t^2) \tag{1}
$$

$$
\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^3) \tag{2}
$$

With FDM, the transient heat equation in 1D can be dispersed into the following:

$$
u^{n+1} = u^n + \frac{\kappa}{\rho c \Delta x^2}(u^n_{i+1} - 2u^n_i + u^n_{i-1}) + \frac{f}{\rho c} \tag{3}
$$

Implicit Euler method

From eq. (1) and eq. (2), we can deduce the following:

$$\rho c \frac{u_i^{n+1} - u_i^n}{\Delta t} - \kappa \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} = f \tag{4}$$

$$-\frac{\kappa}{\rho c \Delta x^2} u_{i+1}^{n+1} + (1 + 2\frac{2\kappa}{\rho c \Delta x^2}) u_i^{n+1} - \frac{\kappa}{\rho c \Delta x^2} u_{i-1}^{n+1} = u_i^n + \frac{f}{\rho c} \tag{5}$$

So we can get a tri-diagonal system $AX = b$:

$$\begin{pmatrix} b & a & 0 & ... & a \\ a & b & a & & \\ & & ... & & \\ & & a & b & a \\ a & 0 & ... & a & b \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ . \\ . \\ u_m^{n+1} \end{pmatrix} = \begin{pmatrix} u_1^n + \frac{f}{\rho c} \\ u_2^n + \frac{f}{\rho c} \\ . \\ . \\ u_m^n + \frac{f}{\rho c} \end{pmatrix}$$

$$a = -\frac{\kappa}{\rho c \Delta x^2}, b = 1 + 2\frac{2\kappa}{\rho c \Delta x^2}$$