Appendices

Day-ahead Trading of Green Hydrogen Guided by Contest Game and Deep Learning

APPENDIX A

PROOF OF THEOREM 2

To facilitate the use of the Lefschetz fixed-point theorem in proving the uniqueness of the Nash equilibrium in the Tullock contest among heterogeneous HRSs, the contest problem is recast within a differential-topology framework. Specifically, the bidding strategy space of the HRSs is transformed into a compact orientable manifold in differential topology.

The compact orientable manifold for the bidding strategy is defined as follows.

$$\Gamma = \prod_{i=1}^{n} C^1\left(\left[c_{\min}, c_{\max}\right], \left[b_{\min}, b_{\max}\right]\right) \cap \left\{\left\|b_i\right\|_{C^1} \le \bar{K}\right\} \tag{A1}$$

where $\bar{K}$ denotes a Lipschitz constant ensuring that the manifold is compact (by the Arzelà-Ascoli theorem), and C^1 denotes a continuously differentiable function of the first order.

The following analyzes the differential structure of the HRS's best response mapping. Let the HRS's optimal bidding strategy $s:\Gamma \to \Gamma$ be defined as follows.

$$s_i\left(\boldsymbol{s}_{-i}, c_i\right) = \arg\max_{b_i} \mathbb{E}_{-c_i}\left[\sum_{k=1}^{m}\left(r - b_i - c_i - \alpha_i\right)\cdot P_{i,k}\left(b_i\right)\cdot Q_k\right] \tag{A2}$$

$$\begin{cases} P_{i,k} = \left(\prod_{g=1}^{k-1}\left(1 - \dfrac{\theta_i b_i}{\theta_i b_i + \sum_{j \in N_g \setminus \{i\}} \theta_j b_j}\right)\right)\bar{p}_{i,k} \\ \bar{p}_{i,k} = \dfrac{\theta_i b_i}{\theta_i b_i + \sum_{j \in N_k \setminus \{i\}} \theta_j b_j} \end{cases} \tag{A3}$$

where the expectation $\mathbb{E}_{-c_i}\left[\cdot\right]$ denotes as an integral over the cost types of the other HRSs.

Let

$$\pi_i\left(s_i, \boldsymbol{s}_{-i}, c_i\right) = \sum_{k=1}^{m}\left(r - b_i - c_i - \alpha_i\right)\cdot P_{i,k}\left(b_i\right)\cdot Q_k \tag{A4}$$

Now, we verify the transversality condition: the best response mapping must have no eigenvalue equal to 1. For HRS *i*, the optimal bidding strategy satisfies

$$\sum_{k=1}^{m}\left[\mathbb{E}\left[\prod_{g=1}^{k-1}\left(1 - P_{i,g}\right)\cdot \bar{p}_{i,k}\right] + \left(r - s_i\left(c_i\right) - c_i - \alpha_i\right)\cdot \mathbb{E}\left[\prod_{g=1}^{k-1}\left(1 - P_{i,g}\right)\cdot \frac{\partial \bar{p}_{i,k}}{\partial s_i\left(c_i\right)}\right]\right]\cdot Q_k = 0 \tag{A5}$$

Let the differential of s, denoted by $D(s)$, be a Fréchet derivative; at the equilibrium point s^*, we obtain that

$$\left[D(s)\right]_{ij} = \frac{\sum_{k=1}^{m} Q_k\left(r - b_i - c_i - \alpha_i\right)\cdot\left(\dfrac{\theta_i\theta_j\left(\theta_i b_i + \sum_{l \in N_k}\theta_l b_l - 2\theta_j b_j\right)}{\left(\theta_i b_i + \sum_{l \in N_k}\theta_l b_l\right)^3}\right)}{\sum_{k=1}^{m} Q_k \cdot \dfrac{\theta_i \sum_{j \in N_k \setminus \{i\}}\theta_j b_j}{\left(\theta_i b_i + \sum_{j \in N_k}\theta_j b_j\right)}\left[2 + \dfrac{2\theta_i\left(r - b_i - c_i - \alpha_i\right)}{\theta_i b_i + \sum_{j \in N_k}\theta_j b_j}\right]} \tag{A6}$$

To ensure $I - D(s)$ has full rank, it is required that $\left\|D(s)\right\|_{\infty} < 1$, where *I* denotes the identity matrix.

Taking the worst-case scenario for $\left[D(s)\right]_{ij}$, the transversality condition holds when Eqs. (A7)-(A8) are satisfied.

$$\frac{\max_i \theta_i}{\min_j \theta_j} < 2 \tag{A7}$$

$$\min_i\left(r - \alpha_i\right) > \frac{2nb_{\max}}{\min_k Q_k} \tag{A8}$$

To verify hyperbolicity, the spectral radius estimate of the bidding strategy is defined as follows.

$$\rho\left(D(s)\right) \le \max_i \sum_{i \ne j}\left|\left[D(s)\right]_{ij}\right| \tag{A9}$$

Using it as an upper bound of the infinity norm to control the spectral radius, we therefore obtain that

$$\left|\left[D(s)\right]_{ij}\right| \le \frac{\sum_{k=1}^{m} Q_k \cdot \theta_j \left(r - b_i - c_i - f_i\right)}{\sum_{k=1}^{m} Q_k \cdot \sum_{j \in N_k \setminus \{i\}} \theta_j b_j} \tag{A10}$$

In the worst-case scenario, taking $\theta_i \le \max_i \theta_i$, $\theta_j \ge \min_j \theta_j$, $Q_k \le \max_k Q_k$, $Q_k \ge \min_k Q_k$, $r - b_i - c_i - \alpha_i \le r - \alpha_i$ and combining with $\sum_{j \in N_k \setminus \{i\}} b_j \ge (n-1) b_{\min}$, we can derive that

$$\rho\left(D(s)\right) < \frac{\max_k Q_k}{\min_k Q_k} \cdot \frac{\max_i \theta_i}{\min_j \theta_j} \cdot \frac{\max_i \left(r - \alpha_i\right)}{(n-1) b_{\min}} \tag{A11}$$

Since the bidding strategy $b_j^* \in C^1$, it follows that

$$\left|\frac{\partial b_j^*}{\partial c_j}\right| < \frac{\max_i \left(r - \alpha_i\right)}{b_{\min}} \tag{A12}$$

Because $\max \theta_i / \min \theta_j < 2$ holds, the inequality $\rho\left(D(s)\right) < 1$ is always satisfied when condition (A13) is met.

$$\sup_{c_i} \left|\frac{\partial b_j^*}{\partial c_j}\right| < \frac{\min_{i,k} \theta_i}{(n-1) \max_{i,k} \theta_i \left(r - \alpha_i\right)} \cdot \frac{\min_k Q_k}{\max_k Q_k} \tag{A13}$$

Since the spectral radius of the optimal bidding strategy is strictly less than 1, $D(s)$ has no eigenvalues of unit modulus; hence, the hyperbolicity condition is satisfied.

We now verify $\left|L(s)\right| = 1$. In the topology of the manifold, the bidding strategy space Γ is convex and contractible; hence, Γ is homotopy equivalent to a single point: $\Gamma \simeq \{pt\}$. By homotopy invariance, the homology groups of Γ are as follows.

$$H_\kappa(\Gamma) = \begin{cases} \mathbb{Z} & \kappa = 0 \\ 0 & \kappa > 0 \end{cases} \tag{A14}$$

According to the Lefschetz fixed-point theorem, the Lefschetz number is

$$L(s) = \sum_{\kappa=0}^{\infty} (-1)^\kappa \, tr\left(H_\kappa(s)\right) = tr\left(H_0(s)\right) \tag{A15}$$

where $H_\kappa(s)$ represents the induced κ-th homology mapping, and $tr(\cdot)$ denotes the trace. Since $H_0(\Gamma) \simeq \mathbb{Z}$, it has $tr\left(H_0(s)\right) = 1$; for $\kappa > 0$, we have $H_k(\Gamma) = 0$, hence $tr\left(H_0(s)\right) = 0$. Therefore, we can obtain that

$$L(s) = (-1)^0 \cdot 1 + \sum_{\kappa=0}^{\infty} (-1)^\kappa \cdot 1 = 1 \tag{A16}$$

Thus, $\left|L(s)\right| = 1$ always holds, and there exists a unique equilibrium bidding strategy. The proof is complete. ■