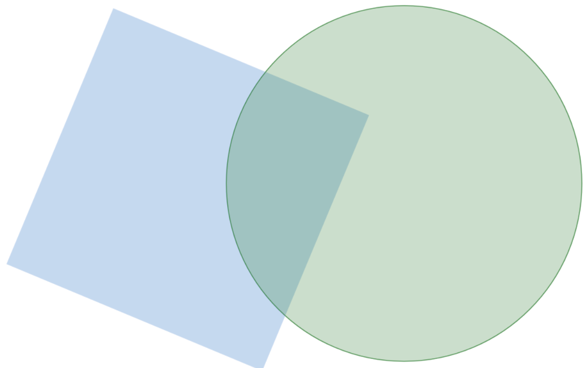
Figure 1: What is the area of the overlapping region?



0.1 Geometry: Determinants, Eigenvalues, and Area []

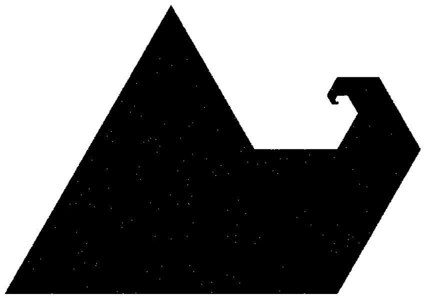
Intuitively, we might think of area as the amount of paint needed in painting a particular shape. The more paint needed, the larger its area, and the larger its area, the more paint needed. To have some sense of what is meant by the area of an object, this intuition is good enough. Larger shapes have larger area while smaller shapes have smaller area, and the area of a shape is some measure of this size.

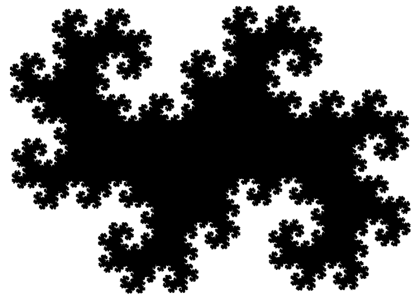
Calculating the areas of shapes (assigning numbers to areas) is another story. We certainly are not going to require that to find the area of an object it needs to be painted and the amount of paint used measured. What paint should be used, by whom, and what instrument should do the measuring? This process would be so imprecise it would lead to a single shape having many areas (depending on the paint, the painter, and the measuring device), a rather undesirable situation. The area of a shape should be uniquely determined. A single shape has but a single size, and so it must have but a single measure of its size—like the area formulas presented in grammar school.

The area of a rectangle is its length times width. The area of a triangle is one half its base times height. The area of a circle is π times the square of its radius. Trapezoids, parallelograms, regular polygons, and unions of such shapes have calculable areas. But what about more complex shapes? For example, take an arbitrary nonempty overlap between a square and circle where neither is the circle contained within the square nor is the square contained within the circle. See figure 1. Calculus provides a method for calculating its area and hints at the complexity of the general question. By slicing the shape into smaller and smaller approximating rectangles and adding up the areas of those rectangles, the area can be approximated more and more accurately. The limit of these areas as the widths of the approximating rectangles approach zero is the area of the overlap. If you've taken calculus, that probably reminds you of integration, and it should! If you have not taken calculus, that probably sounds rather confusing and complicated, and it should! That is really the point. It is not an easy matter to calculate area, even of shapes that are easy to draw.

To stretch the point just a bit further, consider the shapes in figure 2. The figure on the left is the snail of Solomon Golomb[**?**] and features an infinitely spiraling appendage. The figure on the right is referred to as a twin dragon as it is the union of a pair of dragon curves. Neither of these figures can be drawn with perfect precision since each has infinitely small detail. The twin dragon is an example of a self-similar fractal with nonzero area. Its boundary (perimeter) is infinitely long and infinitely intricate. The more one magnifies the boundary, the more detail is revealed. While the snail can be formed by a union of infinitely many nonoverlapping triangles in a straightforward way, making its area calculable, the twin dragon cannot. Even applying calculus to the problem of finding the area of the twin dragon is not a straightforward matter. Does it even have a calculable area? What does having a calculable area

Figure 2: What are the areas of these shapes?





mean? Are there sets whose areas are not calculable? These questions can be followed deep into measure theory, a branch of analysis far outside the reaches of this textbook.

With the very definition of area up in the air,

Crumpet 1: A Definition of Area

The area of a bounded region of the plane, a shape S, can be defined as follows. Let R be a polygonal region containing S, and let $\mathcal{P}_R$ be a primitive partition of R (a finite set of parallelograms and triangles whose interiors do not overlap and whose union is R). Define the norm of a partition, denoted $||\mathcal{P}_R||$, as the maximum of the areas of the primitives in $\mathcal{P}_R$. Then

$$\text{area}(S) = \lim_{||\mathcal{P}_R|| \to 0} \sum_{\substack{p \in \mathcal{P}_R \\ p \subseteq S}} \text{area}(p)$$

whenever such limit exists.

it hardly makes practical sense to expect to prove the ways linear transformations affect the areas of general shapes. The following discussion is inherently incomplete this way. Certain claims regarding area will necessarily remain unproven.

Areas and determinants

In general, the image of a set S is defined as the set of images of all the points in S. That is, if S is a subset of A and $T : A \to B$, then the image of S under T is defined by $T(S) = \{T(s) : s \in S\}$. This definition is typical in all of mathematics, not just linear algebra, and applies no matter the sets A and B.

To understand how the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $T_A(\mathbf{v}) = A\mathbf{v}$ affects areas, it is convenient to write A is a product of elementary matrices, $A = E_p \cdots E_2 E_1$, as we have done before, assuming A is invertible (page **??**). Since $T_A(S) = (T_{E_p} \circ \cdots \circ T_{E_2} \circ T_{E_1})(S)$, if we can understand how linear transformations associated with elementary matrices affect area, we have a chance of understanding how general linear transformations affect area. If E is a row swap matrix, then T_E is a reflection about the line $y = x$, so in this case $\text{area}(T_E(S)) = \text{area}(S)$. Reflections do not change areas. If E is a row replace matrix, then T_E is a shear transformation, and it is a known result of calculus that shear transformations do not affect area, so again $\text{area}(T_E(S)) = \text{area}(S)$. If E is a row scale matrix, then T_E scales shapes either horizontally or vertically—not both!—by a factor of s, so $\text{area}(T_E(S)) = |s| \cdot \text{area}(S)$. In every case,

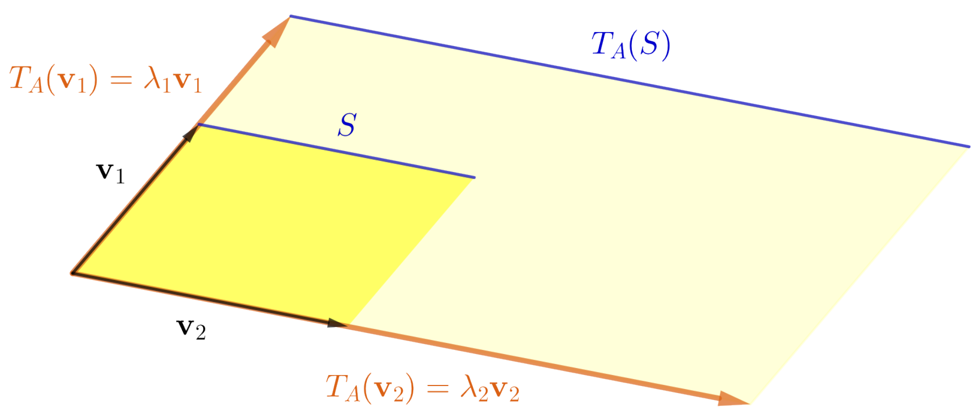
$\text{area}(T_E(S)) = |\det E| \cdot \text{area}(S)$. Remember, the determinant of a row swap matrix is -1, the determinant of a row replace matrix is 1 and the determinant of a row scale matrix with scale factor s is s. Therefore,

$$\begin{aligned}
\text{area}\,(T_A(S)) &= \text{area}\left((T_{E_p} \circ \cdots \circ T_{E_2} \circ T_{E_1})(S)\right) \\
&= \text{area}\left(T_{E_p}(\cdots(T_{E_2}(T_{E_1}(S)))\cdots)\right) \\
&= |\det E_p| \cdots |\det E_2| \cdot |\det E_1| \cdot \text{area}(S) \\
&= |\det A| \cdot \text{area}(S).
\end{aligned}$$

If A is noninvertible, then one of the columns of A is a multiple of the other, so any linear combination of the columns is also a multiple of that column. Therefore, the image of any vector, which is a linear combination of the columns of A, is a multiple of that column. Thus the image of every vector lies on the line determined by that column, giving the image of any shape area zero. The entire image is contained within a line. Of course, $|\det A| = 0$, so again we have $\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$.

Areas and eigenvalues

Let A be a 2×2 matrix with linearly independent eigenpairs $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$. Then $T_A(\mathbf{v}_1) = \lambda_1\mathbf{v}_1$ and $T_A(\mathbf{v}_2) = \lambda_2\mathbf{v}_2$. In fact, if we let $S = \{\mathbf{v}_1 + \alpha\mathbf{v}_2 : 0 \le \alpha \le 1\}$, the line segment from $\mathbf{v}_1$ to $\mathbf{v}_1 + \mathbf{v}_2$, then $T_A(S) = \{T_A(\mathbf{v}_1 + \alpha\mathbf{v}_2) : 0 \le \alpha \le 1\} = \{\lambda_1\mathbf{v}_1 + \alpha\lambda_2\mathbf{v}_2 : 0 \le \alpha \le 1\}$ is the line segment from $T_A(\mathbf{v}_1)$ to $T_A(\mathbf{v}_2)$. Further analysis shows that the image of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the parallelogram determined by $T_A(\mathbf{v}_1)$ and $T_A(\mathbf{v}_2)$. Can you supply this analysis? Answer on page 8.



Letting P be the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$, we see that T_A scales P in the $\mathbf{v}_1$ direction by a factor of λ_1 and in the $\mathbf{v}_2$ direction by factor λ_2. Therefore, the area of $T_A(P)$ equals $|\lambda_1\lambda_2|$ times the area of P. Since we have been arguing that linear transformations scale the areas of all shapes the same way, we have generally that $\text{area}(T_A(S)) = |\lambda_1\lambda_2| \cdot \text{area}(S)$ for any shape whose area is measurable. With respect to the eigenvectors of A, T_A is a simple scaling.

Now we have that

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$
$$\text{and}$$
$$\text{area}(T_A(S)) = |\lambda_1\lambda_2| \cdot \text{area}(S).$$

It must be, then, that $|\det A| = |\lambda_1\lambda_2|$. This is true for any 2×2 matrix including noninvertible ones, but the statement can be made much stronger, as in the following theorem.

Theorem 1. [Determinant and the Product of Eigenvalues] *If A is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its n (possibly complex) eigenvalues, then*

$$\det A = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Some but not all parts of the justification of this theorem are straightforward. For example, if A is upper triangular, then the conclusion follows quickly. As we have seen, $\det A$ is the product of the entries on the main diagonal. That is, $\det A = \prod_{i=1}^{n} A_{i,i}$. The characteristic equation

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} A_{1,1} - \lambda & \star & \cdots & \star \\ 0 & A_{2,2} - \lambda & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{n,n} - \lambda \end{bmatrix}\right) \\ &= (A_{1,1} - \lambda)(A_{2,2} - \lambda) \cdots (A_{n,n} - \lambda) \end{aligned}$$

has solutions $A_{1,1}, A_{2,2}, \ldots, A_{n,n}$, so the eigenvalues of A are the entries on the main diagonal of A. Hence $\prod_{i=1}^{n} A_{i,i} = \prod_{i=1}^{n} \lambda_i$ completing the proof for upper triangular matrices.

If A is any matrix, the conclusion follows from two facts.

1. The determinant and eigenvalues of $P^{-1}AP$ are the same as the determinant and eigenvalues of A for any invertible $n \times n$ matrix P (see theorem **??**).

2. For any $n \times n$ matrix A, there is an $n \times n$ matrix P such that $P^{-1}AP$ is upper triangular (see crumpet 2).

Given these two facts, if $U = P^{-1}AP$, then $\det A = \det U$ and the eigenvalues of A are the eigenvalues of U by fact 1 (theorem **??**). Now if P is that special matrix such that U is upper triangular, as guaranteed to exist by fact 2, then the determinant of U (which equals the determinant of A) and the product of the eigenvalues of U (which equals the product of the eigenvalues of A) are both $\prod_{i=1}^{n} U_{i,i}$ and therefore equal. This concludes the proof of the theorem for general matrices.

Crumpet 2: Triangularization

For a square matrix M, $P^{-1}MP$ is a triangularization of M whenever $P^{-1}MP$ is upper triangular. We wish to show that there is a triangularization of any $n \times n$ matrix. Triangularization of a 1×1 matrix is simple enough since all 1×1 matrices are upper triangular. Choose, $P = \begin{bmatrix} 1 \end{bmatrix}$ for example. Proceeding by induction, assume a triangularization exists for every $(k-1) \times (k-1)$ matrix for some $k \geq 2$, and let M be a particular but arbitrary $k \times k$ matrix. Take any eigenpair $\lambda, \mathbf{v}$ of M and find vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ such that $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}\}$ is linearly independent. This set can always be found since $\mathbf{v}$ must have at least one nonzero entry ($\mathbf{0}$ is not a permissible eigenvector). Assuming $\mathbf{v}_{i,1} \neq 0$, we may take $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}\} = \{\mathbf{v}\} \cup \{I_{:,j} : j \neq i\}$. Setting $Q = \begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix}$, Q is invertible (its columns are linearly independent), and

$$\begin{aligned} Q^{-1}AQ &= Q^{-1}A \begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} Q^{-1}A\mathbf{v} & Q^{-1}A\mathbf{u}_1 & Q^{-1}A\mathbf{u}_2 & \cdots & Q^{-1}A\mathbf{u}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda Q^{-1}\mathbf{v} & Q^{-1}A\mathbf{u}_1 & Q^{-1}A\mathbf{u}_2 & \cdots & Q^{-1}A\mathbf{u}_{n-1} \end{bmatrix}. \end{aligned}$$

While we cannot say much about $Q^{-1}A\mathbf{u}_j$ for any j, we can say $\lambda Q^{-1}\mathbf{v} = \lambda I_{:,1}$ because $Q^{-1}Q = Q^{-1}\begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix} = I$. Q^{-1} times the first column of Q must be the first column of I. Hence we have

$$Q^{-1}AQ = \begin{bmatrix} \lambda & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ \vdots & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \end{bmatrix}.$$

By the inductive hypothesis, there is a triangularization of $(Q^{-1}AQ)_{\backslash 1,1}$. Let R be such that $R^{-1}(Q^{-1}AQ)_{\backslash 1,1}R$ is upper triangular, and set $\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$. Then $\hat{Q}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix}$ and

$$\hat{Q}^{-1}(Q^{-1}AQ)\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix}\begin{bmatrix} \lambda & \star \\ 0 & (Q^{-1}AQ)_{\backslash 1,1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

is upper triangular. Hence $(Q\hat{Q})^{-1}A(Q\hat{Q})$ is a triangularization of A and we set $P = Q\hat{Q}$. This result suffices for our purposes, but the result can be strengthened to specify that $Q\hat{Q}$ have a certain property, a so-called Schur decomposition.

Hence we have two ways to measure the effect of a linear transformation on the plane. In rough terms, a linear transformation expands or compresses areas by a factor equal to the absolute value of the determinant (which is equal to the absolute value of the product of the eigenvalues) of its standard matrix. More precisely a linear transformation expands or compresses areas in the direction of each eigenvector by a factor equal to the absolute value of the associated eigenvalue.

Determinants, eigenvalues, and volumes

The analysis of elementary 3×3 matrices follows much along the same lines as the analysis of 2×2 matrices in section **??**. Vectors in $\mathbb{R}^3$ can be imagined as arrows or points just as they are in $\mathbb{R}^2$. Images of cubes in $\mathbb{R}^3$ under transformations associated with elementary matrices analogous to the images of the coffee cup in $\mathbb{R}^2$ can be derived. They will also be a collection of reflections, shears, and scalings. Rotation in $\mathbb{R}^3$ can be accomplished by a compostion of scalings and shears just as in $\mathbb{R}^2$. Noninvertible 3×3 matrices can be described by compositions of elementary matrices and projections as well. Hence theorem **??** can be proved for operators on $\mathbb{R}^3$. Generally, if the 2's in this section are replaced by 3's and the word area is replaced by the word volume, the discourse still applies with only minor additional modification. In particular, for $3{\times}3$ matrices M with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and three-dimensional regions of space, R,

$$\begin{aligned} \text{volume}(T_M(R)) &= |\det M| \cdot \text{volume}(R) \\ &= |\lambda_1\lambda_2\lambda_3| \cdot \text{volume}(R) \end{aligned}$$

and the concluding paragraph in the discussion of transformations of the plane might be rephrased for transformations of space as follows.

We have two ways to measure the effect of a linear transformation on space, $\mathbb{R}^3$. In rough terms, a linear transformation expands or compresses volumes by a factor equal to the absolute value of the determinant (which is equal to the absolute value of the product of the eigenvalues) of its standard matrix. More precisely a linear transformation expands or compresses volumes in the direction of each eigenvector by a factor equal to the absolute value of the associated eigenvalue.

Crumpet 3: Hyperspace

The main results of this section and the previous are stated and hold for $\mathbb{R}^n$, giving an enterprising individual a basis to extend the ideas of area and volume to dimensions higher than 3! The notion of a hypercube (in hyperspace) is exactly this enterprise.

Affine Transformations

Translations, transformations of the form $T : \mathbb{R}^n \to \mathbb{R}^n$,

$$T(\mathbf{x}) = \mathbf{x} + \mathbf{c},$$

are not linear for any $\mathbf{c} \neq \mathbf{0}$. Can you provide a justification? Answer on page 0.1.3. But because their geometric effect is to simply displace all points by the same distance and direction, they do not change the shapes of figures and therefore do not change areas or volumes of figures. For a translation T, $\text{area}(T(S)) = \text{area}(S)$ for any set S with measurable area.

Affine transformations, compositions of linear transformations with translations, are consequently not linear either, and their effect on areas is predictable. They scale areas in exactly the same manner as their linear parts. For an affine transformation $F : \mathbb{R}^n \to \mathbb{R}^n$,

$$F(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$$

for some matrix A and vector $\mathbf{c}$ and $\text{area}(F(S)) = |\det A| \cdot \text{area}(S)$ for any set S with measurable area.

0.1.1 Key Concepts

set image For any transformation (map or function) $f : A \to B$ and subset S of A,

$$f(S) = \{f(s) : s \in S\}$$

determinant and area For any linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ and any subset S of $\mathbb{R}^2$ with measurable area,

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

determinant and volume For any linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ and any subset S of $\mathbb{R}^3$ with measurable volume, r

$$\text{volume}(T_A(S)) = |\det A| \cdot \text{volume}(S)$$

determinant and eigenvalues The determinant of any square matrix is the product of its eigenvalues.

triangularization For any square matrix M there is an invertible matrix P such that $P^{-1}MP$ is upper triangular.

affine transformation The composition of a linear transformation with a translation.

0.1.2 Exercises

1. Find the area of the parallelogram with vertices

   (a) $(0,0)$, $(2,3)$, $(5,-1)$, $(3,-4)$

   (b) $(0,0)$, $(1,8)$, $(-1,5)$, $(-2,-3)$

   (c) $(0,0)$, $(-5,6)$, $(7,18)$, $(12,12)$

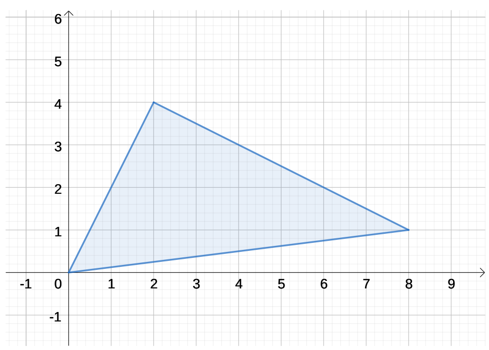
   (d) $(4,5)$, $(8,11)$, $(16,12)$, $(12,6)$

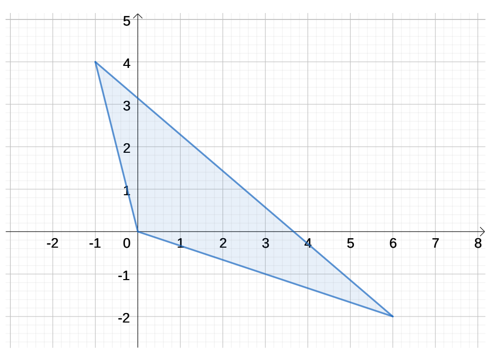
   (e) $(-1,3)$, $(3,-1)$, $(9,-4)$, $(5,0)$

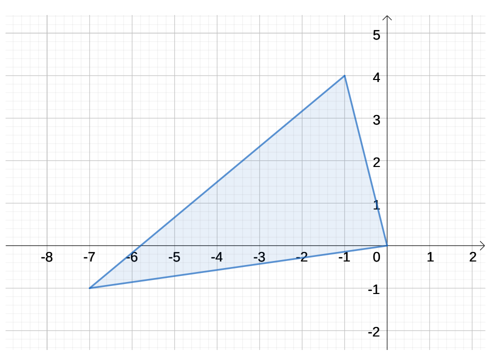
   (f) $(4,-2)$, $(11,-5)$, $(9,-10)$, $(2,-7)$

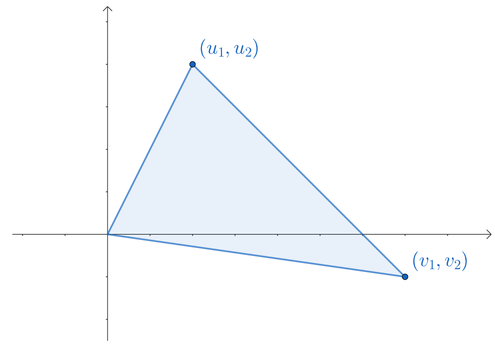
2. Use the fact that $\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$ to justify the claim that the area of the parallelogram determined by the columns of a 2×2 matrix A is $|\det A|$. Alternatively, justify the claim that the area of the parallelogram determined by two vectors (anchored at the origin) is the determinant of the matrix whose columns are the two vectors.

3. Calculate the area of the triangle as half of a determinant. See exercise 2 for a hint.

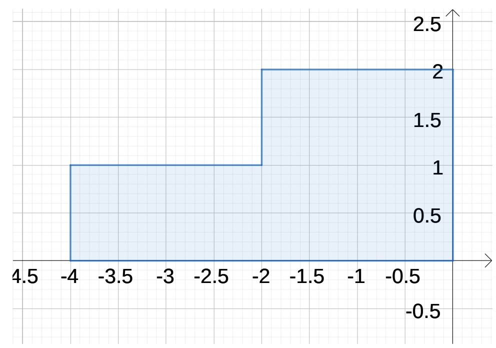
   (a)


   (b)


   (c)


   (d)


4. The image of the hexagon with adjacent vertices $(0,0)$, $(2,0)$, $(2,1)$, $(1,1)$, $(1,2)$, $(0,2)$ under the transformation $T(\mathbf{x}) = A\mathbf{x}$ where A is a 2×2 matrix. What is the determinant of A?



5. What are the eigenvalues of the matrix A of question 4?

6. Suppose M factors as shown. Use the information to find (i) $\det M$ and (ii) the eigenvalues of M.

   (a) $M = \begin{bmatrix} 12 & -10 \\ 10 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -9 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 12 & -10 \\ 10 & 4 \end{bmatrix}$

   (b) $M = \begin{bmatrix} -12 & 10 \\ -7 & 8 \end{bmatrix}^{-1} \begin{bmatrix} -6 & 12 \\ 0 & -8 \end{bmatrix} \begin{bmatrix} -12 & 10 \\ -7 & 8 \end{bmatrix}$

   (c) $M = \begin{bmatrix} 4 & 9 & 11 \\ -5 & 1 & -6 \\ -7 & -3 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -11 & -4 & -7 \\ 0 & 0 & -12 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 4 & \cdots \\ -5 & \cdots \\ -7 & \cdots \end{bmatrix}$

   (d) $M = \begin{bmatrix} 12 & 8 & -12 \\ -6 & -10 & 1 \\ 6 & -11 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -11 & 4 & 11 \\ 0 & -5 & -2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 12 & \cdots \\ -6 & \cdots \\ 6 & \cdots \end{bmatrix}$

7. Use SageMath to verify your answers in question 6 by calculating M and having SageMath compute the determinant and eigenvalues.

   (a) https://sagecell.sagemath.org/?z=eJxTVnDOzy3ISS1
   1

   (b) https://sagecell.sagemath.org/?z=eJxTVnDOzy3ISS1
   2

(c) https://sagecell.sagemath.org/?z=eJxTVnDOzy3ISS1JVSjJSF2Vlzk9J5eUKULBVyE0sKcqs0DDWMdaJNtGx1DE01
gLVBuhl5pWlFhWnamhqhWoFAAAgCh9Jlang=sageinteracts=eJyLjgUAARUAuQ== SageMathCell
3

(d) https://sagecell.sagemath.org/?z=eJxTVnDOzy3ISS1JVSjJSFVlzk9J5eUKULBVyE0sKcqs0DDWMdaJNjTSsdDRB
$sageinteracts = eJyLjgUAARUAuQ ==$ SageMathCell 4

8. Use the fact that volume$(T_A(S)) = |\det A| \cdot$ volume(S) to justify the claim that the volume of the parallelepiped determined by the columns of a 3×3 matrix A is $\det A$.

9. Let S be the unit square with opposite corners $(0,0)$ and $(1,1)$. Sketch the image of S under the affine transformation $T(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$.

   (a) $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$; $\mathbf{c} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}$; $\mathbf{c} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & 1/2 \\ -1/2 & 0 \end{bmatrix}$; $\mathbf{c} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

10. Let S be the triangle with vertices $(0,0)$, $(1,1)$, and $(-1,1)$ and suppose F and G are affine transformations such that $F(S)$ is the triangle with vertices $(0,0)$, $(0,1)$, and $(-1,1)$; and $G(S)$ is the triangle with vertices $(0,0)$, $(1,1)$, and $(0,1)$. Draw S, $F(S)$, and $G(S)$ and use your sketch to determine the determinants of the matrix parts of F and G?

11. Trump 2 outlines a recursive procedure for diagonalizing any matrix. It is constructive, giving an algorithm for finding the diagonalizing matrix P. Use it to find P such that $P^{-1}MP$ is upper triangular. One eigenvalue of M is given.

    (a) $M = \begin{bmatrix} 13 & -8 \\ 20 & -13 \end{bmatrix}$; $\lambda = -3$

    (b) $M = \begin{bmatrix} 22 & -7 \\ 28 & -13 \end{bmatrix}$; $\lambda = -6$

    (c) $M = \begin{bmatrix} 11 & -8 \\ 2 & 1 \end{bmatrix}$; $\lambda = 3$

    (d) $M = \begin{bmatrix} -9 & 7 & 28 \\ -11 & -27 & -28 \\ 8 & 8 & 12 \end{bmatrix}$; $\lambda = 12$

    (e) $M = \begin{bmatrix} -34 & -50 & -24 \\ 37 & 53 & 24 \\ -26 & -35 & -14 \end{bmatrix}$; $\lambda = 4$

    (f) $M = \begin{bmatrix} 78 & -50 & -54 \\ -167 & 135 & 186 \\ 201 & -150 & -193 \end{bmatrix}$; $\lambda = 5$

12. Prove that any square matrix M can be factored as PUP^{-1} for some invertible matrix P and upper triangular matrix U.

0.1.3 Answers

further analysis The parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the set $S = \{\beta\mathbf{v}_1 + \alpha\mathbf{v}_2 : 0 \le \alpha, \beta \le 1\}$ so its image is

$$\begin{aligned} T_A(S) &= T_A\left(\{\beta\mathbf{v}_1 + \alpha\mathbf{v}_2 : 0 \le \alpha,\beta \le 1\}\right) \\ &= \{T_A(\beta\mathbf{v}_1 + \alpha\mathbf{v}_2) : 0 \le \alpha,\beta \le 1\} \\ &= \{\beta T_A(\mathbf{v}_1) + \alpha T_A(\mathbf{v}_2) : 0 \le \alpha,\beta \le 1\} \\ &= \{\beta\lambda_1\mathbf{v}_1 + \alpha\lambda_2\mathbf{v}_2 : 0 \le \alpha,\beta \le 1\} \end{aligned}$$

which is the paralelogram determined by $T_A(\mathbf{v}_1)$ and $T_A(\mathbf{v}_2)$.

translations are not linear On the one hand,

$$T(\mathbf{x} + \mathbf{y}) = \mathbf{x} + \mathbf{y} + \mathbf{c}$$

and on the other hand,

$$\begin{aligned} T(\mathbf{x}) + T(\mathbf{y}) &= (\mathbf{x} + \mathbf{c}) + (\mathbf{y} + \mathbf{c}) \\ &= \mathbf{x} + \mathbf{y} + 2\mathbf{c} \end{aligned}$$

so $T(\mathbf{x} + \mathbf{y}) \neq T(\mathbf{x}) + T(\mathbf{y})$ whenever $\mathbf{c} \neq \mathbf{0}$.