TEA TIME LINEAR ALGEBRA

Explorations in Mathematics

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the second in a series of tea time textbooks

Southern Connecticut State University

To
Victorija, Cecelia, and Amy

Contents

Preface

About Tea Time Linear Algebra

Greetings! And thanks for giving *Tea Time Linear Algebra* a read. The phrase “tea time” is meant to do more than give the book a catchy title. It is meant to describe the general nature of the discourse within. Much of the material will be presented as if it were being told to a student during tea time at University, but with the benefit of careful planning. There will be no big blue boxes highlighting the main points, no stream of examples after a short introduction to a topic, and no theorem. . . proof. . . theorem. . . proof structure. Instead, the necessary terms and definitions and theorems and examples will be woven into a more conversational style. My hope is that this blend of formal and informal mathematics will be easier to digest, and dare I say, students will be more invited to do their reading in this format.

Those who enjoy a more typical presentation might still find this textbook suits their preference to a large extent. There will be a summary of the key concepts at the end of each conversation and a number of the exercises will be solved in complete detail in the appendix. So, one can get a closer-to-typical presentation by scanning for theorems in the conversations, reading the key concepts, and then skipping to the exercises with solutions. I hope most readers won’t choose to do so, but it is an option. In any case, the exercises with solutions will be critical reading for most. Learning by example is often the most effective means. After reading a section, or at least scanning it, readers are strongly encouraged to skip to the statements of the exercises with solutions (marked by $^{[\mathbb{S}]}$or $^{[\mathbb{S}]}$), contemplate their solutions, solve them if they can, and then turn to the back of the book for full disclosure. The hope is that, with their placement in the appendix, readers will be more apt to consider solving the exercises on their own before looking at the solutions.

The topical coverage in *Tea Time Numerical Analysis* is fairly typical. The book starts with an introductory chapter, followed by root finding methods, interpolation (part 1), numerical calculus, interpolation (part 2), and the second edition introduces a chapter on differential equations. The first five chapters cover what, at SCSU, constitutes a first semester course in numerical analysis. As this book is intended for use as a free download or an inexpensive print-on-demand volume, no effort has been made to keep the page count low or to spare copious diagrams and colors. In fact, I have taken the inexpensive mode of delivery as liberty to do quite the opposite. I have added many passages and diagrams that are not strictly necessary for the study of numerical analysis, but are at least peripherally related, and may be of interest to some readers. Most of these passages will be presented as digressions, so they will be easy to identify. For example, Taylor’s theorem plays such a central role in the subject that not only its statement is presented. Its proof and a bit of history are added as “crumpets”. Of course they can be skipped, but are included to provide a more complete understanding of this fundamental theorem of numerical analysis. For another example, as a fan of dynamical systems, I found it impossible to refrain from including a section on visualizing Newton’s method. The powerful and beautiful pictures of Newton’s method as a dynamical system should be eyebrow-raising and question-provoking even if only tangentially important. There are, of course, other examples of somewhat less critical content, but each is there to enhance the reader's understanding or appreciation of the subject, even if the material is not strictly necessary for an introductory study of numerical analysis. While version 2.5 does not introduce any new sections, it contains many corrections, new exercises and modified discourse. As a result, page and exercise numbering has changed.

Implementation of the numerical methods in the form of computer code will also be discussed. While one could simply ignore the programming sections and exercises and still get something out of this text, it is my firm belief that full appreciation for the content can not be achieved without getting one's hands "dirty" by doing some programming. It would be nice if readers have had at least some minimal exposure to programming whether it be Python, Java, C, web programming, or just about anything, but I have made every effort to give enough detail so that even those who have never written a single line of code will be able to participate in this part of the study. In keeping with the desire to produce a completely free learning experience, GNU Octave was chosen as the programming language for this book. GNU Octave (Octave for short) is offered freely to anyone and everyone! It is free to download and use. Its source code is free to download and study. And anyone is welcome to modify or add to the code if so inclined. As an added bonus, users of the much better-known MATLAB will not be burdened by learning a new language. Octave is a MATLAB clone. By design, nearly any program written in MATLAB will run in Octave without modification. So, if you have access to MATLAB and would prefer to use it, you may do so without worry. I have made considerable effort to ensure that every line of Octave in this book will run verbatim under MATLAB. Even with this earnest effort, though, it is possible that some of the code will not run under MATLAB. It has only been tested in Octave! If you find any code that does not run in MATLAB, please let me know. Version 2.5 includes substantial rewriting of details related to Octave from installation to coding, and even includes information about computing in the cloud.

I hope you enjoy your reading of *Tea Time Numerical Analysis*. It was my pleasure to write it. Feedback is always welcome.

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How to Get Sage

Just use sagecell.sagemath.com

How to Get the Code

All the code appearing in the textbook can be downloaded from this textbook's companion website,

`http://lqbrin.github.io/tea-time-numerical/ancillaries.html`.

The code printed within and accompanying Tea Time Numerical Analysis electronically is distributed under the GNU Public License (GPL). Details are available at the website.

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Part I

Matrix Mechanics

Chapter 1

Matrix Calculations

1.1 Matrices

The fundamental object of linear algebra is the **matrix**. A matrix is very much like a table or a spreadsheet, but without headings, labels, or lines. The data in a matrix are separated by space. The whole matrix is enclosed by large parentheses or square brackets, but is otherwise unadorned.

Crumpet 1: Dictionary Definition

ma•trix (mā′trĭks) *n.*, *pl.* **ma•tri•ces** (mā′trĭ-sēz′). *Math.* A rectangular array of algebraic quantities *usu.* delimited by parentheses or square brackets.

The following are all matrices.

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \quad \begin{pmatrix} 3 & -4 & 63 \\ 2 & 8 & -17 \end{pmatrix} \quad \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 6 & \sqrt{3} \\ \sqrt{2} & 13 & e \end{bmatrix}$$

$$\begin{pmatrix} .929 & .988 & .405 & .877 & .752 & .541 & .269 \\ .390 & .595 & .186 & .328 & .315 & .566 & .478 \\ .731 & .224 & .254 & .543 & .575 & .499 & .881 \end{pmatrix}$$

The **size of a matrix** is described by its number of rows "by" its number of columns, and is abbreviated as in 2×3, read "two by three". A 7×5 matrix has seven rows and 5 columns. The number of rows is always listed first. The rows are indexed from top to bottom, and the columns are indexed from left to right. The first column is the leftmost column, and the first row is the topmost row. There are no restrictions on the numbers of rows or columns other than each must be a positive integer. The individual quantities in a matrix are called **entries**. The entry in the i^{th} row (from the top) and j^{th} column (from the left) of a matrix is called the i,j-entry. The row number always precedes the column number.

Matrices are most often labeled by capital letter variables such as A, B, or M. This helps distinguish them from numerical variables such as x, y, z, s, or t. In this text, the i,j-entry of a matrix A is denoted by $A_{i,j}$.

Crumpet 2: Other Notations for Entries

The subscripted lower case counterpart to a matrix variable is often used to represent the entries of a matrix. You will often see $b_{1,2}$ or even b_{12} represent the 1,2-entry of B. Don't be surprised when you run into it!

The 5,1-entry of a matrix M is denoted by $M_{5,1}$. Taking our cue from computer science and the currently very popular Python programming language, the i^{th} row of a matrix B is denoted by $B_{i,:}$, the : indicating that all columns of the row are included. The j^{th} column of the same matrix B is denoted by $B_{:,j}$, where the : indicates that all rows are included.

$$\text{If } B = \begin{bmatrix} 2 & 6 & 1 & 8 \\ -3 & 4 & -2 & 1 \\ -2 & -5 & 4 & 1 \end{bmatrix} \text{ then } B_{2,:} = \begin{bmatrix} -3 & 4 & -2 & 1 \end{bmatrix} \text{ and } B_{:,4} = \begin{bmatrix} 8 \\ 1 \\ 1 \end{bmatrix}$$

A **submatrix** of a matrix M is any matrix derived by deleting some number of rows (less than the total number of rows) and some number of columns (less than the total number of columns) from M.

$$\begin{bmatrix} 2 & 6 & 1 \\ -3 & 4 & -2 \end{bmatrix} \text{ is a submatrix of } \begin{bmatrix} 2 & 6 & 1 & 8 \\ -3 & 4 & -2 & 1 \\ -2 & -5 & 4 & 1 \end{bmatrix}$$

derived by deleting the last row and last column. $\begin{bmatrix} 6 \end{bmatrix}$ is a submatrix of $\begin{bmatrix} 2 & 6 & 1 \\ -3 & 4 & -2 \end{bmatrix}$, derived by deleting the second row, the first column, and the third column. A submatrix derived by deleting one row and one column of a matrix is common enough that we use a special notation for it: $B_{\backslash i,j}$ (read "B without row i and column j").

$$\text{If } B = \begin{bmatrix} 2 & 6 & 1 & 8 \\ -3 & 4 & -2 & 1 \\ -2 & -5 & 4 & 1 \end{bmatrix} \text{ then } B_{\backslash 2,3} = \begin{bmatrix} 2 & 6 & 8 \\ -2 & -5 & 1 \end{bmatrix}.$$

Though we will not make frequent use of it, the : notation can be used to identify submatrices other than single columns or single rows by placing a number before and a number after the colon as in $2:5$, which means rows (or columns) two through five. For example, $B_{1:2,1:3}$ represents the submatrix of B consisting of its first two rows and first three columns. All other rows and columns are excluded. $B_{2:7,3}$ represents the submatrix of B consisting of rows two through seven of column three.

Typically the entries of a matrix will have underlying meaning, but the point of putting quantities in a matrix is to perform some mathematical operation. Labels would only get in the way, so the rows and columns of a matrix are not labeled directly. Their meaning must be gathered some other way. A table or spreadsheet of common grocery items at various stores such as

Price Comparison ($)

		Item		
		eggs	milk	bananas
Store	Food Plus	2.89	4.69	2.07
	Grocer Girl	3.69	4.99	2.37
	Eddie's Eats	2.79	4.29	2.57

or

	A	B	C	D	E
1	**Price Comparison ($)**				
2			eggs	milk	bananas
3	**Store**	Food Plus	2.89	4.69	2.07
4		Grocer Girl	3.69	4.99	2.37
5		Eddie's Eats	2.79	4.29	2.57

would be summarized in a matrix as

$$\begin{bmatrix} 2.89 & 4.69 & 2.07 \\ 3.69 & 4.99 & 2.37 \\ 2.79 & 4.29 & 2.57 \end{bmatrix}.$$

All the descriptive words are stripped. While the meaning of the entries in this case is stated explicitly elsewhere, there are times when meaning will be simply implied or understood from context. In any case, If the numbers in a matrix are to retain their contextual meaning, that information must be supplied separately. View this video[1] (3:13) for more examples where a matrix might be useful.

Key Concepts

matrix A rectangular array of algebraic quantities usually delimited by parentheses or square brackets. Upper case letters are used for variables representing matrices.

(matrix) entry One of the individual quantities in a matrix. Subscripted lower case letters are used to represent entries of a matrix.

(matrix) size Stated as the number of rows "by" the number of columns.

submatrix The matrix resulting from deleting some number of rows (less than the total number of rows) and some number of columns (less than the total number of columns) from a matrix.

Notation

$A, B, \ldots, M, \ldots$ Upper case letters are used for variables representing matrices.

$A_{i,j}$ The entry in row i and column j of matrix A.

$A_{m,:}$ Row m of matrix A.

$A_{:,n}$ Column n of matrix A.

$A_{\backslash m,n}$ The submatrix of A consisting of all entries except those in row m or in column n.

$A_{i:j,k:l}$ The submatrix of A consisting of rows i through j of columns k through l.

$m \times n$ The size of a matrix with m rows and n columns.

SageMath

The matrices and operations of this section (and the entire text) can be handled electronically by SageMath. All you need is the syntax, the proper combinations of words and symbols. Before any operations can be performed, a matrix must be defined. In SageMath, there are several ways to define a matrix, but we will most often use the syntax

```
M = matrix(rows,cols,[list of entries])
```

The `rows` and `cols` stand for the number of rows and number of columns in the matrix, respectively. The `list of entries` must be comma-separated as in `1,2,3`. Entered into SageMath properly, this line creates the variable `M`

[1] https://youtu.be/BZWFkUQ3tco?t=71

Crumpet 3: Instantiation

In computer science, the creation or definition of a variable is called *instantiation*.

as a matrix from which we can extract entries or submatrices, or simply print out. Just as we can on paper, we can name the matrix using any letter. It does not have to be `M`.

Submatrices can be extracted using : notation just as we have been doing on paper. In SageMath, though, subscripts aren't used. Square brackets are. So $M_{3,:}$ would be written `M[2,:]` in SageMath. Yes, that looks like a typo. It is not! On paper, and in mathematics generally, we index the rows and columns of matrices in a way that seems most natural. The first row is row 1, the second row is row 2, and so on. However, SageMath uses the very common computer programming convention of 0-indexing. That is, counting starts with 0 instead of 1 in SageMath. So the first row of a matrix `M` (in SageMath, Python, and many other programming languages) is row 0, the second row is row 1, and so on. The square bracket notation is used to extract entries of a matrix, too. In SageMath the i, j-entry of a matrix `M` is indicated by `M[i-1,j-1]`. Table 1.1 summarizes the extraction of entries and submatrices using SageMath.

Table 1.1: Matrices, entries, and submatrices in SageMath.

	Mathematics	SageMath (0-indexed)
matrix	$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$	`M = matrix(3,2,[1,2,3,4,5,6])`
row	$M_{r,:}$	`M[r-1,:]`
column	$M_{:,c}$	`M[:,c-1]`
submatrix	$M_{i:j,k:l}$	`M[i-1:j,k-1:l]`
submatrix	$M_{\backslash r,c}$	`M.delete_rows([r-1]).delete_columns([c-1])`
entry	$M_{r,c}$	`M[r-1,c-1]`

In SageMath, the lines

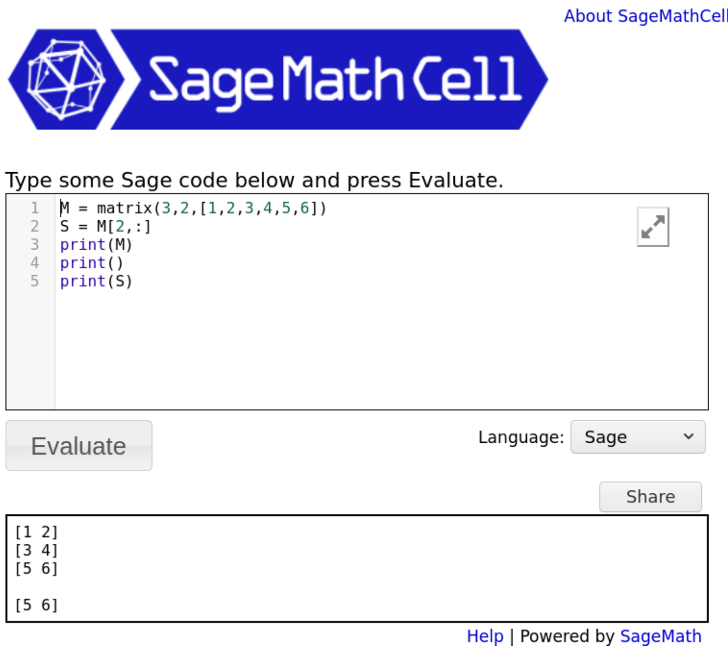
```
M = matrix(3,2,[1,2,3,4,5,6])
S = M[2,:]
```

define a matrix `M` and a submatrix `S`, but do not create any output. When the code is run (evaluated), it seems nothing has happened! (You can try it at `sagecell.sagemath.org`). Rest assured, though, these lines cause SageMath to do things internally. We just aren't seeing the results yet. If we add a couple lines requesting the display of our matrices, we will see the results. In SageMath, this can be done with the `print(object)` statement. In this case, we want to print out two matrices. A little space between them would be good too. Printing "nothing", using `print()`, actually prints a blank line. The following SageMath code creates a matrix `M`, a submatrix `S`, and prints them both with a blank line between.

```
M = matrix(3,2,[1,2,3,4,5,6])
S = M[2,:]
```

```
print(M)
print()
print(S)
```

Here is a screenshot of this code being processed at `SageCell.SageMath.org`.



Try it at `sagecell.sagemath.org`.

Crumpet 4: Nested Statements in SageMath

SageMath statements may be nested. That is, one statement may appear as the argument (inside) of another. For example, the code

```
M = matrix(3,2,[1,2,3,4,5,6])
S = M[2,:]
print(M)
print()
print(S)
```

might also be written as follows.

```
M = matrix(3,2,[1,2,3,4,5,6])
print(M)
print()
print(M[2,:])
```

Notice the extraction of the third row of `M` happens inside the `print` statement. There is no need to produce a variable named `S` since it is not used for any other purpose.

Exercises

1. How many rows does a matrix with the given size have?

   (a) 15×6

   (b) 6×8

   (c) 1×11

   (d) 17×2

2. How many columns does a matrix with the given size have?

   (a) 5×10

   (b) 12×5

   (c) 6×12

   (d) 18×19

3. How many entries does a matrix with the given size have?

   (a) 3×13

   (b) 9×8

   (c) 4×14

   (d) 7×6

4. Identify the requested entry of the given matrix.

   (a) $A_{2,4}$;
   $$A = \begin{bmatrix} 23 & 31 & 44 & -9 & 45 \\ 27 & -6 & 14 & 33 & -33 \\ -22 & 48 & -17 & -48 & 41 \end{bmatrix}$$

   (b) $B_{1,2}$;
   $$B = \begin{bmatrix} -3 & 39 & -1 \\ 3 & -30 & 7 \\ -27 & -48 & 32 \end{bmatrix}$$

   (c) $P_{3,4}$;
   $$P = \begin{bmatrix} 47 & 14 & -10 & 10 & -11 \\ 21 & -29 & -39 & 49 & -26 \\ -22 & 20 & 12 & 37 & 44 \\ -18 & -37 & -30 & -42 & -17 \end{bmatrix}$$

   (d) $M_{3,1}$;
   $$M = \begin{bmatrix} 21 & -14 & 43 & 34 \\ 8 & -32 & -3 & -20 \\ -2 & 50 & -24 & 20 \end{bmatrix}$$

5. Let $N = \begin{bmatrix} -11 & -2 & -6 & -4 & -3 & 5 \\ -5 & 12 & 3 & -2 & -4 & -7 \\ -8 & -12 & 11 & -12 & 4 & -3 \\ 3 & 10 & 9 & 0 & -7 & -10 \\ -2 & 12 & -9 & 3 & -5 & 8 \end{bmatrix}$. What is the size of the submatrix?

   (a) $N_{5,:}$

   (b) $N_{:,2}$

   (c) $N_{1:5,2:4}$

   (d) $N_{\backslash 3,5}$

6. Let $A = \begin{bmatrix} -7 & 2 & 1 & 8 & -1 \\ -8 & -11 & 10 & -6 & 1 \\ 1 & -10 & 12 & -12 & 0 \\ 9 & -1 & 3 & -6 & -8 \\ 6 & -9 & 3 & 4 & -5 \\ 2 & -4 & 10 & -7 & -3 \end{bmatrix}$. Identify the submatrix of A.

   (a) $A_{3,:}$

   (b) $A_{:,2}$

   (c) $A_{2:3,3:5}$

   (d) $A_{\backslash 4,2}$

7. Let $P = \begin{bmatrix} 11 & 7 & 4 \\ -8 & -6 & 1 \\ -2 & -1 & 5 \\ 8 & 3 & 12 \end{bmatrix}$. Supply notation for the entry or submatrix.

   (a) 5

   (b) $\begin{bmatrix} 11 & 4 \\ -8 & 1 \\ -2 & 5 \end{bmatrix}$

   (c) $\begin{bmatrix} -8 & -6 & 1 \\ -2 & -1 & 5 \end{bmatrix}$

   (d) $\begin{bmatrix} 8 & 3 & 12 \end{bmatrix}$

   (e) -2

   (f) $\begin{bmatrix} 11 \\ -8 \\ -2 \\ 8 \end{bmatrix}$

8. Let $B = \begin{bmatrix} 6 & -7 & 1 \\ 11 & -11 & -2 \\ 11 & 12 & 12 \\ -7 & 9 & 10 \end{bmatrix}$. Write SageMath code that will accomplish the following.

(a) Create the matrix B.

(b) Print B.

(c) Extract $B_{2,3}$.

(d) Print $B_{2,3}$.

9. Add code that will print out (i) the third row and (ii) the first column of `D`.

```
D = matrix(3,4,[-2,10,-7,8,-2,7,
                11,-7,-4,5,6,10])
```

What is the output of your code?

10. If you swap the 3 and the 4 in the code of exercise 9, as shown below, what is the new output of your code (third row and first column of `D`)?

```
D = matrix(4,3,[-2,10,-7,8,-2,7,
                11,-7,-4,5,6,10])
```

1.2 Component-wise Matrix Operations

While a sudoku board is not a matrix, if we strip away the color and the lines, it certainly is a rectangular array of numbers, the essence of a matrix. Soon we will do just that, but for now let's have a look at the sudoku board without thinking about matrices. Notice it consists of nine 3×3 blocks.

1	4	7	8	6	5	2	3	9
2	6	3	1	9	4	5	7	8
8	5	9	3	7	2	1	6	4
3	2	1	7	4	6	8	9	5
9	7	5	2	8	3	4	1	6
4	8	6	9	5	1	3	2	7
6	3	4	5	2	7	9	8	1
5	9	2	6	1	8	7	4	3
7	1	8	4	3	9	6	5	2

Pick your favorite two 3×3 blocks and think about how you might add them to one another. Don't just read on. Stop and think about this briefly. [PAUSE HERE AND THINK.] If you are like most students, you probably came up with one of two ways to add the blocks. The first one is to add all the numbers in each block giving a sum of 90. Sum a different pair of blocks, and you will notice you get 90 again. In fact, take any pair of blocks and the sum done this way will give you 90. Can you see why? Answer on page 14. This way of adding is legitimate, but maybe a little unsatisfying since the sum is always 90. A second way of adding supplies some variation in the sums.

What if the sum of the two blocks were another 3×3 block? This way of thinking has a lot of precedent in mathematics. The sum of two integers is an integer. The sum of two rational numbers is a rational number. The sum of two functions is a function. The sum of two areas is an area. The operation of addition always seems to preserve the type of object being added.

Crumpet 5: Operators

In mathematics a **binary operator**, such as +, takes two objects (inputs or addends) from a set and produces a third object (output or sum) from the same set.

With this idea in hand, perhaps the most organized way to proceed is to add the number in the upper-left corner of the first matrix to the number in the upper-left corner of the second matrix to produce the number in the upper-left corner of the sum. Similarly, the other 8 numbers of the sum can be produced by adding the corresponding numbers (by location) of the two blocks being added. Here is an illustration of that process.

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline 2 & 6 & 3 \\ \hline 8 & 5 & 9 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 7 & 4 & 6 \\ \hline 2 & 8 & 3 \\ \hline 9 & 5 & 1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 1+7 & 4+4 & 7+6 \\ \hline 2+2 & 6+8 & 3+3 \\ \hline 8+9 & 5+5 & 9+1 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 8 & 8 & 13 \\ \hline 4 & 14 & 6 \\ \hline 17 & 10 & 10 \\ \hline \end{array}$$

This is exactly the path taken in adding matrices, component-wise (entry-by-entry) addition. In the notation of matrices,

$$\text{if } A \text{ and } B \text{ are matrices, then } (A + B)_{i,j} = A_{i,j} + A_{i,j}$$

for all entries $A_{i,j}$ and $B_{i,j}$ of A and B. The i,j-entry of $A+B$ is the sum of the i,j-entries of A and B. This requires A and B to have at least as many rows and columns as their sum, but in fact, adding matrices requires that both addends and the sum have the exact same size. The sum of matrices of differing size is undefined. Subtraction of matrices is defined analogously.

$$\text{If } A \text{ and } B \text{ are matrices, then } (A-B)_{i,j} = a_{i,j} - b_{i,j}$$

for all entries $A_{i,j}$ and $B_{i,j}$ of A and B. The difference of matrices of differing size is undefined.

Now let's do the same addition exercise using matrices. Transferring the numbers of a sudoku board to a matrix is good practice in creating matrices where there are none, extracting them from their context for mathematical work. Let's start by looking at each 3×3 block of the sudoku board as a matrix.

1	4	7	8	6	5	2	3	9
2	6	3	1	9	4	5	7	8
8	5	9	3	7	2	1	6	4
3	2	1	7	4	6	8	9	5
9	7	5	2	8	3	4	1	6
4	8	6	9	5	1	3	2	7
6	3	4	5	2	7	9	8	1
5	9	2	6	1	8	7	4	3
7	1	8	4	3	9	6	5	2

$$\Rightarrow \quad \begin{matrix} \begin{bmatrix} 1 & 4 & 7 \\ 2 & 6 & 3 \\ 8 & 5 & 9 \end{bmatrix} & \begin{bmatrix} 8 & 6 & 5 \\ 1 & 9 & 4 \\ 3 & 7 & 2 \end{bmatrix} & \begin{bmatrix} 2 & 3 & 9 \\ 5 & 7 & 8 \\ 1 & 6 & 4 \end{bmatrix} \\ \begin{bmatrix} 3 & 2 & 1 \\ 9 & 7 & 5 \\ 4 & 8 & 6 \end{bmatrix} & \begin{bmatrix} 7 & 4 & 6 \\ 2 & 8 & 3 \\ 9 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 8 & 9 & 5 \\ 4 & 1 & 6 \\ 3 & 2 & 7 \end{bmatrix} \\ \begin{bmatrix} 6 & 3 & 4 \\ 5 & 9 & 2 \\ 7 & 1 & 8 \end{bmatrix} & \begin{bmatrix} 5 & 2 & 7 \\ 6 & 1 & 8 \\ 4 & 3 & 9 \end{bmatrix} & \begin{bmatrix} 9 & 8 & 1 \\ 7 & 4 & 3 \\ 6 & 5 & 2 \end{bmatrix} \end{matrix}$$

Previously we added the upper-left block and the middle block of the sudoku board. Now let's add the upper-left matrix and the middle matrix:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 6 & 3 \\ 8 & 5 & 9 \end{bmatrix} + \begin{bmatrix} 7 & 4 & 6 \\ 2 & 8 & 3 \\ 9 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1+7 & 4+4 & 7+6 \\ 2+2 & 6+8 & 3+3 \\ 8+9 & 5+5 & 9+1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 13 \\ 4 & 14 & 6 \\ 17 & 10 & 10 \end{bmatrix}$$

Numerically, it is the same computation.

Multiplying a matrix by a number is also done component-wise. Multiplying the bottom-left 3×3 matrix extracted from our sudoku board by 5 is done as follows.

$$5 \begin{bmatrix} 6 & 3 & 4 \\ 5 & 9 & 2 \\ 7 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 5 \cdot 6 & 5 \cdot 3 & 5 \cdot 4 \\ 5 \cdot 5 & 5 \cdot 9 & 5 \cdot 2 \\ 5 \cdot 7 & 5 \cdot 1 & 5 \cdot 8 \end{bmatrix} = \begin{bmatrix} 30 & 15 & 20 \\ 25 & 45 & 10 \\ 35 & 5 & 40 \end{bmatrix}$$

This is often referred to as scalar[2] multiplication to differentiate it from matrix multiplication, the subject of the next section. In symbols,

$$\text{If } A \text{ is a matrix and } c \text{ is a scalar, then } (cA)_{i,j} = cA_{i,j}$$

for all entries $A_{i,j}$ of A. This means that cA has the same size as A and the i,j-entry of cA is c times the i,j-entry of A. To be complete Ac is defined to equal cA.

[2] In this textbook, the word *scalar* refers to either a real number or a complex number. In more abstract settings, the word scalar refers to any element of a field.

Crumpet 6: Fields

Sets of scalars other than real numbers and complex numbers are permissible in linear algebra as long as matrix entries come from the same field. A field must contain an additive identity, denoted by 0, and a multiplicative identity, denoted by 1. A field with only these two elements can be defined by treating 0 and 1 as integers except that $1 + 1 = 0$. The field of two elements is often denoted $\mathbb{F}_2$ or $\mathbb{Z}_2$.

Key Concepts

binary operator A function with two inputs and one output, all three from the same set.

matrix addition For any matrices A and B of the same size, the sum $A + B$ is defined, has the same size as A and B, and $(A+B)_{i,j} = A_{i,j} + B_{i,j}$ for all entries $A_{i,j}$ and $B_{i,j}$. If A and B differ in size, then $A+B$ is undefined.

matrix subtraction For any matrices A and B of the same size, the difference $A - B$ is defined, has the same size as A and B, and $(A-B)_{i,j} = A_{i,j} - B_{i,j}$ for all entries $A_{i,j}$ and $B_{i,j}$. If A and B differ in size, then $A - B$ is undefined.

scalar An element of a field.

scalar multiplication For any matrix A and scalar c, the scalar product cA is defined, has the same size as A, and $(cA)_{i,j} = cA_{i,j}$ for all entries $A_{i,j}$. Moreover, Ac is defined to equal cA.

SageMath

The syntax for scalar multiplication, matrix addition, and matrix subtraction in SageMath is much like calculator syntax. The plus sign is used for addtion, the minus sign for subtraction, and the asterisk for multiplication. The asterisk is not optional. Typing two quantities with no operator between produces an error. Multiplication is not implied by lack of a symbol. SageMath code that reproduces the calculations of this section follows.

```
A=matrix(3,3,[1,4,7,2,6,3,8,5,9])
B=matrix(3,3,[7,4,6,2,8,3,9,5,1])
print(A+B)
print()
C=matrix(3,3,[6,3,4,5,9,2,7,1,8])
print(5*C)
```

Try it at `sagecell.sagemath.org`. The output is as follows.

```
[ 8  8 13]
[ 4 14  6]
[17 10 10]

[30 15 20]
[25 45 10]
[35  5 40]
```

Exercises

1. Perform the operation if possible.

(a) $\begin{bmatrix} -1 & -6 & 0 \\ -6 & -5 & 10 \end{bmatrix} + \begin{bmatrix} 1 & -10 & 3 \\ 9 & 0 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1.6 & 8.4 \\ 8.16 & -0.33 \end{bmatrix} + \begin{bmatrix} 4.01 & 1.75 \\ 9.35 & 1.49 \\ -0.24 & 0.58 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & -8 & 7 & 5 \\ -9 & -3 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & -4 & 7 & -8 \\ 1 & -2 & 2 & -5 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -10 & -3 \\ 1 & 8 \end{bmatrix}$

(e) $\begin{bmatrix} -6 \\ 0 \\ -6 \end{bmatrix} + \begin{bmatrix} 9 \\ 10 \\ 0 \end{bmatrix}$

(f) $2\begin{bmatrix} 5 & -11 & -2 \\ 14 & 1 & -8 \\ 13 & -1 & 6 \end{bmatrix}$

(g) $\begin{bmatrix} 3.43 & 6.59 \\ -0.96 & 0.16 \end{bmatrix} + \begin{bmatrix} -0.78 & 8.68 \\ 2.14 & 8.79 \end{bmatrix}$

(h) $\begin{bmatrix} -9 & 1 & 5 \\ 10 & 1 & -10 \\ 2 & -3 & -7 \end{bmatrix} + \begin{bmatrix} -2 & 7 & 8 \\ 9 & -4 & 9 \\ 2 & -10 & -10 \end{bmatrix}$

(i) $2\begin{bmatrix} -1 & 6 \\ 8 & 15 \end{bmatrix}$

(j) $\begin{bmatrix} 4.65 & 1.33 & 8.86 \\ 6.03 & 4.56 & 4.8 \end{bmatrix} - \begin{bmatrix} 1.85 & 6.4 & 7.33 \\ 4.58 & 8.39 & 1.89 \end{bmatrix}$

(k) $\begin{bmatrix} 0 & 6 & -8 & -2 \\ 8 & 10 & 7 & -3 \end{bmatrix} + \begin{bmatrix} 9 & 2 \\ 6 & -3 \end{bmatrix}$

(l) $\begin{bmatrix} 4.83 & 7.65 \\ -0.48 & 7.82 \\ 0.25 & 2.53 \end{bmatrix} - \begin{bmatrix} 4.44 & 6.57 \\ 4.22 & 7.17 \end{bmatrix}$

(m) $\begin{bmatrix} 1 & -9 & 6 & 10 \end{bmatrix} - \begin{bmatrix} -2 & -1 & 2 & -7 \end{bmatrix}$

(n) $\begin{bmatrix} 9 & -10 & 4 \\ 10 & 10 & 1 \\ 0 & -8 & 3 \end{bmatrix} - \begin{bmatrix} 10 & -4 & -4 \\ 8 & 9 & 4 \end{bmatrix}$

(o) $\begin{bmatrix} 2 \\ 7 \\ 9 \\ 3 \end{bmatrix} + \begin{bmatrix} -7 & 8 \\ 3 & 9 \\ -4 & 2 \\ 2 & 9 \end{bmatrix}$

(p) $\begin{bmatrix} 10 \\ 4 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 & 10 & -2 \end{bmatrix}$

(q) $3.19\begin{bmatrix} -12.96 \\ -0.96 \\ -7.99 \\ 11.05 \end{bmatrix}$

(r) $\begin{bmatrix} -6 & -4 \\ 7 & -2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} -4 & -6 \\ 10 & 5 \\ -3 & 8 \end{bmatrix}$

2. Suppose M is a 5×5 matrix and $M + N$ is defined (the sum can be computed). How many entries does N have?

3. In your own words, describe how to add or subtract two matrices, and explain how to determine whether the addition or subtraction can be done.

4. Can a matrix with 29 nonzero entries be added to a matrix with 25 nonzero entries?

5. Suppose M and N are matrices such that their sum is defined ($M + N$ can be computed). Is the following true or false?

$$M + N = N + M$$

6. Suppose M and N are matrices such that their difference is defined ($M - N$ can be computed). Is the following true or false?

$$M - N = N - M$$

7. Suppose M is a matrix of size 3×7, c is a scalar, and the matrix computation cM is defined. What is the size of matrix cM?

For the remaining exercises, let

$$A = \begin{bmatrix} 42 & 0 & -47 & -34 & -10 & -48 \\ 8 & 26 & 43 & -18 & -20 & -30 \\ -41 & -40 & -29 & -36 & -44 & 12 \\ -42 & 47 & 28 & 4 & 38 & -22 \\ 18 & -15 & -1 & 29 & 37 & 9 \end{bmatrix} \quad N = \begin{bmatrix} -21 & -33 & 28 & -15 & 34 & 45 \\ 27 & 40 & -13 & -23 & -10 & 15 \\ 43 & -6 & 46 & 17 & 13 & 21 \\ -40 & -46 & 2 & 16 & 22 & -14 \\ 10 & -12 & 29 & 35 & 48 & -31 \end{bmatrix}$$

$$Q = \begin{bmatrix} -17 & -37 & -34 & 20 & -14 & 10 \\ -23 & 44 & 47 & 18 & 19 & 49 \\ 11 & 33 & 35 & -50 & 2 & 9 \\ -36 & -18 & 7 & 17 & -49 & 31 \\ -8 & 16 & 28 & -32 & -2 & 5 \end{bmatrix} \quad T = \begin{bmatrix} 40 & 47 & 13 & -2 & -22 & 3 \\ -45 & 4 & -16 & 6 & -18 & 8 \\ 18 & -26 & -27 & -19 & -48 & -35 \\ 33 & 35 & 9 & 25 & 2 & 7 \\ -8 & 10 & -12 & -34 & 11 & 38 \end{bmatrix}$$

8. Compute $A + Q$

9. Compute $N - 5T$

10. Compute $3.17(1.11Q + .22N)$

You may follow this link or copy and paste the following code into a SageCell to get started.

```
A = matrix(5,6,[42,0,-47,-34,-10,-48,8,26,43,-18,-20,-30,-41,-40,-29,-36,
                -44,12,-42,47,28,4,38,-22,18,-15,-1,29,37,9,])
N = matrix(5,6,[-21,-33,28,-15,34,45,27,40,-13,-23,-10,15,43,-6,46,17,
                13,21,-40,-46,2,16,22,-14,10,-12,29,35,48,-31,])
Q = matrix(5,6,[-17,-37,-34,20,-14,10,-23,44,47,18,19,49,11,33,35,-50,
                2,9,-36,-18,7,17,-49,31,-8,16,28,-32,-2,5,])
T = matrix(5,6,[40,47,13,-2,-22,3,-45,4,-16,6,-18,8,18,-26,-27,-19,
                -48,-35,33,35,9,25,2,7,-8,10,-12,-34,11,38,])
```

Answers

Sudoku sum: Since each block of a sudoku board is required to contain the numbers from 1 through 9 exactly once each, the sum of a single block is $1 + 2 + 3 + \cdots + 9 = \frac{9 \cdot 10}{2} = 45$ making the sum of any pair of blocks 90.

1.3 Matrix Multiplication

Matrix addition, matrix subtraction and scalar multiplication are each done component-wise, something many people find natural. Even those for whom it does not come naturally rarely question why the operations are done the way they are. After explanation, they are acceptable. Devoid of context, however, there is nothing natural or intuitive about matrix multiplication. It's not difficult. It just takes some getting used to. The purpose of the current section is to start the process of familiarization. The reason multiplication is done the way it is will not come up for a little while yet. In the meantime, a little patience and concentration will be enough.

If you can master the product of a **row matrix** (a $1 \times n$ matrix) with a **column matrix** (an $m \times 1$ matrix), you can master the product of any two matrices. The following example illustrates the process.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix} \tag{1.3.1}$$

Given a row matrix R and a column matrix C with the same number of entries, say n, their product is the sum of the products of corresponding entries. That is,

$$RC = \begin{bmatrix} r_{1,1}c_{1,1} + r_{1,2}c_{2,1} + \cdots + r_{1,n}c_{n,1} \end{bmatrix}.$$

The first entry of R (reading from left to right) corresponds with the first entry of C (reading from top to bottom). The second entry of R corresponds with the second entry of C, and so on. The product of the two matrices is the sum of these entry products. As with addition, multiplication is an operator, so the product of two matrices is a matrix. In this case, a 1×1 matrix, as shown in (1.3.1). If R and C differ in length the product RC is undefined.

For matrices with multiple rows and columns, this row-matrix-column-matrix calculation is repeated for each entry of their product. The i, j-entry of AB is the single entry of the product of the i^{th} row of A with the j^{th} column of B (where this makes sense). That is, if A and B are matrices, then the product AB is calculated by setting $(AB)_{i,j}$ equal to the lone entry of $A_{i,:}B_{:,j}$ (where this makes sense). Several conclusions can be drawn from this description.

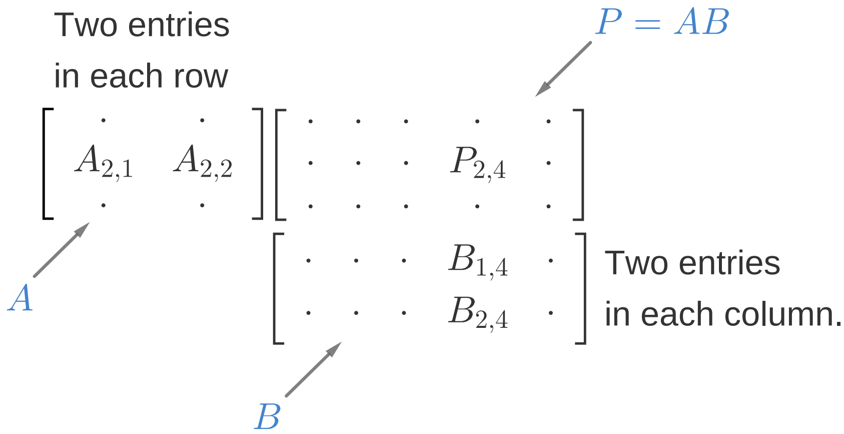
- The rows of A and the columns of B must all have the same length (number of entries). Otherwise $A_{i,:}B_{:,j}$, is undefined.
- P has the same number of rows as A (P and A have the same height).
- P has the same number of columns as B (P and B have the same width).

These last two observations suggest an organizational technique for multiplication. Writing B to the right of A and just below leaves a space above B and to the right of A that's exactly the right size for the product AB. Plus, the row needed for calculating $(AB)_{i,j}$ is directly left of it and the column needed for calculating $(AB)_{i,j}$ is directly below it. See figure 1.3.1.

Transposition and the Dot Product

If A is a matrix, then its transpose is the matrix resulting from turning the rows of A into columns. The first row of the matrix becomes the first column of the transpose. The second row of the matrix becomes the second column of the transpose, and so on. Equivalently, the transpose of a matrix A is the matrix resulting from turning the columns of A into rows. The first column of the matrix becomes the first row

Figure 1.3.1: $(AB)_{2,4} = P_{2,4} = A_{2,1}B_{1,4} + A_{2,2}B_{2,4}$



of the transpose. The second column of the matrix becomes the second row of the transpose, and so on. Can you see why turning rows into columns and turning columns into rows are equivalent?

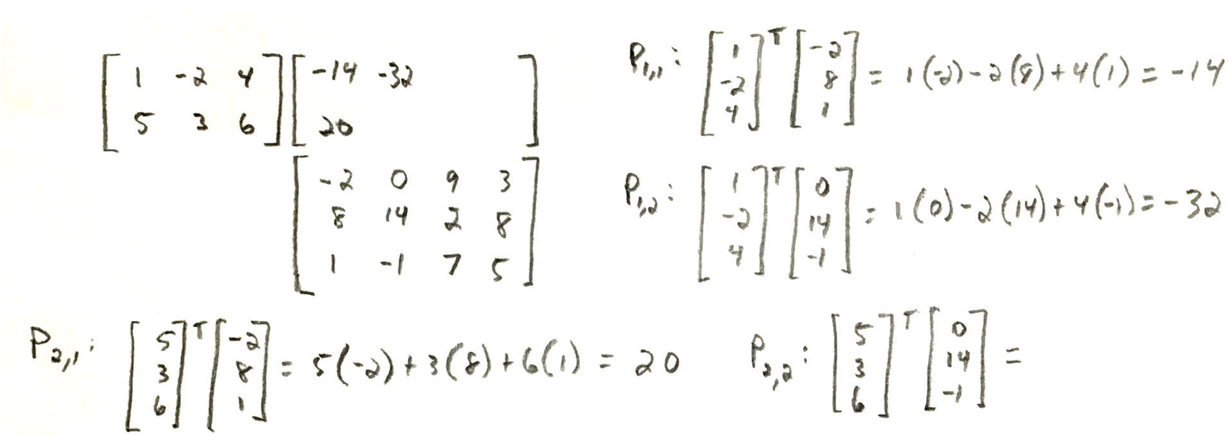
If a matrix has only one row (is a row matrix) then its transpose has one column (is a column matrix), and vice versa. Using a superscript T for transpose the row-matrix-column-matrix product from the beginning of this section can be written

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}^T \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix} \tag{1.3.2}$$

Writing this way may help you keep track of which numbers should be multiplied by which since they are side by side in the expression using the transpose. Combining this observation with the organizational technique of figure 1.3.1, computing the product

$$\begin{bmatrix} 1 & -2 & 4 \\ 5 & 3 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 & 9 & 3 \\ 8 & 14 & 2 & 8 \\ 1 & -1 & 7 & 5 \end{bmatrix}$$

might look like the following on paper (at least to start).



For example, the -32, $P_{1,2}$, is calculated by taking the row directly to its left, $\begin{bmatrix} 1 & -2 & 4 \end{bmatrix}$, and multiplying by the column directly below it, $\begin{bmatrix} 0 \\ 14 \\ -1 \end{bmatrix}$. This product is calculated to the right of the matrices and is just one of the 8 entries of the product. It looks like a lot of work, and it is! Not to worry, though. With some practice, you will become proficient and not have to write down all the individual row-matrix-column-matrix products in such detail. In fact, it will be very important that you acquire such proficiency. This row-matrix-column-matrix calculation sits at the core of linear algebra and its connection to various sciences. If you have seen the dot product, a very similar calculation, in physics or calculus, think of the row-matrix-column-matrix product as the linear algebra equivalent of the dot product.

In physics or calculus (**vectors**): $\langle 5,3,6\rangle \cdot \langle 0,14,-1\rangle = 5\cdot 0 + 3\cdot 14 + 6\cdot -1 = 36$

In linear algebra (matrices): $\begin{bmatrix} 5 \\ 3 \\ 6 \end{bmatrix}^T \begin{bmatrix} 0 \\ 14 \\ -1 \end{bmatrix} = 5\cdot 0 + 3\cdot 14 + 6\cdot -1 = 36$

It's the same calculation! There are enough similarities between column matrices and vectors that we often use column matrix notation to represent vectors and call them **column vectors** or just vectors, and we call the row-matrix-column-matrix calculation the **dot product**.

Crumpet 7: Row Vector

A row matrix is sometimes referred to as a **row vector** and can be used to represent vectors like those in physics or calculus just as a column vector can.

Thus the distinction between the two objects is blurred, but make no mistake, a column matrix is a matrix, and a vector is a vector. They are not the same thing. It is a convenience in linear algebra to represent vectors as column matrices, giving the column matrix notation two meanings, (1) a matrix, and (2) a vector. Though we try not to do this type of thing in mathematics often, it happens sometimes, much like the numerous words in English with multiple meanings. What you can do with a ring depends entirely on what type of ring. A wedding ring might be worn on your ring finger, and a circus ring might contain a tiny car with two dozen clowns in it. Certainly not the other way around!

Crumpet 8: Ring

In mathematics, a ring is a set together with two binary operators that satisfy a number of properties. This is something you will study in abstract algebra.

Analogously, what you can do with a one-column array of numbers depends entirely on what it represents. If it represents a matrix, it might be transposed or used in the solution of a sytem of linear equations. If it represents a vector it might be used in the dot product with another vector or plotted in the Cartesian coordinate system.

Notice the product in equation (1.3.2) is written as a 1×1 matrix, but the same type of matrix product is written as a scalar in the pencil-and-paper calculation of a matrix product. This is another example of a single notation having multiple interpretations, indicated through context. There is no context for equation (1.3.2), so the product is rightfully a matrix. In the calculation of a matrix product, the result of each individual dot product will become an entry—a scalar—not a matrix in the resulting matrix. The square brackets are dropped. The 1×1 matrix is treated as if it were a scalar. In fact, 1×1 matrices and scalars are often used interchangeably, jeopardizing the distinction between these two objects. Again, make no mistake, a 1×1 matrix is a matrix, and a scalar is not a matrix at all. They are different things. It is a convenience to let 1×1 matrix notation (square brackets) and scalar notation (lack of delimiters) each represent both objects, whichever is appropriate for the situation.

Compute the products.

$$\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix}$$

Besides good practice in multiplying matrices, this example shows that

$$\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 7 \end{bmatrix},$$

and more importantly, therefore ***matrix multiplication is not commutative***. Given matrices M and N, we cannot expect MN and NM to be equal even when both products are defined.

Key Concepts

column vector A vector represented as a column matrix.

matrix multiplication For any matrices A and B, if the rows of A and the columns of B have the same number of entries, then the product AB is defined. Moreover, AB has the same number of rows (height) as A and the same number of columns (width) as B, and $(AB)_{i,j}$ equals the lone entry of $A_{i,:}B_{:,j}$ for all entries $(AB)_{i,j}$ of AB. If the rows of A and columns of B do not have the same number of entries, then AB is undefined.

transpose For any $m \times n$ matrix A, the transpose of A, denoted A^T, is defined as the $n \times m$ matrix with $(A^T)_{i,j} = a_{j,i}$ for each entry $a_{j,i}$ of A.

vector A quantity with both magnitude and direction.

dot product the dot product of $m \times 1$ matrices $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{u}^T\mathbf{v}$.

SageMath

If `M` is a matrix in SageMath, then `M.transpose()` is its transpose. The following code defines the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, extracts columns 2 and 3 as column matrices, and finds the (matrix) product $A_{:,2}^T A_{:,3}$.

```
A=matrix(2,3,[1,2,3,4,5,6])
print("Matrix A:")
print(A)
print()
print("Treating columns 2 and 3 as matrices:")
print()
c2 = matrix(2,1,A.column(1))
```

```
c3 = matrix(2,1,A.column(2))
print("column 2:")
print(c2)
print()
print("column 3:")
print(c3)
print()
print("column 2 transpose times column 3:")
print(c2.transpose()*c3)
```

The output of this code is

```
Matrix A:
[1 2 3]
[4 5 6]

Treating columns 2 and 3 as matrices:

column 2:
[2]
[5]

column 3:
[3]
[6]

column 2 transpose times column 3:
[36]
```

Notice the columns are displayed as column matrices, and the product is also displayed as a matrix, using the square brackets. The `.column()` method extracts a column of a matrix as a vector, however, which is why the definitions of `c2` and `c3` explicitly take each column and feed them to the `matrix()` function.

On the other hand, SageMath is perfectly capable of treating the columns as vectors, as seen in the following code. The * operator is used to compute the dot product of two vectors.

```
A=matrix(2,3,[1,2,3,4,5,6])
print("Matrix A:")
print(A)
print()
print("Treating columns 2 and 3 as vectors:")
print()
c2 = A.column(1)
c3 = A.column(2)
print("column 2:")
print(c2)
print()
print("column 3:")
print(c3)
```

```
print()
print("Dot product of columns 2 and 3:")
print(c2*c3)
```

The output of this code is

```
Treating columns 2 and 3 as vectors:

column 2:
(2, 5)

column 3:
(3, 6)

Dot product of columns 2 and 3:
36
```

Notice the notation for a vector (parentheses around a comma-separated list of entries), making it clear SageMath is interpreting the columns as vectors, not matrices. Also notice the dot product is displayed (and indeed interpreted) as a scalar, not a matrix.

Exercises

1. Multiply if possible.

   (a) $\begin{bmatrix} 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$

   (b) $\begin{bmatrix} 7 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -5 \end{bmatrix}$

   (c) $\begin{bmatrix} -1 & 0 & -3 \end{bmatrix}\begin{bmatrix} 6 \\ -2 \\ 5 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}$

   (e) $\begin{bmatrix} -3 & 2 \end{bmatrix}\begin{bmatrix} 7 \\ -1 \end{bmatrix}$

   (f) $\begin{bmatrix} 6.3 \\ 4.1 \\ 3.4 \end{bmatrix}\begin{bmatrix} 2.3 & 4.5 \end{bmatrix}$

   (g) $\begin{bmatrix} 5.8 & 0.2 \end{bmatrix}\begin{bmatrix} 2.5 \\ 3.8 \end{bmatrix}$

   (h) $\begin{bmatrix} -3 & 3 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$

   (i) $\begin{bmatrix} 7 \\ -2 \end{bmatrix}\begin{bmatrix} -2 & 0 \end{bmatrix}$

   (j) $\begin{bmatrix} 7 & 0 & 4 & 6 \end{bmatrix}\begin{bmatrix} -2 \\ 7 \\ 1 \\ 3 \end{bmatrix}$

   (k) $\begin{bmatrix} 1.35 & 4.58 & 7.36 \end{bmatrix}\begin{bmatrix} 3.36 & -0.25 & 1.6 \end{bmatrix}$

   (l) $\begin{bmatrix} 4 \\ -1 \end{bmatrix}\begin{bmatrix} 5 & 1 & 4 & 6 \end{bmatrix}$

2. Multiply if possible.

   (a) $\begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 7 \\ 6 & 3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0.03 & -0.6 \\ 4.25 & 5.09 \end{bmatrix}\begin{bmatrix} -0.3 \\ 4.6 \end{bmatrix}$

   (c) $\begin{bmatrix} 1 & 9 & 10 \\ 3 & 0 & 8 \\ 3 & 8 & 10 \end{bmatrix}\begin{bmatrix} 8 & 4 \\ 10 & 10 \\ 9 & 5 \\ 3 & 8 \end{bmatrix}$

   (d) $\begin{bmatrix} -3 & 0 & 1 \\ 2 & 5 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$

   (e) $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

   (f) $\begin{bmatrix} 6 & 7 & 4 \\ -3 & 0 & 7 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & 1 \\ -1 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 7.94 \\ 1.15 \\ 2.88 \\ 8.95 \end{bmatrix}\begin{bmatrix} 9.98 & 2.91 \\ 1.48 & 8.05 \\ 6.41 & 9.67 \\ 5.16 & 8.88 \end{bmatrix}$

(h) $\begin{bmatrix} 3 & 0 \\ 5 & -4 \end{bmatrix}\begin{bmatrix} 6 & 3 & 1 \\ 6 & 1 & 7 \end{bmatrix}$

(i) $\begin{bmatrix} 2 & 5 \\ -1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 6 & -3 \end{bmatrix}$

(j) $\begin{bmatrix} 8 \\ 0 \\ 9 \end{bmatrix}\begin{bmatrix} 2 & 4 & -1 & 1 \\ 9 & 6 & 10 & 8 \end{bmatrix}$

(k) $\begin{bmatrix} 0 & 0 & 3 & 6 \\ -3 & 0 & 7 & -3 \\ 1 & -1 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \\ 4 \\ 5 \end{bmatrix}$

(l) $\begin{bmatrix} 5 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 6 & -1 \end{bmatrix}$

(m) $\begin{bmatrix} 3.47 & -2.73 \end{bmatrix}\begin{bmatrix} 5.53 & 5.89 \\ 5.24 & 0.82 \end{bmatrix}$

(n) $\begin{bmatrix} 1 & -1 & 6 & 3 \\ 10 & 4 & 8 & 3 \end{bmatrix}\begin{bmatrix} 8 & 3 & 7 & 10 \\ 8 & 1 & 4 & 4 \end{bmatrix}$

3. Suppose A is a matrix of size 2×7, C is a matrix of size 5×7, and the matrix computation $A + BC$ is defined. What is the size of matrix B?

4. Matrices A, B, C, D are such that $(A + B)(CD)$ is defined (all of the operations are possible). If B is a 3×4 matrix and D is a 5×8 matrix, what are the dimensions of A and C?

5. Describe how to multiply two matrices, and explain how to determine whether the multiplication can be done.

6. Find a pair of matrices M and N so that MN is defined, but NM is not, and therefore $MN \neq NM$.

7. Find a pair of matrices M and N such that MN and NM are both defined but are different sizes, and therefore $MN \neq NM$.

8. Find a pair of 3×3 matrices M and N such that $MN \neq NM$.

9. Can you find a pair of 2×2 matrices M and N such that $MN = NM$?

10. Suppose the matrix product MN is defined (the multiplication can be done). Which of the following is true?

    (a) M and N must have the same number of rows.

    (b) M and N must have the same number of columns.

    (c) The number of rows of M must equal the number of columns of N .

    (d) The number of columns of M must equal the number of rows of N .

    (e) None of the above.

For the remaining exercises, let

$$A = \begin{bmatrix} 42 & 0 & -47 & -34 & -10 & -48 \\ 8 & 26 & 43 & -18 & -20 & -30 \\ -41 & -40 & -29 & -36 & -44 & 12 \\ -42 & 47 & 28 & 4 & 38 & -22 \\ 18 & -15 & -1 & 29 & 37 & 9 \end{bmatrix} \quad U = \begin{bmatrix} -21 & -33 & 28 & -15 & 34 & 45 \\ 27 & 40 & -13 & -23 & -10 & 15 \\ 43 & -6 & 46 & 17 & 13 & 21 \\ -40 & -46 & 2 & 16 & 22 & -14 \\ 10 & -12 & 29 & 35 & 48 & -31 \end{bmatrix}$$

$$Q = \begin{bmatrix} -17 & -37 & -34 & 20 & -14 & 10 \\ -23 & 44 & 47 & 18 & 19 & 49 \\ 11 & 33 & 35 & -50 & 2 & 9 \\ -36 & -18 & 7 & 17 & -49 & 31 \\ -8 & 16 & 28 & -32 & -2 & 5 \end{bmatrix} \quad R = \begin{bmatrix} 40 & 47 & 13 & -2 & -22 & 3 \\ -45 & 4 & -16 & 6 & -18 & 8 \\ 18 & -26 & -27 & -19 & -48 & -35 \\ 33 & 35 & 9 & 25 & 2 & 7 \\ -8 & 10 & -12 & -34 & 11 & 38 \end{bmatrix}$$

11. Compute $(A^T)(U)$ and $(U)(A^T)$. Are they the same?

12. Compute $\left(3Q^T - 2R^T\right)^T$ and $3Q - 2R$. What do you notice? Why?

13. Can you determine which of the following computations are defined? Ask SageMath to compute them all. The ones that are undefined will produce long error messages.

$$(QR)^T \qquad AU^T R \qquad QUAR^T \qquad A^T QU^T A$$
$$A + R^T \qquad AR^T \qquad (R - A)^T$$

You may follow this link or copy and paste the following code into a SageCell to get started.

```
A = matrix(5,6,[42,0,-47,-34,-10,-48,8,26,43,-18,-20,-30,-41,-40,-29,-36,
                -44,12,-42,47,28,4,38,-22,18,-15,-1,29,37,9,])
U = matrix(5,6,[-21,-33,28,-15,34,45,27,40,-13,-23,-10,15,43,-6,46,17,
                13,21,-40,-46,2,16,22,-14,10,-12,29,35,48,-31,])
Q = matrix(5,6,[-17,-37,-34,20,-14,10,-23,44,47,18,19,49,11,33,35,-50,
                2,9,-36,-18,7,17,-49,31,-8,16,28,-32,-2,5,])
R = matrix(5,6,[40,47,13,-2,-22,3,-45,4,-16,6,-18,8,18,-26,-27,-19,
                -48,-35,33,35,9,25,2,7,-8,10,-12,-34,11,38,])
```

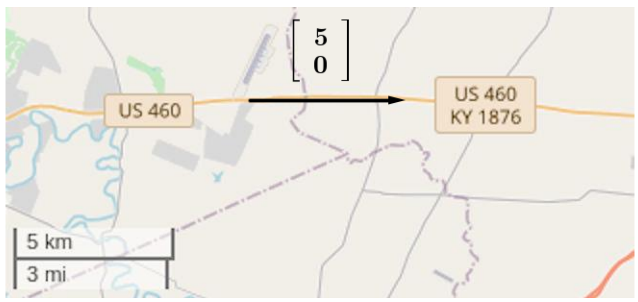
1.4 Length and Orthogonality

Geometric Interpretation of Vectors

One day, my friend Victor took a 5 kilometer drive. When Victor told me this I knew just how long his drive was. It was 5 kilometers. When Victor added that his drive was on a very straight highway headed due east, I knew more. I knew which way Victor was driving. I could imagine tracing out his path on a map by drawing a horizontal arrow pointing to the right (eastward) with a length equivalent to 5 kilometers. The arrow captures both the direction and length of Victor's drive. Vectors can be imagined in the same way. The vector

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

has a 5 as its first entry and a 0 as its second. Thinking of these entries as x- and y- coordinates, the five represents 5 units right (eastward) and the zero represents 0 units up (northward). In this way, the vector represents both the length and direction of Victor's drive, just like the arrow. The vector and the arrow can be interpreted to represent the same thing, blurring any distinction between them. [3]

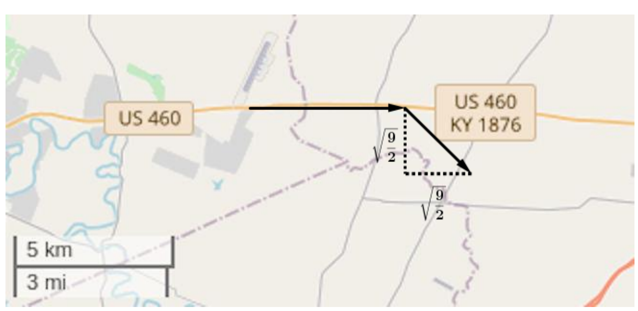


The vector/arrow represents Victor's displacement, or movement, 5 kilometers in the eastward direction.

Notice there is no origin on the map. This is typical of drawing vectors. They are not specified relative to an origin. They only represent a change in location, or displacement, starting anywhere. A vector represents the locations of two points relative to one another. Exactly where those two points lie is not determined by the vector itself. Further information is needed to locate the vector. In the case of Victor's travel, I needed to know on what road and where he was driving to create an accurate picture of his drive.

After driving 5 kilometers east, Victor exited the highway and drove 3 kilometers southeast (using a road that does not appear on the map). When I heard this, I was able to capture this part of Victor's journey by the vector

$$\begin{bmatrix} \sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}.$$

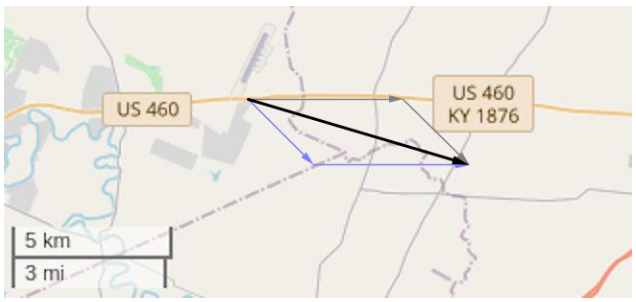


[3]Street map minus vectors © OpenStreetMap contributors

And I knew exactly where to put it since it started just where the previous leg left off. Drawn as an arrow, the vector is the hypotenuse of a right triangle with side lengths $\sqrt{\frac{9}{2}}$, which by the Pythagorean theorem gives it length $\sqrt{\left(\sqrt{\frac{9}{2}}\right)^2+\left(\sqrt{\frac{9}{2}}\right)^2}=3$. It has the right length and it points southeast (and starts where the first leg leaves off), so it accurately represents the second leg of Victor's drive.

As the crow flies, Victor's total displacement or movement for the drive is represented by the sum of the vectors,

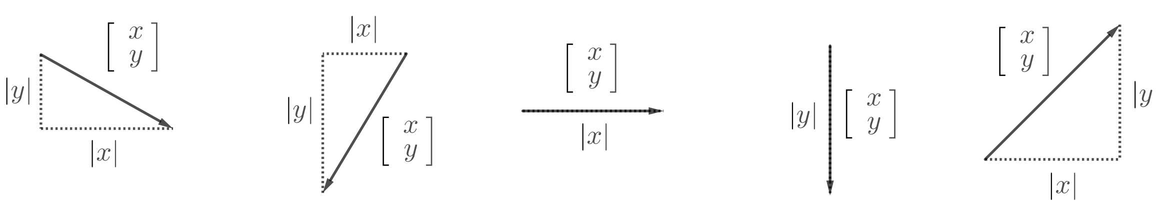
$$\begin{bmatrix} 5+\sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}.$$



Since addition of vectors is commutative, it does not matter which vector is plotted first. After the pair of displacements, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}$, Victor's total displacement is $\begin{bmatrix} 5+\sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}$. In the diagram, the gray vectors represent $\begin{bmatrix} 5 \\ 0 \end{bmatrix}+\begin{bmatrix} \sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}$ and the blue vectors represent $\begin{bmatrix} \sqrt{\frac{9}{2}} \\ -\sqrt{\frac{9}{2}} \end{bmatrix}+\begin{bmatrix} 5 \\ 0 \end{bmatrix}$. The five vectors together illustrate the parallelogram rule for vector addition. The sum of two vectors is a diagonal of the parallelogram determined by the two vectors.

Perpendicularity

The length of a vector, not surprisingly, is defined by the length of its representative arrow. A collection of vectors pointing in various directions, including vertical and horizontal are shown below.



Regardless of which direction the vector $\mathbf{v}=\begin{bmatrix} x \\ y \end{bmatrix}$ points, its length is $\sqrt{|x|^2+|y|^2}$ or simply $\sqrt{x^2+y^2}$. The Pythagorean theorem can be used to calculate lengths of vectors that are not horizontal or vertical.

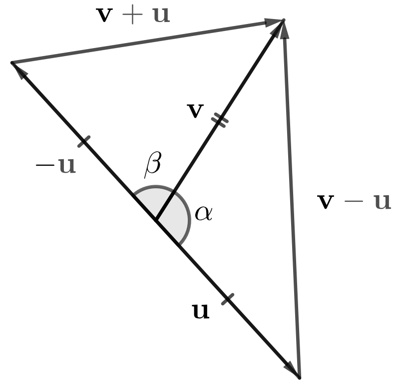
Coincidentally the dot product of $\mathbf{v}$ with itself, $\mathbf{v}^T\mathbf{v}$, is

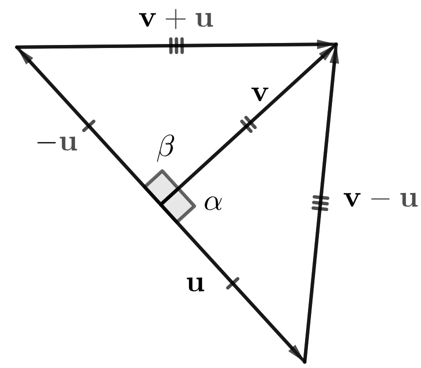
$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = x^2+y^2$$

so the length of **v** can also be written as $\sqrt{\mathbf{v}^T\mathbf{v}}$. This expression has a nice symmetry and is independent of the number of entries in **v**. It could apply to vectors with 3, 8, or 28 entries just as wel as vectors with 2 entries. Indeed, this is how the length, or magnitude, of a vector is defined for any size vector. The **length** of a vector **v**, denoted by $||\mathbf{v}||$, is defined as

$$||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}}.$$

The following diagram illustrates the relationship between the lengths of vectors $\mathbf{v} - \mathbf{u}$ and $\mathbf{v} + \mathbf{u}$. By the side-angle-side theorem from geometry the marked triangles in each figure are congruent if and only if $\alpha = \beta$. Since α and β together form a straight angle, $\alpha = \beta$ if and only if they are both right angles. Consequently the lengths of $\mathbf{v} + \mathbf{u}$ and $\mathbf{v} - \mathbf{u}$ are equal if and only if **u** and **v** are perpendicular.





This observation leads to a very useful property of the dot product, exposed by the following calculation.

$$\begin{aligned}
||\mathbf{v} + \mathbf{u}|| &= ||\mathbf{v} - \mathbf{u}|| \\
\sqrt{(\mathbf{v} + \mathbf{u})^T(\mathbf{v} + \mathbf{u})} &= \sqrt{(\mathbf{v} - \mathbf{u})^T(\mathbf{v} - \mathbf{u})} \\
(\mathbf{v} + \mathbf{u})^T(\mathbf{v} + \mathbf{u}) &= (\mathbf{v} - \mathbf{u})^T(\mathbf{v} - \mathbf{u}) \\
(\mathbf{v}^T + \mathbf{u}^T)(\mathbf{v} + \mathbf{u}) &= (\mathbf{v}^T - \mathbf{u}^T)(\mathbf{v} - \mathbf{u}) \\
\mathbf{v}^T\mathbf{v} + \mathbf{v}^T\mathbf{u} + \mathbf{u}^T\mathbf{v} + \mathbf{u}^T\mathbf{u} &= \mathbf{v}^T\mathbf{v} - \mathbf{v}^T\mathbf{u} - \mathbf{u}^T\mathbf{v} + \mathbf{u}^T\mathbf{u} \\
\mathbf{v}^T\mathbf{u} + \mathbf{u}^T\mathbf{v} &= -\mathbf{v}^T\mathbf{u} - \mathbf{u}^T\mathbf{v} \\
2\mathbf{v}^T\mathbf{u} &= -2\mathbf{u}^T\mathbf{v} \\
2\mathbf{v}^T\mathbf{u} &= -2\mathbf{v}^T\mathbf{u} \\
4\mathbf{v}^T\mathbf{u} &= 0 \\
\mathbf{v}^T\mathbf{u} &= 0 \qquad (1.4.1)
\end{aligned}$$

Passing from the seventh equation to the eighth equation depends on the fact that $\mathbf{v}^T\mathbf{u} = \mathbf{u}^T\mathbf{v}$. Can you show this is true for any vectors of equal size? Answer on page 27. Thus vectors **u** and **v** (with two entries) are perpendicular if and only if their dot product is zero. As with the formula for length, this formula naturally extends to vectors with more than two entries. We say that vectors **u** and **v** of the same size are **orthogonal** if and only if their dot product is zero. For vectors with two or three entries this means the vectors, eminating from the same point, are perpendicular. In this way, orthogonality extends the idea of perpendicularity to dimensions greater than three.

Key Concepts

geometric interpretation of vectors vectors are often thought of as displacements represented by arrows.

geometric interpretation of vector sum the sum of two vectors is represented geometrically by a diagonal of the parallelogram determined by the two vectors.

length the length of a column vector $\mathbf{v}$ is $\sqrt{\mathbf{v}^T\mathbf{v}}$.

orthogonal two vectors whose dot product is defined and is are orthogonal.

SageMath

SageMath distinguishes between vectors and matrices, but just like in mathematics the distinction is blurry. The SageMath code

```
u=vector([1,2,3])
v=matrix(3,1,[3,2,1])
print(u*v)
```

runs even though the third line requests the product of a vector with a matrix. SageMath treats matrix v as if it were a vector, sort of. The output of the code is

```
(10)
```

a vector with one entry—-not a scalar and not a 1×1 matrix. If v is defined as a vector as in the following code, the output is the scalar value 10, not a vector.

```
u=vector([1,2,3])
v=vector([3,2,1])
print(u*v)
```

produces

```
10
```

SageMath's internal process of converting one type of variable to another to avoid throwing an error, a process called coersion, can produce unanticipated results. More predictable results are obtained by explicitly converting one type of variable to another. The SageMath code

```
u=vector([1,2,3])
v=matrix(3,1,[3,2,1])
print(u*vector(v))
```

explicitly tells SageMath to treat v as a vector in the computation of the product so no coersion is needed, and it produces

```
10
```

just as if v were defined as a vector in the first place.

Any row or column matrix can be converted to a vector the same way. In fact, vectors can be converted to row or column matrices just as easily. The following code converts u to a matrix (instead of converting v to a vector) and then computes the dot product.

```
u=vector([1,2,3])
v=matrix(3,1,[3,2,1])
print(matrix(1,3,u)*v)
```

produces the 1×1 matrix

```
[10]
```

since the multiplicands are both matrices. Be aware that vectors and matrices are not equivalent in SageMath. Unexpected results may be seen when the two types are intermingled. To avoid surprises, convert one to the other explicitly as needed.

The length of a vector can be computed using the `.norm()` method. Consistent with the developing theme, the `.norm()` method can be applied to either matrices or vectors, and the results are different! The following code defines the "same" vector as both a SageMath vector and a SageMath matrix and then outputs their lengths, or norms.

```
u_vec=vector([6,5,-3])
u_mat=matrix(1,3,[6,5,-3])
print(u_vec.norm())
print(u_mat.norm())
```

produces

```
sqrt(70)
8.366600265340756
```

The norm of a vector is computed symbolically while the norm of a matrix is computed as an approximate decimal equivalent. $\sqrt{70} \approx 8.366600265340756$.

Exercises

1. Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and set
$$\mathbf{w} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}.$$
   (a) Calculate $\mathbf{u}^T\mathbf{w}$.
   (b) Calculate $\mathbf{v}^T\mathbf{w}$.
   (c) Are $\mathbf{u}$ and $\mathbf{w}$ perpendicular?
   (d) Are $\mathbf{v}$ and $\mathbf{w}$ perpendicular?

2. Justify the claim.
   (a) If $\mathbf{u}$ and $\mathbf{v}$ are placed with their tails at the same point, then $\|\mathbf{u} - \mathbf{v}\|$ is the distance between the heads of $\mathbf{u}$ and $\mathbf{v}$. As such, we define the distance between $\mathbf{u}$ and $\mathbf{v}$ as $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$.

dot product equality Letting $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

$$\mathbf{u}^T\mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}^T \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

and

$$\mathbf{v}^T\mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = v_1u_1 + v_2u_2 + \cdots + v_nu_n.$$

Since multiplication of scalars is commutative, these expressions are equal.

1.5 The Determinant

$b^2 - 4ac$ is "the discriminant", but why? Each quadratic function, $p(x) = ax^2 + bx + c$, has two real roots, one (repeated) real root, or two complex roots. The discriminant discriminates between which quadratics are which. If the coefficients a, b, c, of a quadratic function are such that $b^2 - 4ac > 0$, then the quadratic has two real roots (and no others). If the coefficients are such that $b^2 - 4ac = 0$, then the quadratic has one real root (and no others). If the coefficients are such that $b^2 - 4ac < 0$, then the quadratic has two complex roots (and no others). In this way, the quantity $b^2 - 4ac$ associated with the quadratic function $p(x) = ax^2 + bx + c$ determines what type of roots p has. It is determinative of the types of roots, and in this light might just as well be known as a determinant (which, in English, means determinative). In mathematics, though, that term is reserved for linear algebra. The determinant is a determinative calculation that can be made for any matrix much the same way the discriminant is a determinative calculation that can be made for any quadratic function. Exactly what the determinants determines will have to wait a short while.

The determinant of an $m \times n$ matrix is undefined if $m \neq n$, so determinants are calculated only for **square matrices**, those with the same number of columns as rows. The determinant of a 1×1 matrix is its lone entry. That is, the determinant of $\begin{bmatrix} a \end{bmatrix}$ is a. As such, the determinant is a scalar. The notations $\det A$ or $|A|$ are used to denote the determinant of the matrix A.

The determinant of a square matrix with more than one row, and therefore more than one column, is defined recursively. If A is an $n \times n$ matrix[4], then

$$\det A = (-1)^{1+1} A_{1,1} \det A_{\backslash 1,1} + (-1)^{1+2} A_{1,2} \det A_{\backslash 1,2} + \cdots + (-1)^{1+n} A_{1,n} \det A_{\backslash 1,n}. \tag{1.5.1}$$

For example, if $A = \begin{bmatrix} -12 & 49 & -45 & -10 \\ 28 & 45 & -46 & 23 \\ -15 & -28 & 4 & -48 \\ -1 & 34 & -38 & -18 \end{bmatrix}$, then

$$\begin{aligned} \det A = \begin{vmatrix} -12 & 49 & -45 & -10 \\ 28 & 45 & -46 & 23 \\ -15 & -28 & 4 & -48 \\ -1 & 34 & -38 & -18 \end{vmatrix} &= -12 \begin{vmatrix} 45 & -46 & 23 \\ -28 & 4 & -48 \\ 34 & -38 & -18 \end{vmatrix} - 49 \begin{vmatrix} 28 & -46 & 23 \\ -15 & 4 & -48 \\ -1 & -38 & -18 \end{vmatrix} \\ &\quad - 45 \begin{vmatrix} 28 & 45 & 23 \\ -15 & -28 & -48 \\ -1 & 34 & -18 \end{vmatrix} + 10 \begin{vmatrix} 28 & 45 & -46 \\ -15 & -28 & 4 \\ -1 & 34 & -38 \end{vmatrix} \end{aligned} \tag{1.5.2}$$

The determinant of the 4×4 matrix is written in terms of the determinants of four 3×3 matrices, one application of recursive formula (1.5.1). To this point, the computation is not so bad. It would take a minute to write down this quantity by hand. However, you might feel no closer to the final result, which is $-393,294$ by the way, than before. Now there are four separate determinants to determine. To continue the computation, the determinant of each 3×3 matrix would be written in terms of the determinants of three 2×2 matrices, a second application of formula (1.5.1). Thus the determinant of A would be written in terms of twelve 2×2 determinants. A final application of formula (1.5.1) would yield the determinant of A in terms of twenty-four 1×1 determinants (scalars), at which point the arithmetic could be done and the determinant determined. If nothing else, hopefully you are convinced that completing this calculation by hand would take a while and be prone to error. The real point of this example is to familiarize yourself with the recursive definition. Making sure you get the right signs on the coefficients and extract the right

[4] For this formula to work for 1×1 matrices, we define $\det A_{\backslash 1,1}$ to be 1 for a 1×1 matrix.

submatrices at each step takes some practice. Can you use formula (1.5.1) to find $\det\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$? Answer on page 35.

The quantities $(-1)^{1+j}\det A_{\setminus 1,j}$ of formula (1.5.1) are called **cofactors**. More generally, the quantity $(-1)^{i+j}\det A_{\setminus i,j}$ is called the i,j-cofactor of A. Cofactors can be computed for any row-column combination. Using the notation $C_{i,j}$ for the i,j-cofactor, recursion (1.5.1) can be rewritten

$$\det A = A_{1,1}C_{1,1} + A_{1,2}C_{1,2} + \cdots + A_{1,n}C_{1,n}.$$

While more succinct, this presentation hides the details of the calculation.

The expression $-12\begin{vmatrix} 45 & -46 & 23 \\ -28 & 4 & -48 \\ 34 & -38 & -18 \end{vmatrix} - 49\begin{vmatrix} 28 & -46 & 23 \\ -15 & 4 & -48 \\ -1 & -38 & -18 \end{vmatrix} - 45\begin{vmatrix} 28 & 45 & 23 \\ -15 & -28 & -48 \\ -1 & 34 & -18 \end{vmatrix} + 10\begin{vmatrix} 28 & 45 & -46 \\ -15 & -28 & 4 \\ -1 & 34 & -38 \end{vmatrix}$ fro calculation (1.5.2) is an example of a **linear combination**. More generally, let S be a set of objects on which addition and scalar mutliplication are defined. Sound familiar? For scalars $x_1, x_2, \ldots, x_n$ and objects $b_1, b_2, \ldots, b_n$ of S, the expression

$$x_1b_1 + x_2b_2 + \cdots + x_nb_n$$

is called a linear combination of the objects $b_1, b_2, \ldots, b_n$, and $x_1, x_2, \ldots, x_n$ are called the **coefficients** of the linear combination.

Crumpet 9: Linear Combinations

Linear combinations appear in many contexts.

- A polynomial in t is a linear combination of the monomials $1, t, t^2, t^3, \ldots, t^n$.
- A Riemann sum is a linear combination of certain values of a function.
- The solutions of the differential equation $y'' - 4y' + 3y = 0$ are linear combinations of the functions e^x and e^{3x}.
- Numerical approximations of derivatives, such as $-\frac{3}{2h}f(x_0) + \frac{2}{h}f(x_0+h) - \frac{1}{2h}f(x_0+2h)$, are linear combinations of certain values of a function.
- The left-hand side of the equation $3x - 2y = 7$ is a linear combination of the variables x and y.
- The expected value of a random variable with finitely many possible values is a linear combination.

Addition and scalar multiplication are defined for objects such as functions, variables, numbers, integrals, vectors, and matrices. Each of the following is a linear combination.

$$3\sin(x) - 2\sin(2x) + \sin(3x) \qquad 7x + 2y - \frac{4}{5}z$$

$$6\sqrt{2} - 2\sqrt{7} \qquad \int_0^1 f(x)dx + \int_1^2 f(x)dx + \int_2^3 f(x)dx + \int_3^4 f(x)dx$$

$$\frac{1}{\sqrt{5}}\langle -2, 1\rangle - \frac{1}{\sqrt{13}}\langle 3, -2\rangle \qquad 2\begin{bmatrix} 2 & -6 \\ 0 & 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$$

Can you think of other places where you've seen linear combinations?

Sudoku Row Linear Combinations

If you enjoy solving sudoku puzzles, give this one a shot before reading on. Answer on page 35.

			5	6		1	9	
7						4		5
1	5	6			9		3	
3	7	1	6			9		
9				1				7
		5			2	3	1	6
	2		1			5	8	9
5		7						1
	1	8		4	5			

Treating each row of each block of the sudoku board as a 1×3 (row) matrix, the operations of addition and scalar multiplication are inherited by the sudoku rows. This means to add two rows of a sudoku block, the inherited operation of addition comes from writing each row as a row matrix, adding the matrices and writing the resulting matrix as a sudoku row. For example,

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 7 \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline 2 & 6 & 3 \\ \hline \end{array}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 4 & 7 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 10 & 10 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{array}{|c|c|c|}\hline 3 & 10 & 10 \\ \hline \end{array}$$

Scalar multiplication on sudoku rows is defined analogously. With addition and scalar multiplication defined, linear combinations are defined. Can the third row of the 2,1-block of the completed sudoku board (on page 35) be written as a linear combination of the first two rows? In other words, does the following equation have a solution?

$$a\,\begin{array}{|c|c|c|}\hline 3 & 7 & 1 \\ \hline \end{array} + b\,\begin{array}{|c|c|c|}\hline 9 & 6 & 2 \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline 8 & 4 & 5 \\ \hline \end{array}$$

To answer the question, write the equation in terms of matrices and solve:

$$a\begin{bmatrix} 3 & 7 & 1 \end{bmatrix} + b\begin{bmatrix} 9 & 6 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 4 & 5 \end{bmatrix}$$
$$\begin{bmatrix} 3a+9b & 7a+6b & a+2b \end{bmatrix} = \begin{bmatrix} 8 & 4 & 5 \end{bmatrix}$$

For these two row matrices to be equal corresponding entries must be equal. In other words, the simultaneous equations

$$\begin{aligned} 3a + 9b &= 8 \\ 7a + 6b &= 4 \\ a + 2b &= 5 \end{aligned}$$

must be true. The second and third equations can be solved (as a system) by elimination, for example. The second equation minus 3 times the third equation yields $4a = -11$, so $a = \frac{-11}{4}$. Substituting into the third equation yields $\frac{-11}{4} + 2b = 5$ which means $b = \frac{31}{8}$. These values of a and b (and only these values of a

and b) satisfy the second and third equations. Substituting into the first equation yields $3\left(\frac{-11}{4}\right)+9\left(\frac{31}{8}\right) = 8$ which can be confirmed FALSE! Therefore there is no solution. There is no way to write the third row of the 2,1-block as a linear combination of the first two rows.

By contrast the third row of the 1,3-block can be written as -1 times the first row plus 2 times the second row. The third row is the linear combination of the first two rows with coefficients -1 and 2. Can you verify this? The 1,3-block is

1	9	8
4	6	5
7	3	2

Answer on page 35.

Through inheritance the determinant of any 3×3 sudoku block can also be calculated. For example, the determinant of the 2,1-block is

$$\begin{aligned}
\begin{vmatrix} 3 & 7 & 1 \\ 9 & 6 & 2 \\ 8 & 4 & 5 \end{vmatrix} &= 3\begin{vmatrix} 6 & 2 \\ 4 & 5 \end{vmatrix} - 7\begin{vmatrix} 9 & 2 \\ 8 & 5 \end{vmatrix} + 1\begin{vmatrix} 9 & 6 \\ 8 & 4 \end{vmatrix} \\
&= 3(6 \cdot 5 - 2 \cdot 4) - 7(9 \cdot 5 - 2 \cdot 8) + 1(9 \cdot 4 - 6 \cdot 8) \\
&= 3(22) - 7(29) + 1(-12) \\
&= 66 - 203 - 12 \\
&= -149
\end{aligned}$$

What is the determinant of the 1,3-block? Answer on page 36.

So, for the block with determinant -149 there was no way to write the third row as a linear combination of the first two, and for the block with determinant 0 there was a way to write the third row as a linear combination of the first two. This bears further investigation, requested in the exercises.

Key Concepts

coefficients The scalar quantities of a linear combination.

determinant The determinant of an $n \times n$ matrix A, denoted $\det A$ or $|A|$, is defined by

$$\det A = (-1)^{1+1}A_{1,1} \det A_{\backslash 1,1} + (-1)^{1+2}A_{1,2} \det A_{\backslash 1,2} + \cdots + (-1)^{1+n}A_{1,n} \det A_{\backslash 1,n}$$

for $n > 1$ and $\det A = A_{1,1}$ for $n = 1$. The determinant of an $m \times n$ matrix is undefined if $m \neq n$.

linear combination An expression of the form

$$x_1b_1 + x_2b_2 + \cdots + x_nb_n = \sum_{i=1}^{n} x_ib_i$$

where $x_1, x_2, \ldots, x_n$ are scalars and $b_1, b_2, \ldots, b_n$ are objects from a set on which addition and scalar multiplication are defined.

square matrix A matrix with the same number or columns as rows. An $n \times n$ matrix.

SageMath

If `M` is a matrix in SageMath, then `M.determinant()` is its determinant. The following code computes the determinant of $A = \begin{bmatrix} -12 & 49 & -45 & -10 \\ 28 & 45 & -46 & 23 \\ -15 & -28 & 4 & -48 \\ -1 & 34 & -38 & -18 \end{bmatrix}$, the matrix behind calculation (1.5.2).

```
M = matrix(4,4,[-12,49,-45,-10,28,45,-46,23,-15,-28,4,-48,-1,34,-38,-18])
print(M.determinant())
```

The output of this code is

```
-393294
```

Exercises

1. Calculate the cofactor.

   (a) $C_{1,1}$ of $\begin{bmatrix} -9 & 3 \\ 0 & 6 \end{bmatrix}$

   (b) $C_{1,2}$ of $\begin{bmatrix} 9 & -6 \\ -11 & -9 \end{bmatrix}$

   (c) $C_{2,1}$ of $\begin{bmatrix} -4 & -6 \\ 9 & -11 \end{bmatrix}$

   (d) $C_{2,2}$ of $\begin{bmatrix} 8 & 13 \\ 6 & 9 \end{bmatrix}$

   (e) $C_{1,3}$ of $\begin{bmatrix} -2 & -1 & 6 \\ -5 & -10 & 11 \\ -9 & 8 & 0 \end{bmatrix}$

   (f) $C_{2,1}$ of $\begin{bmatrix} -3 & 0 & 1 \\ 3 & -5 & 7 \\ 4 & 11 & -7 \end{bmatrix}$

   (g) $C_{2,2}$ of $\begin{bmatrix} -12 & -2 & -10 \\ 5 & 0 & 3 \\ 2 & -9 & -4 \end{bmatrix}$

   (h) $C_{3,2}$ of $\begin{bmatrix} 7 & -6 & -7 \\ -2 & -4 & -11 \\ 2 & 0 & 4 \end{bmatrix}$

2. Calculate the determinant if possible.

   (a) $\det A$ where $A = \begin{bmatrix} 30 \end{bmatrix}$

   (b) $|A|$ where $A = \begin{bmatrix} -6 \end{bmatrix}$

   (c) $\det A$ where $A = \begin{bmatrix} -45 \end{bmatrix}$

   (d) $\det A$ where $A = \begin{bmatrix} 44 \end{bmatrix}$

   (e) $\det \begin{pmatrix} 5 & -2 \\ 7 & 2 \end{pmatrix}$

   (f) $\begin{vmatrix} -18 & 19 \\ 6 & -3 \end{vmatrix}$

   (g) $\begin{vmatrix} 18 & 5 \\ 14 & -16 \end{vmatrix}$

   (h) $\begin{vmatrix} -5 & 2 & -4 \\ 9 & 0 & -2 \\ -6 & 8 & 4 \end{vmatrix}$

   (i) $\begin{vmatrix} 0 & 9 & 7 \\ -1 & -6 & -2 \\ 6 & -9 & -5 \end{vmatrix}$

   (j) $\det \begin{pmatrix} -3 & -1 & -9 \\ 1 & -4 & -8 \\ 2 & 9 & 6 \end{pmatrix}$

   (k) $\det \begin{pmatrix} 2 & 8 & -2 & 0 \\ 3 & 8 & 1 & 2 \\ 0 & 0 & 1 & 6 \\ 2 & 0 & -1 & 6 \end{pmatrix}$

   (l) $\begin{vmatrix} 4 & 5 & -2 & 0 \\ 2 & 0 & 4 & 7 \\ 8 & 4 & 5 & -2 \\ 2 & -2 & 0 & 0 \end{vmatrix}$

   (m) $\det \begin{pmatrix} 5 & 0 & 2 & 8 \\ 4 & 8 & 6 & -2 \\ 0 & -1 & 6 & 0 \\ 0 & 3 & -1 & 3 \end{pmatrix}$

3. Formula (1.5.1) reduces the calculation of the determinant of a 4×4 matrix into a linear combination of the determinants of twenty-four 1×1 matrices, as in calculation (1.5.2).

(a) Formula (1.5.1) reduces the calculation of the determinant of a 5×5 matrix into a linear combination of the determinants of how many 1×1 matrices?

(b) Formula (1.5.1) reduces the calculation of the determinant of an $n \times n$ matrix into a linear combination of the determinants of how many 1×1 matrices?

4. True or false?

   (a) The determinant of a matrix is a scalar.

   (b) The determinant of a matrix is always positive since it is the absolute value of a number.

   (c) The determinant of a 5×6 matrix can be written as a linear combination of the determinants of thirty 1×1 matrices.

   (d) If A and B are 1×1 matrices, then $\det A + \det B = \det(A + B)$.

   (e) If A and B are 2×2 matrices, then $\det A + \det B = \det(A + B)$.

5. Compare and contrast (i) scalar, (ii) 1×1 matrix, and (iii) the determinant of a 1×1 matrix.

6. Calculate the determinant. **HINT**: Despite the large sizes of some of the matrices, this does not require a lot of work.

   (a) $\begin{vmatrix} 7 & 0 \\ 6 & -8 \end{vmatrix}$

   (b) $\begin{vmatrix} -2 & 0 & 0 \\ -7 & 4 & 0 \\ 2 & -9 & 7 \end{vmatrix}$

   (c) $\begin{vmatrix} -4 & 0 & 0 & 0 \\ -8 & -9 & 0 & 0 \\ 2 & 7 & 6 & 0 \\ -5 & -7 & 3 & 9 \end{vmatrix}$

   (d) $\begin{vmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -7 & -3 & 0 & 0 & 0 & 0 \\ -6 & -4 & 3 & 0 & 0 & 0 \\ -1 & 5 & 1 & -2 & 0 & 0 \\ 4 & -5 & 8 & 3 & 4 & 0 \\ 8 & 9 & 7 & -9 & 0 & 9 \end{vmatrix}$

7. In your own words, draw a conjecture based on the calculations of question 6.

The remaining exercises refer to the completed sudoku board of this section (page 35).

8. The determinants of the 1,3-block and the 2,1-block are 0 and -149 respectively. Find the determinants of the remaining 7 blocks.

9. For the 1,3-block, the third row can be written as a linear combination of the first two (-1 times the first row plus 2 times the second row). For the 2,1-block, the third row cannot be written as a linear combination of the first two. For the remaning 7 blocks, explore whether there is any row that can be written as a linear combination of the other two.

10. Make a conjecture about the connection between determinant and the possibility of writing one of the rows of a block as a linear combination of the others.

11. Can any of the 9 rows of the sudoku board be written as a linear combination of the other 8?

You may follow this link or copy and paste the following code into a SageCell to get started.

```
var('a','b')

print("block 1,3:")
M = matrix(3,3,[1,9,8,4,6,5,7,3,2])
print("Determinant:", M.determinant())
print("Solution:",solve(list(a*M.row(0)+b*M.row(1)-M.row(2)),a,b))
print()

print("block 2,1:")
M = matrix(3,3,[3,7,1,9,6,2,8,4,5])
print("Determinant:", M.determinant())
```

```
print("Solution:",solve(list(a*M.row(0)+b*M.row(1)-M.row(2)),a,b))
print()
```

The output of this code is

```
block 1,3:
Determinant: 0
Solution: [
[a == -1, b == 2]
]

block 2,1:
Determinant:
-149
Solution: [

]
```

It shows the determinants of the 1,3-block and the 2,1-block: 0 and −149, respectively. Also, for each block the output shows an attempt to find a linear combination of the first two rows that equals the third. Just as in the calculations of the text (see page 31) it shows a solution for the 1,3-block but not for the 2,1-block. The empty square brackets indicate no solution was found.

Answers

determinant:

$$\begin{aligned}\det\begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix} &= (-1)^{1+1}(2)\det(3) + (-1)^{1+2}(4)\det(-1) \\ &= 2(3) - 4(-1) \\ &= 10\end{aligned}$$

sudoku:

2	3	4	5	6	7	1	9	8
7	8	9	2	3	1	4	6	5
1	5	6	4	8	9	7	3	2
3	7	1	6	5	8	9	2	4
9	6	2	3	1	4	8	5	7
8	4	5	7	9	2	3	1	6
4	2	3	1	7	6	5	8	9
5	9	7	8	2	3	6	4	1
6	1	8	9	4	5	2	7	3

linear combination:

$$\begin{aligned}(-1)\begin{bmatrix} 1 & 9 & 8 \end{bmatrix} + 2\begin{bmatrix} 4 & 6 & 5 \end{bmatrix} &= \begin{bmatrix} -1+8 & -9+12 & -8+10 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 3 & 2 \end{bmatrix}\end{aligned}$$

sudoku determinant:

$$
\begin{aligned}
\begin{vmatrix} 1 & 9 & 8 \\ 4 & 6 & 5 \\ 7 & 3 & 2 \end{vmatrix} &= 1\begin{vmatrix} 6 & 5 \\ 3 & 2 \end{vmatrix} - 9\begin{vmatrix} 4 & 5 \\ 7 & 2 \end{vmatrix} + 8\begin{vmatrix} 4 & 6 \\ 7 & 3 \end{vmatrix} \\
&= 1(6 \cdot 2 - 5 \cdot 3) - 9(4 \cdot 2 - 5 \cdot 7) + 8(4 \cdot 3 - 6 \cdot 7) \\
&= 1(-3) - 9(-27) + 8(-30) \\
&= -3 + 243 - 240 \\
&= 0
\end{aligned}
$$

1.6 Matrix "Division"

You may have heard the claim "there's no such thing as subtraction—it's just adding the opposite" or something to that effect. There is a vital concept of linear algebra buried in this addage. The link between addition, opposites, and zero that makes subtraction optional this way is as follows. For any real numbers a and b,

$$a \text{ and } b \text{ are opposites if and only if } a + b = b + a = 0.$$

Why zero, and not some other number here? Zero is that special number that can be added to any number without changing its value, and there is none other like it! In symbols, $a + 0 = 0 + a = a$. This property is so special it has a name. Zero is the **additive identity** for real numbers—the word *identity* to signal this special property and the word *additive* to document the operation. The additive inverse, or opposite, of a real number ***is defined*** by the fact that adding the two yields the additive identity. Two numbers are opposites (additive inverses) if and only if their sum is zero.

Likewise, one is that special number that can be multiplied by any number without changing its value. Consequently, one is the **multiplicative identity** for real numbers. In symbols, $a \cdot 1 = 1 \cdot a = a$. The multiplicative inverse, or reciprocal, of a real number ***is defined*** by the fact that multiplying the two yields the multiplicative identity. Two numbers are reciprocals (multiplicative inverses) if and only if their product is one. This leads us to a link between multiplication, reciprocals, and one that is analogous to the link between addition, opposites, and zero. For any real numbers a and b,

$$a \text{ and } b \text{ are reciprocals if and only if } a \cdot b = b \cdot a = 1.$$

The same relationship holds among addition, opposites, and zero as holds among multiplication, reciprocals, and one. Addition and multiplication are operations, opposites and reciprocals are inverses, and zero and one are identities.

There is an important analogy for matrices. Compute the following products.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -8 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 & -1 \\ 17 & -21 & 55 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{5} & -\pi & 18 & \frac{2}{3} \\ \frac{17}{8} & 34 & \sqrt[5]{19} & \ln 9 \\ \frac{\pi}{4} & 0.34 & e^7 & \sin(1) \\ 2^{2^2} & \tan^{-1}(1) & 12 & 10^{1000} \end{bmatrix}$$

Answers on page 44. Hopefully these exercises have led you to the conclusion that multiplying by matrices such as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

leaves the multiplicand (the matrix being multiplied by it) unchanged. Multiplication by matrices such as these acts very much like multiplying numbers by 1, or adding 0. This is like the properties that made 1 the multiplicative identity and 0 the additive identity for real numbers, and this is the property that makes these matrices identity matrices. They can each be multiplied with any other, as long as the product is defined, without changing the other's value. The $n \times n$ **identity matrix** is denoted by $I_{n\times n}$ or just I when the size of the matrix is determined by context. Identity matrices have ones on the main diagonal, the diagonal of entries running from the 1,1-entry through the n,n-entry, and zeros everywhere else. In symbols, $I_{1,1} = I_{2,2} = \cdots = I_{n,n} = 1$ and $I_{j,k} = 0$ whenever $j \neq k$. Matrices denoted by I are those special matrices such that $A \cdot I_{n\times n} = I_{m\times m} \cdot A = A$.

With an identity, or really set of identities, for multiplication, we are only one element shy of the operation, inverse, identity triumvirate for matrices. Compute the following products.

$$\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix} \qquad \begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 16 \\ 5 & -11 & -36 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & -2 \\ 1 & 1 & -6 & -3 \\ -2 & -3 & 10 & 5 \\ 0 & 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -9 & -3 & -4 \\ -1 & 1 & 0 & 1 \\ -1 & 3 & 1 & 2 \\ 3 & -9 & -3 & -5 \end{bmatrix}$$

Answers on page 44. These exercises demonstrate that there are pairs of matrices A and B such that $AB = I$. But what about BA? In our formulations for real number inverses we had $a + b = b + a = 0$ and $a \cdot b = b \cdot a = 1$, but we observed in section 1.3 that matrix multiplication was not commutative. Compute the following products (the same as above only in the opposite order).

$$\begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & -7 \\ -2 & 5 & 16 \\ 5 & -11 & -36 \end{bmatrix}\begin{bmatrix} -4 & 5 & 3 \\ 8 & -1 & -2 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -9 & -3 & -4 \\ -1 & 1 & 0 & 1 \\ -1 & 3 & 1 & 2 \\ 3 & -9 & -3 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 & -2 \\ 1 & 1 & -6 & -3 \\ -2 & -3 & 10 & 5 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

As you were hopefully able to verify, all of these products are the (appropriate size) identity too! It seems that multiplicative inverse pairs commute. That is, if A and B are multiplicative inverses, then $AB = BA$. We finally have evidence that a matrix analogy for linking multiplication, inverses, and identity matrices exists. For any matrices A and B,

$$A \text{ and } B \text{ are inverses if and only if } A \cdot B = B \cdot A = I.$$

Crumpet 10: Inverses of non-square matrices?

Suppose A is an $m \times n$ matrix and there are matrices L and R such that $LA = I$ and $AR = I$. Because A has n columns and $LA = I$, the identity matrix in this equation must also have n columns. To be precise then, we have $LA = I_{n\times n}$. Because A has m rows and $AR = I$, the identity matrix in this equation must also have m rows. To be precise then, we must have $AR = I_{m\times m}$. Furthermore, for the proucts LA and AR to be defined, it must be that L has m columns and R has n rows. It follows that both products $L(AR)$ and $(LA)R$ are defined and

$$L(AR) = LI_{m\times m} = L \quad \text{and} \quad (LA)R = I_{n\times n}R = R.$$

By the associative property for matrix multiplication (which we will encounter in section 3.1), $L(AR) = (LA)R$, so we have $L = R$. m must equal n because $LA = I$ implies A has a pivot position in each column and $AR = I$ implies A has a pivot position in each row and the only way to have both is for A to be square (see section 2.2 for a definition of pivot position and the proof of theorem 7 for the rest). Thus invertibility only makes sense for square matrices.

This is enough that we could define $\frac{I}{A}$ as the multiplicative inverse (or reciprocal) of A and have the understanding that $A \div B$ means $A \cdot \frac{I}{B}$ much like we have for real numbers, but by convention we do not! For one thing, we would need a second division symbol to mean $\frac{I}{B} \cdot A$ since matrix multiplication is not commutative. In general, $\frac{I}{B} \cdot A$ and $A \cdot \frac{I}{B}$ could be unequal. Instead, we stick with the addage that "there's no such thing as matrix division—it's just multiplying by the inverse". The notation we use for the inverse of A is A^{-1}, borrowing from the algebra of real numbers but not using division bars or division symbols.

A Formula for the Inverse

For any matrix A, if A is invertible then

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{m,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{m,n} \end{bmatrix} \tag{1.6.1}$$

where the $C_{i,j}$ are the cofactors of A. This implies that when A^{-1} exists

1. A must be square since $\det A$ and the $C_{i,j}$ are undefined if A is not square, and
2. $\det A$ must be nonzero since division by 0 is undefined.

When A^{-1} is defined, we say that A is **invertible**. The matrix

$$\begin{bmatrix} C_{1,1} & C_{2,1} & \cdots & C_{n,1} \\ C_{1,2} & C_{2,2} & \cdots & C_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1,n} & C_{2,n} & \cdots & C_{n,n} \end{bmatrix}$$

is called the **adjugate** of A, $\text{adj}A$. With this definition, the formula for the inverse can be summarized as

$$A^{-1} = \frac{1}{\det A}\text{adj}A.$$

One Property of the Inverse

Multiplication by a matrix's inverse "undoes" multiplication by the matrix just as dividing by a number undoes multiplication by that same number. In symbols, if A and B are matrices and B is invertible (has an inverse)

$$(AB)B^{-1} = A$$

for matrices much like $(a \cdot b) \div b = a$ for real numbers. If we used division in linear algebra, the equation $(AB)B^{-1} = A$ might be written $(A \cdot B) \div B = A$, making the comparison clearer. The only potential harm in thinking with division is that $(BA)B^{-1}$ is generally not A, so $(B \cdot A) \div B \neq A$ for matrices even though $(b \cdot a) \div b = a$ for real numbers. Since multiplication of matrices is not commutative, right-multiplication by B^{-1} does not undo left-multiplication by B. Using the notation B^{-1} and paying close attention to right-multiplication versus left-multiplication will help keep this straight.

Table 1.2: ASCII (American Standard Code for Information Interchange) characters

Dec	Hex	Char		Dec	Hex	Char	Dec	Hex	Char	Dec	Hex	Char
0	0	NUL	(null character)	32	20	(space)	64	40	@	96	60	`
1	1	SOH	(start of heading)	33	21	!	65	41	A	97	61	a
2	2	STX	(start of text)	34	22	"	66	42	B	98	62	b
3	3	ETX	(end of text)	35	23	#	67	43	C	99	63	c
4	4	EOT	(end of transmission)	36	24	$	68	44	D	100	64	d
5	5	ENQ	(enquiry)	37	25	%	69	45	E	101	65	e
6	6	ACK	(acknowledge)	38	26	&	70	46	F	102	66	f
7	7	BEL	(bell)	39	27	'	71	47	G	103	67	g
8	8	BS	(backspace)	40	28	(	72	48	H	104	68	h
9	9	HT	(horizontal tab)	41	29	)	73	49	I	105	69	i
10	A	LF	(line feed)	42	2A	*	74	4A	J	106	6A	j
11	B	VT	(vertical tab)	43	2B	+	75	4B	K	107	6B	k
12	C	FF	(form feed)	44	2C	,	76	4C	L	108	6C	l
13	D	CR	(carriage return)	45	2D	-	77	4D	M	109	6D	m
14	E	SO	(shift out)	46	2E	.	78	4E	N	110	6E	n
15	F	SI	(shift in)	47	2F	/	79	4F	O	111	6F	o
16	10	DLE	(data link escape)	48	30	0	80	50	P	112	70	p
17	11	DC1	(device control 1)	49	31	1	81	51	Q	113	71	q
18	12	DC2	(device control 2)	50	32	2	82	52	R	114	72	r
19	13	DC3	(device control 3)	51	33	3	83	53	S	115	73	s
20	14	DC4	(device control 4)	52	34	4	84	54	T	116	74	t
21	15	NAK	(negative acknowledge)	53	35	5	85	55	U	117	75	u
22	16	SYN	(synchronous idle)	54	36	6	86	56	V	118	76	v
23	17	ETB	(end of transmission block)	55	37	7	87	57	W	119	77	w
24	18	CAN	(cancel)	56	38	8	88	58	X	120	78	x
25	19	EM	(end of medium)	57	39	9	89	59	Y	121	79	y
26	1A	SUB	(substitute)	58	3A	:	90	5A	Z	122	7A	z
27	1B	ESC	(escape)	59	3B	;	91	5B	[	123	7B	{
28	1C	FS	(file separator)	60	3C	<	92	5C	\	124	7C	|
29	1D	GS	(group separator)	61	3D	=	93	5D	]	125	7D	}
30	1E	RS	(record separator)	62	3E	>	94	5E	^	126	7E	~
31	1F	US	(unit separator)	63	3F	?	95	5F	_	127	7F	(delete)

To illustrate, let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -4 \\ -5 & 7 \end{bmatrix}$, making

$$AB = \begin{bmatrix} -7 & 10 \\ -11 & 16 \end{bmatrix}.$$

Can you verify this? As seen earlier, $B^{-1} = \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$. Compute $(AB)B^{-1}$ to see that it equals A, and compute $B^{-1}(AB)$ to see that it does not equal A. Answer on page 45.

Inverses and Cryptography

-97 -40 17 -207 -185 -68 -416 -303 -97 -312 -178 -28 -143
-68 10 -190 -93 2 -305 -149 -7 102 134 83 -257 -194 -65

The ability to undo multiplication by an invertible matrix makes it possible to use matrices and their inverses for encrypting and decrypting messages. Given a messge, a system for converting letters to numbers turns the message into a list of numbers. These numbers are grouped into column vectors and multiplied by an encoding matrix, producing a new set of numbers, the encoded message. The list of numbers opening this subsection was created this way! Decoding the message then amounts to grouping the code into column vectors and multiplying by the decoding matrix. As long as the parties on either end of the message transmission each have one of a given pair of inverse matrices, they can each encode and decode with their matrix, send their message securely, and decode received messages. The greatest weakness of this scheme is security of the coding/decoding matrix pair. If either matrix gets out, messages are no longer secure. Without knowledge of the coding or decoding matrix, the message above would be very difficult to decode!

While a basic conversion from letters to numbers would have each letter of the alphabet assigned a number from 1-26 or 0-25, this leaves punctuation, symbols like spaces and hashtags, capitalization, and numbers out. Since the early 1960's the American National Standards Institute has maintained a coding system for the electronic transmission of documents in English called ASCII (pronounced ass-kee) or US-ASCII. Part of that system, largely developed by Bob Bemer [17], is a numeric representation of all the symbols you are likely to find on an English language keyboard. See Table 40. For example, the capital letter "A" has numeric code 65, the lower case letter "a" has numeric code 97, and the space has numeric code 32.

Using the coding matrix $\begin{bmatrix} -7 & 3 & 2 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$, the message "Hello World!" would be encrypted as follows.

1. "Hello World!" is converted to the numeric sequence 72 101 108 108 111 32 87 111 114 108 100 33 using ASCII.

2. Since we are using a 3×3 coding matrix, the numeric sequence is grouped three at a time into the 3-row matrix
$$\begin{bmatrix} 72 & 108 & 87 & 108 \\ 101 & 111 & 111 & 100 \\ 108 & 32 & 114 & 33 \end{bmatrix}$$
If the message did not have a multiple of three characters, 0's (null characters) could be added to the end.

3. The message matrix is multiplied by the coding matrix.
$$\begin{bmatrix} -7 & 3 & 2 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 72 & 108 & 87 & 108 \\ 101 & 111 & 111 & 100 \\ 108 & 32 & 114 & 33 \end{bmatrix} = \begin{bmatrix} 57 & -395 & -295 & -357 \\ 51 & -156 & -145 & -255 \\ 42 & 7 & -8 & -75 \end{bmatrix}$$
This is a good place to use a calculator or SageMath!

4. The coded message is extracted from the product:

   57 51 42 -395 -156 7 -295 -145 -8 -357 -255 -75

The message at the beginning of this section was encoded with the same matrix. Can you decode it (using a calculator or SageMath to assist)? Answer on page 45.

Crumpet 11: Lester S. Hill

The first documented multiple-letter cipher is attributed to Lester S. Hill. His *Mathematical Monthly* article of 1929 [11] outlines a procedure very similar to the one presented here except modular arithmetic is used to make sure all numbers in the encoded message are valid character codes. Thus the encoded message is transmitted as a sequence of letters and symbols, not numbers. His work far predates electronic computing devices so, to be practical, he needed a way to limit the difficulty of doing the computations, a second impetus for using modular arithmetic.

Key Concepts

A^{-1} The inverse of A.

adjugate For a square matrix, the transpose of its matrix of cofactors.

identity matrix A matrix with ones on the main diagonal and zeros elsewhere.

matrix inverse Matrices A and B are inverses of one another if and only if $AB = BA = I$.

invertible matrix A matrix whose inverse is defined.

main diagonal The i,i-entries of a matrix.

SageMath

If `M` is a matrix in SageMath, then `M.inverse()` is its inverse. The following code computes the inverses of $A = \begin{bmatrix} 4 & -9 & -3 & -4 \\ -1 & 1 & 0 & 1 \\ -1 & 3 & 1 & 2 \\ 3 & -9 & -3 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} \sqrt{5} & -\pi & 18 \\ \frac{17}{8} & 34 & \sqrt[5]{19} \\ \frac{\pi}{4} & 0.34 & e^7 \end{bmatrix}$.

```
A = matrix(4,4,[4,-9,-3,-4,-1,1,0,1,-1,3,1,2,3,-9,-3,-5])
print(A.inverse()); print()
B=matrix(3,3,[sqrt(5),-pi,18,17/8,34,19^(1/5),pi/4,0.34,e^7])
print(B.inverse())
```

The output of A^{-1} is

```
[ 1  0 -3 -2]
[ 1  1 -6 -3]
[-2 -3 10  5]
[ 0  0  3  1]
```

but the output of B^{-1} is far too long to fit on the page. Just the 1,1-entry is $1/5(\sqrt{5}\pi - \sqrt{5}(\sqrt{5}\pi^2 + 6.8)/(\sqrt{5}\pi + 80))(\sqrt{5}\pi(153\sqrt{5} - 20 \cdot 19^{1/5}))/(\sqrt{5}\pi + 80) - 153\sqrt{5})/(153\sqrt{5}\pi - (\sqrt{5}\pi^2 + 6.8)(153\sqrt{5} - 20 \cdot 19^{1/5})/(\sqrt{5}\pi + 80) - 170e^7) + 1/5\sqrt{5} - \pi/(\sqrt{5}\pi + 80)$.

Exercises

1. Compute the inverse if possible.

   (a) $\begin{bmatrix} -3 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 \end{bmatrix}$

   (c) $\begin{bmatrix} \frac{2}{\sqrt{3}} \end{bmatrix}$

   (d) $\begin{bmatrix} 4\pi \end{bmatrix}$

   (e) $\begin{bmatrix} 3 & -2 \\ 11 & -7 \end{bmatrix}$

   (f) $\begin{bmatrix} \sqrt{12} & 3 \\ 2 & \sqrt{3} \end{bmatrix}$

   (g) $\begin{bmatrix} 5 & \sqrt{18} \\ \sqrt{8} & 3 \end{bmatrix}$

   (h) $\begin{bmatrix} 5 & -3 \\ -5 & 4 \end{bmatrix}$

   (i) $\begin{bmatrix} 2 & -3 & \sqrt{7} \\ 12 & \sqrt{28} & 5 \end{bmatrix}$

   (j) $\begin{bmatrix} 3 & 4 \\ -7 & 8 \\ -1 & 9 \end{bmatrix}$

(k) $\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ 7 & 8 & 1 \end{bmatrix}$

(l) $\begin{bmatrix} 0 & -3 & 2 \\ 5 & 1 & 2 \\ 5 & -8 & 8 \end{bmatrix}$

(m) $\begin{bmatrix} 6 & 3 & 0 \\ -1 & -1 & 6 \\ 0 & 0 & 7 \end{bmatrix}$

(n) $\begin{bmatrix} 3 & -2 & -6 \\ -1 & 1 & 3 \\ -4 & 3 & 10 \end{bmatrix}$

(o) $\begin{bmatrix} 9 & -7 & -2 & -11 \\ 0 & 12 & 2 & -12 \\ -1 & -9 & 6 & 10 \end{bmatrix}$

(p) $\begin{bmatrix} -4 & -9 & 0 \\ 1 & -8 & -1 \\ -10 & 11 & 9 \\ -3 & 8 & 10 \end{bmatrix}$

(q) $\begin{bmatrix} 2 & 1 & -6 & 2 \\ 1 & 1 & -3 & 0 \\ -4 & -1 & 13 & -7 \\ 3 & 0 & -11 & 9 \end{bmatrix}$

(r) $\begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 0 & 2 & 0 \\ 3 & 1 & 7 & 0 \end{bmatrix}$

(s) $\begin{bmatrix} 4 & 7 & 0 & 8 \\ 0 & 8 & 0 & -1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$

2. Compare and contrast the inverse of a 1×1 matrix with the multiplicative inverse of a real number.

3. True or false? If all the entries in a square matrix M are integers and $\det M = 1$, then all the entries in M^{-1} are integers.

4. When the determinant of an $n \times n$ matrix is zero, (select all that apply)

   (a) exactly one row is a linear combination of the others.

   (b) every row is a linear combination of the others.

   (c) each row after the first one is a linear combination of the rows above it.

   (d) any linear combination of the n rows sums to zero.

   (e) at least one row is a linear combination of the others.

   (f) its inverse is the zero matrix.

   (g) it has no inverse.

5. Suppose A is an invertible 3×3 matrix and

$$A \cdot \begin{bmatrix} 1.4 & -70 \\ -29 & 95 \\ -12 & -43 \end{bmatrix} = \begin{bmatrix} 80 & 4.9 \\ -62 & -52 \\ -32 & 52 \end{bmatrix}.$$

Find $A^{-1} \cdot \begin{bmatrix} 80 & 4.9 \\ -62 & -52 \\ -32 & 52 \end{bmatrix}$.

6. Which matrix is the inverse of

$$\begin{bmatrix} 0 & 0 & 4 & 0 & -1 \\ -2 & 4 & 4 & -1 & 1 \\ -4 & 7 & 10 & -2 & 2 \\ -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}?$$

(a)
$$\begin{bmatrix} -1 & 4 & -2 & -1 & 8 \\ 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 0 & 4 \end{bmatrix}$$

(b)
$$\begin{bmatrix} -1 & 4 & -2 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 2 \\ -1 & 0 & 0 & 4 \end{bmatrix}$$

(c)
$$\begin{bmatrix} 4 & -1 & -2 & -1 & 8 \\ 2 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & 2 & 0 \\ -1 & 0 & 4 & 0 & 0 \end{bmatrix}$$

(d)
$$\begin{bmatrix} 0 & -2 & -4 & -1 & 0 \\ 0 & 4 & 7 & 2 & 0 \\ 4 & 4 & 10 & 0 & 1 \\ 0 & -1 & -2 & 0 & 0 \\ -1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

7. Find x and y so that A and B are inverses.

$$A=\begin{bmatrix}1&1&-5\\-3&1&4\\2&-3&4\end{bmatrix}\qquad B=\begin{bmatrix}16&11&9\\20&14&y\\x&5&4\end{bmatrix}$$

For the remaining exercises, let $A=\begin{bmatrix}7&-5&-2\\-3&3&1\\-3&2&1\end{bmatrix}$ and $B=\begin{bmatrix}1&2&2\\2&8&7\\-3&-5&-5\end{bmatrix}$.

8. Compute
   (a) $A^{-1}(AB)$
   (b) $(AB)A^{-1}$
   (c) $B^{-1}(BA)$
   (d) $(BA)B^{-1}$

   Did you get what you expected?

9. Compute
   (a) $(AB)^{-1}$
   (b) $A^{-1}B^{-1}$
   (c) $B^{-1}A^{-1}$

   What do you notice?

10. Calculate
    (a) $\det A$
    (b) $\det B$
    (c) $\det(AB)$
    (d) $\det(3A)$
    (e) $\det(3B)$

    What do you notice?

11. Decode the message -423 606 256 -353 463 232 -127 47 122 -478 774 255 -446 635 273. It was encoded using

$$\begin{bmatrix}1&-4&-2\\-3&7&3\\0&2&1\end{bmatrix}.$$

Answers

matrix products part 1:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\-8\end{bmatrix}=\begin{bmatrix}3\\-8\end{bmatrix}$$

$$\begin{bmatrix}3&4&-1\\17&-21&55\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&4&-1\\17&-21&55\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\sqrt{5}&-\pi&18&\frac{2}{3}\\\frac{17}{8}&34&\sqrt[5]{19}&\ln 9\\\frac{\pi}{4}&0.34&e^7&\sin(1)\\2^{2^2}&\tan^{-1}(1)&12&10^{1000}\end{bmatrix}=\begin{bmatrix}\sqrt{5}&-\pi&18&\frac{2}{3}\\\frac{17}{8}&34&\sqrt[5]{19}&\ln 9\\\frac{\pi}{4}&0.34&e^7&\sin(1)\\2^{2^2}&\tan^{-1}(1)&12&10^{1000}\end{bmatrix}$$

matrix products part 2:

$$\begin{bmatrix}7&4\\5&3\end{bmatrix}\begin{bmatrix}3&-4\\-5&7\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix}-4&5&3\\8&-1&-2\\-3&1&1\end{bmatrix}\begin{bmatrix}1&-2&-7\\-2&5&16\\5&-11&-36\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-3&-2\\1&1&-6&-3\\-2&-3&10&5\\0&0&3&1\end{bmatrix}\begin{bmatrix}4&-9&-3&-4\\-1&1&0&1\\-1&3&1&2\\3&-9&-3&-5\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

inverse undoes multiplication:

(i)

$$\begin{aligned}
(AB)B^{-1} &= \begin{bmatrix} -7 & 10 \\ -11 & 16 \end{bmatrix} \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A \\
((AB)B^{-1})_{1,1} &= (-7)(7) + (10)(5) = -49 + 50 = 1 \\
((AB)B^{-1})_{1,2} &= (-7)(4) + (10)(3) = -28 + 30 = 2 \\
((AB)B^{-1})_{2,1} &= (-11)(7) + (16)(5) = -77 + 80 = 3 \\
((AB)B^{-1})_{2,2} &= (-11)(4) + (16)(3) = -44 + 48 = 4
\end{aligned}$$

(ii)

$$\begin{aligned}
B^{-1}(AB) &= \begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} -7 & 10 \\ -11 & 16 \end{bmatrix} = \begin{bmatrix} -93 & 134 \\ -68 & 98 \end{bmatrix} \neq A \\
(B^{-1}(AB))_{1,1} &= (7)(-7) + (4)(-11) = -49 - 44 = -93 \\
(B^{-1}(AB))_{1,2} &= (7)(10) + (4)(16) = 70 + 64 = 134 \\
(B^{-1}(AB))_{2,1} &= (5)(-7) + (3)(-11) = -35 - 33 = -68 \\
(B^{-1}(AB))_{2,2} &= (5)(10) + (3)(16) = 50 + 48 = 98
\end{aligned}$$

decoding: The coding matrix $C = \begin{bmatrix} -7 & 3 & 2 \\ -4 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ has determinant one:

$$\begin{aligned}
\det C &= -7 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} - 3 \begin{vmatrix} -4 & 2 \\ -1 & 1 \end{vmatrix} + 2 \begin{vmatrix} -4 & 1 \\ -1 & 0 \end{vmatrix} \\
&= -7(1) - 3(-4 + 2) + 2(0 + 1) \\
&= -7 + 6 + 2 \\
&= 1
\end{aligned}$$

The following are cofactors, not entries:

$$\begin{array}{ccc}
C_{1,1} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 & C_{1,2} = -1 \begin{vmatrix} -4 & 2 \\ -1 & 1 \end{vmatrix} = 2 & C_{1,3} = \begin{vmatrix} -4 & 1 \\ -1 & 0 \end{vmatrix} = 1 \\
C_{2,1} = -1 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = -3 & C_{2,2} = \begin{vmatrix} -7 & 2 \\ -1 & 1 \end{vmatrix} = -5 & C_{2,3} = -1 \begin{vmatrix} -7 & 3 \\ -1 & 0 \end{vmatrix} = -3 \\
C_{3,1} = \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} = 4 & C_{3,2} = -1 \begin{vmatrix} -7 & 2 \\ -4 & 2 \end{vmatrix} = 6 & C_{3,3} = \begin{vmatrix} -7 & 3 \\ -4 & 1 \end{vmatrix} = 5
\end{array}$$

so

$$C^{-1} = \frac{1}{1} \operatorname{adj} C = \begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 6 \\ 1 & -3 & 5 \end{bmatrix}.$$

Decoding is therefore done by multiplying

$$
\begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 6 \\ 1 & -3 & 5 \end{bmatrix}
\begin{bmatrix} -97 & -207 & -416 & -312 & -143 & -190 & -305 & 102 & -257 \\ -40 & -185 & -303 & -178 & -68 & -93 & -149 & 134 & -194 \\ 17 & -68 & -97 & -28 & 10 & 2 & -7 & 83 & -65 \end{bmatrix}
$$
$$
= \begin{bmatrix} 91 & 110 & 114 & 108 & 98 & 93 & 108 & 82 & 107 \\ 76 & 101 & 32 & 103 & 114 & 32 & 8 & 111 & 115 \\ 105 & 97 & 65 & 101 & 97 & 66 & 8 & 99 & 0 \end{bmatrix}
$$

and the numeric message is 91 76 105 110 101 97 114 32 65 108 103 101 98 114 97 93 32 66 108 8 8 82 111 99 107 115 0. The last step is to look these numbers up in the ASCII table.

1.7 Eigenpairs

Let $A = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}$, and compute each product before reading on.

$$A\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad A\begin{bmatrix} -\frac{8}{3} \\ \frac{2}{3} \end{bmatrix}$$
$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

You should find that $A\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ 3 \end{bmatrix}$, $A\begin{bmatrix} -\frac{8}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$, $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and $A\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$. Put another way,

$$A\begin{bmatrix} 4 \\ -1 \end{bmatrix} = -3\begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad A\begin{bmatrix} -\frac{8}{3} \\ \frac{2}{3} \end{bmatrix} = -3\begin{bmatrix} -\frac{8}{3} \\ \frac{2}{3} \end{bmatrix}$$
$$A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad A\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = 2\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

Letting $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, these products can be rewritten as

$$A\mathbf{v}_1 = -3\mathbf{v}_1 \qquad A\left(-\frac{2}{3}\mathbf{v}_1\right) = -3\left(-\frac{2}{3}\mathbf{v}_1\right)$$
$$A\mathbf{v}_2 = 2\mathbf{v}_2 \qquad A\left(-\frac{1}{2}\mathbf{v}_2\right) = 2\left(-\frac{1}{2}\mathbf{v}_2\right)$$

making the relationship between the matrix A and the vectors more apparent. The product of A with each of these vectors gives a scalar multiple of the vector. That's unusual. Previous experience would suggest this does not always happen.

There is nothing magical about the matrix A nor the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. For example,

$$A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

but $\begin{bmatrix} 10 \\ 1 \end{bmatrix}$ is no scalar multiple of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. There is no number λ such that $A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \lambda\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Multiplying a vector by A does not always produce a multiple of the vector. Letting $B = \begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix}$, $B\mathbf{v}_1 = \begin{bmatrix} 3 & 0 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$ but $\begin{bmatrix} -3 \\ -3 \end{bmatrix} \neq \lambda\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ for any value of λ. Similarly $B\mathbf{v}_2 \neq \lambda\mathbf{v}_2$ for any λ. It is not always the case that the product of a 2×2 matrix with $\mathbf{v}_1$ is a multiple of $\mathbf{v}_1$, and similarly for $\mathbf{v}_2$.

So A is not special on its own, nor are $\mathbf{v}_1$ or $\mathbf{v}_2$ special on their own. A and $\mathbf{v}_1$ are only special together just as A and $\mathbf{v}_2$ are only special together. To indicate the special relationship between A and $\mathbf{v}_1$ ($A\mathbf{v}_1 = \lambda\mathbf{v}_1$ for some λ), we call $\mathbf{v}_1$ an **eigenvector** of A. Similarly, $\mathbf{v}_2$ is an eigenvector of A. But that doesn't tell the whole story. $A\mathbf{v}_1 = \lambda\mathbf{v}_1$ and $A\mathbf{v}_2 = \lambda\mathbf{v}_2$ for different values of λ. The value -3 is associated with the eigenvector $\mathbf{v}_1$ and the value 2 is associated with the eigenvector $\mathbf{v}_2$. To mark this relationship, we call -3 an **eigenvalue** of A associated with eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and we call 2 an eigenvalue of A associated with the eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Any eigenvalue together with an associated eigenvector is called an **eigenpair**. For each eigenvector there is an eigenvalue, and for each eigenvalue there is an eigenvector (at least, that would be nice!).

It is true, for any matrix A and any scalar λ, that $A\mathbf{0} = \lambda\mathbf{0}$ where $\mathbf{0}$ is any vector with a zero for each entry, a so-called **zero vector**. Given the truth of this statement for all matrices, it does not tell us anything useful about any given matrix. Moreover, if we allow $\mathbf{0}$ to be an eigenvector, the eigenvalue associated with $\mathbf{0}$ would be ill-defined. It could be any number! Therefore we disallow $\mathbf{0}$ from the definition of eigenvector. With this restriction, every eigenvector has a unique associated eigenvalue.

Given an eigenvector $\mathbf{v}$ of a matrix M, it is easy to calculate the associated eigenvalue. Given an eigenvalue of a matrix, finding an associated eigenvector takes some work. Suppose λ is an eigenvalue of the matrix M. By definition, the associated eigenvector $\mathbf{v}$ satisfies $M\mathbf{v} = \lambda\mathbf{v}$. Equivalently,

$$(M - \lambda I)\mathbf{v} = \mathbf{0}. \tag{1.7.1}$$

We will see why these equations are equivalent later on. The equivalent form (1.7.1) gives us a way to find eigenvectors. Each side of the equation expresses a vector, and for two vectors to be equal, their corresponding entries must be equal. For a vector $\mathbf{v}$ with n entries, this observation yields n linear equations in the n unknown entries of $\mathbf{v}$. Recalling how to solve linear systems of equations gives a solution for $\mathbf{v}$. To illustrate, we find an eigenvector of $\begin{bmatrix} -31 & 14 & 10 \\ -76 & 35 & 26 \\ 18 & -9 & -8 \end{bmatrix}$ associated with the eigenvalue -2. Starting with (1.7.1), we have

$$\left(\begin{bmatrix} -31 & 14 & 10 \\ -76 & 35 & 26 \\ 18 & -9 & -8 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}\right)\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} -29 & 14 & 10 \\ -76 & 37 & 26 \\ 18 & -9 & -6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Multiplying yields

$$\begin{bmatrix} -29v_1 + 14v_2 + 10v_3 \\ -76v_1 + 37v_2 + 26v_3 \\ 18v_1 - 9v_2 - 6v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so we must have

$$\begin{array}{rcrcrcl} -29v_1 & + & 14v_2 & + & 10v_3 & = & 0 \\ -76v_1 & + & 37v_2 & + & 26v_3 & = & 0 \\ 18v_1 & - & 9v_2 & - & 6v_3 & = & 0 \end{array} \tag{1.7.2}$$

Can you find a single set of values for the variables v_1, v_2, v_3 that solves all three equations? Answer on page 52. One solution is $v_1 = v_2 = 2$ and $v_3 = 3$. We can verify that

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \text{ is indeed an eigenvector}$$

by multiplying:

$$\begin{bmatrix} -31 & 14 & 10 \\ -76 & 35 & 26 \\ 18 & -9 & -8 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ -6 \end{bmatrix} = -2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Note that if $\mathbf{v}$ is an eigenvector of A associated with value λ, so is $c\mathbf{v}$. Therefore, solving for an eigenvector will always yield an infinite number of solutions. There is no unique eigenvector associated with a given eigenvalue. We will prove these facts later.

Now we can find an eigenvector of a matrix M given an eigenvalue, and we can find an eigenvalue of a matrix M given an eigenvector, but what if we have neither an eigenvalue nor eigenvector of M? Returning to (1.7.1), we know $(M - \lambda I)\mathbf{v} = \mathbf{0}$. Perhaps this reminds you of solving quadratic equations by factoring. In algebra, if you know that $(x - 2)(x + 5) = 0$, then you know that either $x - 2 = 0$ or $x + 5 = 0$ giving two solutions, $x = 2$ and $x = -5$. Something similar can be said of the equation $(M - \lambda I)\mathbf{v} = \mathbf{0}$. Certainly if $\mathbf{v} = \mathbf{0}$, the equation is true. But we have decided that $\mathbf{0}$ is excluded from being an eigenvector, so we seek solutions where $\mathbf{v} \neq \mathbf{0}$. Suppose $\det(M - \lambda I) \neq 0$, meaning $(M - \lambda I)$ is invertible. Then left-multiplying both sides of (1.7.1) yields

$$(M - \lambda I)^{-1}\left((M - \lambda I)\mathbf{v}\right) = (M - \lambda I)^{-1}\mathbf{0} = \mathbf{0}.$$

On the other hand, we know that multiplication by a matrix's inverse undoes multiplication by that matrix, so $(M - \lambda I)^{-1}\left((M - \lambda I)\mathbf{v}\right) = \mathbf{v}$. Since $(M - \lambda I)^{-1}\left((M - \lambda I)\mathbf{v}\right) = \mathbf{0}$ and $(M - \lambda I)^{-1}\left((M - \lambda I)\mathbf{v}\right) = \mathbf{v}$, we conclude $\mathbf{v} = \mathbf{0}$, which is disallowed as an eigenvector. No solutions come from letting $\det(M - \lambda I) \neq 0$, so it must be that $\det(M - \lambda I) = 0$. Since the determinant of a matrix is a scalar, this equation is a scalar equation (like those you have seen in algebra) in the single variable λ. Solving that equation for λ gives eigenvalues and knowing eigenvalues gives eigenvectors. Given that the equation

$$\det(M - \lambda I) = 0 \tag{1.7.3}$$

is the linchpin in finding eigenvalues and eigenvectors of a matrix, it has a name—the **characteristic equation** (of M). The expression $\det(M - \lambda I)$ is an n^{th} degree polynomial in λ and is called the **characteristic polynomial** (of M). Exercise 17 of section 3.7 requests an argument that $\det(M - \lambda I) = 0$ if and only if λ is an eigenvalue of M.

Key Concepts

characteristic equation $\det(M - \lambda I) = 0$ for any matrix M.

characteristic polynomial $\det(M - \lambda I)$ for any matrix M.

eigenpair An eigenvalue together with an associated eigenvector. $(\lambda, \mathbf{v})$ is an eigenpair for matrix M if $M\mathbf{v} = \lambda\mathbf{v}$.

eigenvalue A value λ is an eigenvalue of the matrix M if there is a nonzero vector $\mathbf{v}$ such that $M\mathbf{v} = \lambda\mathbf{v}$.

eigenvector A nonzero vector $\mathbf{v}$ is an eigenvector of a matrix M if there is a value λ such that $M\mathbf{v} = \lambda\mathbf{v}$.

zero vector Any vector with a zero for each entry.

SageMath

If `M` is a matrix in SageMath, then `M.eigenvectors_right()` lists its eigenvalues and eigenvectors, `M.eigenvalues()` lists only its eigenvalues, and `M.charpoly()` gives its characteristic polynomial. The following code computes the eigenvalues, eigenvectors, and characteristic polynomial of

$$A = \begin{bmatrix} -112 & -21 & -15 & -372 \\ -84 & -13 & -19 & -292 \\ -36 & -13 & -3 & -116 \\ 36 & 7 & 5 & 120 \end{bmatrix}.$$

```
M = matrix(4,4,[-112,-21,-15,-372,-84,-13,-19,-292,
                -36,-13,-3,-116,36,7,5,120])
print(M); print()
print(M.eigenvalues()); print()
print(M.eigenvectors_right()); print()
print(M.charpoly())
```

The output of the code is

```
[-112  -21  -15 -372]
[ -84  -13  -19 -292]
[ -36  -13   -3 -116]
[  36    7    5  120]

[-4, -12, 4, 4]

[(-4, [
(1, -3, -3, 0)
], 1), (-12, [
(1, 2/3, 2/3, -1/3)
], 1), (4, [
(1, -1/3, 1, -1/3)
], 2)]

x^4 + 8*x^3 - 64*x^2 - 128*x + 768
```

Since the characteristic polynomial of an $n \times n$ matrix has degree n it has n eigenvalues counting multiplicities and complex eigenvalues. The output of the `eigenvalues()` method above shows that 4 has multiplicity 2. It is listed twice in the list of eigenvalues (`[-4, -12, 4, 4]`). The `eigenvectors_right()` gives the same information and more. It prints out each eigenvalue, all associated eigenvectors, and finally the multiplicity of the eigenvalue. In the output above, the `eigenvectors_right()` method shows the multiplicity of the eigenvalue 4 (and more) by listing the eigenvalue 4 followed by its associated eigenvector $\langle 1, -1/3, 1, -1/3 \rangle$ and finally its multiplicity 2 (the last 3 lines of output before the characteristic polynomial):

```
], 1), (4, [
(1, -1/3, 1, -1/3)
], 2)]
```

Exercises

1. Use the fact that $\mathbf{v}$ is an eigenvector of A to find an eigenvalue of A.

   (a) $A = \begin{bmatrix} 8 & 6 \\ -9 & -7 \end{bmatrix}; \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

   (b) $A = \begin{bmatrix} -5 & -4 \\ 2 & 1 \end{bmatrix}; \mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & 0 & -2 \\ -4 & -1 & 1 \end{bmatrix}; \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 24 & -8 & 10 \\ 0 & 6 & 0 \\ -45 & 18 & -21 \end{bmatrix}; \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$

(e) $A = \begin{bmatrix} -28 & 0 & 48 & -48 \\ 80 & 6 & -186 & 182 \\ -4 & -1 & -3 & 9 \\ 4 & -1 & -15 & 21 \end{bmatrix}$;

$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

(f) $A = \begin{bmatrix} -24 & -8 & 11 & 27 \\ 216 & 52 & -130 & -258 \\ -52 & -8 & 29 & 57 \\ 52 & 8 & -33 & -61 \end{bmatrix}$;

$\mathbf{v} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$

2. Find the characteristic polynomial.

(a) $\begin{bmatrix} 7 & -10 \\ -11 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} -2 & -4 \\ 7 & 12 \end{bmatrix}$

(c) $\begin{bmatrix} -12 & 12 \\ 9 & -9 \end{bmatrix}$

(d) $\begin{bmatrix} -8 & -3 \\ 3 & -2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 \\ 4 & 5 & 1 \\ -6 & -6 & -4 \end{bmatrix}$

(f) $\begin{bmatrix} -12 & 9 & 6 \\ -15 & 12 & 6 \\ -5 & 3 & 5 \end{bmatrix}$

(g) $\begin{bmatrix} 9 & -15 & 4 \\ 39 & -33 & 28 \\ 33 & -15 & 28 \end{bmatrix}$

3. Find the eigenvalues. They may be complex.

(a) $\begin{bmatrix} 3 & -5 \\ 2 & -4 \end{bmatrix}$

(b) $\begin{bmatrix} 8 & -10 \\ 5 & -7 \end{bmatrix}$

(c) $\begin{bmatrix} -9 & 4 \\ -36 & 15 \end{bmatrix}$

(d) $\begin{bmatrix} -7 & 25 \\ -1 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 9 & 6 \\ -15 & -10 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & -2 \\ 6 & 3 \end{bmatrix}$

(h) $\begin{bmatrix} -2 & 2 \\ -1 & -2 \end{bmatrix}$

(i) $\begin{bmatrix} -1 & 10 & 6 \\ 2 & 3 & 2 \\ -2 & -2 & -1 \end{bmatrix}$

(j) $\begin{bmatrix} 2 & -3 & -2 \\ 12 & -17 & -12 \\ -15 & 21 & 15 \end{bmatrix}$

(k) $\begin{bmatrix} -3 & 0 & 0 \\ 4 & 13 & -12 \\ 4 & 16 & -15 \end{bmatrix}$

(l) $\begin{bmatrix} -4 & 3 & 3 & 0 \\ -16 & 1 & -13 & -24 \\ -4 & 2 & 4 & 0 \\ 6 & -3 & -3 & 2 \end{bmatrix}$

4. Find an eigenvector associated with the given eigenvalue.

(a) $A = \begin{bmatrix} 3 & -10 \\ 8 & -15 \end{bmatrix}; \lambda = -5$

(b) $A = \begin{bmatrix} -6 & 20 \\ -1 & 3 \end{bmatrix}; \lambda = -2$

(c) $A = \begin{bmatrix} -4 & 2 \\ -16 & 8 \end{bmatrix}; \lambda = 0$

(d) $A = \begin{bmatrix} 1 & 6 \\ 3 & -5 \end{bmatrix}; \lambda = -2 + 3\sqrt{3}$

(e) $A = \begin{bmatrix} 2 & 4 \\ -3 & -4 \end{bmatrix}; \lambda = -1 - i\sqrt{3}$

(f) $A = \begin{bmatrix} -5 & 6 & -12 \\ 7 & -8 & 16 \\ 5 & -6 & 12 \end{bmatrix}; \lambda = -2$

(g) $A = \begin{bmatrix} 9 & 1 & -5 \\ 33 & 17 & -25 \\ 36 & 12 & -24 \end{bmatrix}; \lambda = 6$

(h) $A = \begin{bmatrix} -18 & 6 & 6 \\ -19 & 5 & 9 \\ -11 & 5 & 1 \end{bmatrix}; \lambda = 0$

5. Describe a connection between eigenvalues and determinants.

6. True or false?

   (a) The only eigenvector corresponding to a zero eigenvalue is the zero vector.

   (b) An eigenvalue may be any complex number except zero.

   (c) An eigenvector cannot be the zero vector.

   (d) All 2×2 matrices have two different eigenvalues.

   (e) Each eigenvalue has exactly one corresponding eigenvector.

   (f) An eigenvalue may be any number, including zero and complex numbers.

   (g) An $n \times n$ matrix can have $n+1$ eigenvalues.

7. True or false? If all the entries in a square matrix M are integers but one of its eigenvalues is $\sqrt{2}$, then the entries of the corresponding eigenvector cannot all be integers.

8. Suppose matrix A is a 3×3 matrix such that $A \cdot \begin{bmatrix} 20 \\ -16 \\ 8 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$. Find an eigenvalue of A.

9. Let

$$M = \begin{bmatrix} 99 & -135 & -199 & 417 \\ 30 & -36 & -61 & 123 \\ 90 & -135 & -174 & 369 \\ 30 & -45 & -57 & 120 \end{bmatrix}$$

and use SageMath to determine which of the following vectors is an eigenvector of M. Also determine its associated eigenvalue.

$$\begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -6 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -6 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

Click here or copy and paste the following code into a SageCell to get started.

```
M = matrix(4,4,[99,-135,-199,417,
                30,-36,-61,123,
                90,-135,-174,369,
                30,-45,-57,120])
v1= matrix(4,1,[4,3,0,0])
v2= matrix(4,1,[2,1,-6,-3])
v3= matrix(4,1,[-1,-1,-6,5])
v4= matrix(4,1,[-1,2,-6,-2])
v5= matrix(4,1,[4,2,1,-1])
```

10. Use SageMath's `M.eigenvectors_right()` method to compute the eigenvectors of M from question 9. Notice that none of the eigenvectors from that question is listed as an eigenvector by SageMath, yet one of them is. Can you resolve this conundrum? Is it possible that linear combinations of eigenvectors are also eigenvectors?

11. Check your work on question 2 using SageMath's `charpoly()` method.

Answers

system solution: One possible solution of equations (1.7.2) follows. Start by dividing the third equation by 3, which yields

$$\begin{array}{rcrcrcl} -29v_1 & + & 14v_2 & + & 10v_3 & = & 0 \\ -76v_1 & + & 37v_2 & + & 26v_3 & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array}$$

The v_3 variable can be eliminated from the first two equations by adding 5 times the third equation to the first and 13 times the third equation to the second:

$$\begin{array}{rcrcrcl} v_1 & - & v_2 & & & = & 0 \\ 2v_1 & - & 2v_2 & & & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array}$$

Dividing the second equation by 2, we see it is just a repeat of the first equation, $v_1 - v_2 = 0$, which implies that $v_1 = v_2$. Substituting into the third equation, we find

$$6v_2 - 3v_2 - 2v_3 = 0$$

or $3v_2 - 2v_3 = 0$, which means $v_3 = \frac{3}{2}v_2$. This set of equations has infinitely many solutions! They take the form $v_1 = v_2$, $v_3 = \frac{3}{2}v_2$, and v_2 is any number. In terms of the eigenvector, this observation means

$$\mathbf{v} = \begin{bmatrix} v_2 \\ v_2 \\ \frac{3}{2}v_2 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ \frac{3}{2} \end{bmatrix}$$

for some value v_2. If $v_2 = 0$, however, then $\mathbf{v} = \mathbf{0}$, which is disallowed as an eigenvector. Therefore *every nonzero scalar multiple* of $\begin{bmatrix} 1 \\ 1 \\ \frac{3}{2} \end{bmatrix}$ is an eigenvector of A associated with the eigenvalue -2.

Chapter 2

Row Operations

2.1 Systems of Linear Equations

People have been solving systems of linear equations for millenia, since long before the advent of what we know today as algebra. Recorded history of linear systems dates back to about 150 BCE in China! Even the modern process of elimination, often learned in high school algebra and precalculus classes, and demonstrated on page 52, dates back in geometric form to at least the second century CE (circa 100), appearing as a narrative in the Chinese treatise *Nine Chapters on the Mathematical Arts*. Roger Hart proposes that a geometric version of the modern day procedure of using determinants to solve systems is also evident in *Nine Chapters* [10].

Crumpet 12: Yanghui Triangle

In China, the triangular arrangement of binomial coefficients, the first five rows of which are

$$
\begin{array}{c}
1\\
1\ 1\\
1\ 2\ 1\\
1\ 3\ 3\ 1\\
1\ 4\ 6\ 4\ 1
\end{array}
$$

is often called the Yanghui triangle. It was devised in the 11^{th} century CE by Jia Xian and popularized in the 13^{th} century by Yang Hui. Omar Khayyam, an 11^{th} century Persian figure also studied the triangle. [12]

In our modern treatment of systems of linear equations, a **linear equation** is an equation that takes the form

$$c_1v_1 + c_2v_2 + \cdots + c_nv_n = b \tag{2.1.1}$$

for coefficients $c_1, c_2, \ldots, c_n$, variables $v_1, v_2, \ldots, v_n$ and constant b. The (ordered) list $s_1, s_2, \ldots, s_n$ is a **solution of the equation** if and only if substitution of the values $s_1, s_2, \ldots, s_n$ for the variables $v_1, v_2, \ldots, v_n$, respectively, make the equation true. A set of linear equations is called a **linear system** or **system of linear equations**. The (ordered) list $s_1, s_2, \ldots, s_n$ is a **solution of a linear system** if and only if it is a solution of every equation in the system. The set of equations (1.7.2) is an example of a system of linear equations, and the lists $1, 1, \frac{3}{2}$ and $2, 2, 3$ are solutions of the system (given that these lists specify values of the variables v_1, v_2, v_3 in that order).

Calculations like the one on page 52 easily submit to the conciseness of matrices. If you are familiar with synthetic division, you are already familiar with this idea. Only coefficients and constants are retained. All variables and other "extraneous" symbols are unused. On the left is the original calculation from page 52. On the right is an accounting of the coefficients and constants of each equation during each step of the process, maintained in a 3×4 matrix. The first column holds the v_1 coefficients, the second column holds the v_2 coefficients, the third column holds the v_3 coefficients, and the fourth column holds the constants from the righthand sides of the equations.

The given system:

$$\begin{array}{rcrcrcl} -29v_1 & + & 14v_2 & + & 10v_3 & = & 0 \\ -76v_1 & + & 37v_2 & + & 26v_3 & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array} \qquad (2.1.2)$$

Adding 5 times the third equation to the first and 13 times the third equation to the second:

$$\begin{array}{rcrcrcl} v_1 & - & v_2 & & & = & 0 \\ 2v_1 & - & 2v_2 & & & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array}$$

Dividing the second equation by 2, we see it is just a repeat of the first equation, $v_1 - v_2 = 0$, so we can scrap it:

$$\begin{array}{rcrcrcl} v_1 & - & v_2 & & & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array}$$

Substituting $v_1 = v_2$ into the equation $6v_1 - 3v_2 - 2v_3 = 0$:

$$3v_2 - 2v_3 = 0$$

As a matrix:

$$\begin{bmatrix} -29 & 14 & 10 & 0 \\ -76 & 37 & 26 & 0 \\ 6 & -3 & -2 & 0 \end{bmatrix}$$

Adding 5 times the third row to the first and 13 times the third row to the second:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 2 & -2 & 0 & 0 \\ 6 & -3 & -2 & 0 \end{bmatrix}$$

Dividing the second row by 2, we see it is just a repeat of the first row, $1 \ -1 \ 0 \ 0$, so we can zero it out and swap it with the third row:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 6 & -3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtracting 6 times the first row from the second:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In either case, we have reduced the system to the two equations $v_1 - v_2 = 0$ and $3v_2 - 2v_3 = 0$, from which the solution follows.

Much like successful synthetic division is dependent on strict ordering of the coefficients of the polynomial, it should be noted that the success of the matrix process is dependent on strict ordering of the entries of the matrix. Each row of the matrix represents one equation. The rightmost column of the matrix represents the constants from the righthand sides of the equations. Each of the remaining columns represents the coefficients of a single variable from the lefthand sides of the equations. It can easily be verified that the systems

$$\begin{array}{rcrcrcl} -29v_1 & + & 14v_2 & + & 10v_3 & = & 0 \\ -76v_1 & + & 37v_2 & + & 26v_3 & = & 0 \\ 6v_1 & - & 3v_2 & - & 2v_3 & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcl} 14v_2 + 10v_3 & = & 29v_1 \\ 26v_3 + 37v_2 & = & 76v_1 \\ 6v_1 & = & 2v_3 + 3v_2 \end{array}$$

are equivalent. Each equation of the system on the left has simply been rewritten using positive coefficients in the system on the right. However, the matrix representation of the system on the right, which might be written as

$$\begin{bmatrix} 14 & 10 & 29 & 0 \\ 26 & 37 & 76 & 0 \\ 0 & 6 & 2 & 3 \end{bmatrix}$$

is not helpful in solving the system using row operations. Certainly we can subtract 38 times the third row from the second row:

$$\begin{bmatrix} 26 & 37 & 76 & 0 \end{bmatrix} - 38 \begin{bmatrix} 0 & 6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 26 & -191 & 0 & -114 \end{bmatrix}$$

to obtain the matrix

$$\begin{bmatrix} 14 & 10 & 29 & 0 \\ 26 & -191 & 0 & -114 \\ 0 & 6 & 2 & 3 \end{bmatrix}$$

creating a zero in the 2,3-entry just as before. The problem is the -191 and 0 of the resultant matrix have no meaning in terms of the original system. Those parts of the calculation represent $37v_2 - 38(6v_1)$, which simplifies to $37v_2 - 228v_1$ and not necessarily -191 times anything; and $76v_1 - 38(2v_3)$, which simplifies to $76v_1 - 76v_3$ or $76(v_1 - v_3)$ and not necessarily 0 times anything. For row operations to make sense in the context of solving systems of equations, the entries in a single column must all be coefficients of the same variable or constants from the equations of the system. By convention, the rightmost column always holds the constants from the righthand sides of the equations and the remainder of the columns represent the coefficients of one variable each. The order of the columns of coefficients is flexible as long as the order is known.

Besides noting that the order of the entries in the matrix is critical, what should be taken from the matrix solution of (2.1.2) is the fact that three row operations are enough to mirror the process of elimination using matrices. On the matrix side, we swapped rows, multiplied rows by a nonzero constant, and added multiples of one row to another. Everything done in solving a linear system can be modeled by one of these operations on the corresponding matrix. As such, these three operations are called **elementary row operations**. To summarize and name them, the elementary row operations are

1. **Swap**: swap two rows.

2. **Scale**: multiply each entry in a row by a scalar.

3. **Replace**: add a multiple of one row to another.

Any system of linear equations can be translated into a matrix and solved using these three operations. Even systems with no solution reveal themselves as unsolvable under the direction of these three operations. If that were the only purpose of row operations, they would be useful, but as the concepts of linear algebra unfold, the ideas laid out here will have much further reaching consequences.

Elementary Matrices

Any matrix resulting from performing an elementary row operation on an identity matrix is called an **elementary matrix**. For example

$$E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is an elementary matrix since it is just $I_{4\times 4}$ with the first two rows swapped. The feature of interest is that left-multiplying any other matrix by this elementary matrix has the effect of performing the corresponding row operation (swapping the first two rows) on that arbitrary matrix so long as the product is defined. For example, if we let $A = \begin{bmatrix} 0 & -12 & 4 & -10 & 10 \\ 6 & 2 & -8 & -2 & 5 \\ -5 & -4 & -6 & 9 & 7 \\ 1 & -1 & 8 & -9 & 3 \end{bmatrix}$, then

$$EA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -12 & 4 & -10 & 10 \\ 6 & 2 & -8 & -2 & 5 \\ -5 & -4 & -6 & 9 & 7 \\ 1 & -1 & 8 & -9 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 & -8 & -2 & 5 \\ 0 & -12 & 4 & -10 & 10 \\ -5 & -4 & -6 & 9 & 7 \\ 1 & -1 & 8 & -9 & 3 \end{bmatrix}. \tag{2.1.3}$$

Take a short break to verify at least one of the entries in each row. This exercise will help you see why the product works out as shown and will help illuminate the following computations.

The $1, j$-entry of EA is a linear combination of the entries in the j^{th} column of A. To be precise, $(EA)_{1,j} = 0A_{1,j} + 1A_{2,j} + 0A_{3,j} + 0A_{4,j}$. The entries from the first row of E are used as the coefficients of the linear combination needed to compute each entry of the first row of EA. It follows that the first row of EA is a linear combination of the rows of A using these same coefficients. Symbolically, $(EA)_{1,:} = 0A_{1,:} + 1A_{2,:} + 0A_{3,:} + 0A_{4,:}$. In summary, each row of EA is the linear combination of the rows of A with coefficients from the corresponding row of E. To compute the third row of EA, for example, we use the entries from the third row of E as coefficients of a linear combination of the rows of A:

$$\begin{aligned} (EA)_{3,:} = & 0\begin{bmatrix} 0 & -12 & 4 & -10 & 10 \end{bmatrix} + \\ & 0\begin{bmatrix} 6 & 2 & -8 & -2 & 5 \end{bmatrix} + \\ & 1\begin{bmatrix} -5 & -4 & -6 & 9 & 7 \end{bmatrix} + \\ & 0\begin{bmatrix} 1 & -1 & 8 & -9 & 3 \end{bmatrix} \\ = & \begin{bmatrix} -5 & -4 & -6 & 9 & 7 \end{bmatrix}. \end{aligned}$$

All the rows of EA can be computed quickly from this perspective. The first row of E, $\begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, suggests that the first row of EA is 0 times row one of A plus 1 times row 2 of A plus 0 times row 3 of A plus 0 times row 4 of A. Looking at it this way makes it clear that the first row of EA is simply the second row of A. Similarly, the second row of EA is just the first row of A, the third row of EA is the third row of A, and the fourth row of EA is the fourth row of A. In other words, multiplying A by E has the effect of swapping the first two rows of A as claimed. The effect of multiplying by other elementary matrices can be verified similarly.

Key Concepts

elementary row matrix The matrix resulting from performing a single row operation on an identity matrix.

elementary row operation One of swap, scale, or replace.

linear equation An equation of the form $c_1v_1 + c_2v_2 + \cdots + c_nv_n = b$ where $c_1, c_2, \ldots, c_n$ are coefficients, $v_1, v_2, \ldots, v_n$ are variables, and b is a constant.

linear system Another name for a system of linear equations.

(row) swap Swapping two rows of a matrix.

(row) scale Multiplying each entry of a single row of a matrix by a nonzero scalar.

(row) replace Replacing a row of a matrix with the sum of it plus some multiple of another row.

system of linear equations A set of linear equations.

solution of a linear system An (ordered) list of values $s_1, s_2, \ldots, s_n$ that is a solution of every equation in the system.

solution of a linear equation An (ordered) list of values $s_1, s_2, \ldots, s_n$ such that substitution for the variables $v_1, v_2, \ldots, v_n$, respectively, in the linear equation $c_1v_1 + c_2v_2 + \cdots + c_nv_n = b$ make the equation true.

Exercises

1. Represent the linear system as a matrix.

   (a)
   $$\begin{array}{rcrcrcr} 3x & + & 2y & - & 8z & = & 9 \\ & - & 3y & + & 2z & = & 10 \\ -7x & + & y & & & = & -11 \end{array}$$

   (b)
   $$\begin{array}{rcrcrcr} 2x_1 & - & 8 & + & x_3 & = & 0 \\ 3 & + & x_2 & - & x_3 & = & 0 \\ x_1 & + & 2x_2 & - & 5 & = & 0 \end{array}$$

   (c)
   $$\begin{array}{rcrcrcrcr} 3v_1 & + & 2v_2 & + & 7 & = & 3v_3 & & \\ & & 8 & + & v_2 & = & 5v_1 & + & 2v_3 \\ v_1 & + & v_2 & + & v_3 & = & 11 & & \end{array}$$

2. Write the linear system associated with the matrix. Assume the variables of the system are $v_1, v_2, \ldots, v_n$ and their coefficients appear in that order in the matrix.

   (a) $\begin{bmatrix} -14 & -15 & 8 & -8 \\ -13 & 2 & -1 & 13 \\ 15 & -9 & -6 & 12 \end{bmatrix}$

   (b) $\begin{bmatrix} 6 & 0 & -11 \\ 4 & 12 & -1 \\ -5 & 15 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 10 & -9 & -1 & 3 & 15 & 6 \\ -11 & 12 & 13 & 5 & -4 & -2 \end{bmatrix}$

3. The matrix for a linear system is given. Find one solution of the associated system. Assume the variables of the system are $v_1, v_2, \ldots v_n$ and their coefficients appear in that order in the matrix.

   (a) $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 3 & 0 & 8 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

   (b) $\begin{bmatrix} 11 & 0 & 0 & 0 & 9 \\ 0 & 5 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & -13 \\ 0 & 0 & 0 & -2 & 6 \end{bmatrix}$

   (c) $\begin{bmatrix} 9 & -9 & -2 \\ 0 & 3 & 4 \end{bmatrix}$

   (d) $\begin{bmatrix} 11 & 0 & 9 & 12 \\ 0 & -8 & -4 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

4. A matrix representing a linear system is given. Explain why the system has no solution.

   $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

5. A matrix representing a linear system is given. Find one solution of the system, and explain why the system has infinitely many more solutions.

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6. What elementary row operation will transform

(a) $\begin{bmatrix} 1 & -4 & 4 & -9 \\ 1 & 6 & -5 & 1 \\ -2 & 7 & 1 & 1 \end{bmatrix}$ into $\begin{bmatrix} 1 & -4 & 4 & -9 \\ 1 & 6 & -5 & 1 \\ 0 & 19 & -9 & 3 \end{bmatrix}$?

(b) $\begin{bmatrix} -3 & 3 & 3 & -7 \\ 4 & -3 & 3 & 6 \\ -9 & -6 & 9 & -1 \end{bmatrix}$ into $\begin{bmatrix} -3 & 3 & 3 & -7 \\ -9 & -6 & 9 & -1 \\ 4 & -3 & 3 & 6 \end{bmatrix}$?

(c) $\begin{bmatrix} -5 & 3 & -2 & 7 \\ 2 & -6 & 3 & -5 \\ -9 & -9 & 7 & -9 \end{bmatrix}$ into $\begin{bmatrix} -9 & -9 & 7 & -9 \\ 2 & -6 & 3 & -5 \\ -5 & 3 & -2 & 7 \end{bmatrix}$?

(d) $\begin{bmatrix} 5 & -4 & 8 & 2 \\ 9 & 3 & -2 & -2 \\ 1 & -5 & -9 & -2 \end{bmatrix}$ into $\begin{bmatrix} 5 & -4 & 8 & 2 \\ 0 & 48 & 79 & 16 \\ 1 & -5 & -9 & -2 \end{bmatrix}$?

(e) $\begin{bmatrix} 2 & 6 & -1 & -8 \\ -1 & 1 & -6 & 4 \\ 2 & -5 & -6 & 6 \end{bmatrix}$ into $\begin{bmatrix} 2 & 6 & -1 & -8 \\ -2 & 2 & -12 & 8 \\ 2 & -5 & -6 & 6 \end{bmatrix}$?

7. Perform the row operation on the matrix.

(a) $A = \begin{bmatrix} 3 & -8 & -9 \\ -9 & 5 & 2 \\ 3 & -7 & -5 \\ -1 & 1 & -4 \end{bmatrix}; -2A_{3,:} \to A_{3,:}$

(b) $B = \begin{bmatrix} 3 & -1 & 1 & -7 \\ 1 & -2 & 6 & -4 \end{bmatrix}; 5B_{2,:} \to B_{2,:}$

(c) $C = \begin{bmatrix} -6 & 2 \\ 9 & 9 \\ 9 & 7 \\ 1 & -9 \end{bmatrix}; C_{1,:} \leftrightarrow C_{2,:}$

(d) $D = \begin{bmatrix} -9 & -9 & -6 \\ 5 & 6 & -2 \\ -9 & 9 & -4 \end{bmatrix}; D_{1,:} \leftrightarrow D_{3,:}$

(e) $E = \begin{bmatrix} -8 & 5 \\ -7 & -9 \\ -5 & -7 \end{bmatrix}; 3E_{2,:} + E_{3,:} \to E_{3,:}$

(f) $F = \begin{bmatrix} -5 & 1 & 9 & -6 \\ -3 & -5 & 1 & -1 \\ -8 & 3 & 2 & 2 \end{bmatrix}; -5F_{1,:} + F_{2,:} \to F_{2,:}$

8. The matrix T is given. What elementary row operation does left-multiplication by T perform?

(a) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$

9. Carry out a single elementary row operation to secure a 1 in row 1, column 1.

(a) $\begin{bmatrix} 10 & 1 & 7 \\ 3 & -3 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 6 & -4 \\ 2 & 3 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 2 & 3 \\ -2 & 10 & -3 \end{bmatrix}$

10. Execute a single elementary row operation to secure a 0 in row 2, column 1.

(a) $\begin{bmatrix} 1 & 10 & 2 \\ 3 & -3 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -3 \\ 2 & 3 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & -3 \\ -2 & 10 & -3 \end{bmatrix}$

11. Find the first row of the product by computing an appropriate linear combination of the rows of the righthand matrix.

(a) $\begin{bmatrix} -2 & 1 \\ 7 & -5 \end{bmatrix}\begin{bmatrix} 4 & -3 & 2 \\ 12 & 8 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & -2 & 4 \\ 7 & -5 & -8 \end{bmatrix} \begin{bmatrix} 0 & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

12. If the third row of A contains all zeros, what can you say about the third row of AB? Assume the product AB is defined.

13. If the first and second rows of A are multiples of one another, what can you say about the first and second rows of AB? Assume the product AB is defined.

2.2 Row Reduction

The system associated with an augmented matrix of the form

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}$$

has only one solution: $4, 5$, and we can see that without doing any computation. The system associated with this matrix, something you can probably visualize mentally, looks like

$$\begin{aligned} x &= 4 \\ y &= 5 \end{aligned}$$

(though you may have imagined different variables). The system is solved! The system corresponding to the augmented matrix

$$\begin{bmatrix} 3 & -2 & 2 \\ 5 & -3 & 5 \end{bmatrix}$$

corresponds to a system that is not solved. Can you write down the associated system to verify? Answer on page 69. **Row reduction** is the process of using elementary row operations to transform a matrix whose associated linear system is not solved into one whose associated linear system is solved, thus solving the system.

The following algorithm describes the process of row reduction for solving a system of n equations in n unknowns, starting with a matrix representation of the system.

Step 1: Select the leftmost column with at least one nonzero entry. This is a **pivot column**. The topmost position in this column is a **pivot position**. If no such column exists or there are no rows below the pivot position, continue to step 5.

Step 2: If the entry in the pivot position is 0, **swap** rows so the entry in the pivot position is nonzero. This nonzero entry is a **pivot**.

Step 3: **Replace** rows until all entries in the pivot column below the pivot are 0.

Step 4: Take the submatrix consisting of all rows below the pivot and return to step 1.

Step 5: Select the column of the rightmost pivot position and **replace** rows until all entries in that column other than the pivot are 0.

Step 6: Scale the row with the pivot so the pivot in that row is 1.

Step 7: If there are no rows above the pivot, stop. Otherwise, take the submatrix consisting of all rows above the pivot and return to step 5.

Translating the resulting matrix back into a system of equations reveals the solution. Any matrix produced by the completion of steps 1 through 4 is said to be in **row echelon form**. The system corresponding to a matrix in row echelon form can be solved by back-substitution, so these steps of the algorithm are often sufficient. Any matrix produced by the completion of the entire algorithm is said to be in **reduced row echelon form**. The system corresponding to a matrix in reduced row echelon form is solved.

Even though the algorithm may sound rather rigid, there are choices to be made along the way. All choices will lead to the same solution, but different choices may lead to drastically different-looking routes. In the end, all choices in row reduction by hand are a matter of preference.

Crumpet 13: Automated Row Reduction

A computer programmed to perform row reduction will have to make choices just as a human working by hand would. However, the objectives of a computer are slightly different from the objectives of a human. A human is looking to make the computation as easy as possible while a computer should be programmed to make the computation as *accurate* as possible. For a computer, working with fractions or decimal values is just as easy as working with integers, and doing a couple “extra” row operations is usually preferable to doing a lengthy analysis of how to avoid them. But computer computations are subject to round-off error, something that should be minimized whenever possible. Swapping rows so the pivot is the entry of greatest magnitude in the column helps reduce round-off error. The following is essentially computer pseudo-code that reduces a matrix to row echelon form.

Step 1: Translate the system into a matrix A.

Step 2: Let $i = k = 1$.

Step 3: **Swap** row i with row j where $j > i$ and $|A_{j,k}| > |A_{m,k}|$ for all $m \geq i$. If no such row exists, increment k by one and try again as long as $k \leq n$. If k reaches $n + 1$, stop.

Step 4: **Scale** row i by $\frac{1}{A_{i,k}}$.

Step 5: For each j from $i + 1$ through n, **replace** row j by $-\frac{A_{j,k}}{Ai,k}$ times row i plus row j.

Step 6: Increment i and k each by one and return to step 3 as long as $k \leq n$.

The resulting matrix must be returned to the form of a linear system and solved by back-substitution to complete the solution.

To illustrate some of the choices that must be made, consider solving the system

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & - & 2x_3 & = & 5 \\ 2x_1 & + & 4x_2 & + & 8x_3 & = & -13 \\ x_1 & + & 2x_2 & + & 3x_3 & = & -4 \end{array}$$

by row reduction. The following discussion charts the progress of three different approaches—(1) using fractions, (2) avoiding fractions by scaling non-pivot rows, and (3) avoiding fractions by scaling and swapping. All three approaches start from the augmented matrix

$$A = \begin{bmatrix} 3 & -1 & -2 & 5 \\ 2 & 4 & 8 & -13 \\ 1 & 2 & 3 & -4 \end{bmatrix}$$

of the system.

(1) Using fractions	(2) Avoiding fractions by scaling	(3) Avoiding fractions by scaling and swapping
Step 1: The first column is a pivot column. The pivot position is the topmost entry of the pivot column. There is nothing to do besides note this fact. **Step 2**: The pivot position must not contain a zero. This is already the case in all three approaches, but in approach (3) the first and third rows are swapped to secure a 1 as a pivot.		

		$A_{1,:} \leftrightarrow A_{3,:}$ $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 4 & 8 & -13 \\ 3 & -1 & -2 & 5 \end{bmatrix}$
Step 3: Replace rows to secure zeros in all rows below the pivot. Multiples of the row with the pivot are added to the rows below. In approach (1) the first row is scaled by $\frac{1}{3}$, and in approach (2) the second and third rows are scaled by 3 to prepare for replacement.		
$\frac{1}{3}A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \\ 2 & 4 & 8 & -13 \\ 1 & 2 & 3 & -4 \end{bmatrix}$	$3A_{2,:} \to A_{2,:}$ and $3A_{3,:} \to A_{3,:}$ $\begin{bmatrix} 3 & -1 & -2 & 5 \\ 6 & 12 & 24 & -39 \\ 3 & 6 & 9 & -12 \end{bmatrix}$	
$-2A_{1,:} + A_{2,:} \to A_{2,:}$ and $-A_{1,:} + A_{3,:} \to A_{3,:}$ $\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \\ 0 & \frac{14}{3} & \frac{28}{3} & -\frac{49}{3} \\ 0 & \frac{7}{3} & \frac{11}{3} & -\frac{17}{3} \end{bmatrix}$	$-2A_{1,:} + A_{2,:} \to A_{2,:}$ and $-A_{1,:} + A_{3,:} \to A_{3,:}$ $\begin{bmatrix} 3 & -1 & -2 & 5 \\ 0 & 14 & 28 & -49 \\ 0 & 7 & 11 & -17 \end{bmatrix}$	$-2A_{1,:} + A_{2,:} \to A_{2,:}$ and $-3A_{1,:} + A_{3,:} \to A_{3,:}$ $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 0 & 2 & -5 \\ 0 & -7 & -11 & 17 \end{bmatrix}$
Step 4: Take the submatrix consisting of all rows below the pivot and return to step 1. The first row of the matrix is fixed until step 5.		
$\begin{bmatrix} 0 & \frac{14}{3} & \frac{28}{3} & -\frac{49}{3} \\ 0 & \frac{7}{3} & \frac{11}{3} & -\frac{17}{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 14 & 28 & -49 \\ 0 & 7 & 11 & -17 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 2 & -5 \\ 0 & -7 & -11 & 17 \end{bmatrix}$
Step 1: The second column is a pivot column. The pivot position is the topmost entry of the pivot column. There is nothing to do besides note this fact. **Step 2**: The pivot position must not contain a zero. This is already the case in approaches (1) and (2), but in approach (3) the rows must be swapped.		
		$\begin{bmatrix} 0 & -7 & -11 & 17 \\ 0 & 0 & 2 & -5 \end{bmatrix}$
Step 3: Replace rows to secure zeros in all rows below the pivot. Multiples of the row with the pivot are added to the rows below. In approach (1) the first row is scaled by $\frac{1}{2}$; in approach (2) the second row is scaled by 2; and in approach (3) this step is already done.		
$\frac{1}{2}A_{1,:} \to A_1$ $\begin{bmatrix} 0 & \frac{7}{3} & \frac{14}{3} & -\frac{49}{6} \\ 0 & \frac{7}{3} & \frac{11}{3} & -\frac{17}{3} \end{bmatrix}$	$2A_{2,:} \to A_2$ $\begin{bmatrix} 0 & 14 & 28 & -49 \\ 0 & 14 & 22 & -34 \end{bmatrix}$	

<table>
<tr>
<td>$-A_{1,:}+A_{2,:}\to A_{2,:}$
$$\begin{bmatrix} 0 & \frac{7}{3} & \frac{14}{3} & -\frac{49}{6} \\ 0 & 0 & -1 & \frac{5}{2} \end{bmatrix}$$</td>
<td>$-A_{1,:}+A_{2,:}\to A_{2,:}$
$$\begin{bmatrix} 0 & 14 & 28 & -49 \\ 0 & 0 & -6 & 15 \end{bmatrix}$$</td>
<td></td>
</tr>
<tr>
<td colspan="3">Step 4: Take the submatrix consisting of all rows below the pivot and return to step 1. The first row of the submatrix (second row of the original matrix) is fixed until step 5.</td>
</tr>
<tr>
<td>$$\begin{bmatrix} 0 & 0 & -1 & \frac{5}{2} \end{bmatrix}$$</td>
<td>$$\begin{bmatrix} 0 & 0 & -6 & 15 \end{bmatrix}$$</td>
<td>$$\begin{bmatrix} 0 & 0 & 2 & -5 \end{bmatrix}$$</td>
</tr>
<tr>
<td colspan="3">Step 1: The third column is a pivot column. The pivot position is the topmost entry of the pivot column. There is nothing to do besides note this fact. Since there are no entries below the pivot, we move to step 5. At this point, our matrices are as follows. These matrices are all in row echelon form.</td>
</tr>
<tr>
<td>$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{2}{3} & \frac{5}{3} \\ 0 & \frac{7}{3} & \frac{14}{3} & -\frac{49}{6} \\ 0 & 0 & -1 & \frac{5}{2} \end{bmatrix}$$</td>
<td>$$\begin{bmatrix} 3 & -1 & -2 & 5 \\ 0 & 14 & 28 & -49 \\ 0 & 0 & -6 & 15 \end{bmatrix}$$</td>
<td>$$\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -7 & -11 & 17 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$</td>
</tr>
<tr>
<td colspan="3">Step 5: Select the column of the rightmost pivot position and replace rows until all entries in that column other than the pivot are 0. The pivots are $A_{1,1}$, $A_{2,2}$, and $A_{3,3}$, so the pivot postions are in columns 1, 2, and 3, and we select the third column. To prepare for the row replacements in approach (1) there is nothing to do; in approach (2) the third row is scaled by $\frac{1}{3}$; and in approach (3) the first and second rows are scaled by 2.</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{3}A_{3,:}\to A_{3,:}$
$$\begin{bmatrix} 3 & -1 & -2 & 5 \\ 0 & 14 & 28 & -49 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$</td>
<td>$2A_{1,:}\to A_{1,:}$ and $2A_{2,:}\to A_{2,:}$
$$\begin{bmatrix} 2 & 4 & 6 & -8 \\ 0 & -14 & -22 & 34 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$</td>
</tr>
<tr>
<td>$\frac{14}{3}A_{3,:}+A_{2,:}\to A_{2,:}$ and
$-\frac{2}{3}A_{3,:}+A_{1,:}\to A_{1,:}$
$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{7}{3} & 0 & \frac{7}{2} \\ 0 & 0 & -1 & \frac{5}{2} \end{bmatrix}$$</td>
<td>$14A_{3,:}+A_{2,:}\to A_{2,:}$ and
$-A_{3,:}+A_{1,:}\to A_{1,:}$
$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 14 & 0 & 21 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$</td>
<td>$11A_{3,:}+A_{2,:}\to A_{2,:}$ and
$-3A_{3,:}+A_{1,:}\to A_{1,:}$
$$\begin{bmatrix} 2 & 4 & 0 & 7 \\ 0 & -14 & 0 & -21 \\ 0 & 0 & 2 & -5 \end{bmatrix}$$</td>
</tr>
<tr>
<td colspan="3">Step 6: Scale the row with the pivot so the pivot in that row is 1. Fractions at this point are unavoidable.</td>
</tr>
<tr>
<td>$-A_{3,:}\to A_{3,:}$
$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{7}{3} & 0 & \frac{7}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$$</td>
<td>$-\frac{1}{2}A_{3,:}\to A_{3,:}$
$$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 14 & 0 & 21 \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$$</td>
<td>$\frac{1}{2}A_{3,:}\to A_{3,:}$
$$\begin{bmatrix} 2 & 4 & 0 & 7 \\ 0 & -14 & 0 & -21 \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$$</td>
</tr>
</table>

Step 7: Take the submatrix consisting of all rows above the pivot and return to step 5. The third row of the matrix is fixed.

$\begin{bmatrix} 1 & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{7}{3} & 0 & \frac{7}{2} \end{bmatrix}$	$\begin{bmatrix} 3 & -1 & 0 & 0 \\ 0 & 14 & 0 & 21 \end{bmatrix}$	$\begin{bmatrix} 2 & 4 & 0 & 7 \\ 0 & -14 & 0 & -21 \end{bmatrix}$

Step 5: The rightmost pivot postion is in column 2. To prepare for the row replacements in approach (1) there is nothing to do; in approach (2) the second row is scaled by $\frac{1}{7}$ while the first row is scaled by 2; and in approach (3) the second row is scaled by $\frac{1}{7}$.

	$\frac{1}{7}A_{2,:} \to A_{2,:}$ and $2A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 6 & -2 & 0 & 0 \\ 0 & 2 & 0 & 3 \end{bmatrix}$	$\frac{1}{7}A_{2,:} \to A_{2,:}$ $\begin{bmatrix} 2 & 4 & 0 & 7 \\ 0 & -2 & 0 & -3 \end{bmatrix}$
$\frac{1}{7}A_{2,:} + A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{7}{3} & 0 & \frac{7}{2} \end{bmatrix}$	$A_{2,:} + A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 6 & 0 & 0 & 3 \\ 0 & 2 & 0 & 3 \end{bmatrix}$	$2A_{2,:} + A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & -2 & 0 & -3 \end{bmatrix}$

Step 6: Scale the row with the pivot so the pivot in that row is 1. Fractions at this point are unavoidable.

$\frac{3}{7}A_{2,:} \to A_{2,:}$ $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \end{bmatrix}$	$\frac{1}{2}A_{2,:} \to A_{2,:}$ $\begin{bmatrix} 6 & 0 & 0 & 3 \\ 0 & 1 & 0 & \frac{3}{2} \end{bmatrix}$	$-\frac{1}{2}A_{2,:} \to A_{2,:}$ $\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{3}{2} \end{bmatrix}$

Step 7: Take the submatrix consisting of all rows above the pivot and return to step 5. The bottom row of the submatrix (second row of the original matrix) is fixed.

$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 6 & 0 & 0 & 3 \end{bmatrix}$	$\begin{bmatrix} 2 & 0 & 0 & 1 \end{bmatrix}$

Step 5: The rightmost pivot postion is in column 1. There are no entries other than the pivot in that column, so there are no row replacements to do. **Step 6**: Scale the row with the pivot so the pivot in that row is 1. Fractions at this point are unavoidable.

	$\frac{1}{6}A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\frac{1}{2}A_{1,:} \to A_{1,:}$ $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$

Step 7: There are no rows above the pivot, so we are done. At this point, our matrices are as follows. These matrices are all in reduced row echelon form.

$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{5}{2} \end{bmatrix}$

Notice that all three approaches produced the same reduced row echelon form. Writing the linear system corresponding to this reduced row echelon form, we see

$$\begin{aligned} x_1 &= \frac{1}{2} \\ x_2 &= \frac{3}{2} \\ x_3 &= -\frac{5}{2} \end{aligned}$$

The original system has this one solution.

A matrix containing the coefficients and constants of a linear system (one row for each equation, one column for the coefficients of each variable, and the rightmost column for the constants, as usual) is called an **augmented matrix**. A matrix containing the coefficients of a linear system (one row for each equation and one column for the coefficients of each variable, as usual) but not the constants is called a **coefficient matrix**. The coefficient matrix for any linear system is a submatrix of the corresponding augmented matrix. The augmented matrix simply has one more column—the column of constants for each equation.

The system

$$\begin{array}{rcrcr} 3x & - & 2y & = & 0 \\ 5x & - & 3y & = & 0 \end{array}$$

has augmented matrix

$$\begin{bmatrix} 3 & -2 & 0 \\ 5 & -3 & 0 \end{bmatrix}$$

and coefficient matrix

$$\begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}.$$

When the constants of a linear system are all zero, it is not necessary to represent the system as an augmented matrix. The coefficient matrix will do. After all, a column of zeros at the beginning of the row reduction process will be a column of zeros at the end of the process. Row operations do not change the entries of a column of zeros. For example, after swapping two rows with zeros in their j^{th} columns, the j^{th} column still has zeros in those rows. All that happened in column j was two zeros swapped places. The rest of the rows are not involved in the swap, so if their j^{th} columns held zeros before, they hold zeros after as well. Can you explain similarly why row replacement and row scaling also leave a column of zeros unchanged? Answer on page 69. A linear system whose constants are all zero is called a **homogeneous system**. Otherwise it is called a **nonhomogeneous system**.

Crumpet 14: Homogeneous Linear Differential Equations

A linear differential equation is homogeneous if its constant term is zero and nonhomogeneous otherwise.

Key Concepts

augmented matrix A matrix holding a particular arrangement of all the coefficients and constants of a linear system.

coefficient matrix A matrix holding a particular arrangement of all the coefficients but none of the constants of a linear system.

homogeneous system A linear system with constants all equal to zero.

nonhomogeneous system A linear system with at least one nonzero constant.

row reduction The process of using elementary row operations to transform a matrix whose associated linear system is not solved into one whose associated linear system is either solved or could be solved by back-substitution.

row echelon form An arrangement of zero and nonzero entries in a matrix such that the first four steps of the row reduction algorithm do not force any changes.

reduced row echelon form An arrangement of zero, one, and nonzero entries in a matrix such that the row reduction algorithm does not force any changes.

Exercises

1. Reduce the matrix to row echelon form.

   (a) $\begin{bmatrix} -2 & -4 & -10 \\ -5 & 2 & -1 \end{bmatrix}$

   (b) $\begin{bmatrix} 1 & -4 & 0 \\ 3 & 5 & 4 \end{bmatrix}$

   (c) $\begin{bmatrix} 6 & -1 & -4 \\ 4 & -3 & 1 \\ 2 & -5 & -1 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & 0 & 2 \\ 4 & 1 & -3 \\ 1 & 1 & 3 \end{bmatrix}$

2. Reduce the matrix in question 68 to reduced row echelon form.

3. The coefficient matrix for a homogeneous linear system is given. Find one nontrivial solution (not all variables equal to zero) of the associated system if possible. Assume the variables of the system are $x_1, x_2, \ldots x_n$ and their coefficients appear in that order in the matrix.

   (a) $\begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix}$

   (b) $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

   (c) $\begin{bmatrix} -1 & 3 \\ 0 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 2 & -2 & 5 \\ 0 & 7 & -1 \\ 0 & 0 & 3 \end{bmatrix}$

   (e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(f) $\begin{bmatrix} 5 & 0 & -3 \\ 0 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 5 & 3 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. Solve the nonhomogeneous system by row reduction.

(a) $\begin{array}{rcrcr} -7x_1 & - & 2x_2 & = & -6 \\ 4x_1 & + & x_2 & = & -1 \end{array}$

(b) $\begin{array}{rcrcr} v_1 & + & 4v_2 & = & -4 \\ -3v_1 & - & 11v_2 & = & -5 \end{array}$

(c) $\begin{array}{rcrcr} 2x & + & 15y & = & 6 \\ -8x & - & 65y & = & 4 \end{array}$

(d) $\begin{array}{rcrcr} -3x & + & 5y & = & -3 \\ -15x & + & 20y & = & 6 \end{array}$

(e) $\begin{array}{rcrcr} x_1 & - & x_2 & = & 3 \\ -4x_1 & + & 5x_2 & = & -3 \end{array}$

(f) $\begin{array}{rcrcr} -45x & - & 8y & = & -6 \\ -10x & - & 2y & = & 4 \end{array}$

(g) $\begin{array}{rcrcrcr} x & + & 3y & - & z & = & -3 \\ x & + & y & + & 2z & = & -5 \\ -3x & - & 10y & + & 4z & = & 6 \end{array}$

(h) $\begin{array}{rcrcrcr} -3v_1 & - & 35v_2 & + & 10v_3 & = & 2 \\ 9v_1 & + & 130v_2 & - & 40v_3 & = & 2 \\ 9v_1 & + & 120v_2 & - & 35v_3 & = & -4 \end{array}$

(i) $\begin{array}{rcrcrcr} -v_1 & + & 2v_2 & + & 8v_3 & = & -6 \\ -v_1 & + & v_2 & + & 5v_3 & = & -6 \\ -2v_1 & + & 2v_2 & + & 11v_3 & = & -1 \end{array}$

(j) $\begin{array}{rcrcrcr} 12x_1 & - & 9x_2 & - & 14x_3 & = & 3 \\ 24x_1 & - & 15x_2 & - & 21x_3 & = & -3 \\ -9x_1 & + & 6x_2 & + & 7x_3 & = & 2 \end{array}$

(k) $\begin{array}{rcrcrcrcr} 2w & - & x & + & y & + & 4z & = & -3 \\ -w & + & x & & & - & 3z & = & 5 \\ -w & + & x & + & y & - & 2z & = & -2 \\ & & & - & y & - & z & = & -6 \end{array}$

(l) $\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 4v_3 & - & v_4 & = & 1 \\ -2x_1 & + & 6x_2 & - & 10x_3 & + & 5x_4 & = & -3 \\ -x_1 & & & + & 2x_3 & - & 2x_4 & = & 2 \\ 3x_1 & - & 2x_2 & + & 11x_3 & - & 2x_4 & = & 6 \end{array}$

5. Let $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ be the coefficient matrix for a linear system. What can you say about the solution(s) of the system if augmented by the given column?

(a) $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

6. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$. What can you say about the solutions of the associated system if

(a) the system is homogeneous and A is the coefficient matrix?

(b) the system is nonhomogeneous and A is the coefficient matrix?

(c) A is the augmented matrix?

Answers

associated system:

$$\begin{array}{rcrcr} 3x & - & 2y & = & 2 \\ 5x & - & 3y & = & 5 \end{array}$$

column of zeros: (row scale) After scaling a row with a zero in its j^{th} column, the j^{th} column still has a zero in that row since 0 times anything is 0. The rest of the rows are not involved in the scale, so if their j^{th} columns held zeros before, they hold zeros after as well. **(row replace)** After adding a multiple of a row with a zero in its j^{th} column, to a different row with a zero in its j^{th} column, both rows still have zeros in their j^{th} columns. 0 times anything is 0, so a 0 was added to the 0 in the row being replaced. Since 0 plus 0 is 0, the j^{th} column of that row is still zero. The rest of the rows are not involved in the replacment, so if their j^{th} columns held zeros before, they hold zeros after as well.

2.3 Existence, Uniqueness, and Echelon Forms

As augmented matrices,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

represent fundamentally different linear systems. The first matrix represents the system

$$\begin{aligned} x &= 0 \\ 2y &= 3 \end{aligned}$$

which has exactly one solution: $x = 0$, $y = \frac{3}{2}$. The second matrix represents the system

$$\begin{aligned} x + 2y &= 3 \\ 0 &= 0 \end{aligned}$$

which has infinitely many solutions, $x = 3 - 2y$ with y arbitrary. One example is $y = 1$, $x = 1$. The third matrix represents the system

$$\begin{aligned} x + 2y &= 0 \\ 0 &= 3 \end{aligned}$$

which has no solution. Even though the first equation has many solutons, the second equation will not be true for any values of x and y since 0 simply does not equal 3.

The three associated linear systems have different types of solution sets. The first set of solutions contains exactly one element. The second set of solutions contains infinitely many elements. The third set of solutions is empty.

The three matrices

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 10 & -3 \end{bmatrix}, \quad \begin{bmatrix} 7 & 10 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \begin{bmatrix} 7 & 10 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

are similar to the first three in this way. One of them has an associated linear system with no solution, one with infinitely many solutions, and one with exactly one solution. Can you tell which is which? Answer on page 79.

Comparing the first set of three matrices to the second set of three matrices, you may notice that all six matrices have three 0 entries and three nonzero entries, one per column. Further, there are only three arrangements of the zeros. Using # to represent the nonzero numbers, the arrangements are

$$\begin{bmatrix} \# & \# & \# \\ 0 & 0 & 0 \end{bmatrix}, \quad , \begin{bmatrix} \# & 0 & 0 \\ 0 & \# & \# \end{bmatrix} \quad \text{and} \begin{bmatrix} \# & \# & 0 \\ 0 & 0 & \# \end{bmatrix}$$

in order from most to fewest solutions. You can verify that each one is in row echelon form by applying the first 4 steps of the row reduction algorithm. The nonzero numbers may change, but the form (arrangement of zeros and nonzeros) will not! Additionally, no matter what nonzero numbers take the place of the #s, the number of solutions of the associated systems will not change. This is the real power of row echelon form matrices.

More generally, some of the entries of these forms could be replaced by other symbols without disrupting row echelon form. Using $\star$ to represent *any number* (including zero), matrices with the following arrangements of entries are all in row echelon form

$$\begin{bmatrix} \# & \star & \star \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \# & \star & \star \\ 0 & \# & \star \end{bmatrix}, \quad \text{and} \begin{bmatrix} \# & \star & \star \\ 0 & 0 & \# \end{bmatrix}. \tag{2.3.1}$$

These more general forms also have infinitely many, one, and zero solutions, respectively. Imagining the associated linear systems will help you verify the number of solutions, but how can you tell they are in row echelon form?

Step 1 of the row reduction algorithm is applied to the entire matrix and then to each submatrix containing all the rows below the last identified pivot. As such, this step simply identifies the pivot positions. In each of the given matrices, the leftmost column has a nonzero entry, so it is a pivot column. The topmost (row one) position in this column is a pivot position. Returning to step 1 with the submatrix containg "all rows beneath the pivot" (as required in step 4) in this case just means the second row:

$$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \# & \star \end{bmatrix}, \quad \text{and} \begin{bmatrix} 0 & 0 & \# \end{bmatrix}$$

The leftmost nonzero entry in the second row is therefore a pivot position. The pivot positions of matrices (2.3.1) are boxed below.

$$\begin{bmatrix} \boxed{\#} & \star & \star \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{\#} & \star & \star \\ 0 & \boxed{\#} & \star \end{bmatrix}, \quad \text{and} \begin{bmatrix} \boxed{\#} & \star & \star \\ 0 & 0 & \boxed{\#} \end{bmatrix}$$

Step 2 of the row reduction algorithm requires that each pivot is nonzero. This is already the case so no action is required.

Step 3 of the row reduction algorithm requires all entries below a pivot (and in the same column as the pivot) be zero. In all three matrices, the only pivot with entries directly below it are those in the 1,1-entry, and there are zeros beneath them in each case, so no action is required.

Step 4 of the row reduction algorithm sends the algorithm back to the first step. It does not, by itself, provide for any changes to the matrix.

What we can glean from analysis of the algorithm is that in row echelon form,

1. every pivot is to the right of the pivots in rows above it, and
2. all rows of zeros are below all rows with nonzero entries.

Both of these facts are immediate consequences of the algorithm. When a pivot is identified, all entries below it in that column are made zero, so when returning to step 1 with the submatrix containing all rows beneath the pivot, the column with the previously identified pivot contains only zeros. The leftmost nonzero column remaining must be to the right! Requiring that the pivot be nonzero ensures that no row of zeros can appear above a row with nonzeros. As long as these two requirements are satisfied, steps 2 and 3 (the only ones of the first 4 that cause any change to a matrix) are satisfied, so the matrix is in row echelon form.

Noticing that a pivot will always be the leftmost nonzero entry in a row makes determining whether a matrix is in row echelon form a simple matter. Identify the leftmost nonzero entry of each row. These are the pivots. Make sure they are all to the right of the ones above them. Then check to make sure any rows of zeros are at the bottom.

Given this description of row echelon form matrices, there are four row echelon forms for 2×3 matrices besides those in (2.3.1). Can you find them? Answer on page 79. Try any other arrangement of 0s, $\star$s, and #s such as

$$\begin{bmatrix} \# & \# & \star \\ 0 & \# & 0 \end{bmatrix} \text{ or } \begin{bmatrix} \star & \star & \star \\ 0 & \# & \star \end{bmatrix}$$

and you will see that it is either a special case of one these seven forms or there is a substitution of numbers for which the matrix is not in row echelon form. Can you show this is true for these two matrices? Answer on page 79.

A linear system with at least one solution is called **consistent**, perhaps deriving from the fact that the equations do not contradict one another. A linear system with no solution is called **inconsistent**. The corresponding augmented matrix in row echelon form makes telling the difference between the two straightforward. If the row echelon form matrix has a pivot in the last column, the linear system is inconsistent. Otherwise it is consistent. Indeed, a row containing a pivot in the last column has the form $\begin{bmatrix} 0 & 0 & \cdots & 0 & \# \end{bmatrix}$, which translates to the equation $0 = \#$ (zero equals a nonzero number), which of course is nonsense. If a row echelon form matrix does not have a pivot in the rightmost column, the corresponding system can be solved by back-substitution.

The pivots can also be used to determine the number of solutions of a consistent system. If the row echelon form matrix has a pivot in each column (beside the rightmost), the linear system has exactly one solution. Otherwise it has infinitely many solutions. The second case is worth taking a closer look at. All of the following augmented matrices are in row echelon form and are pivot-free in the rightmost column plus at least one other column.

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \quad \begin{bmatrix} -3 & 5 & -2 & -1 \\ 0 & 5 & -4 & 0 \end{bmatrix} \quad \begin{bmatrix} -4 & 1 & 5 & -7 & 3 \\ 0 & 0 & 5 & 1 & -1 \end{bmatrix}$$

This is enough to know they all have infinitely many solutions, but writing down those solutions still takes a little work. The linear system represented by the first matrix is

$$x_1 + 3x_2 = 5,$$

a single equation in two variables. Solving for x_1, this system has solutions of the form $x_1 = 5 - 3x_2$ and x_2 arbitrary. This implies that we are free to choose any value for x_2 and use the relation $x_1 = 5 - 3x_2$ to determine x_1. For example, we may let $x_2 = 7$, forcing $x_1 = 5 - 3(7) = -16$. So $7, -16$ is a solution. In this way, the equation $x_1 = 5 - 3x_2$ (or equivalently the equation $x_1 + 3x_2 = 5$) identifies all the solutions of the system, so we could write $\{x_1 = 5 - 3x_2\}$ as the solution set.

Looking back, it would have been just as well to solve for x_2, yielding $x_2 = \frac{5-x_1}{3}$. This formulation would suggest we are free to choose the value of x_1 and use the formula to determine x_2. Doing so makes the solution set $\left\{x_2 = \frac{5-x_1}{3}\right\}$.

Either variable may be treated as arbitrary, giving what appear to be two different solutions. If this makes you feel a little uneasy, you are not alone. Even if we were never to have seen the second solution, the first one may seem a little unsatisfying. We have a formula for one variable and an implicit understanding that the other is free to take on any value. Perhaps a more satisfying way to write down the set of all solutions is to return to the variable x_2, for which we are free to assign any arbitrary value and let it be r (for **r**-bitrary). Doing this, we have $x_2 = r$ and $x_1 = 5 - 3r$ for a solution set. In the spirit of linear algebra, this solution can be expressed in matrix notation as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 - 3r \\ r \end{bmatrix}$$

or better still as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Take a moment to verify that these matrix representations are equivalent to $x_2 = r$ and $x_1 = 5-3r$. The last presentation of the solution might feel more satisfying. It does not favor one variable over the other and gives an explicit formula for the value of each variable. This form of the solution is called **parametric vector form**. Returning to the variable x_1 and setting it equal to r leads to a similar parametric vector form of the solution. Can you find it? Answer on page 79.

Crumpet 15: Particular and Homogeneous Solutions

The solution of the linear system

$$x_1 + 3x_2 = 5$$

can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_p + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_h$$

where $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_p$, called the particular solution, is any one (particular) solution of the system and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_h$, called the homogeneous solution, is the solution of the corresponding homogeneous system, $x_1 + 3x_2 = 0$. For example, $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ are valid particular solutions. The homogeneous system has solution $x_1 = -3x_2$, so any solution that includes all instances where the first variable is -3 times the second, as in $r\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $r\begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}$, or $r\begin{bmatrix} -3\sqrt{2} \\ \sqrt{2} \end{bmatrix}$, is a valid homogeneous solution. Any combination of one particular solution and one homogeneous solution is an equivalent **general solution** of the linear system. See section

---

Putting the second matrix,

$$\begin{bmatrix} -3 & 5 & -2 & -1 \\ 0 & 5 & -4 & 0 \end{bmatrix},$$

into reduced row echelon form will facilitate writing the solution of the corresponding linear system. Subtracting row 2 from row 1 and scaling each row appropriately yields reduced row echelon form

$$\begin{bmatrix} 1 & 0 & -2/3 & 1/3 \\ 0 & 1 & -4/5 & 0 \end{bmatrix}$$

(you should verify this) suggesting that the simplest way to write the solution is

$$\begin{aligned} x_1 &= \frac{1}{3} + \frac{2}{3}x_3 \\ x_2 &= \frac{4}{5}x_3 \end{aligned}$$

This solution further suggests that we let x_3 be **r**-bitrary and write the solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2/3 \\ 4/5 \\ 1 \end{bmatrix}.$$

For a solution without fractions, we can let $r = 10 + 15s$ (after all, r is arbitrary!), which gives

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + (10 + 15s) \begin{bmatrix} 2/3 \\ 4/5 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1/3 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 20/3 + 10s \\ 8 + 12s \\ 10 + 15s \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ 8 \\ 10 \end{bmatrix} + s \begin{bmatrix} 10 \\ 12 \\ 15 \end{bmatrix}
\end{aligned}$$

where s is the arbitrary variable. There will always be various ways to write the parametric form of the solution of a system with infinitely many solutions.

Putting the third matrix into reduced row echelon form will facilitate writing down its solution as well.

$$\begin{bmatrix} -4 & 1 & 5 & -7 & 3 \\ 0 & 0 & 5 & 1 & -1 \end{bmatrix}$$

is reduced by subtracting the second row from the first and then scaling the rows appropriately. The resulting reduced row echelon form is

$$\begin{bmatrix} 1 & -\frac{1}{4} & 0 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

from which we conclude that $x_1 = -1 + \frac{1}{4}x_2 - 2x_4$ and $x_3 = -\frac{1}{5} - \frac{1}{5}x_4$. Variables x_1 and x_3 are easily written in terms of x_2 and x_4, suggesting we allow x_2 and x_4 to be arbitrary. Letting $x_2 = r$ and $x_4 = s$, it follows that $x_1 = -1 + \frac{1}{4}r - 2s$ and $x_3 = -\frac{1}{5} - \frac{1}{5}s$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{5} \\ 0 \end{bmatrix} + r \begin{bmatrix} \frac{1}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix}.$$

Two variables may be set arbitrarily this time, a consequence of the fact that we have four variables and only two pivots. Each variable column without a pivot gives a variable that may be set arbitrarily, what are known as **free variables**. Variables represented by columns with pivots are called **basic variables**. Other forms of the solution may be obtained by letting different pairs of variables be the arbitrary ones or by making substitutions for the arbitrary r and s in the above solution, such as $r = 4t$ and $s = 4 + 5u$. Can you write the solution that results from this substitution? Answer on page 80.

The observations that free variables lead to infinitely many solutions and a pivot in the rightmost column of an augmented matrix leads to no solution justify the existence and uniqueness theorem for linear systems.

Theorem 1. [Existence and Uniqueness] *A linear system is consistent if and only if the rightmost column of its augmented matrix representation is not a pivot column. Furthermore, a consistent system will have (a) exactly one solution if it admits no free variables; and (b) infinitely many solutions if it admits at least one free variable.*

Thus the nature of the solution set for any linear system can be determined from a row echelon form of its associated augmented matrix. In problems where this is the entire question at hand, the row echelon form suffices, and can save a bit of time compared to using the reduced row echelon form. The reduced row echelon form, being a row echelon form itself, can be used for this purpose too, but better serves as a place from which to write down the solutions of the system. Thus in problems where the solution set for a linear system is needed, it is usually worth the time and effort to find the reduced row echelon form.

Reduced row echelon form requires that the entries above pivots are zero (step 5) and that each pivot is 1 (step 6), so reduced row echelon form matrices are row echelon form matrices with these extra two properties. The reduced row echelon form of a 2×3 matrix will take one of the following seven forms, for example.

$$\begin{bmatrix} 1 & \star & \star \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \star \\ 0 & 1 & \star \end{bmatrix}, \begin{bmatrix} 1 & \star & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \star \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is not by mistake that we refer to ***a*** row echelon form or ***the*** reduced row echelon form of a matrix. Row echelon form is not unique, but reduced row echelon form is unique (see crumpet 23 on page 168).

SageMath

If `M` is a matrix in SageMath, then `M.echelon_form()` returns a row echelon from and `M.rref()` returns the reduced row echelon form. The following code computes a row echelon form and the reduced echelon form of

$$A = \begin{bmatrix} 364 & -238 & 364 & -560 & -1414 \\ -1144 & 1537 & -1786 & 2966 & 7759 \\ -1040 & 1277 & -1490 & 2662 & 7067 \\ -1000 & 1114 & -1528 & 2684 & 6502 \\ 344 & -377 & 530 & -886 & -2087 \end{bmatrix}.$$

```
M = matrix(5,5,[-480,-340,-110,-110,100,242,146,54,54,-60,-721,-673,-277,
                -152,155,968,809,316,191,-215,-1039,-882,-268,-268,170])
print(M); print()
print(M.echelon_form()); print()
print(M.rref())
```

The output of the code is

```
[ -480  -340  -110  -110   100]
[  242   146    54    54   -60]
[ -721  -673  -277  -152   155]
[  968   809   316   191  -215]
[-1039  -882  -268  -268   170]

[   1   13   87 462   45]
[   0   25   25 150   25]
[   0    0 125 625   50]
[   0    0    0 750  150]
[   0    0    0   0    0]
```

```
[    1     0     0     0 -2/5]
[    0     1     0     0  2/5]
[    0     0     1     0 -3/5]
[    0     0     0     1  1/5]
[    0     0     0     0    0]
```

Key Concepts

basic variable a variable corresponding to the column of an augmented matrix with a pivot.

free variable a variable corresponding to the column of an augmented matrix with no pivot.

consistent having at least one solution.

inconsistent having no solution.

existence and uniqueness of solutions of linear systems see theorem 1.

parametric vector form a linear combination of vectors.

reduced row echelon form of a matrix is unique.

Exercises

1. The augmented matrix for a linear system is given. Determine whether it is in row echelon form. If it is, (i) state whether the associated linear system is consistent or inconsistent, and (ii) how many solutions it has.

(a) $\begin{bmatrix} 1 & 4 & 0 & 0 & 3 & -1 \\ 0 & 0 & 3 & 0 & 5 & -6 \\ 0 & 0 & 0 & -1 & 2 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 & 4 & 1 & 1 & 3 \\ 0 & 0 & -4 & 4 & -2 & -4 \\ 0 & -5 & 0 & 3 & 5 & 3 \\ 0 & 0 & 0 & 0 & -5 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 2 & -8 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & -5 \\ 5 & -3 & 0 & -1 \\ -3 & 4 & -1 & -5 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & -2 & 5 & 6 & 7 \\ 0 & -2 & 0 & 7 & -4 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 18 & 37 \end{bmatrix}$

2. The augmented matrix for a linear system is given. Determine whether it is in reduced row echelon form. If it is, find the solution set. If the solution set is infinite give your answer in parametric vector form.

(a) $\begin{bmatrix} 1 & 7 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 6 & 0 & -2 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & -9 & 0 & 4 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

3. The augmented matrix for a linear system is given. Is the system homogeneous? If yes, find

the solution set. If the solution set is infinite give your answer in parametric vector form.

(a) $\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & -2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. The coefficient matrix for a homogeneous linear system is given. Find the solution set. If the solution set is infinite give your answer in parametric vector form.

(a) $\begin{bmatrix} 6 & 7 & -3 \\ 0 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & -7 \\ 0 & -2 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & -6 \\ -4 & -2 & 12 \\ 1 & \frac{1}{2} & -3 \end{bmatrix}$

5. Use row reduction and a reduced row echelon form matrix to find all the eigenvectors. All eigenvalues for the matrix are given.

(a) $\begin{bmatrix} 12 & -8 \\ 15 & -10 \end{bmatrix}; \lambda = 0, 2$

(b) $\begin{bmatrix} 40 & 66 \\ -22 & -37 \end{bmatrix}; \lambda = 7, -4$

(c) $\begin{bmatrix} 6 & -4 & 16 \\ 3 & -7 & 4 \\ -6 & 2 & -14 \end{bmatrix}; \lambda = -6, -3$

(d) $\begin{bmatrix} -7 & -2 & -1 \\ 20 & 6 & 4 \\ -5 & -2 & -3 \end{bmatrix}; \lambda = -2, 0$

6. Using $0, \#, \star$ notation, list all the posible row echelon forms for a 3×3 matrix.

7. Suppose a 4×7 coefficient matrix has 4 pivots. What can you say about the existence and uniqueness of solutions for this system?

8. What conditions on the pivots of an augmented matrix would ensure the associated system had a unique solution?

9. What conditions on the pivots of a coefficient matrix would ensure the associated system had a unique solution?

10. A system of linear equations with fewer equations than unknowns is sometimes called underdetermined. Suppose such a system were consistent. Explain why the system must have an infinite number of solutions.

11. A system of linear equations with more equations than unknowns is sometimes called overdetermined. Can such a system be consistent? Illustrate your answer with an example (if yes) or an argument as to why not (if no).

12. Use SageMath to find (i) a row echelon form (but not reduced row echelon form) and (ii) a reduced row echelon form of

$$\begin{bmatrix} 2049 & -4548 & -511 & -5177 & 6023 \\ -4526 & 10252 & 916 & 11438 & -13292 \\ -6947 & 15538 & 1740 & 17601 & -20614 \\ -1388 & 2866 & 263 & 3166 & -3697 \\ -5781 & 12812 & 1211 & 14321 & -16671 \end{bmatrix}$$

Code to help you get started can be found at the end of the exercises.

13. Use SageMath to find the reduced row echelon form of the augmented matrix, and write down the solution of the corresponding linear system.

$$\begin{bmatrix} 27 & -36 & -4 & 2 & 4 & 58 \\ 12 & -16 & -2 & 1 & 2 & 27 \\ 15 & -20 & -4 & 2 & 4 & 42 \\ -3 & 4 & -2 & 1 & 2 & 7 \end{bmatrix}$$

Code to help you get started can be found at the end of the exercises.

14. Suppose the matrix in question 13 is the coefficient matrix of a homogeneous linear system and repeat the exercise.

SageMath code for exercise 12:

```
M=matrix(5,5,[2049,-4548,-511,-5177,6023,-4526,10252,916,11438,
              -13292,-6947,15538,1740,17601,-20614,-1388,2866,
              263,3166,-3697,-5781,12812,1211,14321,-16671])
M.echelon_form()
```

SageMath code for exercises 13 and 14:

```
M=matrix(4,6,[27,-36,-4,2,4,58,12,-16,-2,1,2,27,
              15,-20,-4,2,4,42,-3,4,-2,1,2,7])
M.rref()
```

Answers

which is which? The three matrices

$$\begin{bmatrix} 7 & 0 & 0 \\ 0 & 10 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \quad , \begin{bmatrix} 7 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{and} \begin{bmatrix} 7 & 10 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

have associated linear systems with one solution, no solution, and infinitely many solutions, respectively.

other row echelon forms The other four possible row echelon forms for 2×3 matrices are

$$\begin{bmatrix} 0 & \# & \star \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \# & \star \\ 0 & 0 & \# \end{bmatrix}, \begin{bmatrix} 0 & 0 & \# \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

cases where the first variable does not appear in either of the equations. The last two forms, where neither variable appears in either equation, have arguable applicability.

substitution The matrix $\begin{bmatrix} \# & \# & \star \\ 0 & \# & 0 \end{bmatrix}$ is in row echelon form for any substitution of numbers, but is a special case of $\begin{bmatrix} \# & \star & \star \\ 0 & \# & \star \end{bmatrix}$. The matrix $\begin{bmatrix} \star & \star & \star \\ 0 & \# & \star \end{bmatrix}$ is not in row echelon form for the substitution $\begin{bmatrix} 0 & 0 & \star \\ 0 & \# & \star \end{bmatrix}$. The pivot in the second column cannot be zero.

parametric vector form Setting $x_1 = r$ yields $x_2 = \frac{5-r}{3}$, which is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} r \\ \frac{5-r}{3} \end{bmatrix}$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix} + r \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix}$$

To make it look a little more like the previous solution, let $r = -3s$, which gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

substitution 2 The solution with substitution is

$$\begin{aligned}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{5} \\ 0 \end{bmatrix} + 4t \begin{bmatrix} \frac{1}{4} \\ 1 \\ 0 \\ 0 \end{bmatrix} + (4+5u) \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} t \\ 4t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{4}{5} - u \\ 4+5u \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 0 \\ -1 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ -1 \\ 5 \end{bmatrix}
\end{aligned}$$

Chapter 3

Matrix Algebra

3.1 Properties of Matrix Operations

Background

If you ever thought that algebra should be renamed "find x" or "how to solve equations", you are not alone. The study of algebra is largely concerned with solving equations. Linear algebra is considerably less concerned with solving equations, but it is still an important feature of the subject. There are many similarities between the rules that govern the manipulation of algebraic expressions involving real numbers and those governing the manipulation of expressions involving matrices, but there are significant differences, all worthy of recording. But first, a few words about the arithmetic of real numbers.

During the late 19^{th} century, the European mathematics community set to the task of answering foundational questions about arithmetic. They debated the questions of how to define the natural numbers and the real numbers; how to define the operations of addition and multiplication; and just as importantly to what extent such definitions were useful. The German mathematician Hermann Günther Grassmann (Graßmann) is generally credited with sparking the debate by showing that properties of the natural numbers that to that point had simply been taken for granted (such as the fact that $a + b = b + a$) could be proved from simpler principles. After a number of developments, the Italian mathematician Giuseppe Peano published his *Arithmetices Principia* [19] (Principles of Arithmetic, 1889) summarizing work to that point and adding his stamp in the form of his five axioms defining the natural numbers.

Crumpet 16: Foundations of Analysis

In 1951 Edmund Landau published the first edition of his *Grundlagen der Analysis* (Foundations of Analysis [14], available at https://b-ok.cc/book/2863641/855790, accessed Feb 9, 2021) where, based on the work of Peano, Dedekind, Cauchy and others, he develops the arithmetic of whole, rational, irrational and complex numbers in a single volume. Peano's five axioms defining the natural numbers appear on page 2 as follows.

> **Axiom 1**: 1 *is a natural number.*
>
> That is, our set is not empty; it contains an object called 1 (read "one").
>
> **Axiom 2**: *For each x there exists exactly one natural number, called the successor of x, which will be denoted by x'.*

In the case of complicated natural numbers x, we will enclose in parentheses the number whose successor is to be written down, since otherwise ambiguities might arise. We will do the same, throughout this book, in the case of $x + y$, xy, $x - y$, $-x$, x^y,etc.

Thus, if

$$x = y$$

then

$$x' = y'.$$

Axiom 3: *We always have*

$$x' \neq 1.$$

That is, there exists no number whose successor is 1.

Axiom 4: *If*

$$x' = y'$$

then

$$x = y.$$

That is, for any given number there exists either no number or exactly one number whose successor is the given number.

Axiom 5 (Axiom of Induction): *Let there be given a set $\mathfrak{R}$ of natural numbers, with the following properties:*

I) 1 *belongs to* $\mathfrak{R}$.

II) *If x belongs to $\mathfrak{R}$ then so does x'.*

Then $\mathfrak{R}$ contains all the natural numbers.

Kurt Friedrich Gödel's incompleteness theorems, published in 1931 [8], essentially concluded the debate. He proved that any consistent axiomatic system sufficient to describe arithmetic on the natural numbers (including Peano's five axioms) will admit statements that cannot be proven nor disproven from within the system. Despite this deficiency, Peano's axioms are sufficient to define natural numbers and prove the familiar properties of the operations of addition and multiplication. With the comfort of knowing these facts rest on solid foundation, we will assume their veracity. That is not to say we will simply take them for granted, however.

$1 + 3$ equals 4, and $3 + 1$ equals 4. $7 + 9$ equals 16, and $9 + 7$ equals 16. The more general statement that $a + b = b + a$ for any numbers a and b is called the commutative property for addition of real numbers. Though this property is one of the basic principles that can be proven based on even more basic principles, we will take the viewpoint that it had to turn out this way! The counting numbers, $1, 2, 3, \ldots$, represent how many of a thing we have (quantity). This is a fact engrained in our minds as we learn to count—at a very young age. Addition models what happens when two quantities are merged—another concept engrained in our minds early on in our mental development. If you add three apples to a basket initially holding one apple, afterward it will contain four apples. This merger is modeled by the statement $1 + 3 = 4$ (one apple plus three more apples gives you four apples). Similarly if you add one apple to a basket initially holding three apples, afterward it will contain four apples. This merger is modeled by the statement $3 + 1 = 4$. The fact that a pair of natural numbers can be added in either order with the same result is simply the way natural numbers work. Any mathematical axiom, theorem, or proof suggesting otherwise is simply not the number system we were taught as youngsters.

Table 3.1: Some Properties of Real Numbers

For all real numbers a, b, c

a. $a+b=b+a$	Commutative property for addition
b. $(a+b)+c=a+(b+c)$	Associative property for addition
c. $a+0=a=0+a$	Additive identity
d. $a(bc)=(ab)c$	Associative property for multiplication
e. $a(b+c)=ab+ac$	Distributive property
f. $1\cdot a=a=a\cdot 1$	Multiplicative identity

Crumpet 17: An Interesting Addition Table

If we completely abandon the usual notions of natural numbers (that there are infinitely many of them and that they represent quantities, for example), addition might be defined as follows.

+	1	2	3	4	5	6
1	2	6	4	5	3	1
2	6	1	5	3	4	2
3	5	4	6	2	1	3
4	3	5	1	6	2	4
5	4	3	2	1	6	5
6	1	2	3	4	5	6

This sytem of addition retains some of the familiar notions of arithmetic such as associativity and the existence of an identity (can you tell which symbol acts as the identity?), but not commutativity. According to this table, $1+3=4$ but $3+1=5$! Addition in this system is not commutative, but that does not make this number system "wrong". It simply means this system, an example of a finite group, does not represent the numbers as we commonly understand them. It makes a poor model for the measurement of quantity.

Some Properties

Table 3.1 summarizes the properties of real numbers of interest to our study of linear algebra, each of which has a matrix analog as shown in the following theorem. An $m\times n$ matrix whose entries are all zero is called a **zero matrix** and is denoted by $0_{m\times n}$ or just 0 when its size is discernible through context.

Theorem 2. [Matrix Properties Part 1] *For all matrices A, B, C*

1. $A+B=B+A$ *(commutative property for addition)*

2. $(A+B)+C=A+(B+C)$ *(associative property for addition)*

3. $A+0=A=0+A$ *(additive identity)*

4. $A(BC)=(AB)C$ *(associative property for multiplication)*

5. $A(B+C)=AB+AC$ *(left distributive property)*

6. *$IA = A = AI$ (multiplicative identity)*

whenever the indicated operations are defined.

Claim 1 can be proven by noting that

$$(A+B)_{i,j} = A_{i,j} + B_{i,j}$$

and

$$(B+A)_{i,j} = B_{i,j} + A_{i,j}$$

by definition of matrix addition. Since $A_{i,j}$ and $B_{i,j}$ are numbers, the commutative property for addition of real numbers allows us to use the fact that $A_{i,j} + B_{i,j} = B_{i,j} + A_{i,j}$ to deduce that $(A+B)_{i,j} = (B+A)_{i,j}$. Since the entries of $A+B$ and $B+A$ are equal, the matrices are equal.

In more words than symbols, the argument goes thus. The i,j-entry of $A+B$ is the i,j-entry of A plus the i,j-entry of B. The i,j-entry of $B+A$ is the i,j-entry of B plus the i,j-entry of A. But since the entries of A and B are numbers, the i,j-entry of A plus the i,j-entry of B is the same as the i,j-entry of B plus the i,j-entry of A. Therefore the i,j-entry of $A+B$ is the same as the i,j-entry of $B+A$. Hence $A+B = B+A$.

Using still fewer symbols, we might argue as follows. The i,j-entries of $A+B$ and $B+A$ are calculated by adding the same two entries of A and B only in different orders. Since addition of real numbers (entries) is commutative, the i,j-entries are equal. Hence $A+B = B+A$.

In yet a fourth way to see that theorem 2 claim 1 is true, consider the following computation without words.

$$\begin{aligned}
A+B &= \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{bmatrix} + \begin{bmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,n} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,n} \end{bmatrix} \\
&= \begin{bmatrix} A_{1,1}+B_{1,1} & A_{1,2}+B_{1,2} & \cdots & A_{1,n}+B_{1,n} \\ A_{2,1}+B_{2,1} & A_{2,2}+B_{2,2} & \cdots & A_{2,n}+B_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1}+B_{m,1} & A_{m,2}+B_{m,2} & \cdots & A_{m,n}+B_{m,n} \end{bmatrix} \\
&= \begin{bmatrix} B_{1,1}+A_{1,1} & B_{1,2}+A_{1,2} & \cdots & B_{1,n}+A_{1,n} \\ B_{2,1}+A_{2,1} & B_{2,2}+A_{2,2} & \cdots & B_{2,n}+A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1}+A_{m,1} & B_{m,2}+A_{m,2} & \cdots & B_{m,n}+A_{m,n} \end{bmatrix} \\
&= B+A
\end{aligned}$$

In essence, the fact that matrices obey the rule $A+B = B+A$ follows directly from the commutative property for addition of real numbers and the fact that addition of matrices is computed component-wise. The rest of the claims in theorem 2 can also be justified by use of the corresponding property for real numbers and careful application of the definitions of matrix addition and multiplication.

As noted in section 1.3 matrix multiplication is not commutative. The familiar commutative property for multiplication of real numbers, $ab = ba$, does not have a matrix analog. Thus it is important to point out, for example, that the distributive property for matrices holds for both left (as in theorem 2 claim 5) and right multiplication. Additionally, the interplay between scalar multiplication and both matrix addition and matrix multiplication must be documented, as seen in the following theorem.

Theorem 3. [Matrix Properties Part 2] *For all matrices A, B, C*

1. $r(A+B) = rA + rB$ *(distributivity of a scalar over matrices)*
2. $(r+s)A = rA + sA$ *(distributivity of a matrix over scalars)*
3. $(rs)A = r(sA)$ *(associativity of multiplication between two scalars and one matrix)*
4. $r(AB) = (rA)B = A(rB)$ *(associativity of multiplication between a scalar and two matrices)*
5. $(B+C)A = BA + CA$ *(right distributive property)*

whenever the indicated operations are defined.

Again these claims can be justified by use of the corresponding property for real numbers and careful application of the definitions of matrix addition and multiplication.

One final theorem for this section contains a list of identities concerning the matrix transpose and inverse. Claims concerning only the transpose can be proven by comparing i, j-entries as before while those concerning the inverse are most easily proven without reference to individual entries.

Theorem 4. [Matrix Properties Part 3] *For all matrices A and B*

1. $(A^T)^T = A$ *(transpose of the transpose)*
2. $(rA)^T = rA^T$ *(transpose of a scalar multiple)*
3. $(A+B)^T = A^T + B^T$ *(transpose of a sum)*
4. $(AB)^T = B^T A^T$ *(transpose of a product)*
5. $(A^{-1})^{-1} = A$ *(inverse of the inverse)*
6. $(AB)^{-1} = B^{-1}A^{-1}$ *(inverse of a product)*
7. $(A^T)^{-1} = (A^{-1})^T$ *(inverse of the transpose)*

whenever the indicated operations are defined.

To make the point that the claims involving inverses can be proven without reference to entries, consider theorem 4 claim 5. In words it says the inverse of the inverse of a matrix is the matrix itself. Or in other words, if you find the inverse of a matrix, then find the inverse of that matrix you get the original matrix. More in the words of the definition of inverse (matrices A and B are inverses if and only if $AB = BA = I$), any statement about inverses can be rephrased as a statement about a product. To show that two matrices are inverses, it is often best to show that their products, in both orders, each equal the identity. As for claim 5, it suffices to show $AA^{-1} = A^{-1}A = I$ (the inverse of A^{-1} is the matrix B such that $BA^{-1} = A^{-1}B = I$), but that equality is true due to the definition of inverse—end of proof. The claim is more a matter of perspective than a claim of something new.

While a list of 18 claims over 3 theorems may seem a bit overwhelming to digest, there are only a small few that require great attention. The claims of theorem 2 are replicas of properties of real numbers with which you are hopefully familiar. As such, it is the differences between the algebra of numbers and the algebra of matrices that should gain your focus. Primarily there is no commutative property for multiplication of matrices! This has consequences such as the appearance of claim 5 of theorem 3. The right distributive property is not necessary to enumerate separately for real numbers because it follows from the combination of commutativity for multiplication and distributivity of real numbers. The rest of theorem 3 is documentation of what you would probably expect to be true about scalar multiplication.

Besides the right distributive property, theorem 4 is worth careful scrutiny. It contains facts about the relatively new concepts of inverses and transposes. In particular, notice that (claim 4) the transpose of a product is the product of the transposes in the opposite order! Similarly, (claim 6) the inverse of a product is the product of the inverses in the opposite order. Can you justify this claim? Answer on page 87.

Applications to eigenpairs

As an example of the utility of the properties of theorems 2 through 4, the claim *if* $\mathbf{v}$ *is an eigenvector of* A *associated with value* λ, *so is* $c\mathbf{v}$ of section 1.7, page 49, can now be justified. Assuming $\mathbf{v}$ is an eigenvector of A, we know that $A\mathbf{v} = \lambda\mathbf{v}$ (by definition). To justify the claim, we need to demonstrate that $A(c\mathbf{v}) = \lambda(c\mathbf{v})$:

$$\begin{aligned} A(c\mathbf{v}) &= c(A\mathbf{v}) && \text{theorem 3 claim 4} \\ &= c(\lambda\mathbf{v}) && \text{definition of eigenpair} \\ &= (c\lambda)\mathbf{v} && \text{theorem 3 claim 3} \\ &= (\lambda c)\mathbf{v} && \text{commutative property for multiplication of real numbers} \\ &= \lambda(c\mathbf{v}) && \text{theorem 3 claim 3} \end{aligned}$$

Each algebraic manipulation must be supported by one of the theorems or a property of the real numbers.

As a second example of the utility of these theorems, we are also prepared to prove the claim that if $(\lambda, \mathbf{v})$ is an eigenpair for the matrix M, then $(M - \lambda I)\mathbf{v} = \mathbf{0}$, also from section 1.7. Technically the first line of the justification should itself be justified in some general form before it is used. Can you supply such a proof? Answer on page 87.

$$\begin{aligned} \mathbf{0} &= M\mathbf{v} - M\mathbf{v} && \text{the difference between a matrix and itself is a zero matrix} \\ &= M\mathbf{v} - \lambda\mathbf{v} && \text{definition of eigenpair (substitution of } \lambda\mathbf{v} \text{ for } M\mathbf{v}) \\ &= M\mathbf{v} - \lambda(I\mathbf{v}) && \text{definition of identity matrix (substitution of } I\mathbf{v} \text{ for } \mathbf{v}) \\ &= M\mathbf{v} - (\lambda I)\mathbf{v} && \text{theorem 3 claim 4} \\ &= (M - \lambda I)\mathbf{v} && \text{theorem 3 claim 5} \end{aligned}$$

Key Concepts

properties of matrix operations see theorems 2, 3, and 4.

zero matrix an $m \times n$ matrix whose entries are all zero, denoted by $0_{m\times n}$ or just 0 when its size is discernible through context.

Exercises

1. Illustrate the rule by example. Then explain in your own words why the rule is true.
   (a) $(A + B) + C = A + (B + C)$ (theorem 2 claim 2).
   (b) $A + 0 = A$ (theorem 2 claim 3).
   (c) $r(A + B) = rA + rB$ (theorem 3 claim 1).
   (d) $(r + s)A = rA + sA$ (theorem 3 claim 2).
   (e) $(rs)A = r(sA)$ (theorem 3 claim 3).
   (f) $(rA)B = A(rB)$ (theorem 3 claim 4).
   (g) $(A^T)^T = A$ (theorem 4 claim 1).
   (h) $(rA)^T = rA^T$ (theorem 4 claim 2).
   (i) $(A + B)^T = A^T + B^T$ (theorem 4 claim 3).
2. Justify the rule by showing that the i, j-entry of the lefthand side of the equation equals the ij-entry of the righthand side.
   (a) $A + 0 = A$ (theorem 2 claim 3).
   (b) $A(BC) = (AB)C$ (theorem 2 claim 4).

(c) $A(B + C) = AB + AC$ (theorem 2 claim 5).

(d) $(r + s)A = rA + sA$ (theorem 3 claim 2).

(e) $(rA)B = A(rB)$ (theorem 3 claim 4).

(f) $(B + C)A = BA + CA$ (theorem 3 claim 5).

(g) $(A^T)^T = A$ (theorem 4 claim 1).

(h) $(A + B)^T = A^T + B^T$ (theorem 4 claim 3).

(i) $(AB)^T = B^T A^T$ (theorem 4 claim 4).

3. Justify the claim by a string of equalities where each equality is supported by a definition or theorem or claim you justify separately.

   (a) $(AB)^{-1} = B^{-1}A^{-1}$ (theorem 4 claim 6).

   (b) $(A^T)^{-1} = (A^{-1})^T$ (theorem 4 claim 7).

4. Let $A = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix}$.

   (a) Compute AB^T.

   (b) Without any further computation, find BA^T and explain how you got it.

5. Let $A = \begin{bmatrix} 4 & 7 \\ 3 & 5 \end{bmatrix}$.

   (a) Compute A^{-1}.

   (b) Without any further computation, find $(A^T)^{-1}$ and explain how you got it.

6. Justify the claim

$$(A + B)(C + D) = AC + AD + BC + BD$$

   using a series of equalities, each one supported by a theorem.

7. Justify the claim by arguing that the i,j-entry on the lefthand side of the conclusion equals the i,j-entry on the righthand side of the conclusion.

   (a) If $A = B$ then $A - B = 0$. (The conclusion is $A - B = 0$. Use the assumption that $A = B$ in your argument.)

   (b) If $A - B = 0$ then $A = B$. (The conclusion is $A = B$. Use the assumption that $A - B = 0$ in your argument.)

8. Let A be an $m \times n$ matrix. Justify the claim by arguing that the i,j-entry on the lefthand side equals the i,j-entry on the righthand side.

   (a) $A0_{n\times\ell} = 0_{m\times\ell}$ for any positive integer ℓ.

   (b) $0_{\ell\times m}A = 0_{\ell\times n}$ for any positive integer ℓ.

9. Show that, for any matrix A, $-A$ is the additive inverse of A. That is,

   (a) $-A + A = 0$; and

   (b) $A + (-A) = 0$.

   $-A$ has the common meaning $-1 \cdot A$.

Answers

inverse of a product $(B^{-1}A^{-1})(AB) = ((B^{-1}A^{-1})A)B = (B^{-1}(A^{-1}A))B = (B^{-1}I)B = B^{-1}B = I$ using the associative property multiple times, the definition of inverse multiple times, and the definition of the multiplicative identity once. The second half of the justification, that $(AB)(B^{-1}A^{-1}) = I$, can be made by a similar string of equalities. You are encouraged to try it.

difference of a matrix with itself The general statement that if A is any matrix then $A - A = 0$ can be proven by noting that

$$(A - A)_{i,j} = A_{i,j} - A_{i,j} = 0$$

for all entries $(A - A)_{i,j}$.

3.2 Matrix Equations

The algebraic equation

$$x - 7 = 5$$

is commonly solved by adding 7 to both sides of the equation. The reason this works is because $-7+7 = 0$ (7 is the additive inverse of -7) and $x + 0 = x$ (0 is the additive identity). In linear algebra there is an additive identity matrix (theorem 2 claim 3) and there are additive inverses (section 3.1 exercise 9), so analogous equations ought to be solvable similarly. Indeed they are! The matrix equation

$$X - \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix} \tag{3.2.1}$$

can be solved by the same process. Adding $\begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix}$ to both sides of the equation gives

$$\begin{aligned} X - \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} &= \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} \\ X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} 5+7 & -3+1 \\ 2-3 & 4+8 \end{bmatrix} \\ X &= \begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix} \end{aligned}$$

Substituting $\begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix}$ for X in equation (3.2.1) yields a true statement, so $\begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix}$ is a solution.

The slightly more advanced equation

$$8x - 7 = 5$$

is commonly solved by first adding 7 to both sides and then dividing both sides by 8 (or equivalently multiplying both sides by $\frac{1}{8}$, the multiplicative inverse of 8). This might be demonstrated as follows.

$$\begin{aligned} 8x - 7 &= 5 \\ 8x - 7 + 7 &= 5 + 7 \\ 8x &= 12 \\ \frac{8x}{8} &= \frac{12}{8} \\ x &= \frac{3}{2} \end{aligned}$$

This method works because, in addtion to the facts that -7 and 7 are additive inverses and 0 is the additive identity, $\frac{8}{8} = 1$ (8 and $\frac{1}{8}$ are multiplicative inverses) and $1x = x$ (1 is the multiplicative identity). In linear algebra, there are multiplicative identity matrices and there are multiplicative inverses (section 1.6), so analogous equations ought to be solvable similarly. Indeed they are! The matrix equation

$$\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix} X - \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix}$$

can be solved by the same process. Adding $\begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix}$ to both sides of the equation and then (left) multiplying both sides by $\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}^{-1}$ gives

$$\begin{aligned}
\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix} X - \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} &= \begin{bmatrix} 5 & -3 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 7 & 1 \\ -3 & 8 \end{bmatrix} \\
\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix} X &= \begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix} \\
\begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix} X &= \begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix} \\
\begin{bmatrix} 1 & -2 \\ -2 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 9 & 4 \\ 4 & 2 \end{bmatrix} X &= \begin{bmatrix} 1 & -2 \\ -2 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 12 & -2 \\ -1 & 12 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X &= \begin{bmatrix} 14 & -26 \\ \frac{-57}{2} & 58 \end{bmatrix} \\
X &= \begin{bmatrix} 14 & -26 \\ \frac{-57}{2} & 58 \end{bmatrix}
\end{aligned}$$

Substituting $\begin{bmatrix} 14 & -26 \\ \frac{-57}{2} & 58 \end{bmatrix}$ for X in the original equation yields a true statement, so $\begin{bmatrix} 14 & -26 \\ \frac{-57}{2} & 58 \end{bmatrix}$ is a solution.

It seems as though the familiar ideas of adding, subtracting, multiplying, and "dividing" both sides of an equation are valid steps in solving matrix equations, but better not to take it for granted on the back of a pair of examples where it works. In adding a matrix to both sides of an equation (or subtracting a matrix from both sides of an equation) in an attempt to solve it, we are using the principle that for matrices A, B, C

$$\text{if } A = B \text{ then } A + C = B + C \tag{3.2.2}$$

whenever the indicated operations are defined. Since the veracity of this proposition is critical to the logical validity of the solutions above, solid proof is warranted.

The principle of equality suggested in exercise 7 of section 3.1 is useful. It says that $A = B$ if and only if $A - B = 0$. That is, if we know that $A = B$, we can safely say that $A - B = 0$. And if we know that $A - B = 0$, we can safely say that $A = B$. Hopefully that sounds logical whether you have completed exercise 7 or not.

Getting back to proposition (3.2.2), note that it begins with the assumption that $A = B$. By the principle of equality we can immediately deduce that $0 = A - B$. Now because $C - C = 0$ for any matrix C and 0 is the additive identity, we can proceed as follows.

$$\begin{aligned}
0 &= A - B \\
&= (A - B) + 0 \\
&= (A - B) + (C - C) \\
&= ((A - B) + C) - C \\
&= (A + (-B + C)) - C \\
&= (A + (C - B)) - C \\
&= ((A + C) - B) - C \\
&= (A + C) + (-B - C) \\
&= (A + C) - (B + C)
\end{aligned}$$

We have used associativity and commutativity for addition of matrices as well as the distributive property as well. Now that we have $0 = (A + C) - (B + C)$ we conclude that $A + C = B + C$.

Equally critical to the second solution above is the principle that for matrices A, B, C

$$\text{if } A = B \text{ then } CA = CB \tag{3.2.3}$$

whenever the indicated operations are defined. The proof is very similar to the proof of (3.2.2). Starting with $A = B$ allows us to proceed with $0 = A - B$, but this time we employ the fact that $C0 = 0$ for any matrix C (exercise 8 of section 3.1):

$$\begin{aligned} 0 &= A - B \\ &= C(A - B) \\ &= CA - CB \end{aligned}$$

and therefore $CA = CB$. It is also true that

$$\text{if } A = B \text{ then } AC = BC \tag{3.2.4}$$

whenever the indicated operations are defined. Why is this claim needed? Can you justify this claim? Answers on page 95.

Symbolic equations

The need to solve a wholly symbolic equation often arises in the study of mathematics. Principles (3.2.2), (3.2.3), and (3.2.4) are often used to solve such equations. Suppose $XA - I = B$ and we are interested in solving for X, for example. The process is the same as we would use if all the symbols represented numbers. We isolate the X by adding to or multiplying both sides of the equation by appropriate matrices:

$$\begin{aligned} XA - I + I &= B + I \\ XA &= B + I \\ XAA^{-1} &= (B + I)A^{-1} \\ X &= (B + I)A^{-1} \end{aligned}$$

Notice the careful right multiplication of both sides in the third line. It is not valid to left multiply one side of the equation while right multiplying the other. Also, we should note that this solution is only good as long as A is invertible!

The most important equation in linear algebra

For essentialy the entire rest of this textbook, we wil be concerned with solving equations of the form

$$M\mathbf{v} = \mathbf{b} \tag{3.2.5}$$

for $\mathbf{v}$. Symbolically the solution is straightforward when M is invertible. Using associativity of multiplication, principle (3.2.3), and the definitions of inverse and identity matrices:

$$\begin{aligned} M^{-1}(M\mathbf{v}) &= M^{-1}\mathbf{b} \\ (M^{-1}M)\mathbf{v} &= M^{-1}\mathbf{b} \\ I\mathbf{v} &= M^{-1}\mathbf{b} \\ \mathbf{v} &= M^{-1}\mathbf{b} \end{aligned} \tag{3.2.6}$$

but understanding it and its ramifications is not. Plus, what if M is not invertible?

On page 58 it was discovered that the product of a matrix A left multiplied by an elementary matrix E could be calculated one row at a time by noting that each row of EA is the linear combination of the rows of A with coefficients from the corresponding row of E. In symbols, row r of EA can be computed as $(EA)_{r,:} = E_{r,1}A_{1,:} + E_{r,2}A_{2,:} + \cdots + E_{r,n}A_{n,:}$. But this computation holds for any matrix product. Given any matrices B and A where BA is defined—that is, B has the same number of columns as A has rows, say n—row r of BA can be computed as

$$(BA)_{r,:} = B_{r,1}A_{1,:} + B_{r,2}A_{2,:} + \cdots + B_{r,n}A_{n,:}. \tag{3.2.7}$$

This fact is helpful in understanding matrix products such as the one in (3.2.5). For example, suppose the third row of M is twice the first row of M. Then the third row of $\mathbf{b}$ must be twice the first row of $\mathbf{b}$. From (3.2.7)

$$\begin{aligned}\mathbf{b}_{1,:} &= M_{1,1}\mathbf{v}_{1,:} + M_{1,2}\mathbf{v}_{2,:} + \cdots + M_{1,n}\mathbf{v}_{n,:}\\ \mathbf{b}_{3,:} &= M_{3,1}\mathbf{v}_{1,:} + M_{3,2}\mathbf{v}_{2,:} + \cdots + M_{3,n}\mathbf{v}_{n,:}\end{aligned}$$

but $M_{3,:} = 2M_{1,:}$, which means $M_{3,1} = 2M_{1,1},\ M_{3,2} = 2M_{1,2}, \ldots, M_{3,n} = 2M_{1,n}$, so

$$\begin{aligned}\mathbf{b}_{3,:} &= 2M_{1,1}\mathbf{v}_{1,:} + 2M_{1,2}\mathbf{v}_{2,:} + \cdots + 2M_{1,n}\mathbf{v}_{n,:}\\ &= 2(M_{1,1}\mathbf{v}_{1,:} + M_{1,2}\mathbf{v}_{2,:} + \cdots + M_{1,n}\mathbf{v}_{n,:})\\ &= 2\mathbf{b}_{1,:}\end{aligned}$$

As a consequence, for such a matrix M, equation (3.2.5) can only have solutions if the third row of $\mathbf{b}$ happens to be twice the first row of $\mathbf{b}$. There are choices of $\mathbf{b}$ for which the equation has no solution!

Exercises 8 through 10 of section 1.5 provide evidence that any matrix M in which some row is a linear combination of the others (this is certainly the case for a matrix where the third row is twice the first) will have determinant zero. Formula (1.6.1) suggests that matrices with zero determinant are not invertible. Stringing all this together, it seems the following concepts are interconnected.

- Some row of M is a linear combination of the others.
- M has determinant zero.
- M is not invertible.
- The equation $M\mathbf{v} = \mathbf{b}$ has no solution for certain choices of $\mathbf{b}$.

We are not quite ready to completely clarify the connection, but the pieces of the puzzle are falling into place, and taking a slightly different perspective on matrix multiplication will add one more item to the list.

Just as we can imagine the product of two matrices as a collection of linear combinations of the rows of the righthand matrix, we can also imagine the product as a collection of linear combinations of the columns of the lefthand matrix. Thinking generally again, suppose we are given an arbitrary pair of matrices B and A where BA is defined—that is, B has the same number of columns as A has rows, say n. By definition the i,c-entry of BA is $B_{i,1}A_{1,c} + B_{i,2}A_{2,c} + \cdots + B_{i,n}A_{n,c}$. Swapping the order of each product, $(BA)_{i,c} = A_{1,c}B_{i,1} + A_{2,c}B_{i,2} + \cdots + A_{n,c}B_{i,n}$. In particular,

$$\begin{aligned}(BA)_{1,c} &= A_{1,c}B_{1,1} + A_{2,c}B_{1,2} + \cdots + A_{n,c}B_{1,n}\\ (BA)_{2,c} &= A_{1,c}B_{2,1} + A_{2,c}B_{2,2} + \cdots + A_{n,c}B_{2,n}\\ &\ \ \vdots\\ (BA)_{m,c} &= A_{1,c}B_{m,1} + A_{2,c}B_{m,2} + \cdots + A_{n,c}B_{m,n}\end{aligned} \tag{3.2.8}$$

Reading these equations together (as columns of numbers) from left to right, they imply that column c of BA (the lefthand sides of the equations) equals $A_{1,c}$ times column 1 of B (the first terms of the righthand sides) plus $A_{2,c}$ times column 2 of B (the second terms of the righthand sides) plus $A_{3,c}$ times column 3 of B (the third terms of the righthand sides), and so on. In symbols,

$$\begin{bmatrix} (BA)_{1,c} \\ (BA)_{2,c} \\ \vdots \\ (BA)_{m,c} \end{bmatrix} = A_{1,c}\begin{bmatrix} B_{1,1} \\ B_{2,1} \\ \vdots \\ B_{m,1} \end{bmatrix} + A_{2,c}\begin{bmatrix} B_{1,2} \\ B_{2,2} \\ \vdots \\ B_{m,2} \end{bmatrix} + \cdots + A_{n,c}\begin{bmatrix} B_{1,n} \\ B_{2,n} \\ \vdots \\ B_{m,n} \end{bmatrix}$$

In other words, column c of BA is a linear combination of the columns of B where the coefficients for the linear combination come from column c of A. In short,

$$(BA)_{:,c} = A_{1,c}B_{:,1} + A_{2,c}B_{:,2} + \cdots + A_{n,c}B_{:,n}. \tag{3.2.9}$$

In the special case of equation (3.2.5),

$$(M\mathbf{v})_{:,c} = \mathbf{v}_{1,c}M_{:,1} + \mathbf{v}_{2,c}M_{:,2} + \cdots + \mathbf{v}_{n,c}M_{:,n}$$

but $\mathbf{v}$ is a vector (column matrix) so it has only one column. Accordingly, $M\mathbf{v}$ has only one column and we can write

$$M\mathbf{v} = \mathbf{v}_{1,1}M_{:,1} + \mathbf{v}_{2,1}M_{:,2} + \cdots + \mathbf{v}_{n,1}M_{:,n}. \tag{3.2.10}$$

Revisiting the case where the third row of M is twice the first, this means

$$\begin{aligned} \mathbf{b} = M\mathbf{v} &= \mathbf{v}_{1,1}M_{:,1} + \mathbf{v}_{2,1}M_{:,2} + \mathbf{v}_{3,1}(2M_{:,1}) + \cdots + \mathbf{v}_{n,1}M_{:,n} \\ &= (\mathbf{v}_{1,1} + 2\mathbf{v}_{3,1})M_{:,1} + \mathbf{v}_{2,1}M_{:,2} + \cdots + \mathbf{v}_{n,1}M_{:,n} \end{aligned}$$

Letting $\mathbf{v}_{1,1} = -2$, $\mathbf{v}_{3,1} = 1$ and $\mathbf{v}_{j,1} = 0$ for all j not equal to 1 or 3, it turns out $\mathbf{b} = \mathbf{0}$. In other words, the associated homogeneous equation $M\mathbf{v} = \mathbf{0}$ has a solution where $\mathbf{v} \neq \mathbf{0}$, a **nontrivial solution**. Finally, we add

- $M\mathbf{v} = \mathbf{0}$ has a nontrivial solution.

to the list of related concepts. But wait, there's more!

Equation (3.2.8) has exactly the same form as a system of linear equations with variables $A_{1,c}, A_{2,c}, \ldots, A_{n,c}$. Can you show that the equation $M\mathbf{v} = \mathbf{b}$ is equivalent to a linear system of equations whose augmented matrix is $\begin{bmatrix} M & \mathbf{b} \end{bmatrix}$ and variables are $\mathbf{v}_{1,1}, \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{n,1}$? Answer on page 3.2. That makes the last item in our list of related concepts

- The linear system represented by the augmented matrix $\begin{bmatrix} M & \mathbf{b} \end{bmatrix}$ has no solution for certain choices of $\mathbf{b}$.

Wow. That seems to make six ways of understanding the same phenomenon.

Key Concepts

addition property of equality for all matrices A, B, C if $A = B$ then $A + C = B + C$ whenever the sums are defined.

left multiplication property of equality for all matrices A, B, C if $A = B$ then $CA = CB$ whenever the products are defined.

matrix form of a linear system if $\mathbf{v}$ is a (variable) vector with n entries, the matrix equation $M\mathbf{v} = \mathbf{b}$ is equivalent to the linear system with augmented matrix $\begin{bmatrix} M & \mathbf{b} \end{bmatrix}$ and variables $\mathbf{v}_{1,1}, \ \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{n,1}$.

matrix product as a linear combination of rows given matrices A and B, if BA is defined then row r of BA can be computed as a linear combination of the rows of A using row r of B as coefficients:

$$(BA)_{r,:} = B_{r,1}A_{1,:} + B_{r,2}A_{2,:} + \cdots + B_{r,n}A_{n,:}.$$

matrix product as a linear combination of columns given matrices A and B, if BA is defined then column c of BA can be computed as a linear combination of the columns of B using column c of A as coefficients:

$$(BA)_{:,c} = A_{1,c}B_{:,1} + A_{2,c}B_{:,2} + \cdots + A_{n,c}B_{:,n}.$$

In the special case of a matrix M times a vector $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$,

$$M\mathbf{v} = v_1 M_{:,1} + v_2 M_{:,2} + \cdots + v_n M_{:,n}.$$

nontrivial solution a solution $\mathbf{v} \neq \mathbf{0}$ of the equation $M\mathbf{v} = \mathbf{0}$.

right multiplication property of equality for all matrices A, B, C if $A = B$ then $AC = BC$ whenever the products are defined.

Exercises

1. Solve

   (a) $X + \begin{bmatrix} -8 & -4 \\ 19 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 11 \\ 16 & 9 \end{bmatrix}$

   (b) $\begin{bmatrix} -4 & -7 \\ -14 & 3 \end{bmatrix} + X = \begin{bmatrix} 12 & -20 \\ -16 & 0 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & -19 \\ -1 & 8 \end{bmatrix} - X = \begin{bmatrix} -2 & 19 \\ -14 & 20 \end{bmatrix}$

   (d) $5X + \begin{bmatrix} 1 & 0 \\ -13 & 11 \end{bmatrix} = \begin{bmatrix} -20 & -17 \\ 2 & -5 \end{bmatrix}$

2. Solve

   (a) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} X = \begin{bmatrix} -18 & 2 \\ -2 & -8 \end{bmatrix}$

   (b) $\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix} X = \begin{bmatrix} 13 & -13 \\ -19 & 7 \end{bmatrix}$

   (c) $\begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix} X + \begin{bmatrix} -13 & 18 \\ -11 & 10 \end{bmatrix} = \begin{bmatrix} -20 & 14 \\ 15 & -2 \end{bmatrix}$

   (d) $\begin{bmatrix} -13 & 1 \\ 19 & 4 \end{bmatrix} - \begin{bmatrix} -5 & 9 \\ -3 & 5 \end{bmatrix} X = \begin{bmatrix} -16 & 4 \\ 1 & -6 \end{bmatrix}$

3. Solve for the specified variable. Assume all indicated operations are defined. Make a note whenever you assume a matrix is invertible.

   (a) $XYZ = B$ for Y

   (b) $XYZ = B$ for Z

   (c) $A(3B + I) = C$ for B

   (d) $PDP^{-1} = A$ for D

   (e) $2A(B^{-1} + C^T) = D$ for C

   (f) $(3C)^T + 2B^{-1} = A$ for B

4. Write the matrix equation as an equivalent linear system.

   (a) $A\mathbf{x} = \mathbf{c}$; $A = \begin{bmatrix} -6 & 2 & 19 \\ -14 & 1 & -10 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$; $\mathbf{c} = \begin{bmatrix} 7 & 3 \end{bmatrix}$

   (b) $A\mathbf{x} = \mathbf{0}$; $A = \begin{bmatrix} -34 & -3 & 37 \\ -118 & 9 & 109 \\ 26 & -3 & -23 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x & y & z \end{bmatrix}^T$;

   (c) $T\mathbf{r}_1 = \mathbf{r}_2$; $T = \begin{bmatrix} 10 & 14 \\ -6 & -3 \end{bmatrix}$; $\mathbf{r}_1 = \begin{bmatrix} r \\ s \end{bmatrix}$; $\mathbf{r}_2 = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$

   (d) $M\mathbf{v} = \mathbf{b}$; $M = \begin{bmatrix} -4 & -14 \\ -17 & 6 \\ -16 & -12 \end{bmatrix}$; $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$;

$$\mathbf{b} = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

5. Specify the matrix M, vector $\mathbf{v}$, and vector $\mathbf{b}$ so that the matrix equation $M\mathbf{v} = \mathbf{b}$ is equivalent to the linear system.

   (a) $$\begin{array}{rcrcrcr} -12x & - & 6y & - & 7z & = & 16 \\ & - & 5y & + & 18z & = & 2 \\ -15x & + & 10y & + & 8z & = & -11 \end{array}$$

   (b) $$\begin{array}{rcrcrcr} -15x_1 & + & 4x_2 & & & = & -14 \\ -4x_1 & - & 17x_2 & - & 7x_3 & = & 3 \end{array}$$

   (c) $$\begin{array}{rcrcr} 15r & - & 11s & = & 0 \\ 3r & + & 10s & = & 0 \end{array}$$

   (d) $$\begin{array}{rcrcr} 14v_1 & - & 17v_2 & = & -11 \\ 2v_1 & & & = & -4 \\ 9v_1 & - & 2v_2 & = & -8 \end{array}$$

6. Compute the third row of the product without computing the rest of the product.

$$\begin{bmatrix} 2 & 2 & 1 & 5 \\ 0 & 1 & -5 & -4 \\ 5 & 0 & 3 & -3 \\ 3 & -2 & -4 & 4 \\ 4 & -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 2 & -4 \\ -5 & 5 \\ 0 & 1 \end{bmatrix}$$

7. Compute the second row of the product by summing an appropriate linear combination of row vectors.

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & -3 \\ 1 & -4 & 5 \\ 3 & 2 & -1 \end{bmatrix}$$

8. Compute the second column of the product without computing the first.

$$\begin{bmatrix} -4 & 5 & -3 & -4 \\ 0 & -3 & 3 & 2 \\ 4 & 1 & 4 & -5 \end{bmatrix} \begin{bmatrix} -3 & 3 \\ 2 & -4 \\ -5 & 5 \\ 0 & 1 \end{bmatrix}$$

9. Compute the third column of the product by summing an appropriate linear combination of column vectors.

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & -3 \\ 1 & -4 & 5 \\ 3 & 2 & -1 \end{bmatrix}$$

10. Suppose the third column of A contains all zeros. What can you say about the third column of BA? Why? Assume BA is defined.

11. Suppose the second row of B contains all ones. What can you say about the second row of BA? Why? Assume BA is defined.

12. Suppose the fifth column of A is three times the second column of A. What can you say about the second column of BA? Why? Assume BA is defined.

13. Demonstrate that the zero product rule for real numbers does not hold for matrices. That is, show that the claim "if A and B are matrices such that $AB = 0$, then $A = 0$ or $B = 0$" is false by finding matrices A and B such that $A \neq 0$ and $B \neq 0$ yet $AB = 0$ where

    (a) A and B are nonsquare matrices.

    (b) A is square but B is not square.

    (c) B is square but A is not square.

    (d) A and B are both square.

14. Argue that if $AB = 0$ then one of the following must be true.

    - $\det A = 0$
    - $\det B = 0$
    - A is not square
    - B is not square

    Use the fact that for a square matrix M, $\det M = 0$ if and only if M is noninvertible.

15. Show that the converse of (3.2.2) is true. That is, justify the claim that for all matrices A, B, C, if $A + C = B + C$ then $A = B$ whenever the sums are defined.

16. Show that the converse of (3.2.2) is false by supplying matrices A, B, C such that $CA = CB$ but $A \neq B$.

Answers

multiplication property of equality part 2 Claims (3.2.3) and (3.2.4) are distinct, and therefore both needed, because matrix multiplication is not commutative. Claim (3.2.4) can be proven as follows. Since $A = B$, $0 = A - B$. Hence, if $A = B$ then

$$\begin{aligned} 0 &= A - B \\ &= (A - B)C \\ &= AC - BC \end{aligned}$$

and therefore $AC = BC$.

$M\mathbf{v} = \mathbf{b}$ as a linear system Let M be an $m \times n$ matrix and suppose $M\mathbf{v} = \mathbf{b}$ (making $\mathbf{v}$ a vector with n entries and $\mathbf{b}$ a vector with m entries). Then

$$\begin{aligned} M\mathbf{v} &= \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m,1} & M_{m,2} & \cdots & M_{m,n} \end{bmatrix} \begin{bmatrix} \mathbf{v}_{1,1} \\ \mathbf{v}_{2,1} \\ \vdots \\ \mathbf{v}_{n,1} \end{bmatrix} \\ &= \begin{bmatrix} M_{1,1}\mathbf{v}_{1,1} + M_{1,2}\mathbf{v}_{2,1} + \cdots + M_{1,n}\mathbf{v}_{n,1} \\ M_{2,1}\mathbf{v}_{1,1} + M_{2,2}\mathbf{v}_{2,1} + \cdots + M_{2,n}\mathbf{v}_{n,1} \\ \vdots \\ M_{m,1}\mathbf{v}_{1,1} + M_{m,2}\mathbf{v}_{2,1} + \cdots + M_{m,n}\mathbf{v}_{n,1} \end{bmatrix} \end{aligned}$$

setting this vector equal to $\mathbf{b}$ gives

$$\begin{bmatrix} M_{1,1}\mathbf{v}_{1,1} + M_{1,2}\mathbf{v}_{2,1} + \cdots + M_{1,n}\mathbf{v}_{n,1} \\ M_{2,1}\mathbf{v}_{1,1} + M_{2,2}\mathbf{v}_{2,1} + \cdots + M_{2,n}\mathbf{v}_{n,1} \\ \vdots \\ M_{m,1}\mathbf{v}_{1,1} + M_{m,2}\mathbf{v}_{2,1} + \cdots + M_{m,n}\mathbf{v}_{n,1} \end{bmatrix} = \begin{bmatrix} \mathbf{b}_{1,1} \\ \mathbf{b}_{2,1} \\ \vdots \\ \mathbf{b}_{m,1} \end{bmatrix}$$

which can only be true if corresponding entries are equal. In other words,

$$\begin{aligned} M_{1,1}\mathbf{v}_{1,1} + M_{1,2}\mathbf{v}_{2,1} + \cdots + M_{1,n}\mathbf{v}_{n,1} &= \mathbf{b}_{1,1} \\ M_{2,1}\mathbf{v}_{1,1} + M_{2,2}\mathbf{v}_{2,1} + \cdots + M_{2,n}\mathbf{v}_{n,1} &= \mathbf{b}_{2,1} \\ &\vdots \\ M_{m,1}\mathbf{v}_{1,1} + M_{m,2}\mathbf{v}_{2,1} + \cdots + M_{m,n}\mathbf{v}_{n,1} &= \mathbf{b}_{m,1} \end{aligned}$$

Therefore, the equation $M\mathbf{v} = \mathbf{b}$ is equivalent to the system with augmented matrix $\begin{bmatrix} M & \mathbf{b} \end{bmatrix}$ and variables $\mathbf{v}_{1,1}, \mathbf{v}_{2,1}, \ldots, \mathbf{v}_{n,1}$.

3.3 Linear Independence

The matrices

$$A = \begin{bmatrix} 2 & -4 \\ 3 & -6 \end{bmatrix}, \ B = \begin{bmatrix} 4 & 2 & 6 \\ -1 & -9 & 7 \end{bmatrix}, \ C = \begin{bmatrix} 3 & 3 & -5 \\ -8 & 2 & 0 \\ 5 & -4 & 7 \end{bmatrix},$$

$$D = \begin{bmatrix} -11 & 10 & -5 & 3 & 7 & -1 \\ -8 & -1 & 6 & 6 & 0 & 1 \\ -2 & 8 & 7 & -9 & -15 & -3 \\ 8 & 5 & -10 & -7 & 5 & 2 \end{bmatrix}$$

have something in common. Each matrix has a column that can be written as a linear combination of the other columns in the matrix. Not all matrices have this property, and there is an important distinction between those that do and those that do not.

Compare $E = \begin{bmatrix} 2 & -8 \\ 1 & 5 \end{bmatrix}$ to A, for example. In E, neither column is a multiple of the other so neither column can be written as a linear combination of the other. However, the second column of A is -2 times the first:

- $A_{:,2} = -2A_{:,1}$

You may be struggling a little bit to see this as a linear combination, but nothing in the definition of linear combination requires more than one term. So if an object is a multiple of another it is a linear combination of it.

Notice that $\det A = 0$ while $\det E \neq 0$. The matrix with one column that can be written as a linear combination of the others has 0 determinant while the matrix whose columns can not be written as linear combinations of the others has nonzero determinant. We made a similar observation about linear combinations of the rows of a matrix and its determinant in section 1.5.

For matrices B, C, D it is less clear that one column is a linear combination of the others, but you can check that

- $B_{:,3} = 2B_{:,1} - B_{:,2}$
- $C_{:,1} = -4C_{:,2} + 3C_{:,3}$
- $D_{:,5} = 2D_{:,1} + 2D_{:,2} + 0D_{:,3} + 3D_{:,4} + 0D_{:,6}$

Don't be misled by the suggestion that "one column" is a linear combination of the others, however. It is true, but does not tell the whole story. In no case is the column written in terms of the others special. For matrix A, for example, we could have easily pointed out that the first column is $-\frac{1}{2}$ the second. The first column is a linear combination of the second, and the second column is a linear combination of the first. Neither one should take priority.

A little algebra will show that all the following equations are also true.

- $B_{:,2} = 2B_{:,1} - B_{:,3}$
- $B_{:,1} = \frac{1}{2}B_{:,2} + \frac{1}{2}B_{:,3}$
- $C_{:,3} = \frac{1}{3}C_{:,1} + \frac{4}{3}C_{:,2}$
- $C_{:,2} = -\frac{1}{4}C_{:,1} + \frac{3}{4}C_{:,3}$

- $D_{:,1} = -D_{:,2} - \frac{3}{2}D_{:,4} + \frac{1}{2}D_{:,5}$
- $D_{:,2} = -D_{:,1} - \frac{3}{2}D_{:,4} + \frac{1}{2}D_{:,5}$
- $D_{:,4} = -\frac{2}{3}D_{:,1} - \frac{2}{3}D_{:,2} + \frac{1}{3}D_{:,5}$

The entire set of columns involved (with nonzero coefficient) in the linear combination is special. Any such column can be written as a linear combination of the others.

In an atempt to emphasize that the *set of columns* is special, not that one particular column within the matrix is special, each of the equations above can be rearranged so one side of the equation becomes 0. As a result, instead of having the 11 equations above, where in each case one column is spotlighted as the "special" column being writen in terms of the others, we have

- $2A_{:,1} + A_{:,2} = 0$
- $2B_{:,1} - B_{:,2} - B_{:,3} = 0$
- $C_{:,1} + 4C_{:,2} - 3C_{:,3} = 0$
- $2D_{:,1} + 2D_{:,2} + 0D_{:,3} + 3D_{:,4} - D_{:,5} + 0D_{:,6} = 0$

The fact that columns within the matrix can be written as linear combinations of the others is captured, but no particular column is prominent, motivating the following definition.

Let S be a set of objects on which addition and scalar mutliplication are defined and which contains an additive identity, called 0. For scalars $x_1, x_2, \dots, x_n$ and objects $b_1, b_2, \dots, b_n$ of S, we say that $b_1, b_2, \dots, b_n$ are **linearly independent** or that $\{b_1, b_2, \dots, b_n\}$ is a linearly independent set if the only solution of

$$x_1b_1 + x_2b_2 + \cdots + x_nb_n = 0 \tag{3.3.1}$$

is the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. Otherwise the objects $b_1, b_2, \dots, b_n$ are **linearly dependent** and $\{b_1, b_2, \dots, b_n\}$ is a linearly dependent set. Note that the 0 in (3.3.1) is the object 0, not necessarily the number 0.

In addition to being a statement about the set of objects rather than one special member of the set, this definition handles the case when one of the objects in the set is the 0 object (additive identity) itself. In this case, that particular object is special. It can be written as a linear combination of the others (with all coefficients equal to zero) but it is not necessarily the case that any of the other objects can be written as a linear combination of the remaining ones. To illustrate, suppose

$$E = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 5 & 0 \\ 3 & -1 & 0 \end{bmatrix}$$

The third column is a zero vector. Accordingly,

$$E_{:,3} = 0E_{:,1} + 0E_{:,2}$$

so the third column is a linear combination of the first two. However, neither of the first two columns is a linear combination of the others. This is clear since the first two columns are not multiples of one another. In the context of the definition, we have

$$0E_{:,1} + 0E_{:,2} + E_{:,3} = 0,$$

a nontrivial linear combination of the columns that sums to 0. No fanfare. No notes of special cases. The definition of linear independence is clean and direct.

Matrix Characterization Part 1

Free variables, solution sets of linear systems, and pivot positions of matrices are all directly connected to the concept of linear independence.

Theorem 5. [Characterization of Matrices Part 1] *Suppose M is an $m \times n$ matrix, $\mathbf{v}$ has n entries, and $\mathbf{b}$ has m entries. Then the following are equivalent.*

(i) The columns of M are linearly independent.

(ii) No column of M is a linear combination of the others.

(iii) $M\mathbf{v} = \mathbf{0}$ has only the trivial solution.

(iv) M has a pivot position in every column.

(v) $M\mathbf{v} = \mathbf{b}$ has no free variables.

(vi) $M\mathbf{v} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$.

The following list of arguments will show that if one of the statements is true, so is another...and if that one is true so is a third...and if that one is true so is the next...and so on until all the statements have been justified. Such a series of justifications means that if the first statement is true, they are all true since they all followed logically from the first. Proving they are equivalent requires one more step. The last statement will be shown to imply the first, completing a logical path from any one of the statements to any other. Closing the loop this way means that if *any one* of the statements is true, they are all true, the very meaning of **equivalent**!

Crumpet 18: Proof by Contraposition

Suppose you are trying to prove that if some statement, call it p, is true, then some other statement, call it q is true. In short, you are trying to prove that if p is true then q is true. Then it is just as good to prove the contrapositive claim, if q is false then p is false, because if the contrapositive is true then it is impossible to have both q false and p true. In other words, if p is true so is q because q cannot be false at the same time p is true (and that means if p is true then q is true). By similar logic, if any statement in a list of equivalent statements is false, they are all false.

The following arguments demonstrate that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i). More succinctly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) $\Rightarrow$ (iii) $\Rightarrow$ (i), and diagrammatically,

$$\begin{array}{ccccccc}
\text{(i)} & & & & \text{(iv)} & & \\
\Downarrow & \Nwarrow & & \Nearrow & & \Searrow & \\
\text{(ii)} & \Leftarrow & \text{(iii)} & \Leftarrow & \text{(vi)} & \Leftarrow & \text{(v)}
\end{array}$$

The diagram illustrates a logical path from any one of the statements to any other. Therefore, justifying each statement as claimed shows that the statements are equivalent.

(i) $\Rightarrow$ (ii) Requested in exercise 19.

(ii) ⇒ (iii) Suppose $M\mathbf{v} = \mathbf{0}$ has a nontrivial solution, $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. Then $M\mathbf{x} = x_1 M_{:,1} + x_2 M_{:,2} + \cdots + x_n M_{:,n} = \mathbf{0}$. Since $\mathbf{x}$ is a nontrivial solution, one of the entries of $\mathbf{x}$ is nonzero, say x_i. Therefore, $x_i M_{:,i} = -x_1 M_{:,1} - \ldots - x_{i-1} M_{:,i-1} - x_{i+1} M_{:,i+1} - \ldots - x_n M_{:,n}$ and more to the point,

$$M_{:,i} = -\frac{x_1}{x_i} M_{:,1} - \ldots - \frac{x_{i-1}}{x_i} M_{:,i-1} - \frac{x_{i+1}}{x_i} M_{:,i+1} - \ldots - \frac{x_n}{x_i} M_{:,n}$$

making column i a linear combination of the other columns.

(iii) ⇒ (iv) Suppose M does not have a pivot position in every column. Then $M\mathbf{v} = \mathbf{0}$, a consistent system with solution $\mathbf{v} = \mathbf{0}$, has free variables. By theorem 1, $M\mathbf{v} = \mathbf{0}$ has more than one solution.

(iv) ⇒ (v) By definition, a free variable comes from a variable column without a pivot. Since all columns of M have pivot postions, $M\mathbf{v} = \mathbf{b}$ cannot have free variables.

(v) ⇒ (vi) By theorem 1 a linear system with no free variables is either inconsistent or has exactly one solution. Therefore $M\mathbf{v} = \mathbf{b}$ has zero or one (in other words, at most one) solution.

(vi) ⇒ (iii) $M\mathbf{v} = \mathbf{0}$ is a special case of $M\mathbf{v} = \mathbf{b}$, so if $M\mathbf{v} = \mathbf{0}$ has a solution it is unique. The trivial solution, $\mathbf{v} = \mathbf{0}$, being a solution of $M\mathbf{v} = \mathbf{0}$ is therefore the only one.

(iii) ⇒ (i) Let $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$. By assumption, $M\mathbf{v} = v_1 M_{:,1} + v_2 M_{:,2} + \cdots + v_n M_{:,n} = \mathbf{0}$ has only the trivial solution. By definition of linear indpendence, $M_{:,1}, M_{:,2}, \cdots, M_{:,n}$ (the columns of M) are linearly independent.

Later, we will see that the determinant, row equivalence, invertibility, and function concepts are also directly connected to these statements.

Key Concepts

characterization of matrices see theorem 5.

equivalent statements a list of statements such that if one of them is true they are all true.

linearly dependent not linearly independent.

linearly independent objects $b_1, b_2, \ldots, b_n$ are linearly independent whenever the trivial solution is the only solution of (3.3.1).

linearly independent set the set $\{b_1, b_2, \ldots, b_n\}$ is a linearly independent set whenever the objects $b_1, b_2, \ldots, b_n$ are linearly independent.

linearly dependent set the set $\{b_1, b_2, \ldots, b_n\}$ is a linearly dependent set whenever the objects $b_1, b_2, \ldots, b_n$ are linearly dependent.

trivial solution the solution $x_1 = x_2 = \cdots = x_n = 0$ of the equation $x_1 b_1 + x_2 b_2 + \cdots + x_n b_n = 0$ is called the trivial solution.

Exercises

1. Show that the vectors are linearly independent.

   (a) $\begin{bmatrix} 5 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} -5 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 3 \\ 3 \end{bmatrix}$

2. Show that the linear system has at most one solution for any values b_1, b_2, b_3, b_4.

(a) $\begin{array}{rcrcl} 4x & + & 11y & = & b_1 \\ 5x & + & 12y & = & b_2 \end{array}$

(b) $\begin{array}{rcrcl} -x_1 & + & x_2 & = & b_1 \\ -8x_1 & + & 7x_2 & = & b_2 \\ 5x_1 & + & x_2 & = & b_3 \end{array}$

(c) $\begin{array}{rcrcrcl} 2v_1 & - & v_2 & - & v_3 & = & b_1 \\ 7v_1 & + & 2v_2 & & & = & b_2 \\ -5v_1 & - & v_2 & + & v_3 & = & b_3 \end{array}$

(d) $\begin{array}{rcrcrcl} x & + & 5y & + & 7z & = & b_1 \\ 6x & + & 7y & + & 2z & = & b_2 \\ -6x & + & 3y & - & z & = & b_3 \\ 5x & + & y & & & = & b_4 \end{array}$

3. Show that the homogeneous system has only one solution, the trivial solution.

(a) $\begin{bmatrix} 1 & 8 \\ 1 & -5 \end{bmatrix} \mathbf{v} = \mathbf{0}$

(b) $\begin{bmatrix} -2 & 5 \\ 5 & -8 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 6 & 1 \\ 6 & -7 & -2 \\ -1 & -4 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 3 & -1 \\ 5 & 0 & 1 \\ 1 & -4 & 1 \\ 5 & 7 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{0}$

4. Show that the functions are linearly independent.

(a) $1 + t$, $t + t^2$, and $1 + t^2$

(b) $\sin^2 t$ and $\cos^2 t$

5. Show that the functions are linearly dependent.

(a) $1 + 3t - 2t^2$, $-9 - 23t + 21t^2$, and $1 + 7t + t^2$

(b) 1, $\sin^2 t$, and $\cos^2 t$

(c) $\sin^2 t$, $\cos^2 t$, and $\cos(2t)$

6. Do the columns of the matrix form a linearly independent set?

(a) $\begin{bmatrix} 12 & 21 \\ -24 & 58 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 9 \\ 11 & 1 \\ -6 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} -8 & -5 & -7 \\ 5 & -9 & -6 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & -5 & -10 \\ 1 & -8 & 6 \\ -9 & -4 & 10 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & -1 & 5 & 4 \\ -2 & -11 & 9 & -7 \\ 6 & -5 & 11 & -6 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & -2 & -9 \\ 0 & -11 & -8 \\ 9 & 6 & -5 \\ 1 & -7 & 8 \end{bmatrix}$

(g) $\begin{bmatrix} -54 & -30 & -96 & 9 & 6 & -74 \\ -24 & 4 & -93 & -68 & 21 & 15 \\ 70 & -89 & 78 & 26 & -78 & 0 \\ -46 & 68 & -87 & -88 & -39 & 67 \end{bmatrix}$

7. Determine whether the set is linearly independent.

(a) $\{-10t + t^2 - 5t^3, 2t + 6t^2 - 2t^3, 31t + 14t^3\}$

(b) $\{\begin{bmatrix} 1 & 8 & -11 \end{bmatrix}, \begin{bmatrix} 9 & 4 & -7 \end{bmatrix}, \begin{bmatrix} 4 & -2 & 2 \end{bmatrix}\}$

(c) $\left\{\begin{bmatrix} 3 & -4 \\ -6 & -3 \end{bmatrix}, \begin{bmatrix} 21 & 6 \\ -18 & 13 \end{bmatrix}, \begin{bmatrix} 6 & 9 \\ 0 & 11 \end{bmatrix}\right\}$

8. A 9×6 matrix has linearly independent columns. How many pivot positions does it have?

9. A 7×6 matrix has 5 pivot positions. What can you say about the linear indpendence of its columns?

10. Give an example of a 3×2 matrix M such that $M\mathbf{v} = \mathbf{0}$

(a) has only the trivial solution

(b) has a nontrivial solution

11. What are the possible reduced row echelon forms of a 4×3 matrix with

    (a) linearly independent columns?

    (b) linearly dependent columns?

12. What are the possible row echelon forms of a 2×2 matrix with

    (a) linearly independent columns?

    (b) linearly dependent columns?

13. If M is an $m \times n$ matrix with linearly independent columns, what can you say about the relationship between m and n? HINT: Can a matrix with linearly independent columns have more columns than rows?

14. Find the value(s) of x for which the matrix has linearly independent columns.

    (a) $\begin{bmatrix} 2 & -6 \\ 3 & x \end{bmatrix}$

    (b) $\begin{bmatrix} x & 4 \\ -2 & 7 \end{bmatrix}$

    (c) $\begin{bmatrix} 1 & 8 & 0 \\ 6 & x & 1 \\ -3 & -2 & 5 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & 8 & 2 \\ 6 & 45 & 1 \\ -3 & -20 & x \end{bmatrix}$

15. Find the value(s) of x for which the matrix has linearly dependent columns.

    (a) $\begin{bmatrix} 3 & x \\ -5 & 4 \end{bmatrix}$

    (b) $\begin{bmatrix} 2 & 7 \\ x & 5 \end{bmatrix}$

    (c) $\begin{bmatrix} x & -6 & 27 \\ -2 & 8 & -30 \\ -1 & 5 & -18 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & 5 & -4 \\ -5 & 3 & x \\ -7 & -11 & 4 \end{bmatrix}$

16. Find a nontrivial solution of $M\mathbf{v} = \mathbf{0}$ using the fact that the first and second columns of M are identical. Do not use row operations.

$$M = \begin{bmatrix} 94 & 94 & 85 \\ -97 & -97 & 83 \\ 6 & 6 & 24 \\ 5 & 5 & -77 \end{bmatrix}$$

17. Argue that the statements are equivalent.

    (a) $x = 8$

    (b) x is a perfect cube between 6 and 20.

18. Argue that the statements are equivalent.

    (a) The graph of f is a line with slope 3 and y-intercept -5.

    (b) $f(x) = 3x - 5$.

    (c) f is a first degree polynomial passing through $(-10, -35)$ and $(10, 25)$.

For exercises 19 and 20 assume that addition and scalar multiplication are defined on the set and that there is a zero object in the set.

19. Argue that if a set of objects is linearly independent then none of the objects is a linear combination of the others. HINT: Try proof by contraposition with multiple cases. Suppose one of the objects in the set is a linear combination of the others, and logically conclude that the set is linearly dependent.

20. Argue that if none of the objects of a set is a linear combination of the others, then the set is linearly independent. HINT: Try proof by contraposition. Suppose the set is linearly dependent, and logically conclude that one of the objects in the set is a linear combination of the others.

3.4 Characterization of $m \times n$ Matrices

Theorem 5 of section 3.3 has a counterpart phrased in terms of the rows of M. Parts (i), (ii), (iii), and (v) of the following theorem can be justified through reference to the previous, but parts (iv) and (vi) cannot.

Theorem 6. [Characterization of Matrices Part 2] *Suppose M is an $m \times n$ matrix, $\mathbf{v}$ and $\mathbf{c}$ have n entries, and $\mathbf{b}$ and $\mathbf{w}$ have m entries. Then the following are equivalent.*

(i) The rows of M are linearly independent.

(ii) No row of M is a linear combination of the others.

(iii) $\mathbf{w}^T M = \mathbf{0}^T$ has only the trivial solution.

(iv) M has a pivot position in every row.

(v) $\mathbf{w}^T M = \mathbf{c}^T$ has no free variables.

(vi) $M\mathbf{v} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.

The justification for this theorem proceeds by logically connecting the statements according to the following diagram.

$$\begin{array}{ccccc} & & \text{(v)} & & \\ & \swarrow & & \nwarrow & \\ \text{(i)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} \\ & \searrow & & & \Uparrow \\ & & \text{(iv)} & \Rightarrow & \text{(vi)} \end{array}$$

Though the first several implications can be proven without reference to theorem 5, such reference will be used to emphasize the direct connection between the two theorems.

(i) $\Rightarrow$ (ii) The rows of M are linearly independent, so the columns of M^T are linearly independent. By theorem 5, none of the columns of M^T can be written as a linear combination of the others, so none of the rows of M can be writen as a linear combination of the others.

(ii) $\Rightarrow$ (iii) No row of M can be written as a linear combination of the others, so no column of M^T can be written as a linear combination of the others. By theorem 5, $M^T\mathbf{w} = \mathbf{0}$ has only the trivial solution. Since $M^T\mathbf{w} = \mathbf{0}$ is equivalent to $(M^T\mathbf{w})^T = \mathbf{0}^T$ (transpose both sides), which is equivalent to $\mathbf{w}^T M = \mathbf{0}^T$ (simplifying the lefthand side), the conclusion follows.

(iii) $\Rightarrow$ (v) Since $\mathbf{w}^T M = \mathbf{0}^T$ has only the trivial solution, the equivalent equations $(\mathbf{w}^T M)^T = (\mathbf{0}^T)^T$ and $M^T\mathbf{w} = \mathbf{0}$ have only the trivial solution. By theorem 5, $M^T\mathbf{w} = \mathbf{c}$ has no free variables. Therefore, the equivalent equation $\mathbf{w}^T M = \mathbf{c}^T$ has no free variables.

(v) $\Rightarrow$ (i) Since $\mathbf{w}^T M = \mathbf{c}^T$ has no free variables, the equivalent equation $M^T\mathbf{w} = \mathbf{c}$ has no free variables. By theorem 5, the columns of M^T are linearly independent. Therefore, the rows of M are linearly independent.

(i) $\Rightarrow$ (iv) Suppose M does not have a pivot in every row. Then any row echelon form of M has a row of zeros. Since that row of zeros is the result of a nontrivial linear combination of the rows of M, there is a nontrivial linear combination of the rows of M that sum to $\mathbf{0}^T$. Therefore the rows of M are not linearly independent.

(iv) ⇒ (vi) Since M has a pivot in every row, no row echelon form of M has a row of zeros. Therefore $\begin{bmatrix} M & \mathbf{b} \end{bmatrix}$ cannot have a pivot in the rightmost column, and by theorem 1 the system $M\mathbf{v} = \mathbf{b}$ is consistent (has at least one solution) for any $\mathbf{b}$.

(vi) ⇒ (iii) In particular, $M\mathbf{w}_i = (I_{n\times n})_{:,i}$ has at least one solution. Let $\mathbf{w}_i$ be such a solution for each i. It follows (by the matrix product as a linear combination of the columns) that for $R = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_n \end{bmatrix}$, $MR = I_{n\times n}$. Supposing $\mathbf{w}^T M = \mathbf{0}^T$, the following equations are deduced by matrix algebra.

$$\begin{aligned} (\mathbf{w}^T M)R &= \mathbf{0}^T R \\ \mathbf{w}^T (MR) &= \mathbf{0}^T \\ \mathbf{w}^T I &= \mathbf{0}^T \\ \mathbf{w}^T &= \mathbf{0}^T \end{aligned}$$

Hence $\mathbf{w}^T M = \mathbf{0}^T$ has only the trivial solution.

The justification for many of the upcoming claims relies heavily on induction (see axiom 5 of crumpet 16 on page 82). The principle behind induction is to show that (i) the claim is actually true for some particular integer, and (ii) if the claim is true for some integer at least as large, then it is also true for the next integer. This way, part (i) establishes the claim for a particular integer, say k. Then part (ii) establishes the claim for the next integer, $k + 1$. Part (i) also establishes the claim for the next integer, $k+2$. Applying part (i) again establishes the claim for the next integer, $k+3$, and so on, part (i) establishing the claim for all integers greater than k. Induction is often the most practical way to show that a statement is true for all integers and is particularly useful in proving claims about matrices of size n (for all n). Proofs of this nature will be shown, but this is not a course on proof technique, so it is up to you or your instructor to decide how deeply you need to understand these proofs. Even if you are not prepared to write your own induction proof or fully understand one, reading them is a good way to get a feel for the technique.

An **upper triangular matrix** is one in which all entries below the main diagonal are zero, and a **lower triangular matrix** is one in which all entries above the main diagonal are zero. Using $\star$ to represent any number (as in the notation of section 2.3), a square upper triangular matrix looks like

$$\begin{bmatrix} \star & \star & \star & \star & \cdots & \star \\ 0 & \star & \star & \star & \cdots & \star \\ 0 & 0 & \star & \star & \cdots & \star \\ 0 & 0 & 0 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \ddots & \star \\ 0 & 0 & 0 & 0 & \cdots & \star \end{bmatrix}$$

and a square lower triangular matrix looks like

$$\begin{bmatrix} \star & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & 0 & 0 & \cdots & 0 \\ \star & \star & \star & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ \star & \star & \star & \star & \cdots & \star \end{bmatrix}.$$

All the nonzero entries are above or on the main diagonal (the upper triangle) for an upper triangular matrix, and all the nonzero entries are below or on the main diagonal (the lower triangle) for a lower triangular matrix.

The determinant of a lower triangular matrix is the product of the entries on its diagonal. Now is a good time to work out a couple examples on your own and think about why this is true in general. We will prove it by induction, but the proof may not resonate with you the way your own thoughts about it will.

Claim. If L is a lower triangular $n \times n$ matrix, then $\det L = L_{1,1}L_{2,2}\cdots L_{n,n}$.

Proof. If L is a 1×1 matrix, it is upper triangular and $\det L = \det([L_{1,1}]) = L_{1,1}$. This establishes part (i) of the proof. The claim is true for the particular value $n = 1$. Now we assume that the claim is true for some (arbitrary) value $n = k$ greater than or equal to one. That is, if L is a lower triangular $k \times k$ matrix and $k \geq 1$, then $\det L = L_{1,1}L_{2,2}\cdots L_{k,k}$. To complete the proof, we must use this information to prove that if L is a $(k+1)\times(k+1)$ matrix, the next integer size up, then $\det L = L_{1,1}L_{2,2}\cdots L_{k+1,k+1}$, so suppose L is a $(k+1)\times(k+1)$ matrix. By definition,

$$\det L = (-1)^{1+1}L_{1,1}\det L_{\backslash 1,1} + (-1)^{1+2}L_{1,2}\det L_{\backslash 1,2}\cdots + (-1)^{1+3}L_{1,3}\det L_{\backslash 1,3} \tag{3.4.1}$$

Since L is lower triangular, $L_{1,j} = 0$ whenever $j > 1$. Therefore, all the terms of the determinant are zero except the first one. That is, the determinant simplifies to

$$\det L = L_{1,1}\det L_{\backslash 1,1}. \tag{3.4.2}$$

But $L_{\backslash 1,1}$ is a $k \times k$ matrix, so its determinant is the product of the entries on its diagonal (this is our inductive hypothesis). So $\det L_{\backslash 1,1} = L_{2,2}L_{3,3}\cdots L_{k+1,k+1}$. Substituting this expression into (3.4.2), we have $\det L = L_{1,1}L_{2,2}L_{3,3}\cdots L_{k+1,k+1}$, and the proof is complete. □

Exercises 18 through 20 request a proof that the determinant of a square upper triangular matrix is also the product of the entries on the main diagonal.

Key Concepts

characterization of matrices see theorem 6.

upper triangular matrix a matrix in which all entries below the main diagonal are zero.

lower triangular matrix a matrix in which all entries above the main diagonal are zero.

determinant of a (square) lower triangular matrix the product of the entries on the main diagonal.

determinant of a (square) upper triangular matrix the product of the entries on the main diagonal.

proof by induction showing that (i) the claim is true for some particular integer, and (ii) if the claim is true for some integer at least as large, then it is also true for the next integer. These together prove that the claim is true for all integers greater than or equal to the particular integer of part (i).

Exercises

1. The size and number of pivot positions of a matrix M are given. Answer the folowing questions as completely as you can. (i) Are the rows of M linearly independent? (ii) Are the columns of M linearly independent? (iii) How many solutions does $M\mathbf{v} = \mathbf{0}$ have? (iv) How many solutions does $M\mathbf{v} = \mathbf{b}$ have for arbitrary $\mathbf{b}$?

   (a) 5×8; 5

   (b) 3×3; 2

   (c) 7×7; 7

(d) 9×6; 6

2. The size of a matrix M is given. What is the maximum number of pivot positions M could have? Assume it has that maximum number and answer the following questions. (i) Are the rows of M linearly independent? (ii) Are the columns of M linearly indpendent? (iii) How many solutions does $M\mathbf{v} = \mathbf{0}$ have? (iv) How many solutions does $M\mathbf{v} = \mathbf{b}$ have for arbitrary $\mathbf{b}$?

   (a) 13×5

   (b) 12×12

   (c) 9×29

3. Redo question 2 assuming M has less than the maximum number of pivot positions.

4. Show that the linear system has at least one solution for any values b_1, b_2, b_3, b_4.

   (a) $\begin{array}{rcrcl} 4x & + & 11y & = & b_1 \\ 5x & + & 12y & = & b_2 \end{array}$

   (b) $\begin{array}{rcrcrcl} -x_1 & - & 8x_2 & + & 5x_3 & = & b_1 \\ x_1 & + & 7x_2 & + & x_3 & = & b_2 \end{array}$

   (c) $\begin{array}{rcrcrcl} 2v_1 & - & v_2 & - & v_3 & = & b_1 \\ 7v_1 & + & 2v_2 & & & = & b_2 \\ -5v_1 & - & v_2 & + & v_3 & = & b_3 \end{array}$

   (d) $\begin{array}{rcrcrcrcl} w & + & 6x & - & 6y & + & 5z & = & b_1 \\ 5w & + & 7x & + & 3y & + & z & = & b_2 \\ 7w & + & 2x & - & y & & & = & b_3 \end{array}$

5. Show that the homogeneous system has only one solution, the trivial solution.

   (a) $\mathbf{v}^T \begin{bmatrix} 1 & 8 \\ 1 & -5 \end{bmatrix} = \mathbf{0}^T$

   (b) $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -2 & 5 & -2 \\ 5 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

   (c) $\mathbf{x}^T \begin{bmatrix} 5 & 6 & 1 \\ 6 & -7 & -2 \\ -1 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} 6 & 5 & 1 & 5 \\ 3 & 0 & -4 & 7 \\ -1 & 1 & 1 & -4 \end{bmatrix} = \mathbf{0}^T$

6. Show that the functions are linearly independent.

   (a) $1 + t$, $t + t^2$, and $1 + t^2$

   (b) $\sin^2 t$ and $\cos^2 t$

7. Do the rows of the matrix form a linearly independent set?

   (a) $\begin{bmatrix} 12 & 21 \\ -16 & -28 \end{bmatrix}$

   (b) $\begin{bmatrix} -3 & 6 \\ 4 & 11 \\ -6 & 7 \end{bmatrix}$

   (c) $\begin{bmatrix} -3 & 6 & -12 \\ -2 & 2 & -11 \end{bmatrix}$

   (d) $\begin{bmatrix} -1 & -3 & -10 \\ -7 & 9 & 5 \\ 2 & -5 & 0 \end{bmatrix}$

   (e) $\begin{bmatrix} -18 & 1 & -1 & 7 \\ -6 & 2 & 6 & 3 \\ 12 & -5 & 0 & 4 \end{bmatrix}$

   (f) $\begin{bmatrix} 5 & 2 & -3 \\ -4 & -6 & 7 \\ 12 & 9 & -5 \\ 0 & -12 & -11 \end{bmatrix}$

   (g) $\begin{bmatrix} -38 & -5 & 30 & 25 & -44 \\ 4 & -28 & 44 & -39 & 43 \\ 21 & 42 & 1 & -11 & -29 \\ 47 & 12 & 26 & -10 & -8 \\ -13 & -15 & 39 & -9 & 22 \\ 9 & -3 & 3 & -41 & 49 \end{bmatrix}$

8. A 5×8 matrix has linearly independent rows. How many pivot positions does it have?

9. A 6×7 matrix has 5 pivot positions. What can you say about the linear indpendence of its rows?

10. Give an example of a 2×3 matrix M such that $\mathbf{v}^T M = \mathbf{0}^T$

    (a) has only the trivial solution

    (b) has a nontrivial solution

11. What are the possible reduced row echelon forms of a 3×4 matrix with

    (a) linearly independent rows?

    (b) linearly dependent rows?

12. What are the possible row echelon forms of a 2×2 matrix with

    (a) linearly independent rows?

    (b) linearly dependent rows?

Compare your answer with the answer to section 3.3 exercise 12.

13. Find the determinant.

(a) $\begin{bmatrix} 2 & 0 \\ -137 & -3 \end{bmatrix}$

(b) $\begin{bmatrix} -8 & 0 & 0 \\ -15 & \frac{1}{4} & 0 \\ -29 & -1 & -41 \end{bmatrix}$

(c) $\begin{bmatrix} -4 & 0 & 0 & 0 \\ -15 & -\frac{5}{12} & 0 & 0 \\ 32 & -4 & 3 & 0 \\ 27 & 37 & 41 & 5 \end{bmatrix}$

14. If M is an $m \times n$ matrix with linearly independent rows, what can you say about the relationship between m and n? HINT: Can a matrix with linearly independent rows have more rows than columns?

15. Find the value(s) of x for which the matrix has linearly independent rows.

(a) $\begin{bmatrix} 2 & -5 \\ 3 & x \end{bmatrix}$

(b) $\begin{bmatrix} x & 4 \\ -2 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 8 & 0 \\ 6 & x & 1 \\ -3 & -2 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 8 & 2 \\ 6 & 45 & 1 \\ -3 & -20 & x \end{bmatrix}$

16. Find the value(s) of x for which the matrix has linearly dependent rows.

(a) $\begin{bmatrix} 3 & x \\ -5 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 7 \\ x & 5 \end{bmatrix}$

(c) $\begin{bmatrix} x & -6 & 27 \\ -2 & 8 & -30 \\ -1 & 5 & -18 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 5 & -4 \\ -5 & 3 & x \\ -7 & -11 & 4 \end{bmatrix}$

Compare your answer with the answer to section 3.3 exercise 15.

17. Find a nontrivial solution of $\mathbf{v}^T M = \mathbf{0}^T$ using the fact that the first and second rows of M are identical. Do not use row operations.

$$M = \begin{bmatrix} -14 & -29 & 49 & -32 \\ -14 & -29 & 49 & -32 \\ 44 & -25 & 13 & -35 \end{bmatrix}$$

18. Prove that if U is upper triangular, so is $U_{\setminus 1,j}$.

19. Prove that if U is upper triangular, then $U_{\setminus 1,j}$ has a zero on its main diagonal whenever $j > 1$.

20. Prove that if U is an upper triangular $n \times n$ matrix, then $\det U = U_{1,1}U_{2,2}\cdots U_{n,n}$. HINT: Use the facts proven in exercises 18 and 19.

3.5 The Determinant Revisited

Pick a number. Any number.
Add 6.
Multiply (your new number) by 6.
Subtract 9.
Divide by 3.
Subtract 9.
⋮
Tell me your latest number, and I'll tell you your starting number. It's half of your latest number!

Matrices and row operations work in a similar fashion ... and you can be the magician. I'll pick a matrix. Any matrix. After swapping the first two rows, my matrix is

$$\begin{bmatrix} 23 & 12 & 22 \\ 13 & -11 & -19 \\ -16 & 19 & -5 \end{bmatrix}.$$

What was my original matrix? Answer on page 118.

How about another? After scaling the third row of my matrix by 2, my matrix is

$$\begin{bmatrix} 2 & -13 & -6 \\ -7 & -9 & 21 \\ -14 & -18 & 24 \end{bmatrix}.$$

What was my original matrix? Answer on page 118.

And one last one...after replacing the first row of my matrix by the first row plus three times the third, my matrix is

$$\begin{bmatrix} 11 & 9 & -1 \\ 3 & 5 & -8 \\ 3 & 8 & 1 \end{bmatrix}.$$

What was my original matrix? Answer on page 119.

In each case the row operation can be undone to recover the original matrix. This is exactly the concept of an inverse! The six elementary matrices corresponding to the six row operations (the row operations that gave the matrices above and the row operations used to recover the original matrices), in the order encountered are

$$\begin{array}{cc} \text{operation} & \text{recovery} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

It must therefore be that each operation matrix is invertible and

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying the operation matrix by the inverse matrix will verify the inverse pairs. Each elementary matrix has an inverse elementary matrix of the same type, and in the case of a row swap, the elementary matrix is its own inverse.

Now notice a couple of things about the determinants:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$
$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2 \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$
$$\begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \qquad \begin{vmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Come to think of it, the determinant of a scale matrix will always be the scale factor! Can you justify this claim? Answer on page 119. Wait a minute! the determinant of a replace matrix is always 1. Can you justify this claim? Answer on page 119.

Is the determinant of a swap matrix always -1? There is no such thing as a swap matrix with one row. There is only one swap matrix with two rows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and there are only three swap matrices with three rows:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easy enough to check that all four of these matrices have determinant -1, so maybe all swap matrices do have determinant -1.

Crumpet 19: Binomial Coefficients

The number of swap matrices with n rows, $n > 1$, is $\frac{n(n-1)}{2}$. For example, there are $\frac{4(3)}{2} = 6$ swap matrices with 4 rows, and there are $\frac{100(99)}{2} = 4950$ swap matrices with 100 rows. The formula $\frac{n(n-1)}{2}$ is a special case of the "choose formula", which is a formula for the number of ways to choose k objects from a set of n objects, with $0 \le k \le n$. This number is also known as a binomial coefficient and there are several notations for it. Common notations and the formula for "n choose k" (a binomial coefficient) are

$$\left(\begin{array}{c} n \\ k \end{array}\right) = {}_nC_k = C(n,k) = \frac{n!}{k!(n-k)!}.$$

Proving it for all $n \times n$ swap matrices requires induction, but before we can do it cleanly, we need one more fact: if M is an $n \times n$ matrix with $n > 1$, then

$$\begin{aligned} \det M &= (-1)^{i+1} M_{i,1} \det M_{\backslash i,1} + (-1)^{i+2} M_{i,2} \det M_{\backslash i,2} + \cdots + (-1)^{i+n} M_{i,n} \det M_{\backslash i,n} \\ &= (-1)^{1+j} M_{1,j} \det M_{\backslash 1,j} + (-1)^{2+j} M_{2,j} \det M_{\backslash 2,j} + \cdots + (-1)^{n+j} M_{n,j} \det M_{\backslash n,j}. \end{aligned} \tag{3.5.1}$$

for any i from 1 through n or any j from 1 through n. This formula implies that determinants may be computed by expanding along any row or any column, not just the first row. To illustrate,

expansion along row 1

$$\begin{aligned} &\begin{vmatrix} -5 & 4 & -1 \\ 2 & -2 & 3 \\ 5 & -2 & -3 \end{vmatrix} \\ &= -5\begin{vmatrix} -2 & 3 \\ -2 & -3 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 5 & -3 \end{vmatrix} - 1\begin{vmatrix} 2 & -2 \\ 5 & -2 \end{vmatrix} \\ &= -5(6+6) - 4(-6-15) - 1(-4+10) \\ &= -60 + 84 - 6 \\ &= 18 \end{aligned}$$

expansion along row 3

$$\begin{aligned} &\begin{vmatrix} -5 & 4 & -1 \\ 2 & -2 & 3 \\ 5 & -2 & -3 \end{vmatrix} \\ &= 5\begin{vmatrix} 4 & -1 \\ -2 & 3 \end{vmatrix} + 2\begin{vmatrix} -5 & -1 \\ 2 & 3 \end{vmatrix} - 3\begin{vmatrix} -5 & 4 \\ 2 & -2 \end{vmatrix} \\ &= 5(12-2) + 2(-15+2) - 3(10-8) \\ &= 50 - 26 - 6 \\ &= 18 \end{aligned}$$

expansion along column 2

$$\begin{aligned} &\begin{vmatrix} -5 & 4 & -1 \\ 2 & -2 & 3 \\ 5 & -2 & -3 \end{vmatrix} \\ &= -4\begin{vmatrix} 2 & 3 \\ 5 & -3 \end{vmatrix} - 2\begin{vmatrix} -5 & -1 \\ 5 & -3 \end{vmatrix} + 2\begin{vmatrix} -5 & -1 \\ 2 & 3 \end{vmatrix} \\ &= -4(-6-15) - 2(15+5) + 2(-15+2) \\ &= 84 - 40 - 26 \\ &= 18 \end{aligned}$$

Formula (3.5.1) makes it relatively straightforward to prove that all swap matrices have determinant -1. We have already shown (assuming you did the calculation above) that the determinant of any 2×2 matrix (and there is only one of them) is -1. Proceeding by induction, assume that for some $k \ge 2$, the determinant of all $k \times k$ swap matrices is -1, and let S be a particular but arbitrary $(k+1) \times (k+1)$ matrix where rows i and j have been swapped. Since $k + 1 \geq 3$, there must be a row of S that is not involved in the swap, say row ℓ. In other words, $\ell \neq i$ and $\ell \neq j$. Expanding the determinant of S along row ℓ,

$$\begin{aligned} \det S &= (-1)^{\ell+1} S_{\ell,1} \det S_{\setminus \ell,1} + (-1)^{\ell+2} S_{\ell,2} \det S_{\setminus \ell,2} + \cdots + (-1)^{\ell+k+1} S_{\ell,k+1} \det S_{\setminus \ell,k+1} \\ &= (-1)^{\ell+\ell} S_{\ell,\ell} \det S_{\setminus \ell,\ell} \end{aligned}$$

because $S_{\ell,:}$ is the ℓ^{th} row of the identity matrix (not being involved in the swap), meaning $S_{\ell,m} = 0$ whenever $\ell \neq m$. Since the swapped rows are both in $S_{\setminus \ell,\ell}$, $S_{\setminus \ell,\ell}$ is a $k \times k$ swap matrix and the inductive hypothesis implies $\det S_{\setminus \ell,\ell} = -1$. Therefore,

$$\det S = (-1)^{\ell+\ell} S_{\ell,\ell} \det S_{\setminus \ell,\ell} = (1)(1)(-1) = -1,$$

completing the proof.

Crumpet 20: Proof of Formula (3.5.1)

Proving formula (3.5.1) requires a bit of work. One way to do it is to prove that (i) there is only one function G taking $n \times n$ matrices as inputs and returning scalars as outputs with the following four properties and (ii) the expressions in formula (3.5.1) have these four properties. Thus, each expression must give the same result.

1. $G(I) = 1$
2. $G(A) = 0$ whenever A has two identical columns.
3. If A, B, and C are identical except in their k^{th} columns where $C_{:,k} = A_{:,k} + B_{:,k}$, then $G(C) = G(A) + G(B)$.
4. If A and B are identical except in their k^{th} columns where $A_{:,k} = cB_{:,k}$, then $G(A) = cG(B)$.

To begin, suppose G is a function from $n \times n$ matrices to scalars satisfying the four properties above. Then G also has the following two properties.

5. $G(A) = 0$ whenever the columns of A are linearly dependent. *Proof:* Because the columns of A are linearly dependent, one of them, say column k, can be written as a linear combination of the others. That is, $A_{:,k} = \sum_{j \neq k} c_j A_{:,j}$ for some constants c_j. Then

$$\begin{aligned} G(A) &= G\left(\begin{bmatrix} A_{:,1} & \cdots & A_{:,k-1} & \sum_{j \neq k} c_j A_{:,j} & A_{:,k+1} & \cdots & A_{:,n} \end{bmatrix}\right) \\ &= \sum_{j \neq k} G\left(\begin{bmatrix} A_{:,1} & \cdots & A_{:,k-1} & c_j A_{:,j} & A_{:,k+1} & \cdots & A_{:,n} \end{bmatrix}\right) \\ &= \sum_{j \neq k} c_j G\left(\begin{bmatrix} A_{:,1} & \cdots & A_{:,k-1} & A_{:,j} & A_{:,k+1} & \cdots & A_{:,n} \end{bmatrix}\right) \\ &= \sum_{j \neq k} c_j \cdot 0 = 0 \end{aligned}$$

by applying properties 3, 4, and 2, respectively.

6. $G(B) = -G(A)$ whenever B is the result of swapping two columns of A. *Proof:* Suppose B is the result of swapping columns i and j of A, and without loss of generality, assume $i < j$. Then
$$B = \begin{bmatrix} A_{:,1} & \cdots & A_{:,j} & \cdots & A_{:,i} & \cdots & A_{:,n} \end{bmatrix}.$$
Now let $C = \begin{bmatrix} A_{:,1} & \cdots & A_{:,i} + A_{:,j} & \cdots & A_{:,i} + A_{:,j} & \cdots & A_{:,n} \end{bmatrix}$. Then by repeated application of property 3,
$$\begin{aligned} G(C) = G(A) + G(B) \\ + G\left(\begin{bmatrix} A_{:,1} & \cdots & A_{:,i} & \cdots & A_{:,i} & \cdots & A_{:,n} \end{bmatrix}\right) \\ + G\left(\begin{bmatrix} A_{:,1} & \cdots & A_{:,j} & \cdots & A_{:,j} & \cdots & A_{:,n} \end{bmatrix}\right) \end{aligned}$$
and by property 2 the last two terms are zero as is $G(C)$. Hence, $0 = G(A) + G(B)$, concluding the proof.

To begin the induction proof that G is unique, note that $G([a]) = G(a[1]) = aG([1]) = aG(I) = a$ by properties 4 and 1, so G is uniquely determined for 1×1 matrices. Now suppose G is unique for all $(k-1) \times (k-1)$ matrices for some $k > 1$, and let M be a particular but arbitrary $k \times k$ matrix. If the columns of M are linearly dependent, then property 5 implies $G(M) = 0$, so $G(M)$ is uniquely determined. Now suppose the columns of M are linearly independent. By theorem 5, M has a pivot in every column. Since M is square, M has a pivot in every row. Therefore M has a nonzero entry in row k, say $M_{k,j} \neq 0$. Letting
$$A = \begin{bmatrix} M_{:,1} - \frac{M_{k,1}}{M_{k,j}} M_{:,j} & \cdots & M_{:,j-1} - \frac{M_{k,j-1}}{M_{k,j}} M_{:,j} & M_{:,j+1} - \frac{M_{k,j+1}}{M_{k,j}} M_{:,j} & \cdots & M_{:,n} - \frac{M_{k,k}}{M_{k,j}} M_{:,j} & M_{:j} \end{bmatrix},$$
repeated application of properties 3 and 4 plus property 6 if needed implies $G(A) = \pm G(M)$ depending on whether $j = k$. Either way, $G(M)$ is uniquely determined if $G(A)$ is. Note that $A_{k,:} = \begin{bmatrix} 0 & \cdots & 0 & M_{k,j} \end{bmatrix}$, making it sufficient to show that $H(B)$ defined on $(k-1) \times (k-1)$ matrices by
$$H(B) = \frac{1}{M_{k,j}} G\left(\begin{bmatrix} B & M_{1:k-1,j} \\ \mathbf{0}^T & M_{k,j} \end{bmatrix}\right)$$
is uniquely determined. But $H(B)$ inherits properties 2, 3, and 4 from G, so it only remains to establish that $H(I) = 1$. Then, by the inductive hypothesis, H is uniquely determined. To that end,
$$\begin{aligned} H(I) &= \frac{1}{M_{k,j}} G\left(\begin{bmatrix} I & M_{1:k-1,j} \\ \mathbf{0}^T & M_{k,j} \end{bmatrix}\right) \\ &= \frac{1}{M_{k,j}} G\left(\begin{bmatrix} I_{:,1} & \cdots & I_{:,k-1} & \sum_{i=1}^{k} M_{i,j} I_{:,i} \end{bmatrix}\right) \\ &= \frac{1}{M_{k,j}} \sum_{i=1}^{k} M_{i,j} G\left(\begin{bmatrix} I_{:,1} & \cdots & I_{:,k-1} & I_{:,i} \end{bmatrix}\right) \\ &= \frac{1}{M_{k,j}} \left(M_{k,j} G(I)\right) \\ &= 1 \end{aligned}$$
concluding the proof that G is unique.

To complete the proof of formula (3.5.1), it remains to show that each of the two expressions for $\det M$ has properties 1 through 4. This is because formula (1.5.1), the definition of determinant, is one of the expressions, so by uniqueness they all produce the same result (as the determinant).

Proceeding by induction, note that $\det([a]) = a$ satisfies all four properties, so the determinant has all four properties on 1×1 matrices. Now suppose $\det M$ has all four properties on $(\ell - 1) \times (\ell - 1)$ matrices for some $\ell > 1$ and consider, for any fixed $i = 1, 2, \ldots \ell$, the formula

$$\det M = (-1)^{i+1} M_{i,1} \det M_{\setminus i,1} + (-1)^{i+2} M_{i,2} \det M_{\setminus i,2} + \cdots + (-1)^{i+\ell-1} M_{i,\ell} \det M_{\setminus i,\ell}$$

on $\ell \times \ell$ matrices.

1. $$\begin{aligned} \det I_{\ell\times\ell} &= (-1)^{i+1} I_{i,1} \det I_{\setminus i,1} + (-1)^{i+2} I_{i,2} \det I_{\setminus i,2} + \cdots + (-1)^{i+\ell} I_{i,\ell} \det I_{\setminus i,\ell} \\ &= (-1)^{i+i} I_{i,i} \det I_{\setminus i,i} \end{aligned}$$

   since $I_{i,j} = 0$ whenever $j \neq i$. But $I_{\setminus i,i} = I_{(\ell-1)\times(\ell-1)}$, so by the inductive hypothesis $\det I_{\setminus i,i} = 1$ and we have $\det I_{\ell\times\ell} = (-1)^{2i} I_{i,i}(1) = (1)(1)(1) = 1$.

2. Suppose M is a particular but arbitrary $\ell \times \ell$ matrix with columns j and k identical, and without loss of generality assume $j < k$. Then

   $$\begin{aligned} \det M &= (-1)^{i+1} M_{i,1} \det M_{\setminus i,1} + (-1)^{i+2} M_{i,2} \det M_{\setminus i,2} + \cdots + (-1)^{i+\ell} M_{i,\ell} \det M_{\setminus i,\ell} \\ &= (-1)^{i+j} M_{i,j} \det M_{\setminus i,j} + (-1)^{i+k} M_{i,k} \det M_{\setminus i,k} \end{aligned}$$

   since $M_{\setminus i,m}$ has two identical columns whenever $m \notin \{j, k\}$ and therefore $\det M_{\setminus i,m} = 0$ by the inductive hypothesis. But because columns j and k of M are identical, $M_{i,j} = M_{i,k}$, so we can rewrite $\det M = (-1)^{i+j} M_{i,j} \left[\det M_{\setminus i,j} + (-1)^{k-j} \det M_{\setminus i,k}\right]$. Now if we swap columns j and $j+1$ of $M_{\setminus i,k}$, and then columns $j+1$ and $j+2$, and so on to column k, a total of $k - j - 1$ swaps, the result is $M_{\setminus i,j}$, so by the inductive hypothesis $\det M_{\setminus i,j} = (-1)^{k-j-1} \det M_{\setminus i,k}$. Substituting into the latest expression for $\det M$, we have $\det M = (-1)^{i+j} M_{i,j} \left[(-1)^{k-j-1} \det M_{\setminus i,k} + (-1)^{k-j} \det M_{\setminus i,k}\right] = (-1)^{i+k-1} \left[\det M_{\setminus i,k} - \det M_{\setminus i,k}\right] = 0$.

3. Suppose A, B, and C are identical $\ell\times\ell$ matrices except in their k^{th} columns where $C_{:,k} = A_{:,k} + B_{:,k}$. Observe that $C_{\setminus i,k} = B_{\setminus i,k} = A_{\setminus i,k}$, $C_{i,j} = B_{i,j} = A_{i,j}$, and $\det C_{\setminus i,j} = \det A_{\setminus i,j} + \det B_{\setminus i,j}$ (by the inductive hypothesis) for $j \neq k$ and all i. It then follows that

   $$\begin{aligned} \det C &= (-1)^{i+1} C_{i,1} \det C_{\setminus i,1} + (-1)^{i+2} C_{i,2} \det C_{\setminus i,2} + \cdots + (-1)^{i+\ell} C_{i,\ell} \det C_{\setminus i,\ell} \\ &= (-1)^{i+k} C_{i,k} \det C_{\setminus i,k} + \sum_{j\neq k} (-1)^{i+j} C_{i,j} \det C_{\setminus i,j} \\ &= (-1)^{i+k} (A_{i,k} + B_{i,k}) \det C_{\setminus i,k} + \sum_{j\neq k} (-1)^{i+j} C_{i,j} \left[\det A_{\setminus i,j} + \det B_{\setminus i,j}\right] \\ &(-1)^{i+k} A_{i,k} \det C_{\setminus i,k} + (-1)^{i+k} B_{i,k} \det C_{\setminus i,k} \\ &\qquad + \sum_{j\neq k} (-1)^{i+j} \left[C_{i,j} \det A_{\setminus i,j} + C_{i,j} \det B_{\setminus i,j}\right] \\ &= (-1)^{i+k} A_{i,k} \det A_{\setminus i,k} + (-1)^{i+k} B_{i,k} \det B_{\setminus i,k} \\ &\qquad + \sum_{j\neq k} (-1)^{i+j} \left[A_{i,j} \det A_{\setminus i,j} + B_{i,j} \det B_{\setminus i,j}\right] \\ &= \sum_{j=1}^{\ell} (-1)^{i+j} \left[A_{i,j} \det A_{\setminus i,j} + B_{i,j} \det B_{\setminus i,j}\right] = \det A + \det B \end{aligned}$$

4. Suppose A and B are identical $\ell \times \ell$ matrices except in their k^{th} columns where $A_{:,k} = cB_{:,k}$. Observe that $B_{\backslash i,k} = A_{\backslash i,k}$, $B_{i,j} = A_{i,j}$, and $\det A_{\backslash i,j} = c \det B_{\backslash i,j}$ (by the inductive hypothesis) for $j \neq k$ and all i. It then follows that

$$\begin{aligned}
\det A &= (-1)^{i+1} A_{i,1} \det A_{\backslash i,1} + (-1)^{i+2} A_{i,2} \det A_{\backslash i,2} + \cdots + (-1)^{i+\ell} A_{i,\ell} \det A_{\backslash i,\ell} \\
&= (-1)^{i+k} A_{i,k} \det A_{\backslash i,k} + \sum_{j\neq k} (-1)^{i+j} A_{i,j} \det A_{\backslash i,j} \\
&= (-1)^{i+k} cB_{i,k} \det B_{\backslash i,k} + \sum_{j\neq k} (-1)^{i+j} B_{i,j} \left(c \det B_{\backslash i,j}\right) \\
&= c\left[(-1)^{i+k} B_{i,k} \det B_{\backslash i,k} + \sum_{j\neq k} (-1)^{i+j} B_{i,j} \det B_{\backslash i,j}\right] \\
&= c \sum_{j=1}^{\ell} (-1)^{i+j} B_{i,j} \det B_{\backslash i,j} = c \det B
\end{aligned}$$

Hence the determinant may be calculated by expansion along any row.

As for expansion along any column, we begin by showing that the function $f(M) = \det M^T$ (where $\det M$ is defined by row expansion) has all four properties for any size matrix, so must equal $\det M$. Note that if M is a 1×1 matrix, $M^T = M$. Therefore $f(M) = \det M^T = \det M = M_{1,1}$, so $f(M)$ has all four properties. Observe that if M is an $n \times n$ matrix and $n > 1$, then by definition of f and the row expansion formula for determinant,

$$f(M) = \sum_{j=1}^{n} (-1)^{i+j} M_{j,i} \det(M^T)_{\backslash i,j} \tag{3.5.2}$$

for any fixed $i = 1, 2, \ldots n$. We now proceed to show that f has all four properties for $n \times n$ matrices where $n > 1$.

1. For any n, $I_{n\times n}^T = I_{n\times n}$, so $f(I_{n\times n}) = \det(I_{n\times n}^T) = \det(I_{n\times n}) = 1$.

2. Suppose M is a 2×2 matrix with two identical columns. Then $M = \begin{bmatrix} a & a \\ b & b \end{bmatrix}$ for some constants a, b, and

$$f(M) = \det M^T = \det\left(\begin{bmatrix} a & b \\ a & b \end{bmatrix}\right) = ab - ba = 0.$$

Now set $n > 2$, suppose $f(A) = 0$ for any $(n-1)\times(n-1)$ matrix A with two identical columns, and let M be an $n \times n$ matrix with identical columns k and ℓ. Because M has at least three columns, there is an i, $1 \leq i \leq n$ such that $i \neq k$ and $i \neq \ell$. Then, for this particular i,

$$\begin{aligned}
f(M) &= \sum_{j=1}^{n} (-1)^{i+j} M_{j,i} \det(M^T)_{\backslash i,j} \\
&= \sum_{j=1}^{n} (-1)^{i+j} M_{j,i} \det(M_{\backslash j,i})^T \\
&= \sum_{j=1}^{n} (-1)^{i+j} M_{j,i} f(M_{\backslash j,i})
\end{aligned}$$

By the inductive hypothesis, $f(M_{\backslash j,i}) = 0$ for all j, so $f(M) = 0$.

3. Let $n > 1$ and suppose A, B, and C are identical $n \times n$ matrices except in their k^{th} columns where $C_{:,k} = A_{:,k} + B_{:,k}$. Then $(C^T)_{\backslash k,j} = (A^T)_{\backslash k,j} = (B^T)_{\backslash k,j}$ for all $j = 1, \ldots, n$ because A^T, B^T, and C^T are all identical except in their k^{th} rows. Now, applying (113) with $i = k$,

$$\begin{aligned} f(C) &= \sum_{j=1}^{n} (-1)^{k+j} C_{j,k} \det(C^T)_{\backslash k,j} \\ &= \sum_{j=1}^{n} (-1)^{k+j} \left(A_{j,k} + B_{j,k}\right) \det(C^T)_{\backslash k,j} \\ &= \sum_{j=1}^{n} (-1)^{k+j} \left(A_{j,k} \det(C^T)_{\backslash k,j} + B_{j,k} \det(C^T)_{\backslash k,j}\right) \\ &= \sum_{j=1}^{n} (-1)^{k+j} \left(A_{j,k} \det(A^T)_{\backslash k,j} + B_{j,k} \det(B^T)_{\backslash k,j}\right) \\ &= \sum_{j=1}^{n} (-1)^{k+j} A_{j,k} \det(A^T)_{\backslash k,j} + \sum_{j=1}^{\ell} (-1)^{k+j} B_{j,k} \det(B^T)_{\backslash k,j} \\ &= f(A) + f(B) \end{aligned}$$

4. Let $n > 1$ and suppose A and B are identical $n \times n$ matrices except in their k^{th} columns where $A_{:,k} = cB_{:,k}$. Observe that $(A^T)_{\backslash k,j} = (B^T)_{\backslash k,j}$ for all $j = 1, \ldots, n$ because A^T and B^T are identical except in their k^{th} rows. Now applying (113) with $i = k$,

$$\begin{aligned} f(A) &= \sum_{j=1}^{n} (-1)^{k+j} A_{j,k} \det(A^T)_{\backslash k,j} \\ &= \sum_{j=1}^{n} (-1)^{k+j} c B_{j,k} \det(B^T)_{\backslash k,j} \\ &= c \sum_{j=1}^{n} (-1)^{k+j} B_{j,k} \det(B^T)_{\backslash k,j} \\ &= c f(B) \end{aligned}$$

Finally, the expression

$$(-1)^{1+j} M_{1,j} \det M_{\backslash 1,j} + (-1)^{2+j} M_{2,j} \det M_{\backslash 2,j} + \cdots + (-1)^{n+j} M_{n,j} \det M_{\backslash n,j}$$

from (3.5.1) equals

$$\begin{aligned} \sum_{i=1}^{n} (-1)^{i+j} M_{i,j} \det M_{\backslash i,j} &= \sum_{i=1}^{n} (-1)^{j+i} M^T_{j,i} \det \left(M^T_{\backslash j,i}\right)^T \\ &= \sum_{i=1}^{n} (-1)^{j+i} M^T_{j,i} \det M^T_{\backslash j,i} \\ &= \det M^T \\ &= \det M \end{aligned}$$

by the row expansion formula and the fact that $\det M^T = \det M$.

This proof is an adaptation of the presentation in sections 6.1 and 6.2 of [16].

To recap,

- the determinant of a swap matrix is -1,
- the determinant of a scale matrix is the scale factor, and
- the determinant of a replace matrix is 1.

Interestingly, within the proof of (3.5.1) lies the proofs that

- if B is the result of swapping two columns of a square matrix A, then $\det B = -\det A$, and
- if B is the result of scaling a columns of a square matrix A by c, then $\det B = c\det A$, and
- for any square matrix M, $\det M^T = \det M$.

Putting these three facts together, it is easy to justify the following two facts.

- If B is the result of swapping two rows of a square matrix A, then $\det B = -\det A$.

  [$\det B = \det B^T = \det A^T = -\det A$ since B^T is the result of swapping two columns of A^T and $\det M^T = \det M$.]

- If B is the result of scaling a rows of a square matrix A by c, then $\det B = c\det A$.

  Can you justify this? Answer on page 3.5.

The relevance of all these observations is mounting evidence that $\det(EA) = \det E \cdot \det A$ for any square matrix A and elementary matrix E. This will be an important point soon enough. We already have that $\det(EA) = -\det A = \det E \cdot \det A$ when E is a swap matrix and $\det(EA) = c\det A = \det E \cdot \det A$ when E is a scale matrix. We are only missing this fact for elementary replacement matrices.

The proof of (3.5.1) does not provide direct proof that if B is the result of a row replacement in a square matrix A, then $\det B = \det A$, but it provides the right tools for the job. Besides the facts already noted, we learn from the proof that

- if A, B, and C are identical square matrices except in one column, say the k^{th}, where $C_{:,k} = A_{:,k} + B_{:,k}$, then $\det(C) = \det A + \det B$, and
- if C is a square matrix with two identical columns, then $\det C = 0$.

As we just encountered, statements about the columns of a matrix and its determinant can generally be restated in terms of rows since $\det M^T = \det M$. It is safe to conclude that

- if A, B, and C are identical square matrices except in one row, say the k^{th}, where $C_{k,:} = A_{k,:} + B_{k,:}$, then $\det(C) = \det A + \det B$, and
- if C is a square matrix with two identical rows, then $\det C = 0$.

Justifications are requested in exercises 16 and 17. These two facts plus the fact that if B is the result of scaling a row of a square matrix A by c, then $\det B = c\det A$ make it a straighforward matter to prove that if B is the result of a row replacement in a square matrix A, then $\det B = \det A$.

Proof. Let A be an $n \times n$ matrix and suppose B is the result of adding c times the j^{th}row to the k^{th} row of A, $j \neq k$. Then

$$\det B = \begin{vmatrix} A_{1,:} \\ \vdots \\ A_{k-1,:} \\ A_{k,:} + cA_{j,:} \\ A_{k+1} \\ \vdots \\ A_{n,:} \end{vmatrix} = \begin{vmatrix} A_{1,:} \\ \vdots \\ A_{k-1,:} \\ A_{k,:} \\ A_{k+1} \\ \vdots \\ A_{n,:} \end{vmatrix} + \begin{vmatrix} A_{1,:} \\ \vdots \\ A_{k-1,:} \\ cA_{j,:} \\ A_{k+1} \\ \vdots \\ A_{n,:} \end{vmatrix} = \det A + c \begin{vmatrix} A_{1,:} \\ \vdots \\ A_{k-1,:} \\ A_{j,:} \\ A_{k+1} \\ \vdots \\ A_{n,:} \end{vmatrix} = \det A + 0 = \det A.$$

□

Therefore, $\det(EA) = 1 \det A = \det E \cdot \det A$ when E is a replacement matrix.

Key Concepts

elementary matrices are invertible and $\det(EA) = \det E \cdot \det A$ for any square matrix A and elementary matrix E.

determinant by expansion the determinant of an $n \times n$ matrix M may be calculated by expansion along any row or any column:

$$\begin{aligned} \det M &= (-1)^{i+1} M_{i,1} \det M_{\setminus i,1} + (-1)^{i+2} M_{i,2} \det M_{\setminus i,2} + \cdots + (-1)^{i+n} M_{i,n} \det M_{\setminus i,n} \\ &= (-1)^{1+j} M_{1,j} \det M_{\setminus 1,j} + (-1)^{2+j} M_{2,j} \det M_{\setminus 2,j} + \cdots + (-1)^{n+j} M_{n,j} \det M_{\setminus n,j} \end{aligned}$$

for any $i = 1, \ldots, n$ or any $j = 1, \ldots, n$.

determinant of replacement matrix if E is an elementary replacement matrix, $\det E = 1$.

determinant of swap matrix if E is an elementary swap matrix, $\det E = -1$.

deteminant of scale matrix if E is an elementary scale matrix, $\det E = s$ where s is the scale factor.

determinant of the transpose for any square matrix A, $\det A^T = \det A$.

Exercises

1. Take advantage of the fact that the determinant may be expanded along any row or any column to compute the determinant.

   (a) $\begin{bmatrix} 0 & -2 & 0 & 9 \\ -4 & 0 & 1 & 0 \\ 4 & -9 & 0 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

   (b) $\begin{bmatrix} 0 & -5 & 0 & -2 \\ 0 & 7 & 0 & 0 \\ 4 & -3 & 0 & 0 \\ 2 & 0 & 2 & -3 \end{bmatrix}$

   (c) $\begin{bmatrix} 3 & 0 & 6 & 0 \\ 0 & 5 & 0 & 3 \\ 5 & 0 & -6 & -3 \\ 0 & -9 & 0 & 0 \end{bmatrix}$

   (d) $\begin{bmatrix} 0 & 9 & 3 & -2 \\ 0 & -2 & 0 & 0 \\ -6 & -4 & 0 & 0 \\ 0 & 4 & 2 & 0 \end{bmatrix}$

2. Find the determinant of the triangular matrix.

   (a) $\begin{bmatrix} 6 & 0 & 0 \\ -1 & -1 & 0 \\ -1 & 3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 & 6 \\ 0 & 8 & -1 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & 8 & 0 & 0 \\ 1 & 8 & 4 & 0 \\ -2 & -2 & 8 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 7 & 7 & 6 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

3. Use the fact that

$$\begin{vmatrix} -2 & 0 & 0 \\ 8 & -2 & 2 \\ 8 & 0 & 8 \end{vmatrix} = 32$$

to compute the determinant of

(a) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 8 & -2 & 2 \\ 8 & 0 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{128} \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 8 & -2 & 2 \\ 8 & 0 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 8 & -2 & 2 \\ 8 & 0 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 18 & 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 8 & -2 & 2 \\ 8 & 0 & 8 \end{bmatrix}$

4. Use the fact that

$$\begin{vmatrix} 5 & 3 & 7 \\ -1 & 4 & 3 \\ 2 & 6 & 8 \end{vmatrix} = 14$$

to compute the determinant of

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 5 & 3 & 7 \\ -1 & 4 & 3 \\ 2 & 6 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 7 \\ -1 & 4 & 3 \\ 2 & 6 & 8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \sqrt{17} & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 7 \\ -1 & 4 & 3 \\ 2 & 6 & 8 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 7 \\ -1 & 4 & 3 \\ 2 & 6 & 8 \end{bmatrix}$

5. Use the fact that

$$\begin{vmatrix} 4 & 4 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 4 \end{vmatrix} = 38$$

to compute the determinant of

(a) $\begin{bmatrix} 4 & 4 & -1 \\ -1 & 3 & 4 \\ 1 & 7 & 3 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 4 & -1 \\ 1 & 7 & 3 \\ -\frac{1}{2} & \frac{3}{2} & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 4 & -1 \\ 0 & 10 & 7 \\ -1 & 3 & 4 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 35 & 15 \\ 4 & 4 & -1 \\ -1 & 3 & 4 \end{bmatrix}$

6. Use the fact that

$$\begin{vmatrix} 1 & 6 & 5 \\ 1 & -1 & 8 \\ 7 & 0 & 5 \end{vmatrix} = 336$$

to compute the determinant of

(a) $\begin{bmatrix} 1 & 6 & 5 \\ 1 & -1 & 8 \\ \frac{1}{2} & 0 & \frac{5}{14} \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 0 & 5 \\ 1 & -1 & 8 \\ 1 & 6 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 6 & 5 \\ 1 & -1 & 8 \\ 9 & -2 & 21 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & -1 & 8 \\ \frac{1}{84} & \frac{1}{14} & \frac{5}{84} \\ 7 & 0 & 5 \end{bmatrix}$

7. Use the fact that

$$\begin{vmatrix} 6 & -2 & 1 \\ 6 & 5 & -2 \\ 7 & 7 & 8 \end{vmatrix} = 455$$

to compute the determinant of

(a) $\begin{bmatrix} 6 & 6 & 7 \\ -2 & 5 & 7 \\ 1 & -2 & 8 \end{bmatrix}$

(b) $\begin{bmatrix} 6 & 7 & 6 \\ -2 & 7 & 5 \\ 1 & 8 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} 6 & 6 & 7 \\ -2 & 5 & 7 \\ 1 & -2 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ \pi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 6 & 7 \\ -2 & 5 & 7 \\ 1 & -2 & 8 \end{bmatrix}\begin{bmatrix} \frac{1}{35} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. Use the facts that

$$\det(E_3E_2E_1A) = 1$$

and

(a) E_1 is a swap matrix.

(b) E_2 is a scale matrix with scale factor -2.

(c) E_3 is a replacement matrix.

to determine $\det A$.

9. Use the facts that

$$\det(E_4E_3E_2E_1A) = 1$$

and

(a) E_1 is a scale matrix with scale factor 2.

(b) E_2 is a scale matrix with scale factor 3.

(c) E_3 is a replacement matrix.

(d) E_4 is a scale matrix with scale factor $\frac{1}{36}$.

to determine $\det A$.

10. Let A be a 4×4 matrix with $\det A = 3$. Find $\det(2A)$.

11. Let M be an $n \times n$ matrix with $\det M = \frac{1}{3}$. Find $\det(7M)$.

12. Let A be a square matrix and E a scale matrix with scale factor $\frac{2}{3}$. Find $\det(E^3A)$. That is, $\det(EEEA)$.

13. Suppose A is a square matrix, E is a swap matrix, and $\det(EA) = 33$. Find

(a) $\det A$

(b) $\det E$

(c) $\det A^T$

14. Suppose M is a square matrix, E is a replacement matrix, and $\det(ME) = -\frac{1}{2}$. Find $\det(M^T)$.

15. Prove that if A is a square matrix and E is an elementary matrix, then $\det(AE) = \det E \cdot \det A$.

16. Use the fact that if A, B, and C are identical square matrices except in one column, say the k^{th}, where $C_{:,k} = A_{:,k} + B_{:,k}$, then $\det(C) = \det A + \det B$ to prove that *if A, B, and C are identical square matrices except in one row, say the k^{th}, where $C_{k,:} = A_{k,:} + B_{k,:}$, then* $\det(C) = \det A + \det B$.

17. Use the fact that if C is a square matrix with two identical columns, then $\det C = 0$ to prove that *if C is a square matrix with two identical rows, then* $\det C = 0$.

18. Prove that *the determinant of a square upper triangular matrix is the product of the entries on its diagonal* by expanding along the first column.

Answers

what is my matrix (swap)? The original matrix can be recovered by swapping the first two rows back:

$$\begin{bmatrix} 13 & -11 & -19 \\ 23 & 12 & 22 \\ -16 & 19 & -5 \end{bmatrix}$$

what is my matrix (scale)? The original matrix can be recoverd by scaling the third row by $\frac{1}{2}$ (the multiplicative inverse of 2):

$$\begin{bmatrix} 2 & -13 & -6 \\ -7 & -9 & 21 \\ -7 & -9 & 12 \end{bmatrix}$$

what is my matrix (replace)? The original matrix can be recoverd by replacing the first row by the first row plus negative three (the additive inverse of 3) times the third:

$$\begin{bmatrix} 2 & -15 & -4 \\ 3 & 5 & -8 \\ 3 & 8 & 1 \end{bmatrix}$$

determinant of a scale matrix A scale matrix is lower triangular with ones on the diagonal everywhere except the row that it scales, where the entry equals the scale factor. Since its determinant is the product of the entries on its diagonal, the determinant equals the scale factor.

determinant of a replace matrix A replace matrix is either lower triangular or upper triangular with ones on the main diagonal. Therefore its determinant is one. NOTE: This argument uses the fact in exercise 20 of section 3.4.

determinant of a scaled matrix $\det B = \det B^T = \det A^T = c \det A$ since B^T is the result of scaling a column of A^T and $\det M^T = \det M$.

3.6 Characterization of Square Matrices

For square matrices, all twelve of the statements in theorems 5 and 6 are equivalent. A square matrix with a pivot in every row has a pivot in every column, and vice versa—end of justification. Square matrices have an additional property to discuss, though—invertibility. It turns out that, for a square matrix, the conditions in theorems 5 and 6 plus a few that don't appear in those theorems are equivalent to invertibility. Consider the following.

1. M has a pivot position in every row and column.
2. $\det M \neq 0$.
3. M can be row reduced to I.
4. M is invertible.
5. There is a matrix L such that $LM = I$.
6. There is a matrix R such that $MR = I$.

You may or may not have considered these statements equivalent up to this point, and there is no harm done either way. It turns out they are equivalent to one another and equivalent to the statements in theorems 5 and 6. All this will be summarized in one last matrix characterization theorem, justified by the following narrative that shows (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (thm 5) $\Rightarrow$ (thm 6) $\Rightarrow$ (6) $\Rightarrow$ (thm 6) $\Rightarrow$ (1). Until the statement of the theorem, where this information will be repeated, assume that M is an $n \times n$ matrix.

Suppose M has a pivot position in every row and every column. **Record the elementary row operations, and more importantly the corresponding elementary matrices, $E_1, E_2, \ldots, E_k$, that reduce M to any row echelon form R. Then $E_k \cdots E_2 E_1 M = R$ where R is in row echelon form. Because all elementary matrices are invertible, $M = E_1^{-1} E_2^{-1} \cdots E_k^{-1} R$ and therefore** $\det M = \det(E_1^{-1} E_2^{-1} \cdots E_k^{-1} R) = \det E_1^{-1} \cdot \det E_2^{-1} \cdots \det E_k^{-1} \cdot \det R$ **(a result of section 3.5)**. Because the inverse of an elementary matrix is an elementary matrix itself and all elementary matrices have nonzero determinant, all the $\det E_i^{-1}$ are nonzero. Because M has a pivot position in every row, R must be upper triangular with nonzero entries (the pivots) on the diagonal, making $\det R$ equal to the product of these nonzero entries. Hence $\det R \neq 0$ and it follows that $\det M \neq 0$.

Suppose $\det M \neq 0$. The reduced row echelon form, R, can be represented by $R = E_k \cdots E_2 E_1 M$ for some elementary matrices $E_1, E_2, \ldots, E_k$. Because $\det R = \det(E_k \cdots E_2 E_1 R) = \det E_k \cdots \det E_2 \cdot \det E_1 \cdot \det M$ and $\det M \neq 0$, $\det R$ must also have nonzero determinant. But the only reduced row echelon form (of a square matrix) with nonzero determinant is the identity (all others have a row of zeros, putting a zero on the main diagonal). Therefore M can be reduced to I.

Supposing M can be reduced to I, we have $E_k \cdots E_2 E_1 M = I$ for some elementary matrices $E_1, E_2, \ldots, E_k$. Letting $E = E_k \cdots E_2 E_1$, we have $EM = I$. But elementary matrices are invertible, so E is invertible and therefore $M = E^{-1}$. Since E^{-1} is invertible (with inverse E), M is invertible.

Supposing M is invertible, there is a matrix L such that $LM = I$. Let $L = M^{-1}$.

Suppose there is a matrix L such that $LM = I$, and let $\mathbf{b}$ have n entries. Then, if $M\mathbf{v} = \mathbf{b}$ has a solution, it must be $L\mathbf{b}$ since $M\mathbf{v} = \mathbf{b} \Rightarrow L(M\mathbf{v}) = L\mathbf{b} \Rightarrow (LM)\mathbf{v} = L\mathbf{b} \Rightarrow \mathbf{v} = L\mathbf{b}$. Hence $M\mathbf{v} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$.

Supposing $M\mathbf{v} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$, theorem 5 implies that M has a pivot position in each column. Since M is square, M has a pivot position in each row, too. By theorem 6, $M\mathbf{v} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$. In particular, there are vectors $\mathbf{v}_i$ such that $M\mathbf{v}_i = I_{:,i}$ for each $i = 1, 2, \ldots, n$. Letting $R = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$, we have R such that $MR = I$.

Supposing there is a matrix R such that $MR = I$, $\mathbf{v} = R\mathbf{b}$ is a solution of $M\mathbf{v} = \mathbf{b}$ since $M(R\mathbf{b}) = (MR)\mathbf{b} = I\mathbf{b} = \mathbf{b}$. Hence $M\mathbf{v} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$, and by theorem 6 M has a pivot position in each row. Since M is square, M has a pivot position in each column as well.

Crumpet 21: Proving Real Numbers are Equal

Every so often, it is convenient to prove that two real numbers, x and y, are equal by showing both $x \leq y$ and $x \geq y$. The only way x can be both less than or equal to y and simultaneously greater than or equal to y is for x to equal y. This technique is implicitly used to justify part (ix) of theorem 7. Theorem 5 implies $M\mathbf{v} = \mathbf{b}$ has at most one solution (the number of solutions is less than or equal to one) and theorem 6 implies $M\mathbf{v} = \mathbf{b}$ has at least one solution (the number of solutions is greater than or equal to one). Together, then $M\mathbf{v} = \mathbf{b}$ has exactly one solution.

We now have justification for the following theorem.

Theorem 7. [Invertible Matrix Theorem] *Suppose M is an $n \times n$ matrix, and $\mathbf{b}$ and $\mathbf{v}$ have n entries. Then the following are equivalent.*

(i) *The columns of M are linearly independent.*

(ii) *The rows of M are linearly independent.*

(iii) *No column of M is a linear combination of the others.*

(iv) *No row of M is a linear combination of the others.*

(v) $M\mathbf{v} = \mathbf{0}$ *has only the trivial solution.*

(vi) *M has a pivot position in every column.*

(vii) *M has a pivot position in every row.*

(viii) $M\mathbf{v} = \mathbf{b}$ *has no free variables.*

(ix) $M\mathbf{v} = \mathbf{b}$ *has exactly one solution for every* $\mathbf{b}$.

(x) *M can be row reduced to I.*

(xi) *There is a matrix L such that $LM = I$.*

(xii) *There is a matrix R such that $MR = I$.*

(xiii) $\det M \neq 0$.

(xiv) *M is invertible.*

This theorem gives 13 ways to detect whether a square matrix is invertible, impressive in itself. But we can also draw two separate, significant conclusions from all this. Parts (xi) and (xii) suggest we only need to check that $AB = I$ ***or*** $BA = I$, ***not both*** as required by the definition, to conclude that B is the inverse of A. The theorem gives the other equality. Additionally, the bolded section of the justification, near the middle of page 120, provides an algorithm for calculating the determinant of a square matrix! Can you follow the instructions to compute the determinant of

$$\begin{bmatrix} 6 & 3 & 6 \\ -2 & 1 & -1 \\ 3 & 4 & 6 \end{bmatrix}?$$

Answer on page 124.

If you concluded in exercise 10 of section 1.5 that one row of a matrix could only be written as a linear combination of the others when the determinant of the matrix was zero, you were correct, and we finally have the theory to support it.

Key Concepts

characterization of invertible matrices see theorem 7.

algorithm for computing the determinant reduce the matrix to row echelon form, noting the row operations used. The product of the determinants of the inverses of the associated elementary matrices with the determinant of the reduced matrix is the desired determinant.

Exercises

1. The row operations that reduce a matrix A to

$$\begin{bmatrix} -5 & 15 & -10 \\ 0 & 12 & -14 \\ 0 & 0 & -2 \end{bmatrix}$$

are given. Find $\det A$.

   (a) Ten row replacements.

   (b) Five row replacements and three row swaps.

   (c) Nine row replacements, a row scale by 6, and a row scale by 5.

   (d) Four row replacements, a row scale by 10, and two row swaps.

2. The row operations that reduce a matrix A to

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 13 & 10 & 5 & 0 \\ 0 & 2 & 14 & 6 \end{bmatrix}$$

are given. Find the possible values of $\det A$.

   (a) Row replacements only.

   (b) Row replacements and row swaps only.

   (c) Row replacements and row scales by 3, 10, and 14.

   (d) Row replacements, row swaps, and row scales.

3. The row operations that reduce a matrix A to

$$\begin{bmatrix} -21 & -2 & 6 \\ 0 & 0 & -5 \\ 0 & 0 & -9 \end{bmatrix}$$

are given. Find $\det A$.

   (a) Five row replacements and three row swaps.

   (b) Four row replacements, two row swaps, and a row scaling by -5.

   (c) 36 row replacements, 13 row swaps, and scaling by 12, -13, and $-\frac{17}{93}$.

4. Row reduce to a triangular matrix to compute the determinant.

   (a) $\begin{bmatrix} -3 & 10 \\ -15 & -10 \end{bmatrix}$

   (b) $\begin{bmatrix} -12 & 12 \\ 14 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 9 & -7 \\ 4 & 7 \end{bmatrix}$

(d) $\begin{bmatrix} 16 & -3 & -2 \\ -8 & 4 & -2 \\ -8 & 1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} -10 & 12 & -50 \\ 20 & -18 & 80 \\ -30 & 18 & -80 \end{bmatrix}$

(f) $\begin{bmatrix} -11 & -15 & 4 \\ 8 & 9 & -4 \\ -3 & -3 & 2 \end{bmatrix}$

(g) $\begin{bmatrix} 3 & 90 & -308 & -6 \\ -3 & -140 & 484 & 10 \\ 6 & 210 & -737 & -16 \\ 3 & 70 & -231 & -4 \end{bmatrix}$

(h) $\begin{bmatrix} -80 & -161 & -18 & 55 \\ 80 & 154 & 27 & -66 \\ 0 & 0 & 9 & -11 \\ -24 & -49 & 27 & -22 \end{bmatrix}$

5. Compute the determinant using a judicious combination of row expansion, column expansion, and row reduction.

(a) $\begin{bmatrix} -21 & 9 & 0 \\ -24 & 9 & 2 \\ -11 & 4 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 5 & -1 \\ 0 & 6 & -1 \\ 4 & -10 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & 2 & 1 \\ -9 & 1 & -1 \\ -32 & 4 & -3 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & -4 & -3 & 3 \\ -1 & -3 & 1 & 9 \\ 13 & 40 & -5 & -113 \\ 0 & 2 & 14 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} -9 & 12 & -87 & -25 \\ -5 & 5 & -47 & -14 \\ -3 & 8 & -28 & 12 \\ 1 & -1 & 10 & 3 \end{bmatrix}$

6. Is the matrix invertible? Explain.

(a) $\begin{bmatrix} -6 & -3 \\ 0 & 19 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -7 \\ 0 & -20 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 \\ -5 & -4 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 11 \\ 0 & 3 & -14 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 3 \\ 4 & 7 & 11 \\ 0 & 3 & 20 \end{bmatrix}$

(f) $\begin{bmatrix} 3 & 1 & -2 \\ -7 & -16 & -9 \\ 8 & 5 & -3 \end{bmatrix}$

7. Suppose M is not invertible yet there is a matrix R such that $MR = I$. How is this possible?

8. Suppose M is square and $3M_{:,2} = 2M_{:,1} - 8M_{:,5} + \frac{1}{2}M_{:,6}$. What is $\det M$?

9. Suppose the rows of M are linearly independent but M is not invertible. How can this be?

10. Explain why a matrix with a pivot position in every row and every column must be invertible.

11. Suppose G is square and $G\mathbf{v} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$. What can you say about solutions of $G\mathbf{v} = \mathbf{0}$?

12. If G is square and $G\mathbf{v} = \mathbf{0}$ has infinitely many solutions, what can you say about solutions of $G\mathbf{v} = \mathbf{b}$?

13. If M is invertible, then the rows of M^T are linearly indepedent. Explain why.

14. If H is 7×7 and $H\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$, how many pivot positions does H have?

15. If a square matrix B cannot be reduced to the identity matrix, what can you say about

    (a) its columns?

    (b) the equation $B\mathbf{v} = \mathbf{0}$?

    (c) the equation $AB = I$?

16. Describe the row echelon form of an invertible matrix.

17. Recall that $\lambda, \mathbf{v}$ is an eigenpair for M whenever $\mathbf{v} \neq \mathbf{0}$ yet $(M - \lambda I)\mathbf{v} = \mathbf{0}$. Use theorem 7 to prove that the following statements are equivalent.

    (a) λ is an eigenvalue of M.

    (b) The rows of $M - \lambda I$ are linearly dependent.

    (c) $\det(M - \lambda I) = 0$.

Answers

determinant The instructions are, in brief: **Record the elementary row operations, and more importantly [note] the corresponding elementary matrices, $E_1, E_2, \ldots, E_k$, that reduce M to any row echelon form. Then** $\det M = \det E_1^{-1} \cdot \det E_2^{-1} \cdots \det E_k^{-1} \cdot \det R$**.**

Recording the elementary row operations as $\begin{bmatrix} 6 & 3 & 6 \\ -2 & 1 & -1 \\ 3 & 4 & 6 \end{bmatrix}$ is reduced to row echelon form:

$$\begin{bmatrix} 6 & 3 & 6 \\ -2 & 1 & -1 \\ 3 & 4 & 6 \end{bmatrix} \xrightarrow[M_{3,:}\to -2M_{3,:}]{M_{2,:}\to 3M_{2,:}} \begin{bmatrix} 6 & 3 & 6 \\ -6 & 3 & -3 \\ -6 & -8 & -12 \end{bmatrix} \xrightarrow[M_{3,:}\to M_{3,:}+M_{1,:}]{M_{2,:}\to M_{2,:}+M_{1,:}}$$

$$\begin{bmatrix} 6 & 3 & 6 \\ 0 & 6 & 3 \\ 0 & -5 & -6 \end{bmatrix} \xrightarrow{M_{2,:}\to M_{2,:}+M_{3,:}} \begin{bmatrix} 6 & 3 & 6 \\ 0 & 1 & -3 \\ 0 & -5 & -6 \end{bmatrix} \xrightarrow{M_{3,:}\to M_{3,:}+5M_{2,:}}$$

$$\begin{bmatrix} 6 & 3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -21 \end{bmatrix}$$

The determinant of $\begin{bmatrix} 6 & 3 & 6 \\ -2 & 1 & -1 \\ 3 & 4 & 6 \end{bmatrix}$ is the determinant of $\begin{bmatrix} 6 & 3 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -21 \end{bmatrix}$, which is -126, times the determinants of the inverse elementary matrices (listed in reverse order):

$$\begin{aligned} \begin{vmatrix} 6 & 3 & 6 \\ -2 & 1 & -1 \\ 3 & 4 & 6 \end{vmatrix} &= (-126)(1)(1)(1)(1)\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right) \\ &= 21. \end{aligned}$$

3.7 The Inverse Revisited

As if we haven't already extracted enough information from theorem 7, we also have the rather significant following theorem as a consequence.

Theorem 8. [Determinant of a Product] *If A and B are $n \times n$ matrices, then* $\det(AB) = \det A \cdot \det B$.

Proof. First suppose AB is noninvertible. By theorem 121, $\det(AB) = 0$. If both A and B are invertible, then $(AB)(B^{-1}A^{-1}) = I$, so AB is invertible. Therefore we must have that either A or B is noninvertible, from which it follows $\det A = 0$ or $\det B = 0$. Either way, $\det A \cdot \det B = 0$ and we have shown $\det(AB) = \det A \cdot \det B$. Now suppose AB is invertible, and let $M = (AB)^{-1}$. Then $I = (AB)M = A(BM)$, so $A^{-1} = BM$ and A is invertible. As in the justification of 3.⇒4. on page 120, we may therefore write A as a product of elementary matrices, $E_k^{-1} \cdots E_2^{-1} E_1^{-1}$. Hence $\det(AB) = \det(E_k^{-1} \cdots E_2^{-1} E_1^{-1} B) = (\det E_k^{-1} \cdots \det E_2^{-1} \det E_1^{-1}) \det B = \det A \det B$. □

As a direct consequence, we can relate the determinants of inverse matrices. If M is invertible, then $\det M \cdot \det M^{-1} = \det I = 1$ and therefore $\det M^{-1} = \frac{1}{\det M}$.

There is more! The proof of theorem 7 also provides an algorithm for finding the inverse of a matrix. Given that M is invertible, it is reducible to the identity matrix, meaning there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_1, E_2, \ldots, E_k M = I$. Therefore $M^{-1} = E_1 E_2 \cdots E_k I$, so the same sequence of elementary row operations that reduces M to the identity also transforms I into M^{-1}! Hence, if we augment M with the identity matrix and reduce to reduced row echelon form, the augmented columns will hold M^{-1}. To illustrate, let

$$M = \begin{bmatrix} 3 & 0 & 5 & 0 \\ 5 & 1 & 0 & 2 \\ 6 & 2 & 0 & 7 \\ 0 & 0 & -1 & -2 \end{bmatrix}$$

Augmenting the identity and reducing,

$$\begin{bmatrix} 3 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 6 & 2 & 0 & 7 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{1,:} \leftrightarrow M_{3,:}} \begin{bmatrix} 6 & 2 & 0 & 7 & 0 & 0 & 1 & 0 \\ 5 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{1,:} \to M_{1,:} - M_{2,:}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 5 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & 5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[M_{3,:} \to M_{3,:} - 3M_{1,:}]{M_{2,:} \to M_{2,:} - 5M_{1,:}} \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & -4 & 0 & -23 & 0 & 6 & -5 & 0 \\ 0 & -3 & 5 & -15 & 1 & 3 & -3 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{2,:} \to M_{2,:} - M_{3,:}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & -1 & -5 & -8 & -1 & 3 & -2 & 0 \\ 0 & -3 & 5 & -15 & 1 & 3 & -3 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{2,:} \to -1M_{2,:}} \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 5 & 8 & 1 & -3 & 2 & 0 \\ 0 & -3 & 5 & -15 & 1 & 3 & -3 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{3,:} \to M_{3,:} + 3M_{2,:}}$$

$$\begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 5 & 8 & 1 & -3 & 2 & 0 \\ 0 & 0 & 20 & 9 & 4 & -6 & 3 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{3,:} \to M_{3,:} + 20M_{4,:}} \begin{bmatrix} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 5 & 8 & 1 & -3 & 2 & 0 \\ 0 & 0 & 0 & -31 & 4 & -6 & 3 & 20 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{M_{3,:} \leftrightarrow M_{4,:}}$$

$$\left[\begin{array}{cccccccc} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 5 & 8 & 1 & -3 & 2 & 0 \\ 0 & 0 & -1 & -2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -31 & 4 & -6 & 3 & 20 \end{array}\right] \xrightarrow[M_{4,:}\to\frac{-1}{31}M_{4,:}]{M_{3,:}\to -1M_{3,:}} \left[\begin{array}{cccccccc} 1 & 1 & 0 & 5 & 0 & -1 & 1 & 0 \\ 0 & 1 & 5 & 8 & 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -\frac{4}{31} & \frac{6}{31} & -\frac{3}{31} & -\frac{20}{31} \end{array}\right] \xrightarrow[M_{1,:}\to M_{1,:}-M_{2,:}]{M_{2,:}\to M_{2,:}-5M_{3,:}}$$

$$\left[\begin{array}{cccccccc} 1 & 0 & 0 & 7 & -1 & 2 & -1 & -5 \\ 0 & 1 & 0 & -2 & 1 & -3 & 2 & 5 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -\frac{4}{31} & \frac{6}{31} & -\frac{3}{31} & -\frac{20}{31} \end{array}\right] \xrightarrow{M_{3,:}\to M_{3,:}-2M_{4,:}} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 7 & -1 & 2 & -1 & -5 \\ 0 & 1 & 0 & -2 & 1 & -3 & 2 & 5 \\ 0 & 0 & 1 & 0 & \frac{8}{31} & -\frac{12}{31} & \frac{6}{31} & \frac{9}{31} \\ 0 & 0 & 0 & 1 & -\frac{4}{31} & \frac{6}{31} & -\frac{3}{31} & -\frac{20}{31} \end{array}\right]$$

$$\xrightarrow[M_{1,:}\to M_{1,:}-7M_{4,:}]{M_{2,:}\to M_{2,:}+2M_{4,:}} \left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & -\frac{3}{31} & \frac{20}{31} & -\frac{10}{31} & -\frac{15}{31} \\ 0 & 1 & 0 & 0 & \frac{23}{31} & -\frac{81}{31} & \frac{56}{31} & \frac{115}{31} \\ 0 & 0 & 1 & 0 & \frac{8}{31} & -\frac{12}{31} & \frac{6}{31} & \frac{9}{31} \\ 0 & 0 & 0 & 1 & -\frac{4}{31} & \frac{6}{31} & -\frac{3}{31} & -\frac{20}{31} \end{array}\right]$$

so

$$M^{-1} = \frac{1}{31}\left[\begin{array}{cccc} -3 & 20 & -10 & -15 \\ 23 & -81 & 56 & 115 \\ 8 & -12 & 6 & 9 \\ -4 & 6 & -3 & -20 \end{array}\right]$$

Crumpet 22: Inverses via Row Reduction

We could have seen that inverses could be computed with the help of row reduction long ago. After all, if A is an $n \times n$ matrix and B is its inverse, then $AB = I$. By thinking of this product one column at a time, this means

$$AB_{:,1} = I_{:,1},\ AB_{:,2} = I_{:,2},\ \ldots,\ AB_{:,n} = I_{:,n}.$$

Solving these equations for the $B_{:,i}$ could be done one at a time by row reduction. Putting the solutions together into a matrix would give B. Reducing A n times would be repetitive and time consuming, though. Better, the solutions could be found simultaneously by augmenting all of the $I_{:,i}$ together—in effect, augmenting by the identity matrix—and reducing once (the algorithm presented in this section).

While this process is still tedious for large matrices, it certainly beats the alternative of using formula (1.6.1). Ironically the ideas presented recently give us the tools to prove that (1.6.1) correctly computes the inverse. Let M be an $n \times n$ matrix and consider modifying M by replacing row j with a copy of row i, $i \neq j$. Call the modified matrix $\tilde{M}$. Then

$$\left|\tilde{M}\right| = (-1)^{j+1}\tilde{M}_{j,1}\left|\tilde{M}_{\backslash j,1}\right| + (-1)^{j+2}\tilde{M}_{j,2}\left|\tilde{M}_{\backslash j,2}\right| + \cdots + (-1)^{j+n}\tilde{M}_{j,n}\left|\tilde{M}_{\backslash j,n}\right|.$$

But $\tilde{M}_{j,k} = M_{i,k}$ and $\tilde{M}_{\backslash j,k} = M_{\backslash j,k}$ by construction, so

$$\left|\tilde{M}\right| = (-1)^{j+1}M_{i,1}\left|M_{\backslash j,1}\right| + (-1)^{j+2}M_{i,2}\left|M_{\backslash j,2}\right| + \cdots + (-1)^{j+n}M_{i,n}\left|M_{\backslash j,n}\right|.$$

On the other hand, $\left|\tilde{M}\right| = 0$ since $\tilde{M}$ has two identical rows. We conclude that

$$(-1)^{j+1}M_{i,1}\left|M_{\backslash j,1}\right| + (-1)^{j+2}M_{i,2}\left|M_{\backslash j,2}\right| + \cdots + (-1)^{j+n}M_{i,n}\left|M_{\backslash j,n}\right| \tag{3.7.1}$$

equals 0 whenever $i \neq j$. Observe that when $i = j$, (3.7.1) is $\det M$ expanded along row i (or row j depending on your perspective). The proof of formula (1.6.1) then lies in noticing that for any square matrix A, the entries of the product

$$A \cdot \mathrm{adj} A$$

all take the form (3.7.1). Accordingly $A \cdot \mathrm{adj} A = (\det A) I$ for any square matrix A. If A is invertible, $\det A \neq 0$ and we have $A \cdot \frac{1}{\det A} \mathrm{adj} A = I$, so $A^{-1} = \frac{1}{\det A} \mathrm{adj} A$.

Another place where row reduction could help ease an earlier burden is finding eigenvectors. Unless you happened to work through exercise 5 of section 2.3, the last time you were asked to compute an eigenvector, you were expected to write out a linear system of equations without using matrix notation and to solve the system using elimination or substitution, not row operations. With the introduction of the parametric vector form for writing solution sets of linear systems with infinitely many solutions, there is no reason not to apply matrix techniques to the task of finding eigenvectors. Can you use row reduction to find the eigenvectors of

$$M = \begin{bmatrix} -17 & 49 \\ -21 & 53 \end{bmatrix}$$

given that its eigenvalues are 4 and 32? Answer on page 128.

Key Concepts

determinant of an inverse if M is invertible, $\det M^{-1} = \frac{1}{\det M}$.

determinant of a product if A and B are $n \times n$ matrices, then $\det(AB) = \det A \cdot \det B$.

inverses by row reduction if A is invertible, then $\begin{bmatrix} A & I \end{bmatrix}$ row reduces to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.

eigenvectors by row reduction if λ is an eigenvalue of M, then corresponding eigenvectors can be found by row reducing $M - \lambda I$.

Exercises

1. Find the inverse by row reduction.

   (a) $\begin{bmatrix} -3 & 10 \\ -15 & -10 \end{bmatrix}$

   (b) $\begin{bmatrix} -12 & 12 \\ 14 & 6 \end{bmatrix}$

   (c) $\begin{bmatrix} 9 & -7 \\ 4 & 7 \end{bmatrix}$

   (d) $\begin{bmatrix} 16 & -3 & -2 \\ -8 & 4 & -2 \\ -8 & 1 & 2 \end{bmatrix}$

   (e) $\begin{bmatrix} -10 & 12 & -50 \\ 20 & -18 & 80 \\ -30 & 18 & -80 \end{bmatrix}$

   (f) $\begin{bmatrix} -11 & -15 & 4 \\ 8 & 9 & -4 \\ -3 & -3 & 2 \end{bmatrix}$

   (g) $\begin{bmatrix} 3 & 90 & -308 & -6 \\ -3 & -140 & 484 & 10 \\ 6 & 210 & -737 & -16 \\ 3 & 70 & -231 & -4 \end{bmatrix}$

2. If $\det M = 2$ and $\det R = \frac{1}{3}$ and M and R are the same size, find

   (a) $\det(MR^T)$

   (b) $\det(M^{-1}R)$

   (c) $\det(MR^{-1})^T$

3. Suppose L, A, M, B are square matrices such that $\det(LA) = 6$, $\det(AM) = 24$, and $\det MB = 48$. Find

   (a) $\det(LA^T)$

   (b) $\det(LM^{-1})$

   (c) $\det(LAMB)$

   (d) $\det(LB)$

4. For a square matrix M, explain why the determinant of $M^T M$ must be nonnegative.

5. Suppose M is invertible. Explain why PMP^{-1} is invertible for any (invertible) matrix P.

6. Support the claim that the product of invertible matrices is invertible.

7. Explain why $\det(PMP^{-1}) = \det M$ for any matrices M and P, assuming both sides of the equation are defined.

8. Suppose
$$AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$
Is A necessarily invertible? What if A is square?

9. If λ is an eigenvalue of M, what can you say about the pivot positions of $M - \lambda I$?

10. Suppose $M - cI$ has linearly independent columns. Can c be an eigenvalue of M? Explain.

11. Use row reduction to find the eigenvectors corresponding to the given eigenvalue. Write your answer in parametric vector form.

(a) $A = \begin{bmatrix} 3 & -10 \\ 8 & -15 \end{bmatrix}; \lambda = -5$

(b) $A = \begin{bmatrix} -4 & 2 \\ -16 & 8 \end{bmatrix}; \lambda = 0$

(c) $A = \begin{bmatrix} 2 & 4 \\ -3 & -4 \end{bmatrix}; \lambda = -1 - i\sqrt{3}$

(d) $A = \begin{bmatrix} 9 & 1 & -5 \\ 33 & 17 & -25 \\ 36 & 12 & -24 \end{bmatrix}; \lambda = 6$

(e) $A = \begin{bmatrix} 14 & 9 & 18 \\ 12 & 17 & -18 \\ 12 & -9 & 8 \end{bmatrix}; \lambda = 26$

(f) $A = \begin{bmatrix} -5 & 6 & -12 \\ 7 & -8 & 16 \\ 5 & -6 & 12 \end{bmatrix}; \lambda = -2$

(g) $A = \begin{bmatrix} 45 & -51 & -24 & -60 \\ 15 & 107 & 18 & 0 \\ 15 & 17 & 98 & 20 \\ -30 & -34 & -16 & 50 \end{bmatrix}; \lambda = 90$

(h) $A = \begin{bmatrix} -10 & -2 & -2 & -6 \\ -26 & -79 & -1 & -93 \\ -116 & -106 & -4 & -138 \\ 26 & 61 & 1 & 75 \end{bmatrix}; \lambda = -18$

Answers

eigenvectors if λ is an eigenvalue of M, then the associated eigenvector, $\mathbf{v}$, satisfies $(M - \lambda I)\mathbf{v} = \mathbf{0}$. For
$$M = \begin{bmatrix} -17 & 49 \\ -21 & 53 \end{bmatrix}$$
and $\lambda = 4$, that means the unknown eigenvector satisfies
$$\begin{bmatrix} -21 & 49 \\ -21 & 49 \end{bmatrix} \mathbf{v} = \mathbf{0}.$$
This system can be solved by reducing the augmented matrix
$$\begin{bmatrix} -21 & 49 & 0 \\ -21 & 49 & 0 \end{bmatrix}.$$
Subtracting row 1 from row 2 yields
$$\begin{bmatrix} -21 & 49 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$
v_2 is a free variable and $v_1 = \frac{-49}{-21} v_2 = \frac{7}{3} v_2$. In parametric vector form,
$$\mathbf{v} = r \begin{bmatrix} \frac{7}{3} \\ 1 \end{bmatrix}$$

or equivalently

$$\mathbf{v} = r\begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

for any r. Speeding up the process for the eigenvalue $\lambda = 32$, we need to reduce the augmented matrix

$$\begin{bmatrix} -49 & 49 & 0 \\ -21 & 21 & 0 \end{bmatrix}.$$

Again the second row disappears with one row operation, leaving

$$\begin{bmatrix} -49 & 49 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

from which we deduce $v_1 = v_2$. The solution is therefore

$$\mathbf{v} = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for any r.

Part II

Matrix Abstraction

Chapter 4

Vector Spaces and Inner Product Spaces

Abstraction is at the heart of most of mathematics. It is an essential vehicle for the development of new ideas. Take natural numbers (one, two, three, and so on), for example. These numbers have a "natural" meaning—quantity. If you have a number of objects before you, then there are one or two or maybe twenty-two. The objects are there. They can be counted. But what does it mean to "count" the number of objects before you when there are none? The very idea is an abstraction of the notion of counting, and it leads to the number zero. To the natural numbers, we add this number we call zero, and say that it represents the number of objects you have when you have none. You cannot explain zero in the same concrete way you can explain one or two or three. It requires an intellectual leap in one's understanding of counting.

The use of variables requires another intellectual leap. It is one thing to say that $4 + 5 = 5 + 4$, but it is quite another to say that $x + y = y + x$. The unspecified quantities (variables x and y) are abstractions of numbers. Much like zero, they represent something that is not readily available to see or put in other concrete terms. The very idea of using unspecified quantities gives rise to an entire branch of mathematics—algebra! Many branches of mathematics revolve around similar abstractions. A body of objects or ideas is stripped down to its essence, which then gives rise to similar but new objects or ideas.

Thinking of the numbers one, two, three, and so on as counting numbers allows the abstraction of "counting" zero objects. Using symbols other than numbers to stand for numbers allows the abstraction of unspecified quantities in an expression. Abstraction provides a new perspective from which new mathematics can bloom. This idea is the foundation for many branches of mathematics. Abstract algebra is built upon abstraction of binary operators such as addition and multiplication, both of which are associative, admit an identity element and inverses, and are closed on certain sets of real numbers. Topology is built upon abstraction of open intervals of the real line, arbitrary unions of which are also open, and closed intervals of the real line, finite intersections of which are also closed. Non-Euclidean geometry is built upon abstraction of the idea of a line. Analysis is built upon abstraction of finite quantity and size to infinite quantity and infinitesimal size. At some level, each branch of mathematics is based upon abstraction.

This part of the book proceeds in this vein. The essential ingredients of objects and ideas already explored and understood in a concrete way (vectors, matrix multiplication, and dot product to be specific) will be extracted from their concrete settings, opening doors to new mathematics.

4.1 Vector Spaces and Span

Back in section 1.5, the definition of linear combination contained a proviso: "let S be a set of objects on which addition and scalar mutliplication are defined." At the time several examples of such sets were given, but not much was made of other properties the set should have. Then in section 3.3, the definition of linear independence contained an extended proviso: "[l]et S be a set of objects on which addition and scalar mutliplication are defined and which contains an additive identity, called 0." Existence of an additive identity was necessary to write the equation in the definition. Yet again, little was made of additional properties the set S should possess.

Implicit in the assumptions that addition and scalar multiplication are defined is that sums and scalar products of objects in the set S are also in the set S. This property is called closure of the operation. The set $T = \{t, t^2, 1 + t^2\}$ is not closed under addition nor scalar multiplication. Can you verify this? Answer on page 138. The set $S = \{\alpha t + \beta t^2 + \gamma(1 + t^2) : \alpha, \beta, \gamma \in \mathbb{R}\}$ containing all linear combinations of elements of T (with scalars from the set of real numbers) is closed under addition and scalar multiplication. Can you verify this? Answer on page 138. Closure is an essential (and until now unmentioned) property of linear combinations and linear independence, and there are others.

We are so used to the basic properties of real numbers, such as associativity and commutativity, it is easy to take the subtleties of computations for granted. In order for $r(\alpha t + \beta t^2 + \gamma(1 + t^2))$ to equal $(r\alpha)t + (r\beta)t^2 + (r\gamma)(1 + t^2)$, for example, we need to distribute the r and associate scalars. Distribution alone yields $r(\alpha t) + r(\beta t) + r(\gamma(1 + t^2))$. The distributive and associative properties are critical to the claim that S is closed under addition and scalar multiplication. In order for $\left[\alpha_1 t + \beta_1 t^2 + \gamma_1(1 + t^2)\right] + \left[\alpha_2 t + \beta_2 t^2 + \gamma_2(1 + t^2)\right]$ to equal $(\alpha_1 + \alpha_2)t + (\beta_1 + \beta_2)t^2 + (\gamma_1 + \gamma_2)(1 + t^2)$ we need a slightly different distributive property, an associative property, and commutativity of addition! These properties are also essential.

It is time to make explicit all the properties of matrices we have taking advantage of when discussing linear combinations. This exercise will provide an abstract footing from which to explore, revealing similarities between certain sets of objects that could otherwise easily go unnoticed.

A set V on which addition and scalar multiplication are defined is called a **vector space** if for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars s, t

1. $\mathbf{u} + \mathbf{v}$ is in V (V is closed under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (addition is commutative)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (addition is associative)
4. there is an element $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ (an additive identity exists)
5. there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (every element has an additive inverse)
6. $s\mathbf{v}$ is in V (V is closed under scalar multiplication)
7. $1\mathbf{v} = \mathbf{v}$
8. $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ (scalars distribute over elements of V)
9. $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ (elements of V distribute over scalars)
10. $s(t\mathbf{u}) = (st)\mathbf{u}$ (scalar multiplication is associative)

The set of all $n \times 1$ matrices (vectors of size n) with real entries is the model from which this list is derived. We use the symbol $\mathbb{R}$ for the set of all real numbers, and use the symbol $\mathbb{R}^n$ for the set of all ordered lists of n real numbers, their representation as $n \times 1$ matrices with real entries giving meaning to the operations of addition and scalar multiplication. For $V = \mathbb{R}^n$, properties 2,3,4,8,9, and 10 are taken directly from the theorems of section 3.1; property 5 is addressed in exercise 9 of the same section; and the other four follow from properties of real numbers. Property 7 is clear in $\mathbb{R}^n$ since $1 \cdot r = r$ for any real number r. Closure (properties 1 and 6) are also clear in $\mathbb{R}^n$ since sums and products of real numbers are real. Hence, these ten properties describe the essential features of $\mathbb{R}^n$, making $\mathbb{R}^n$ the canonical vector space and allowing abstraction.

Elements of an arbitrary vector space V are called **vectors** (even if they are functions, matrices, sequences, or some other type of object). In this abstract sense, even polynomials are vectors! The set $\mathbb{P}_n(\mathbb{R})$, the set of all polynomials with real coefficients and degree n or less, is a vector space[1]. Proof of this claim is mostly an exercise of *observing* that polynomials have the right properties (based on familiarity with polynomials) rather than *demonstrating* that polynomials have the right properties.

1. The sum of two polynomials of degree n or less is a polynomial of degree n or less. (closure under addition)

2. For any polynomials $p(x)$ and $q(x)$, $p(x) + q(x) = q(x) + p(x)$. (addition is commutative)

3. For any polynomials $p(x)$, $q(x)$, and $r(x)$, $p(x) + (q(x) + r(x)) = (p(x) + q(x)) + r(x)$. (addition is associative)

4. The polynomial $z(x) = 0$ is in $\mathbb{P}_n(\mathbb{R})$ and has the property that $z(x) + q(x) = q(x)$ for any polynomial $q(x)$. (an additive identity exists)

5. Given any polynomial $p(x)$, $p(x) + (-1 \cdot p(x)) = z(x)$ and $-1 \cdot p(x)$ is in $\mathbb{P}_n(\mathbb{R})$. (every element has an additive inverse)

6. For any polynomial $p(x)$ of degree n or less, $rp(x)$ is a polynomial of degree n or less. (closure under scalar multiplication)

7. $1 \cdot p(x) = p(x)$ for any polynomial $p(x)$.

8. $r(p(x) + q(x)) = rp(x) + rq(x)$ for any real number r and polynomials $p(x)$ and $q(x)$. (real numbers distribute over polynomials)

9. $(r + s)p(x) = rp(x) + sp(x)$ for any real numbers r and s and polynomial p. (polynomials distribute over real numbers)

10. $r(sp(x)) = (rs)p(x)$ for any real numbers r and s and polynomial p. (scalar multiplication is associative)

If the properties of polynomials were not so well known, each one would ned to be accompanied by reference to a theorem or axiom for support. Even with great familiarity, though, the same four properties that derived from properties of real numbers (1,4,5,6) require further attention. Closure (properties 1 and 6) cannot be taken for granted. These properties only follow because scalar multiplication and polynomial addition do not increase the degree of a polynomial. Properties 4 and 5 assert that an additive identity is

[1] with the understanding that adding polynomials and multiplying polynomials by real numbers work as in high school algebra.

in the set and for every element of the set, its additive inverse is also in the set. These properies are not automatic for arbitrary subsets of polynomials. It is critical to note that $\mathbb{P}_n(\mathbb{R})$ contains them.

Coincidentally, given a subset H of any vector space V, the same four properties are the only ones that need to be shown true (in H) to prove that H is itself a vector space. The other six properties are inherited by H through the fact that they hold for *all* elements of V (including those in H). In fact property 5 can be deduced once properties 1 and 6 have been established.

Suppose H is a subset of a vector space V, properties 1 and 6 hold for H, and $\mathbf{h}$ is a particular but arbitrary element of H. Because H is closed under scalar multiplication (property 6), $-1 \cdot \mathbf{h}$ is in H. Because H is closed under addition (property 1), $\mathbf{h} + (-1 \cdot \mathbf{h})$ is in H. But $\mathbf{h} + (-1 \cdot \mathbf{h}) = (1 + (-1))\mathbf{h} = 0\mathbf{h}$ by property 9. If it were true that $0\mathbf{h} = \mathbf{0}$ (like it is in $\mathbb{R}^n$), we would be done as $-1 \cdot \mathbf{h}$ would then have been shown to be an additive inverse of $\mathbf{h}$ in H. Can you prove that $0\mathbf{h} = \mathbf{0}$ using only the ten properties of a vector space? Answer on page 138. Hence a subset of a vector space is a vector space itself if it satisfies properties 1, 4, and 6 (closure and containment of the zero vector).

Any subset of a vector space that is itself a vector space is called a **subspace**. One common way to define a subspace is through the collection of all linear combinations of a set of vectors (reminiscent of how $T = \{t, t^2, 1 + t^2\}$ is not closed under addition or multiplication but the collection of all linear combinations of elements of T is). Such a set is called the **span** of those vectors. To be precise, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are elements of a vector space (vectors), the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, denoted $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, is given by

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are scalars}\}.$$

Can you show that given any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of a vector space, $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a subspace? Answer on page 138.

Key Concepts

vector space A set V on which addition and scalar multiplication are defined is called a **vector space** if for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars (real numbers or complex numbers) s, t

1. $\mathbf{u} + \mathbf{v}$ is in V (V is closed under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (addition is commutative)
3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (addition is associative)
4. there is an element $\mathbf{0}$ in V such that $\mathbf{0} + \mathbf{u} = \mathbf{u}$ (an additive identity exists)
5. there is an element $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. (every element has an additive inverse)
6. $s\mathbf{v}$ is in V (V is closed under scalar multiplication)
7. $1\mathbf{v} = \mathbf{v}$
8. $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$ (scalars distribute over elements of V)
9. $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$ (elements of V distribute over scalars)
10. $s(t\mathbf{u}) = (st)\mathbf{u}$ (scalar multiplication is associative)

vector an element of a vector space.

span for vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of a vector space, the span of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$, denoted $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, is the colection of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. That is,

$$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are scalars}\}.$$

subspace a subset of a vector space that is itself a vector space.

subspace conditions a subset of a vector space is a subspace if it contains $\mathbf{0}$ and is closed under addition and scalar multiplication.

span as subspace given any vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ of a vector space, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a subspace.

Notation

Some subsets of the complex numbers are denoted as follows.

$\mathbb{Z} = \{z : z \text{ is an integer}\}$
$\mathbb{Z}^+ = \{z : z \text{ is a positive integer}\}$
$\mathbb{Q} = \{q : q \text{ is a rational number}\}$
$\mathbb{R} = \{r : r \text{ is a real number}\}$
$\mathbb{C} = \{c : c \text{ is a complex number}\}$

Some sets, each of which has a natural definition as a vector space, are denoted as follows. $\mathbb{F}$ is taken to be one of[2] $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$, and D is taken to be a subset of $\mathbb{R}$.

$\mathbb{R}^n = \{\mathbf{v} : \mathbf{v} \text{ is an ordered list of } n \text{ real numbers}\}$
$\mathbb{R}^{\mathbb{N}} = \{s : s \text{ is a sequence of real numbers}\}$
$\mathcal{M}_{m\times n}(\mathbb{F}) = \{M : M \text{ is an } m \times n \text{ matrix with entries in the set } \mathbb{F}\}$
$GL_n(\mathbb{F}) = \{M \text{ in } \mathcal{M}_{n\times n}(\mathbb{F}) : M^{-1} \text{ exists in } \mathcal{M}_{n\times n}(\mathbb{F})\}$
$\mathbb{P}_n(\mathbb{F}) = \{p : p \text{ is a polynomial of degree } n \text{ or less with coefficients in } \mathbb{F}\}$
$F(D) = \{f : f \text{ is a function from } D \text{ to } \mathbb{R}\}$
$C(D) = \{f : f \text{ is a continuous function from } D \text{ to } \mathbb{R}\}$
$C^n(D) = \{f : f \text{ is a continuous function from } D \text{ to } \mathbb{R} \text{ with } n \text{ continuous derivatives}\}$

Exercises

1. Verify that various vector spaces are truly vector spaces.

2. Verify that various subsets are subspaces.

3. Show that various subsets are not subspaces.

4. Describe geometrically $\text{Span}\{\mathbf{v}\}$.

5. Describe geometrically $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

6. Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ both have the property of being a zero vector. That is, for any vector $\mathbf{v}$, $\mathbf{u}_1 + \mathbf{v} = \mathbf{v}$ and $\mathbf{u}_2 + \mathbf{v} = \mathbf{v}$. Supply a reason for each line of the string of equalities proving that $\mathbf{u}_1 = \mathbf{v}_1$, thus proving that the zero vector is unique.

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{u}_2 + \mathbf{u}_1\\ &= \mathbf{u}_1 + \mathbf{u}_2\\ &= \mathbf{u}_2\end{aligned}$$

7. Prove that for any vector $\mathbf{v}$ in a vector space, $-1\cdot\mathbf{v}$ is an additive inverse. Use the fact that $0\mathbf{v} = \mathbf{0}$.

8. Suppose $\mathbf{u}_1$ and $\mathbf{u}_2$ both have the property of being an additive inverse of $\mathbf{v}$. That is, for this particular vector $\mathbf{v}$, $\mathbf{u}_1 + \mathbf{v} = \mathbf{0}$ and $\mathbf{u}_2 + \mathbf{v} = \mathbf{0}$. Supply a reason for each line of the string of equalities proving that $\mathbf{u}_1 = \mathbf{v}_1$, thus proving that additive inverses are unique [and therefore $-1\cdot\mathbf{v}$ is *the* additive inverse of $\mathbf{v}$].

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{u}_1 + \mathbf{0}\\ &= \mathbf{u}_1 + (\mathbf{u}_2 + \mathbf{v})\\ &= \mathbf{u}_1 + (\mathbf{v} + \mathbf{u}_2)\\ &= (\mathbf{u}_1 + \mathbf{v}) + \mathbf{u}_2\\ &= \mathbf{0} + \mathbf{u}_2\\ &= \mathbf{u}_2\end{aligned}$$

9. Let c be any scalar. Supply a reason for each line

[2] Z could actually be any field.

of the string of equalities proving that $c\mathbf{0} = \mathbf{0}$.

$$\begin{aligned} c\mathbf{0} &= c\mathbf{0} + \mathbf{0} \\ &= c\mathbf{0} + (c\mathbf{0} + (-(c\mathbf{0}))) \\ &= (c\mathbf{0} + c\mathbf{0}) + (-(c\mathbf{0})) \\ &= c(\mathbf{0} + \mathbf{0}) + (-(c\mathbf{0})) \\ &= c\mathbf{0} + (-(c\mathbf{0})) \\ &= \mathbf{0} \end{aligned}$$

10. Suppose $c\mathbf{u} = \mathbf{0}$ for some nonzero scalar c. Create a string of equalities showing the $\mathbf{u} = \mathbf{0}$ and justify each equality in the string.

Answers

not closed With $T = \{t, t^2, 1 + t^2\}$, $t \in T$ and $t^2 \in T$, but $t + t^2$, the sum of two elements in T, is not itself in T. $17t^2$, a scalar multiple of an element of T, is not itself in T.

closure of linear combinations With $S = \{\alpha t + \beta t^2 + \gamma(1 + t^2) : \alpha, \beta, \gamma \in \mathbb{R}\}$, an arbitrary element of S has the form $\alpha t + \beta t^2 + \gamma(1 + t^2)$ for some scalars α, β, γ. The scalar multiple of an arbitrary element of S, $r(\alpha t + \beta t^2 + \gamma(1 + t^2)) = (r\alpha)t + (r\beta)t^2 + (r\gamma)(1 + t^2)$ has the form of an element of S and is therefore in S. The sum of two elements of S, $\left[\alpha_1 t + \beta_1 t^2 + \gamma_1(1 + t^2)\right] + \left[\alpha_2 t + \beta_2 t^2 + \gamma_2(1 + t^2)\right] = (\alpha_1 + \alpha_2)t + (\beta_1 + \beta_2)t^2 + (\gamma_1 + \gamma_2)(1 + t^2)$ has the form of an element of S and is therefore in S too.

the zero vector Proving that $0\mathbf{v} = \mathbf{0}$ for any vector $\mathbf{v}$ of any vector space V is a very abstract and subtle chore. Where to start is a particularly befuddling question. Proofs of this nature are very difficult when seen for the first time. You are in good company if you did not come up with a proof of your own. One way to proceed is to start with one side and produce a list of equalities that flow logically from the properties and lead to the other side. Let $\mathbf{v}$ be a particular but arbitrary element of a vector space V.

$$\begin{aligned} \mathbf{0} &= 0\mathbf{v} + (-0\mathbf{v}) && \text{[existence of additive inverses]} \\ &= (0 + 0)\mathbf{v} + (-0\mathbf{v}) && \text{[substitution of } 0 + 0 \text{ for } 0] \\ &= (0\mathbf{v} + 0\mathbf{v}) + (-0\mathbf{v}) && \text{[scalars distribute over vectors]} \\ &= 0\mathbf{v} + (0\mathbf{v} + (-0\mathbf{v})) && \text{[addition is associative]} \\ &= 0\mathbf{v} + \mathbf{0} && \text{[property of additive inverses]} \\ &= 0\mathbf{v} && [\mathbf{0} \text{ is the additive identity}] \end{aligned}$$

Note that this proof could be written in precisely the reverse order and it would remain valid.

span is a subspace Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be elements of a vector space and set $H = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$. Then

1. [property 4] $0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_k = \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}$ is in H.
2. [property 1] for any elements $\mathbf{u}$ and $\mathbf{v}$ of H, there are scalars $b_1, b_2, \ldots, b_k$ and $c_1, c_2, \ldots, c_k$ such that

$$\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k \quad \text{and} \quad \mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

so

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k) + (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) \\ &= b_1\mathbf{v}_1 + c_1\mathbf{v}_1 + b_2\mathbf{v}_2 + +c_2\mathbf{v}_2 \cdots + b_k\mathbf{v}_k + c_k\mathbf{v}_k \\ &= (b_1 + c_1)\mathbf{v}_1 + (b_2 + c_2)\mathbf{v}_2 + \cdots + (b_k + c_k)\mathbf{v}_k \end{aligned}$$

is in H.

3. [property 6] for any element $\mathbf{u}$ of H, there are scalars $b_1, b_2, \ldots, b_k$ such that

$$\mathbf{u} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k$$

so given any scalar s,

$$\begin{aligned} s\mathbf{u} &= s\,(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k) \\ &= s(b_1\mathbf{v}_1) + s(b_2\mathbf{v}_2) + \cdots + s(b_k\mathbf{v}_k) \\ &= (sb_1)\mathbf{v}_1 + (sb_2)\mathbf{v}_2 + \cdots + (sb_k)\mathbf{v}_k \end{aligned}$$

is in H.

4.2 Basis and Dimension

Every vector in $\mathbb{R}^n$ can be written as a unique linear combination of the columns of the $n \times n$ identity matrix. Think about it for a moment. Can you justify this claim? Answer on page 143.

Taking the perspective that the matrix-column product $I\mathbf{v}$ is a linear combination of the columns of I with coefficients from $\mathbf{v}$, what we are claiming is that for any vector $\mathbf{b}$, the equation

$$I\mathbf{v} = \mathbf{b} \tag{4.2.1}$$

has exactly one solution $\mathbf{v}$. By the invertible matrix theorem, this is equivalent to claiming that I is invertible, which of course is true!

Noting that the solution of (4.2.1) is $\mathbf{v} = \mathbf{b}$ we see the vector $\mathbf{b}$ itself holds the coefficients of the proclaimed linear combination. There is nothing ground-breaking about the calculation itself. However, it strikes at the essence of elements of $\mathbb{R}^n$, paving the way for abstraction to arbitrary vector spaces.

Given any set of vectors (elements of a vector space), we can now imagine elements of $\mathbb{R}^n$ as holding coefficients of a linear combination of these vectors. This perspective would not be terribly useful, however, if the columns of I were the only special collection of vectors (from which each vector could be written as a unique linear combination). As it turns out, there are many such sets. For example, $T = \{t, t^2, 1 + t^2\}$ forms such a set in $\mathbb{P}_2(\mathbb{R})$. Can you verify this? Answer on page 143. In $\mathbb{R}^n$ the columns of any invertible matrix M form such a set.

If M is invertible, the invertible matrix theorem tells us that for any vector $\mathbf{b}$ there is exactly one solution $\mathbf{v}$ of the equation

$$M\mathbf{v} = \mathbf{b}.$$

In terms of linear combinations, every vector $\mathbf{b}$ can be written as a unique linear combination of the columns of M. In the terminology of the previous section, the fact that $M\mathbf{v} = \mathbf{b}$ always has a solution is to say that the columns of M span $\mathbb{R}^n$. The linear independence of the columns of M provides for uniqueness (by theorem 5). These are the characteristics of the columns of an invertible matrix that makes it a special set—linear independence and span. Conveniently, the terms *linear independence* and *span* have meaning in *any* vector space, not just $\mathbb{R}^n$, motivating the following definitions.

A subset S of a vector space V is called a **spanning set** if $\operatorname{span} S = V$. A subset of a vector space V is called a **basis** (of V) if it is a linearly independent spanning set.

From the discussion that led to the definition of a basis, we can be sure that a basis of $\mathbb{R}^n$ has the special property that every vector in $\mathbb{R}^n$ can be written as a unique linear combination of the basis vectors. However, this property does not automatically transfer to a basis of an arbitrary vector space. We will need to argue that if $\mathcal{B}$ is a basis of a vector space V and $\mathbf{v}$ is in V, then $\mathbf{v}$ is expressible as a unique linear combination of the vectors in $\mathcal{B}$.

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V and let $\mathbf{v}$ be a vector in V. Because $\mathcal{B}$ is a spanning set, every vector in V can be written as a linear combination of the elements of $\mathcal{B}$, including $\mathbf{v}$. Thus $\mathbf{v}$ is expressible as at least one linear combination of the vectors in $\mathcal{B}$. It remains to show $\mathbf{v}$ is expressible as at most one linear combination of the vectors in $\mathcal{B}$. To that end, suppose $\mathbf{v}$ has two representations as linear combinations of the elements of $\mathcal{B}$. That is,

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n \quad \text{and} \quad \mathbf{v} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_n\mathbf{v}_n.$$

Then $\mathbf{0} = \mathbf{v}-\mathbf{v} = (a_1\mathbf{v}_1+a_2\mathbf{v}_2+\cdots+a_n\mathbf{v}_n)-(b_1\mathbf{v}_1+b_2\mathbf{v}_2+\cdots+b_n\mathbf{v}_n) = (a_1-b_1)\mathbf{v}_1+(a_2-b_2)\mathbf{v}_2+\cdots+(a_n-b_n)\mathbf{v}_n$. Since $\mathcal{B}$ is a linearly independent set, the only solution of this equation is

$$a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$$

and therefore the two linear combinations are equal. So, given a basis of a vector space, every element of the vector space can be written as a unique linear combination of the elements of the basis.

Bases (the plural of basis) have another important property. If there is a basis of a vector space with n elements, then any subset of the vector space with more than n elements is linearly dependent.

Suppose $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V and $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ where $p > n$. Since $\mathcal{B}$ is a spanning set, each element of S can be written as a linear combination of the elements of $\mathcal{B}$. Let

$$\begin{aligned}
\mathbf{u}_1 &= M_{1,1}\mathbf{v}_1 + M_{2,1}\mathbf{v}_2 + \cdots + M_{n,1}\mathbf{v}_n \\
\mathbf{u}_2 &= M_{1,2}\mathbf{v}_1 + M_{2,2}\mathbf{v}_2 + \cdots + M_{n,2}\mathbf{v}_n \\
&\vdots \\
\mathbf{u}_p &= M_{1,p}\mathbf{v}_1 + M_{2,p}\mathbf{v}_2 + \cdots + M_{n,p}\mathbf{v}_n
\end{aligned}$$

Then

$$M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,p} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,p} \end{bmatrix}$$

has more columns than rows, so M cannot have a pivot in each column. By theorem 5, there is a nonzero vector $\mathbf{w}$ such that $M\mathbf{w} = \mathbf{0}$. Let $\mathbf{w} = \begin{bmatrix} w_1 & w_2 & \cdots & w_p \end{bmatrix}^T$ be such a vector. Then

$$\begin{aligned}
w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_p\mathbf{u}_p = \;& w_1M_{1,1}\mathbf{v}_1 + w_1M_{2,1}\mathbf{v}_2 + \cdots + w_1M_{n,1}\mathbf{v}_n \\
& + w_2M_{1,2}\mathbf{v}_1 + w_2M_{2,2}\mathbf{v}_2 + \cdots + w_2M_{n,2}\mathbf{v}_n \\
& \vdots \\
& + w_pM_{1,p}\mathbf{v}_1 + w_pM_{2,p}\mathbf{v}_2 + \cdots + w_pM_{n,p}\mathbf{v}_n \\
= \;& (M_{1,1}w_1 + M_{1,2}w_2 + \cdots + M_{1,p}w_p)\mathbf{v}_1 \\
& + (M_{2,1}w_1 + M_{2,2}w_2 + \cdots + M_{2,p}w_p)\mathbf{v}_2 \\
& \vdots \\
& + (M_{n,1}w_1 + M_{n,2}w_2 + \cdots + M_{n,p}w_p)\mathbf{v}_n \\
= \;& 0\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_n \\
= \;& \mathbf{0}.
\end{aligned}$$

Since $\mathbf{w} \neq \mathbf{0}$, we have demonstrated a nontrivial linear combination of the vectors in S that sum to $\mathbf{0}$, showing that S is linearly dependent. This fact is important enough to repeat as a theorem.

Theorem 9. *If a vector space V has a basis with n elements, then any subset of V containing more than n elements is linearly dependent.*

To understand the implications of this theorem, let V be a vector space with basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and consider two subsets of V—F containing fewer than n vectors and G containing greater than n vectors. G cannot be a basis because it has more than n elements (which makes it a linearly dependent set). F cannot be a basis for V because if it were, that would make $\mathcal{B}$ linearly dependent thereby contradicting the fact that $\mathcal{B}$ is a basis. In other words, if a vector space V admits a basis with n elements, all bases of V have n elements. This number n is thus a characteristic of V and therefore deserves a name. We call the number of elements in a basis the **dimension** of a vector space. The trivial vector space, $\{\mathbf{0}\}$ contains only the one vector $\mathbf{0}$ and, by definition, has dimension 0. Observe that

1. the dimension of $\mathbb{R}^n$ is n, and
2. the dimension of $\mathbb{P}_2(\mathbb{R})$ is 3

because

1. the columns of $I_{n\times n}$ form a basis of $\mathbb{R}^n$, and
2. $T = \{t, t^2, 1 + t^2\}$ forms a basis of $\mathbb{P}_2(\mathbb{R})$.

$\{I_{:,1}, I_{:,2}, \ldots, I_{:,n}\}$ is called the **standard basis** of $\mathbb{R}^n$ and $\{1, t, \ldots, t^n\}$ is called the **standard basis** of $\mathbb{P}_n(\mathbb{R})$.

Key Concepts

spanning set S is a spanning set of a vector space V if $\mathrm{span}S = V$.

basis a linearly independent spanning set of a vector space.

dimension the number of vectors in a basis of a vector space.

standard basis of $\mathbb{R}^n$ the columns of $I_{n\times n}$.

standard basis of $\mathbb{P}_n(\mathbb{R})$ $\{1, t, \ldots, t^n\}$.

linear dependence if a vector space V has dimension n, any subset of V with more than n elements is linearly dependent.

spanning if a vector space V has dimension n, any subset of V with less than n elements does not span V.

Exercises

1. Justify the claim.
   (a) $\{1, t, \ldots, t^n\}$ is a basis of $\mathbb{P}_n(\mathbb{R})$.
   (b) If a vector space V has dimension n, any subset of V with less than n elements does not span V.

Answers

unique linear combination For an arbitrary vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ in $\mathbb{R}^n$,

$$\begin{aligned}
\mathbf{b} &= \begin{bmatrix} b_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + b_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\
&= b_1 I_{:,1} + b_2 I_{:,2} + \cdots + b_n I_{:,n}
\end{aligned}$$

and no other linear combination of the columns of I equals $\mathbf{b}$ (because each coefficient of the linear combination affects one and only one entry of $\mathbf{b}$).

unique linear combinations of polynomials An arbitrary element of $\mathbb{P}_2(\mathbb{R})$ takes the form $p(t) = at^2 + bt + c$. To write p as a linear combination of the elements of $T = \{t, t^2, 1 + t^2\}$, we need real numbers α, β, γ such that

$$\alpha t + \beta t^2 + \gamma(1 + t^2) = at^2 + bt + c$$

or $\gamma + \alpha t + (\gamma + \beta)t^2 = at^2 + bt + c$. From here, it is clear that we need $\gamma = c$, $\alpha = b$ and $\gamma + \beta = a$. Solving this last equation for β, we find $\beta = a - \gamma = a - c$. The algebra shows that not only is this a solution to the problem, it is the only one!

4.3 Functions and Transformations

Background

Give yourself a moment to recall everything you can about functions before reading on. Go ahead. Think about it. Close your eyes. Close the book and just think. There are no right or wrong answers. You remember what you remember.

What did you come up with? Common answers include things like "inputs and outputs", "black box", "slope-intercept", "graphing", "zeros", "$f(x)$", "there's a domain", and so on. Whatever came to mind is okay—it was probably related. Now try to divorce yourself from all those ideas. Free your mind from past experiences, and start over. We'll come back to the familiar ideas of function notation and graphing later. For now try to think more generally, more abstractly about functions.

A function is made from three ingredients—three sets, really. Two of the sets may contain any types of objects—real numbers in each, fruits in one and colors in the other, car companies in one and countries in the other, subsets of real numbers in one and polynomials in the other, matrices in one and integers in the other—no restrictions. The definition of function does not specify. These two sets are called the domain and codomain. Any set can be the domain of a function, and any set can be the codomain of a function. The only requirements of a function are placed on the third set. This set must contain exactly one ordered pair for each element of the domain. The order of the elements is important too. The first component of each ordered pair must be an element of the domain and the second component must be an element of the codomain.

A relation, like a function is made from three sets, a domain, a codomain, and a set of ordered pairs where the first component of the orderd pair is an element of the domain and the second component is an element of the codomain. Unlike a function, there are no further requirements. Thus, a relation can be thought of as a **rela**xed func**tion**. It has to adhere to fewer rules. Remember, the set of ordered pairs defining a function must contain exactly one ordered pair for each element of the domain.

To be precise, even though a relation is composed of three sets, A, B, C, where C is any subset of $\{(a, b) : a \text{ is in } A \text{ and } b \text{ is in } B\}$, the set C is the **relation** itself. The sets A and B are just ingredients. To simplify the notation, the set $\{(a, b) : a \text{ is in } A \text{ and } b \text{ is in } B\}$ is denoted by $A \times B$, read "A cross B". Given sets A and B, a relation, then, is any subset of $A \times B$, and a **function** is a subset of $A \times B$ containing exactly one ordered pair for each element of A.

A relation C might be given using the rather compact set notation $C = \{(a, b) \in A \times B : \text{rule of correspondence}\}$ such as in

$$C = \{(a, b) \in \mathbb{R} \times \mathbb{R} : a^2 + b^2 = 1\}.$$

The same relation might be simplified to just $a^2 + b^2 = 1$ if it is understood, without writing explicitly, that a and b are real numbers.

When a relation is a function, we may emphasize this point using the notation $f : A \to B$, read "f is a function from A to B". Implied in this notation is that f is a function and there exists a rule of correspondence specifying exactly which ordered pairs $(a, b) \in A \times B$ are in f. To define a specific function, the familiar function notation is often used, as in $f : A \to B$, $f(a) = \text{"insert formula here"}$, meaning $f = \{(a, b) \in A \times B : b = \text{"insert formula here"}\}$.

Explicit mention of the domain and codomain of a function is often not made. The domain and codomain often fade into the background in favor of focusing on the rule of correspondence, such as in $f(x) = 3x + 11$. It is just assumed or implied that the codomain is the set of all real numbers and the domain is some subset of the real numbers—with good reason. It would be rather repetitive to write $f : \mathbb{R} \to \mathbb{R}$ every time a function on the real numbers comes up. More importantly, though, the domains of many functions with succinct formulas are not all real numbers. It would be a difficult distraction to have to write $f : A \to \mathbb{R}$ and get the set A correct every time a function is mentioned. The domain only includes elements that correspond to some element of the codomain, so to be complete, the definition of a function should include this information. To compensate for the lack of completeness in practice, a classic problem in algebra is to find the *implied* domain of functions such as $f(x) = \frac{3x+7}{2x-5}$—the subset of real number inputs that correspond to real number outputs according to the formula.

The discussion of inverse functions is often obfuscated by not introducing the notion of a relation. if C is a relation with domain A and codomain B, the relation with domain B, codomain A given by $C^{-1} = \{(b, a) : (a, b) \text{ is in } C\}$ is called the **inverse relation** of C. Every relation has an inverse relation. By extension, every function has an inverse *relation*. That is not to say that every function has an inverse function, however. There are plenty that do not.

Maps and Transformations

Map, **mapping**, and **transformation** are all synonyms for **function**. All four words share the same definition. When the codomain is not the set of real numbers, the word "function" is often supplanted by one of the others. Getting used to this fact is a matter of experience with it.

The $3n + 1$ conjecture is that iteration (computing the sequence $n, R(n), R(R(n)), R(R(R(n))), \ldots$) of the *map* $R : \mathbb{Z}^+ \to \mathbb{Z}^+$ defined by

$$R(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

always ends with the cycle $4, 2, 1, 4, 2, 1, \ldots$. Try it starting with $n = 13$, for example. How many iterations does it take to get the first 4? Answer on page 148.

The *transformation* $\mathcal{A} : C(D) \to C^1(D)$ given by $\mathcal{A}(f) = \int_0^x f(t)\,dt$ is a standard of calculus, though it is uncommon to discuss the antiderivative as a transformation during a calculus class. The derivative provides another example of a *transformation* from some set of functions to another set of functions. For example, $D : \mathbb{P}^n \to \mathbb{P}^{n-1}$ defined by $D(p) = p'(x)$ is a *map* from the set of polynomials of degree at most n to the set of polynomials of degree at most $n - 1$. For example, if $p(x) = 3x^2 - 2x + 1$, then $D(p) = p'(x) = 6x - 2$. Mechanically there is no advantage to writing the derivative as a *transformation*, but conceptually it gives a certain perspective on the process of differentiation. The process itself can be thought of as a function!

The *map* $l : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ given by $l(s_0, s_1, s_2, \ldots) = s_1, s_2, s_3, \ldots$ is sometimes called the (left) shift operator. Its cousin, the left bit shift operator (called left-shift) plays a huge role in computing.

The determinant is a function or *map* $\det : \mathcal{M}_{n\times n}(\mathbb{F}) \to \mathbb{F}$ for each n since each $n\times n$ matrix has exactly one determinant. The *transformation* $V : GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ *mapping* an invertible matrix to its inverse provides motivation to think of finding the inverse of a matrix as a function as well. The most common *transformation* considered in linear algebra, however, is the *transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = M\mathbf{x}$ for some matrix M. Transformations T defined this way have two properties:

1. $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for any $\mathbf{x}$ and any $\mathbf{y}$ in $\mathbb{R}^n$.

2. $T(c\mathbf{x}) = cT(\mathbf{x})$ for any c in $\mathbb{R}$ and $\mathbf{x}$ in $\mathbb{R}^n$.

These two properties mean that performing addition and scalar multiplication in the domain and then transforming (the lefthand sides of the equations) gives the same result as transforming first and then performing the addition and scalar multiplication in the codomain (the righthand sides of the equations). We say that this type of transformation preserves the operations of addition and scalar multiplication. Its properties form the essence for the abstraction of this idea to arbitrary vector spaces.

Linear Transformations

Given vector spaces V and W, a **linear transformation** is any transformation $L : V \to W$ such that for every x, y in V and scalar c,

1. $L(x + y) = L(x) + L(y)$ and
2. $L(cx) = cL(x)$.

This definition is modeled after $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{x}) = M\mathbf{x}$, making this particular transformation the canonical example of a linear transformation. Some of the other transformations mentioned previously are linear transformations and some are not. For example, R is not since $R(2 + 3) = 16$ but $R(2) + R(3) = 1 + 10 = 11$, and $11 \neq 16$. It is easy to find positive integers that violate property 1. The derivative is a linear transformation since two basic results of calculus are that $(f + g)' = f' + g'$ and $(cf)' = cf'$. These rules of differentiation say precisely that properties 1 and 2 hold when differentiation is viewed as a mapping. In other words, $D(f + g) = D(f) + D(g)$ and $D(cf) = cD(f)$. What about $\mathcal{A}, l, \det,$and V? Answers on page 148.

Vocabulary of Transformations

If $T : A \to B$ is a transformation, then

- $T(a)$ is called **the image** of a.
- The set of all images, $\{T(a) : a \text{ is in } A\} \subseteq B$, is called **the range** of T, denoted range(T).
- a is called **a preimage** of b whenever $T(a) = b$.
- The set of all preimages of an element b of B is also called **the preimage** of b.
- If B is a vector space, the **kernel** is the preimage of $\mathbf{0}$.
- When the inverse relation of T is a function, it is denoted by $T^{-1} : \text{range}(T) \to A$ and T^{-1} is called the **inverse function** of T or simply **the inverse** of T.

It is not an accident that T^{-1} is called the inverse of T. The notion of an inverse function extends the idea of additive and multiplicative inverses. By definition, if T has an inverse function T^{-1}, then $T^{-1}(T(a)) = a$ for any a in A. Start with a, end with a. Composing T^{-1} with T returns the starting value, much like $M^{-1}(MA) = A$—left multiplying the matrix A by invertible matrix M and then left multiplying the result by the matrix M^{-1} results in the starting value A. Start with A, end with A.

Key Concepts

Cartesian product given any sets A and B, the Cartesian product of A and B is the set $\{(a, b) : a \in A \text{ and } b \in B\}$, denoted $A \times B$.

relation given any sets A and B, a relation is a subset C of $A \times B$.

function given any sets A and B, a function is a relation C such that if $(a, b_1) \in C$ and $(a, b_2) \in C$ then $b_1 = b_2$.

domain the set A in the definitions of relation and function.

codomain the set B in the definitions of relation and function.

rule of correspondence the rule that defines the set C in the definitions of relation and function.

inverse relation (of a relation T with domain A and codomain B) the relation with domain B, codomain A, and rule of correspondence $\{(T(a), a) : a \text{ is in } A\}$.

inverse function the inverse relation of a function whenever it happens to be a function.

inverse an inverse function. If $T^{-1} : \text{range}(T) \to A$ is the inverse of $T : A \to B$, then $T^{-1}(T(a)) = a$ for all a in A.

invertible a function is called invertible if its inverse relation is a function.

map a function, usually used when the codomain is not $\mathbb{R}$.

mapping a function, usually used when the codomain is not $\mathbb{R}$.

transformation a function, usually used when the codomain is not $\mathbb{R}$.

image an element, $T(a)$, of the codomain of a transformation T.

preimage if b is an element of the codomain of a transformation T, then an element a in the domain of T is a preimage of b if $T(a) = b$. Also, the set of all elements a in the domain of T such that $T(a) = b$.

range the set of all images, $\text{range} f = \{f(a) : a \text{ is in } A\}$ for any function $f : A \to B$.

linear transformation given vector spaces V and W, a transformation $L : V \to W$ is linear (is a linear transformation) if for any x, y in V and any scalar c,

1. $L(x + y) = L(x) + L(y)$ and
2. $L(cx) = cL(x)$.

Exercises

1. Verify that if M is an $m \times n$ matrix, then $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{v}) = M\mathbf{v}$ is a linear transformation.

2. Find the image of $\mathbf{a}$ under the mapping $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{v}) = M\mathbf{v}$.

   (a) M = stuff; $\mathbf{a}$ = stuff

3. Perhaps ironically, what is known as a linear function in algebra is not necessarily a linear transformation. Verify that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + b$

   (a) is a linear transformation when $b = 0$

   (b) is not a linear transformation when $b \neq 0$

4. The range of the function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$ is $[0, \infty)$. Therefore, the inverse relation of g is
$$g^{-1} = \{(y, x) \in [0, \infty) \times \mathbb{R} : y = x^2\}.$$
   Verify that g is a function and argue that g^{-1} is not.

5. Justify the claim.

   (a) For any function $C \subseteq A \times B$, if $(a, b_1) \in C$ and $(a, b_2) \in C$ then $b_1 = b_2$.

   (b) If the mapping $L : V \to W$ is linear, then $L(\mathbf{0}) = \mathbf{0}$. Note: the $\mathbf{0}$ on the lefthand side is the zero vector of V and the $\mathbf{0}$ on the righthand side is the zero vector in W. They are not necessarily equal.

   (c) The composition of linear transformations is linear.

   (d) A map $L : V \to W$ is linear if and only if $L(x + cy) = L(x) + cL(y)$ for every x, y in V and scalar c.

Answers

$3n+1$ **problem** $R(13) = 3(13) + 1 = 40$ since 13 is odd. $R(R(13)) = R(40) = \frac{40}{2} = 20$ since 40 is even. $R(R(R(13))) = R(R(40)) = R(20) = 10$, and so on. The sequence $n, R(n), R(R(n)), R(R(R(n))), \ldots$ is

$$13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, \ldots$$

so it takes 7 iterations to get the first 4.

linear or not? Another two results of calculus are that $\int(f+g) = \int f + \int g$ and $\int(cf) = c\int f$, so $\mathcal{A}(f+g) = \mathcal{A}(f) + \mathcal{A}(g)$ and $\mathcal{A}(cf) = c\mathcal{A}(f)$. $\mathcal{A}$ is a linear transformation. On one hand,$l(s+t) = l((s_0, s_1, s_2, \ldots) + (t_0, t_1, t_2, \ldots)) = l(s_0 + t_0, s_1 + t_1, s_2 + t_2, \ldots) = s_1 + t_1, s_2 + t_2, \ldots$. On the other hand, $l(s) + l(t) = l(s_0, s_1, s_2, \ldots) + l(t_0, t_1, t_2, \ldots) = (s_1, s_2, \ldots) + (t_1, t_2, \ldots) = s_1 + t_1, s_2 + t_2, \ldots$. Therefore $l(s+t) = l(s) + l(t)$. Property 1 holds for l. To check that property 2 holds, note that $l(cs) = l(cs_0, cs_1, cs_2, \ldots) = cs_1, cs_2, \ldots$ and $cl(s) = cl(s_0, s_1, s_2, \ldots) = c(s_1, s_2, \ldots) = cs_1, cs_2, \ldots$ too. det is not a linear transformation. For example, let $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $B = -A$. Then $\det A = \det B = 1$, so $\det(A) + \det(B) = 2$ while $\det(A+B) = 0$. Matrix inversion is also not a linear transformation. Using the same matrices A and B, $A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -5 & 3 \\ 3 & -2 \end{bmatrix}$ so the sum of the inverses is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but the inverse of the sum does not exist. In other words, $V(A) + V(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ but $V(A+B)$ does not exist, so they are not equal.

4.4 Linear Transformations on Vectors

Geometric Interpretation of Linear Transformations

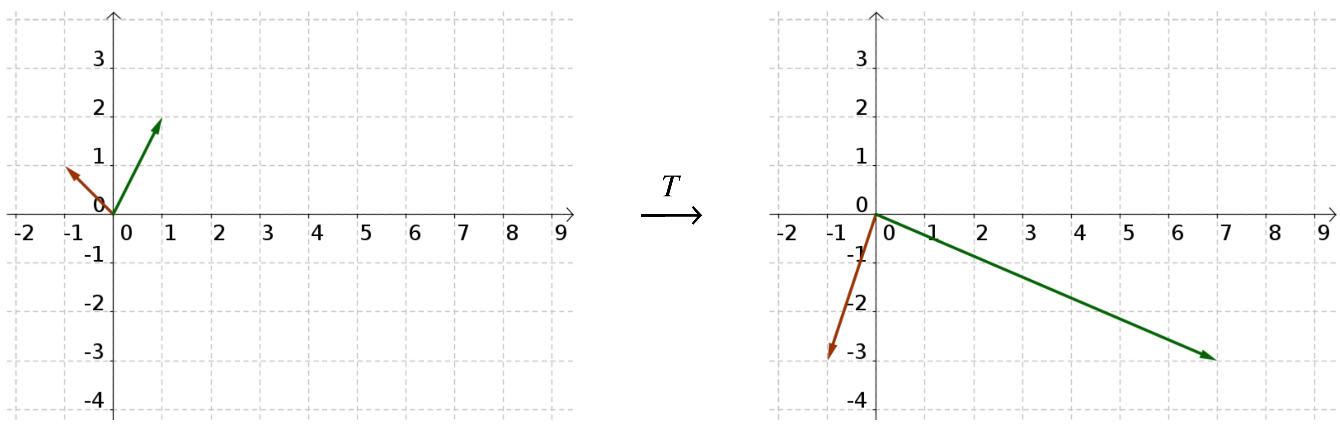
Drawing vectors as arrows, as in section 1.4, gives us a way to picture linear transformations. We can draw any vector and its image to help understand the action of the map geometrically. For example, if every vector we draw has an image pointing in the same direction but twice as long, that gives us a clear picture of how it transforms vectors. It doubles their lengths.

Since vectors do not have an inherent starting location, we can always imagine them starting anywhere. In the case of picturing linear transformations, it is helpful to imagine vectors rooted at the origin. Much like plotting points in the plane, these vectors are marked off starting at $(0, 0)$. Given this special use of the vector, there is little distinction between the point at the head of an arrow and the arrow itself. For this reason, it is just as common to imagine a vector as a point in the plane as it is to imagine it as an arrow.

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$,

$$T(\mathbf{v}) = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix} \mathbf{v}.$$

The image of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is $\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$ and the image of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$. Geometrically these facts are captured by the diagram



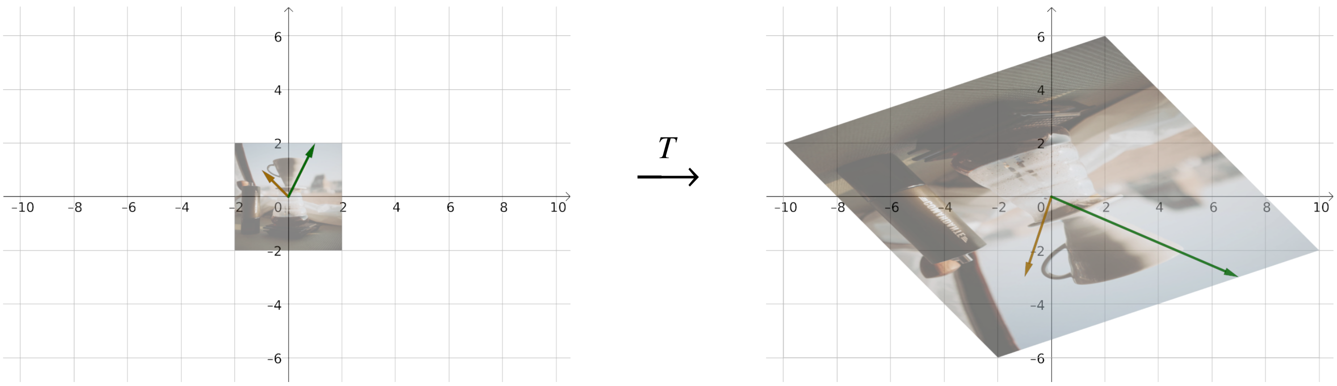
where the vectors have been represented by arrows rooted at the origin and the T indicates that the change is due to the transformation T. The vectors have also been color coded so the image of the brown vector is brown and the image of the green vector is green. From just these two sample vectors, it is hard to describe just what the transformation does in general. This is where it is helpful to interpret vectors as points. If we color a bunch of points, transform each of them, one at a time, giving their images the same color, we get a much clearer picture of the action of the transformation.

Consider again the transformation

$$T(\mathbf{v}) = \begin{bmatrix} 3 & 2 \\ 1 & -2 \end{bmatrix} \mathbf{v}$$

but this time imagine the vectors it acts on and their images as points. Coloring all the points in the square with opposite corners at $(-2, -2)$ and $(2, 2)$ to manifest as a photo of a coffee mug[3] and coloring their images accordingly is summarized in the folowing diagram.

[3] Photo by Dziana Hasanbekava from Pexels



The same brown and green vectors as before are superimposed on the picture to help relate back to this interpretation. The point at the end of the green arrow is sky blue and lands on the boundary of the picture before transforming, so the point at the end of the green arrow is sky blue and lands on the boundary of the picture after transforming, too. A large portion of the green vector runs up the side of the coffee mug both before and after transformation. The point at the end of the brown arrow is sky blue and lands just next to the handle of the coffee mug before transforming, so the point at the end of the brown arrow is sky blue and lands just next to the handle of the coffee mug after transforming, too. In all, the transformed image is larger than the original, rotated, reflected (the handle is on the left of the coffee mug in one picture and on the right in the other), and sheared (the transformed picture covers a parallelogram, not a square). These are the words we use to describe the action of the transformation. It scales, rotates, reflects, and shears the plane, and objects in it. Actually, these are the only invertible actions a linear transformation can take on the plane, as we will see.

The Matrix of a Linear Transformation

Suppose a map $G : \mathbb{R}^n \times \mathbb{R}^m$ is given by $G(\mathbf{v}) = M\mathbf{v}$ for some matrix M. Then, by theorem 2 part 5, $G(\mathbf{u}+\mathbf{v}) = M(\mathbf{u}+\mathbf{v}) = M\mathbf{u}+M\mathbf{v} = G(\mathbf{u})+G(\mathbf{v})$ and by theorem 3 part 4, $G(c\mathbf{u}) = M(c\mathbf{u}) = c(M\mathbf{u}) = cG(\mathbf{u})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ and any scalar c. Therefore G is a linear transformation.

Suppose a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by matrix multiplication by an $m \times n$ matrix A. That is, $T(\mathbf{v}) = A\mathbf{v}$. Then it is easy to calculate its action on the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

the columns of the $n \times n$ identity matrix. Thinking of matrix multiplication of a vector as a linear combination of the columns of the matrix, it is clear that

$$A \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A_{:,1},\ A \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A_{:,2},\ A \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} = A_{:,3}, \ldots, A \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = A_{:,n}.$$

In short, $AI_{:,j} = A_{:,j}$. The j^{th} column of A is the image of the j^{th} column of I.

On the other hand, suppose you know the images of the columns of I but are not given T as a matrix product. In other words, all you know is

$$T(I_{:,1}) = \mathbf{c}_1,\ T(I_{:,2}) = \mathbf{c}_2, \ldots, T(I_{:,n}) = \mathbf{c}_n$$

for some $m \times 1$ vectors $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$. It turns out this is enough information to determine T, and T can be represented by matrix multiplication! Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be an arbitrary $n \times 1$ vector. Due to linearity,

$$\begin{aligned}
T(\mathbf{v}) &= T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{bmatrix}\right) \\
&= T\left(v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}\right) \\
&= T\left(v_1 I_{:,1} + v_2 I_{:,2} + \cdots + v_n I_{:,n}\right) \\
&= T\left(v_1 I_{:,1}\right) + T\left(v_2 I_{:,2}\right) + \cdots + T\left(v_n I_{:,n}\right) \\
&= v_1 T\left(I_{:,1}\right) + v_2 T\left(I_{:,2}\right) + \cdots + v_n T\left(I_{:,n}\right) \\
&= v_1 \mathbf{c}_1 + v_2 \mathbf{c}_2 + \cdots + v_n \mathbf{c}_n \\
&= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} \mathbf{v}.
\end{aligned}$$

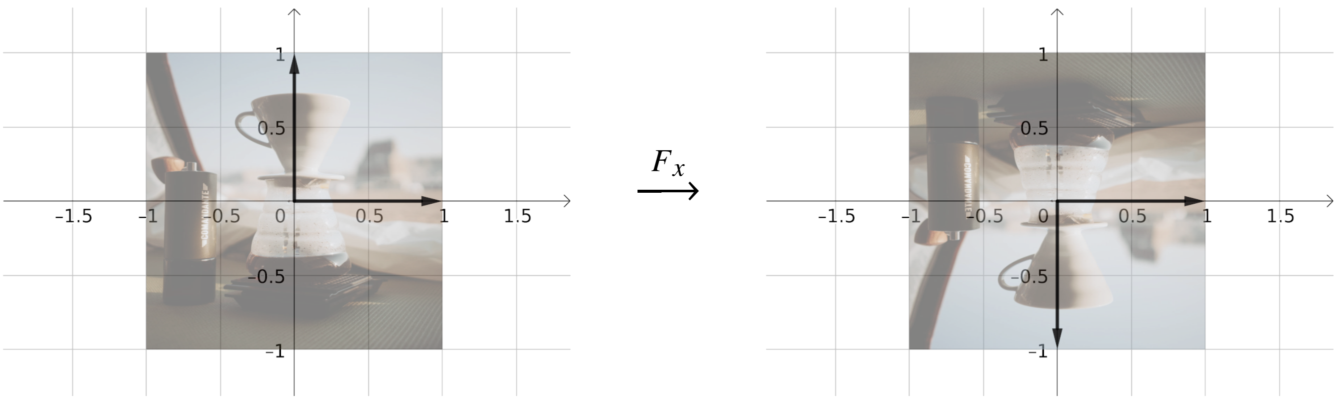
In other words, to represent T as a matrix product, form the matrix with columns equal to the images of the columns of I.

These calculations justify the folowing theorem.

Theorem 10. [The Standard Matrix of a Linear Transformation] *Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, T is linear if and only if $T(\mathbf{v}) = M\mathbf{v}$ where $M_{:j} = T(I_{:,j})$, $j = 1, 2, \ldots, n$. M is called the standard matrix of T.*

In words, a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is linear if and only if it can be represented by matrix multiplication by a matrix whose columns are the images of the columns of the identity matrix.

This observaton can be applied immediately to write down the algebraic (matrix) representation of transformations with which you may already be familiar. For example, consider reflection about the x-axis in the plane, call it F_x. Geometrically, this transformation can be illustrated as in the following diagram.



The image of $I_{:,1}$ is $I_{:,1}$ and the image of $I_{:,2}$ is $-I_{:,2}$, so *if* this is a linear transformation,

$$F_x(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v}.$$

It is easy to verify that F_x acts on the entire plane (not just the columns of I) in the way expected.

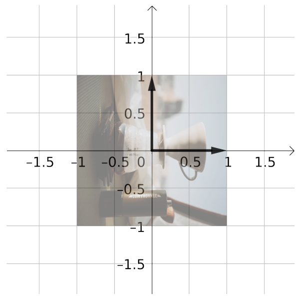
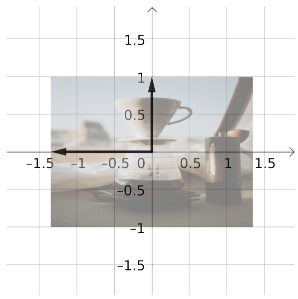
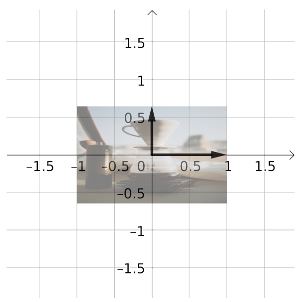
$$F_x(\mathbf{v}) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{v} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

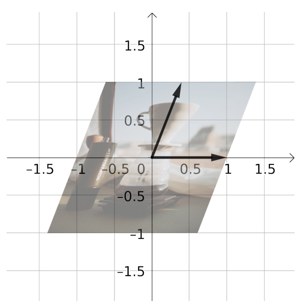
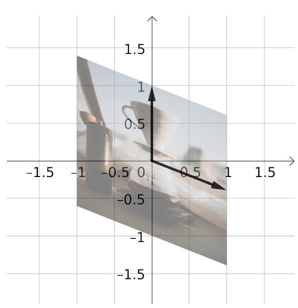
so the image of $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is $\begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$, the reflection of $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ about the x-axis. It must be that reflection about the x-axis is a linear transformation. Can you justify this claim? Answer on page 156.

Notice that $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is an elementary matrix (scale the second row by -1). Every elementary 2×2 matrix takes one of the following five forms—swap, scale first row, scale second row, replace first row, replace second row, respectively—for some scalar r or $s \neq 0$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}.$$

The following series of diagrams illustrates the types of transformations attainable by multiplication by these elementary matrices. Geometrically they are reflection, scaling, scaling with reflection, and shearing.

matrix M	image under $T(\mathbf{v}) = M\mathbf{v}$	Notes
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$		• reflection about the line $y = x$ • $I_{:,1}$ and $I_{:,2}$ swap places
$\begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}$		• horizontal scale by factor s • includes reflection about the y-axis when $s < 0$ • expansion when $\lvert s\rvert > 1$; compression when $\lvert s\rvert < 1$
$\begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}$		• vertical scale by factor s • includes reflection about the x-axis when $s < 0$ • expansion when $\lvert s\rvert > 1$; compression when $\lvert s\rvert < 1$

matrix M	image under $T(\mathbf{v}) = M\mathbf{v}$	Notes
$\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}$		• horizontal shear by factor r • $I_{:,1}$ is uneffected
$\begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$		• vertical shear by factor r • $I_{:,2}$ is uneffected

As we have seen previously, any invertible matrix can be written as a product of elementary matrices. If there were some connection between matrix multiplication and linear transformations, we would be on our way to a comprehensive characterization of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. By theorem 2 part 4 $A(B\mathbf{v}) = (AB)\mathbf{v}$. In terms of linear transformations, the left side, $A(B\mathbf{v})$, represents applying the transformation whose associated matrix is B first and then applying the transformation whose associated matrix is A to the result. In other words, $A(B\mathbf{v})$ represents composing the two transformations whose associated matrices are B and A. The right side, $(AB)\mathbf{v}$, represents applying the transformation whose associated matrix is the product AB. It must be that matrix multiplication corresponds to function composition! To facilitate the following discussion, which puts these words into symbols and expands from there, for any matrix M we adopt the notation T_M for the linear transformation $T_M : \mathbb{R}^n \to \mathbb{R}^m$, $T_M(\mathbf{v}) = M\mathbf{v}$.

Letting T_A and T_B be arbitrary linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$, the following calculation encapsulates the idea that matrix multiplication corresponds to function composition.

$$\begin{aligned}(T_A \circ T_B)(\mathbf{v}) &= T_A(T_B(\mathbf{v}))\\ &= T_A(B\mathbf{v})\\ &= A(B\mathbf{v})\\ &= (AB)\mathbf{v}\\ &= T_{AB}(\mathbf{v}).\end{aligned}$$

This calculation has two important consequences. First, if M is invertible, then $T_M \circ T_{M^{-1}} = T_{M^{-1}} \circ T_M = T_I$. In other words, $(T_M \circ T_{M^{-1}})(\mathbf{v}) = (T_{M^{-1}} \circ T_M)(\mathbf{v}) = T_I(\mathbf{v}) = I\mathbf{v} = \mathbf{v}$, so T_M is invertible and $(T_M)^{-1} = T_{M^{-1}}$. Second, if M is invertible and we write M as the product of elementary matrices, $E_1E_2\cdots E_p$, then

$$\begin{aligned}T_M(\mathbf{v}) &= M\mathbf{v}\\ &= (E_1E_2\cdots E_p)\mathbf{v}\\ &= \left(T_{E_1} \circ T_{E_2} \circ \cdots \circ T_{E_p}\right)(\mathbf{v}),\end{aligned}$$

so T_M is a composition of linear transformations whose associated matrices are elementary matrices.

Finally, because the action of transformations defined by 2×2 elementary matrices include only reflection, scaling, and shearing, these actions and compositions of them are the only actions of invertible linear transformations on $\mathbb{R}^2$. But what about T_M where M is noninvertible? Noninvertible matrices have linearly dependent columns and linearly dependent rows. In the case of 2×2 matrices that means the rows are multiples of one another or one of the rows contains two zeros. Likewise, either its columns are multiples of one another or one of the columns contains two zeros. In any case, it has the form

$$N = \begin{bmatrix} ka & \ell a \\ kb & \ell b \end{bmatrix}$$

for some scalars a, b, k, ℓ. Can you verify this form covers all four cases? Answer on page 156. If $k \neq 0$

$$N = \begin{bmatrix} ka & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ kb & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & \frac{\ell}{k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ka & \ell a \\ kb & \ell b \end{bmatrix}$$

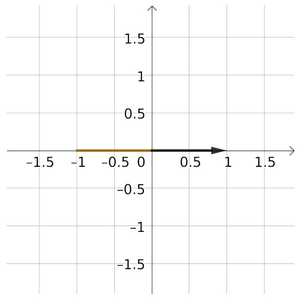
and if $k = 0$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & \ell b \end{bmatrix}\begin{bmatrix} 1 & \ell a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \ell a \\ 0 & \ell b \end{bmatrix}.$$

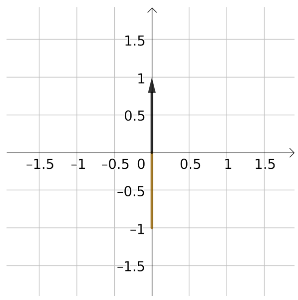
Either way, N can be written as the product of elementary matrices and either

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

These matrices are called projection matrices. Their action is to squash (or project) the entire plane onto the x-axis or the y-axis, respectively, as shown below. The brown line segment indicates the part of the axis in the image that is not covered by the vector.

<table>
<tr><th>matrix M</th><th>image under $T(\mathbf{v}) = M\mathbf{v}$</th><th>Notes</th></tr>
<tr><td>$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$</td><td>
</td><td>

- projection onto the x-axis
- $I_{:,1}$ is uneffected
- $I_{:,2}$ is squashed to the origin

</td></tr>
<tr><td>$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$</td><td>
</td><td>

- projection onto the y-axis
- $I_{:,2}$ is uneffected
- $I_{:,1}$ is squashed to the origin

</td></tr>
</table>

These projections complete the characterization of linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, also called linear operators on $\mathbb{R}^2$. Because every 2×2 matrix can be written as a product of the matrices

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & s \end{bmatrix}, \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and multiplication by these matrices represents reflection, scaling, shearing, and/or projection, we have the following theorem.

Theorem 11. [Characterization of Linear Transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$] *A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear if and only if it is a composition of some sequence of reflections, scalings, shearings, and projections.*

Key Concepts

geometric interpretation of vectors vectors in $\mathbb{R}^n$ are often thought of as points.

matrices and linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if $T(\mathbf{v}) = M\mathbf{v}$ where $M_{:j} = T(I_{:,j}),\ j = 1, 2, \ldots, n$.

elementary matrices as linear transformations of the plane the action of a swap matrix is reflection about the line $y = x$; the action of a scale matrix is scaling and possibly reflection; the action of a replacement matrix is shearing.

noninvertible linear transformations of the plane a linear transformation of the plane, T_M, is noninvertible if and only if M is the product of a projection matrix with elementary matrices.

geometric characterization of linear transformations of the plane see theorem 11.

standard matrix see theorem 10.

Exercises

1. Argue that rotation in the plane is linear and write it algebraically as matrix multiplication by the following steps. Let R_θ be rotation through angle θ counterclockwise about the origin.
   (a) Argue in words that R_θ is a linear transformation. Your argument need not be a proof, just enough reason to believe R_θ is (likely) linear.
   (b) Find the images of $I_{:,1}$ and $I_{:,2}$ under R_θ.
   (c) Write R_θ as a matrix product using the results of 1b.
2. Let S be the transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$, $S\left(\begin{bmatrix} x & y \end{bmatrix}^T\right) = \begin{bmatrix} x^2 & y^2 \end{bmatrix}^T$ and repeat parts (b) and (c) of exercise 1 on S. Use this calculation to explain why S is not linear.
3. Prove that the matrix derived in exercise 1c affects rotation through angle θ about the origin on an arbitrary vector $\begin{bmatrix} x \\ y \end{bmatrix}$. This proves that rotation is properly represented by matrix multiplication, making it a linear transformation by theorem 10.
4. Find a sequence of elementary matrices whose product is the matrix derived in exercise 1c. Hint: try a product of the form
$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & n \end{bmatrix}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$
5. Show that reflection about the origin of the plane, which maps $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} -x \\ -y \end{bmatrix}$, is linear by finding a matrix M such that $M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$.
6. Show that $F_{xy} : \mathbb{R}^2 \to \mathbb{R}^2$, $F_{xy}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}$ (reflection about the origin of the plane) is linear by showing that $F_{xy}(\mathbf{u}+\mathbf{v}) = F_{xy}(\mathbf{u}) + F_{xy}(\mathbf{v})$ and $F_{xy}(c\mathbf{u}) = cF_{xy}(\mathbf{u})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ and scalar c.
7. Find standard matrix for reflection about the vertical y-axis.

Answers

reflection about x-axis Using F_x as the name for reflection about the x-axis, note that $F_x\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = F_x\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ -(u_2+v_2) \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ -u_2-v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} = F_x\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + F_x\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$ and $F_x\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = F_x\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} cu_1 \\ -cu_2 \end{bmatrix} = c\begin{bmatrix} u_1 \\ -u_2 \end{bmatrix} = cF_x\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$. This shows that $F_x(\mathbf{u}+\mathbf{v}) = F_x(\mathbf{u}) + F_x(\mathbf{v})$ and $F_x(c\mathbf{u}) = cF_x(\mathbf{u})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ and scalar c, so F_x is linear by definition.

four cases
- rows are multiples of one another: if a and b are nonzero, then $N_{1,:} = \frac{a}{b}N_{2,:}$ and $\frac{b}{a}N_{1,:} = N_{2,:}$
- one of the rows contains two zeros: $a = 0$ or $b = 0$ while k and ℓ are arbitrary
- columns are multiples of one another: if k and ℓ are nonzero, then $N_{:,1} = \frac{k}{\ell}N_{:,2}$ and $\frac{\ell}{k}N_{:,1} = N_{:,2}$
- one of the columns contains two zeros: $k = 0$ or $\ell = 0$ while a and b are arbitrary

4.5 Isomorphisms

The word *vector* has been used in a number of different ways in this book. In section 1.3 the word vector was said to have the understood meaning from physics or calculus and *represented* using the angled bracket notation $\langle x, y \rangle$. $n \times 1$ matrices were called column vectors, or just vectors, and were said to *represent* vectors despite being different objects. The calculus/physics idea of a vector was brought to life geometrically in section 1.4 when a vector was *represented* by an arrow with both magnitude and direction. In section 4.2 it was noted that vectors in a vector space have unique *representations* as linear combinations of basis vectors. Most recently, vectors (with tails at the origin, in section 4.4) were *represented* by points. In all instances, these were *representations* of vectors, not vectors outright. Only in section 4.1, where the word *vector* was used to refer to any element of a vector space, did we have a definition. To be clear, this is the one and only definition of vector. All other uses will have to be justified from within this umbrella.

By definition, $\mathbb{R}^n$ is the set of all ordered lists of n real numbers. It is not, on the surface, a vector space at all. Elements of $\mathbb{R}^n$ are therefore not inherently vectors! It is only once addition and scalar multiplication are defined (and adhere to the ten properties outlined in section 4.1) that $\mathbb{R}^n$ becomes a vector space. When nothing is said to the contrary, addition and scalar multiplication in $\mathbb{R}^n = \{r_1, r_2, \ldots, r_n : r_i \in \mathbb{R}, i = 1, 2, \ldots, n\}$ are understood to be defined element-wise. That is, for any elements $r_1, r_2, \ldots, r_n \in \mathbb{R}^n$ and $s_1, s_2, \ldots, s_n \in \mathbb{R}^n$

$$r_1, r_2, \ldots, r_n \; + \; s_1, s_2, \ldots, s_n \; = \; r_1 + s_1, r_2 + s_2, \ldots, r_n + s_n$$

and for any element $r_1, r_2, \ldots, r_n \in \mathbb{R}^n$ and scalar $c \in \mathbb{R}$,

$$c \times r_1, r_2, \ldots, r_n \; = \; cr_1, cr_2, \ldots, cr_n.$$

These definitions should remind you of the definitions of matrix addition and scalar multiplication, which are defined entry-wise. For $n \times 1$ matrices, addition and scalar multiplication are defined as follows. For any elements $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$ and $\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} r_1 + s_1 \\ r_2 + s_2 \\ \vdots \\ r_n + s_n \end{bmatrix}$$

and for any element $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$ and scalar $c \in \mathbb{R}$,

$$c \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} cr_1 \\ cr_2 \\ \vdots \\ cr_n \end{bmatrix}.$$

So what is the difference between elements of $\mathbb{R}^n$ and elements of $\mathcal{M}_{n\times 1}(\mathbb{R})$? Functionally there is no difference! There is no way to distinguish elements of $\mathbb{R}^n$ and elements of $\mathcal{M}_{n\times 1}(\mathbb{R})$ based purely on their properties. Each ordered list of real numbers $r_1, r_2, \ldots, r_n$ could just as easily be written as a column matrix

$$\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

and vice versa. The sum of two ordered lists of real numbers could just as easily be written as a sum of two column matrices and vice versa. Each scalar multiple of an ordered list of real numbers could just as easily be written as a scalar multiple of a column matrix and vice versa. When two sets are interchangeable in form and function, we say they are isomorphic.

Formally, two sets are isomorphic if there exists an isomorphism between them. What defines an isomorphism depends on the structure of the sets. A vector space is a set endowed with two operations. The set defines the elements of the vector space and the operations define the structure. An isomorphism between vector spaces maps each element of one vector space to exactly one element of the other and preserves vector addition and scalar multiplication. Such an isomorphism can be understood as the mathematical formalization allowing the free flow between one representation of a vector and another. It supplies the rigor behind using row vectors, column vectors, ordered lists, arrows, points, linear combinations, and vectors in the sense of calculus or physics as if they were all the same thing. For example, the map $T : \mathbb{R}^n \to \mathcal{M}_{n\times 1}(\mathbb{R})$,

$$T(r_1, r_2, \ldots, r_n) = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

is an isomorphism. Can you verify this claim? Answer on page 160. To complete the formalism, the following definitions are introduced. A map $T : A \to B$ is called

1. **onto** if for each $b \in B$ the equation $T(a) = b$ has at least one solution $a \in A$.

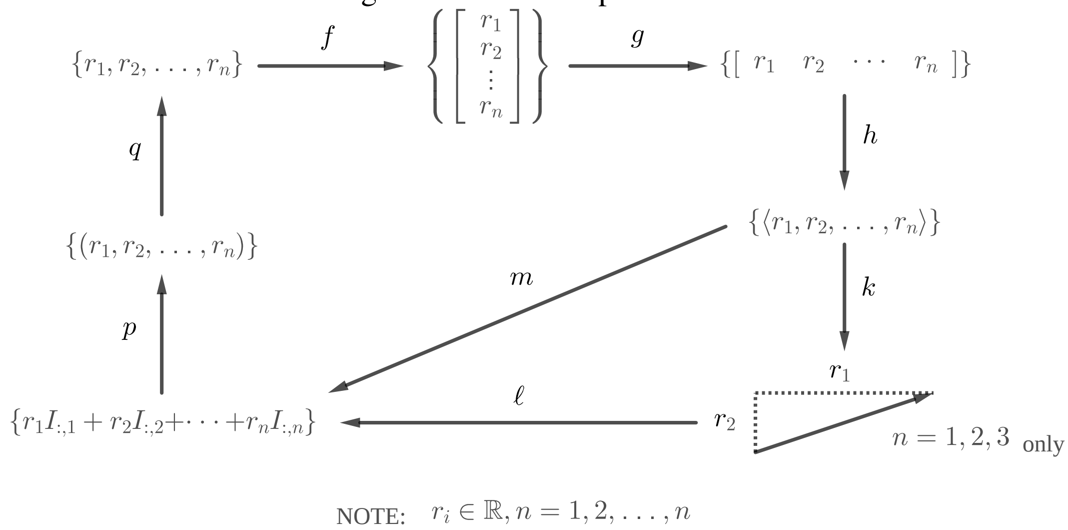
2. **one-to-one** if for each $b \in B$ the equation $T(a) = b$ has at most one solution $a \in A$.

If A and B are vector spaces, then T is an **isomorphism** if it is one-to-one, onto, and linear. Being one-to-one and onto assures "each element of one vector space to exactly one element of the other" and being linear assures it "preserves vector addition and scalar multiplication".

When we use the various representations of elements of $\mathbb{R}^n$ interchangeably, we are relying on the existence of an isomorphism from each one to each other. Much like showing that a list of statements are equivalent by showing a path of implications from any statement to any other, this can be done by showing a path of isomorphisms from any vector space to any other. This is because the composition of isomorphisms is an isomorphism. Can you justify this claim? Answer on page 161. See figure 4.5.1. Once isomorphisms $f, g, h, k, \ell, m, p, q$ between the sets are demonstrated to exist, each vector space is isomorphic to each other by composition. For example, $g : \mathcal{M}_{n\times 1}(\mathbb{R}) \to \mathcal{M}_{1\times n}(\mathbb{R})$ defined by

$$g\left(\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}\right) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

Figure 4.5.1: Isomorphism Path



is an isomorphism. Can you justify this? Answer on page 161.

Maybe more surprising is the claim that all n-dimensional vector spaces over the real numbers (those defined for real number scalars) are isomorphic. In different words, we might say up to isomorphism, there is only one vector space over the real numbers. Different vector spaces may look different and contain different objects, but they all have the same structure and therefore are interchangeable. This claim can be proven by leaning on the fact that an n-dimensional vector space has a basis with n elements.

Let V and W be n-dimensional vector spaces. By definition, each has a basis with n elements. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be bases for V and W, respectively, and define

$$f : V \to \mathbb{R}^n, f(\mathbf{v}) = r_1, r_2, \ldots, r_n \text{ where } \mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$$
$$g : W \to \mathbb{R}^n, g(\mathbf{w}) = s_1, s_2, \ldots, s_n \text{ where } \mathbf{w} = s_1\mathbf{w}_1 + s_2\mathbf{w}_2 + \cdots + s_n\mathbf{w}_n$$

Since the expression of an element of a vector space as a linear combination of basis vectors is unique, f and g are well-defined (they are actually functions, not simply relations). Given an arbitrary element $\mathbf{r} = r_1, r_2, \ldots, r_n$ of $\mathbb{R}^n$, let $\mathbf{v} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$. Since vector spaces are closed under linear combinations, $\mathbf{v}$ is in V. Furthermore, $f(\mathbf{v}) = r_1, r_2, \ldots, r_n$ so f is onto. Now suppose there is a second element $\mathbf{u}$ in V such that $f(\mathbf{u}) = r_1, r_2, \ldots, r_n$. By definition of f it must be that $\mathbf{u} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$ so $\mathbf{u} = \mathbf{v}$ and f is one-to-one. Now let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary elements of V, and write $\mathbf{u} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n$ and $\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n$. Letting c be an arbitrary scalar,

$$\begin{aligned}
f(\mathbf{u} + c\mathbf{v}) &= f(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n + c(s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_n\mathbf{v}_n)) \\
&= f(r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_n\mathbf{v}_n + cs_1\mathbf{v}_1 + cs_2\mathbf{v}_2 + \cdots + cs_n\mathbf{v}_n) \\
&= f((r_1 + cs_1)\mathbf{v}_1 + (r_2 + cs_2)\mathbf{v}_2 + \cdots + (r_n + cs_n)\mathbf{v}_n) \\
&= r_1 + cs_1, r_2 + cs_2, \ldots, r_n + cs_n \\
&= r_1, r_2, \ldots, r_n \;+\; cs_1, cs_2, \ldots, cs_n \\
&= r_1, r_2, \ldots, r_n \;+\; c \;\times\; s_1, s_2, \ldots, s_n \\
&= f(\mathbf{u}) + cf(\mathbf{v})
\end{aligned}$$

so f is linear. Because f is one-to-one, onto, and linear, f is an isomorphism. By similar argument, g is also an isomorphism. By exercise 1e g^{-1} is an isomorphism. Since the composition of isomorphisms is an isomorphism, $g^{-1} \circ f : V \to W$ is an isomorphism.

Key Concepts

onto a map $T : A \to B$ such that for each $b \in B$ the equation $T(a) = b$ has at least one solution $a \in A$.

one-to-one a map $T : A \to B$ such that for each $b \in B$ the equation $T(a) = b$ has at most one solution $a \in A$.

isomorphism a one-to-one, onto, linear transformation between vector spaces.

isomorphic vector spaces between which there exists an isomorphism.

composition of isomorphisms is an isomorphism.

Exercises

1. Justify the claim.

   (a) the statement "$T_M : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one" may be added to the list of equivalent statements of theorem 5.

   (b) the statement "$T_M : \mathbb{R}^n \to \mathbb{R}^m$ is onto" may be added to the list of equivalent statements of theorem 6.

   (c) the statements "$T_M : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one" and "$T_M : \mathbb{R}^n \to \mathbb{R}^m$ is onto" may be added to the list of equivalent statements of theorem 7.

   (d) If $f : V \to W$ is an isomorphism between vector spaces V and W, then f is invertible.

   (e) If $f : V \to W$ is an isomorphism between vector spaces V and W, then f^{-1} is an isomorphism.

Answers

first isomorphism An isomorphism between vector spaces maps each element of one vector space to exactly one element of the other and preserves vector addition and scalar multiplication. The map $T : \mathbb{R}^n \to \mathcal{M}_{n\times 1}(\mathbb{R})$,

$$T(r_1, r_2, \dots, r_n) = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$$

does just that because

1. the arbitrary element $r_1, r_2, \dots, r_n \in \mathbb{R}^n$ maps via T to $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$ and only $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$.

2. the arbitrary element $\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \in \mathcal{M}_{n\times 1}(\mathbb{R})$ maps via T^{-1} to $r_1, r_2, \dots, r_n \in \mathbb{R}^n$ and only $r_1, r_2, \dots, r_n$.

3. $T(r_1, r_2, \dots, r_n + s_1, s_2, \dots, s_n) = T(r_1+s_1, r_2+s_2, \dots, r_n+s_n) = \begin{bmatrix} r_1 + s_1 \\ r_2 + s_2 \\ \vdots \\ r_n + s_n \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix} =$

   $T(r_1, r_2, \dots, r_n) + T(s_1, s_2, \dots, s_n)$.

4. $T(c \times r_1, r_2, \ldots, r_n) = T(cr_1, cr_2, \ldots, cr_n) = \begin{bmatrix} cr_1 \\ cr_2 \\ \vdots \\ cr_n \end{bmatrix} = c \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = cT(r_1, r_2, \ldots, r_n)$.

composition of isomorphisms Let A, B, C be vector spaces and $T : A \to B$ and $S : B \to C$ be isomorphisms. We need to show that $S \circ T : A \to C$ is an isomorphism.

1. Let c be an element of C. Then because S is onto, there is at least one $b \in B$ such that $S(b) = c$. Let b be such a solution. Because T is onto, there is at least one $a \in A$ such that $T(a) = b$. Let a be such a solution. Then $S \circ T(a) = S(T(a)) = S(b) = c$ so $S \circ T(a) = c$ has at least one soution and $S \circ T$ is onto. Generally, this shows that **the composition of onto mappings is an onto mapping**.
2. Suppose $S \circ T(a_1) = c$ and $S \circ T(a_2) = c$. Equivalently $S(T(a_1)) = c$ and $S(T(a_2)) = c$. But S is one-to-one, so the equation $S(b) = c$ has at most one solution. Therefore, $T(a_1) = T(a_2) = b$ for the same $b \in B$. Since T is one-to-one, the equation $T(a) = b$ has at most one solution. Therefore $a_1 = a_2$, which shows that for each $c \in C$, the equation $S \circ T(a_1) = c$ has at most one solution and $S \circ T$ is one-to-one. Generally, this shows that **the composition of one-to-one mappings is a one-to-one mapping**.
3. In exercise 5c of section 4.3 you are asked to show that the composition of linear transformations is linear. This completes the proof.

isomorphism g Let $g : \mathcal{M}_{n\times 1}(\mathbb{R}) \to \mathcal{M}_{1\times n}(\mathbb{R})$ be defined by

$$g\left(\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}\right) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}.$$

Then

1. Given $\begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$ in $\mathcal{M}_{1\times n}(\mathbb{R})$,

$$g\left(\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}\right) = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$$

   so g is onto.

2. Given $r = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$ in $\mathcal{M}_{1\times n}(\mathbb{R})$, suppose $g(u) = r$ and $g(v) = r$. Then

$$u = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} = v$$

   so g is one-to-one.

3. Let $x = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$ and $y = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$ be in $\mathcal{M}_{n\times 1}(\mathbb{R})$ and c be a scalar. Using the result of exercise 5d of section 4.3, the following calculation shows the linearity of L.

$$\begin{aligned}
L(x+cy) &= L\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + c\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}\right) \\
&= L\left(\begin{bmatrix} x_1 + cy_1 \\ x_2 + cy_2 \\ \vdots \\ x_n + cy_n \end{bmatrix}\right) \\
&= \begin{bmatrix} x_1 + cy_1 & x_2 + cy_2 & \cdots & x_n + cy_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} + c\begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \\
&= L(x) + cL(y)
\end{aligned}$$

4.6 Inner Product Spaces

Section 4.5 ended by showing that all n-dimensional vector spaces are isomorphic. That means, for example, $\mathbb{R}^6$, $\mathbb{P}_5(\mathbb{R})$, $\mathcal{M}_{2\times 3}(\mathbb{R})$, and the vector space generated by $G = \{\sin t, \cos t, \sin 2t, \cos 2t, \sin 3t, \cos 3t\}$ are all isomorphic. The essential ingredient of elements of each space is an ordered list of six real numbers. Their placement as coefficients or arrangement in a matrix can be tossed away without losing any information.

Yet it might be nice to draw a distinction between odered lists of real numbers, polynomials, 2×3 matrices, and linear combinations of the elements of G. After all, they are different types of objects. Polynomials and elements of G are functions. A list of real numbers is just a list. A matrix is yet a different type of object.

One important feature of $\mathbb{R}^n$ not included in the definition of a vector space is the dot product. We have seen that the dot product allows us to define the magnitudes and orthogonality of elements of $\mathbb{R}^n$. What if we could extend this idea to any vector space? We would need to write down the essential features of the dot product and hope that, as an abstract list of requirements, other vector spaces would admit similar operators. Representing elements of $\mathbb{R}^n$ by column vectors, the dot product of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is given by $\mathbf{u}^T\mathbf{v}$ (see section 1.3). In other words, if $\mathbf{u} = u_1, u_2, \ldots, u_n$ and $\mathbf{v} = v_1, v_2, \ldots, v_n$, then the dot product of $\mathbf{u}$ and $\mathbf{v}$, which we will begin to denote by $\mathbf{u} \cdot \mathbf{v}$, is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Thus

$$\begin{aligned}\mathbf{u} \cdot \mathbf{u} &= u_1u_1 + u_2u_2 + \cdots + u_nu_n \\ &= u_1^2 + u_2^2 + \cdots + u_n^2\end{aligned}$$

so $\mathbf{u} \cdot \mathbf{u}$ is nonnegative (the squares of real numbers are nonnegative and the sum of nonegative numbers is nonnegative). This seems like an important property of the dot product since it is what led to using the dot product to define magnitude.

In section 1.4 it was shown that $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ implying that the dot product is commutative. Commutativity is a fundamental property of addition and multiplication, so it seems like it is probably an important property of the dot product too. In fact, nonnegativity and commutativity are the only essential features of the dot product other than some sense of linearity, which follows from properties of real numbers. For the dot product of elements of $\mathbb{R}^n$,

$$(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}$$

and

$$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}).$$

Can you show these identities are true? Answer on page 165.

One other subtle point about the dot product that cannot be proven from these four is that

$$\mathbf{u} \cdot \mathbf{u} = 0 \text{ if and only if } \mathbf{u} = \mathbf{0}.$$

Though we did not discuss it before, this fact shows that the dot product can be used to determine when a vector is the zero vector. Can you argue that the equivalence is true? Answer on page 165.

It turns out these are the five features of the dot product necessary for abstraction of the notion of a dot product. Accordingly, we define an **inner product** on a real vector space V to be any operator $\langle , \rangle : V \times V \to \mathbb{R}$ such that

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u}$ in V
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$
3. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v}$ in V
4. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V
5. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v}$ in V and all scalars c

Any real vector space on which an inner product is defined is called an **inner product space**. Extending the ideas of magnitude, distance, and orthogonality is then a simple matter since, in $\mathbb{R}^n$, they are all based on the dot product (of which the inner product is a generalization).

	in $\mathbb{R}^n$	in an n-dimensional inner product space
norm[a]	$\lVert\mathbf{u}\rVert = \sqrt{\mathbf{u} \cdot \mathbf{u}}$	$\lVert\mathbf{u}\rVert = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$
distance[b]	$d(\mathbf{u}, \mathbf{v}) = \lVert\mathbf{u} - \mathbf{v}\rVert$	$d(\mathbf{u}, \mathbf{v}) = \lVert\mathbf{u} - \mathbf{v}\rVert$
orthogonality[c]	$\mathbf{u} \cdot \mathbf{v} = 0$	$\langle \mathbf{u}, \mathbf{v} \rangle = 0$

[a] replacement for the word *magnitude* in an arbitrary inner product space.
[b] see exercise 2a in section 1.4
[c] calculation (1.4.1) proceeds identically if each dot product is replced by an inner product.

One reason for using the words norm and orthogonal instead of magnitude and perpendicular is because magnitude and perpendicular have ingrained geometric meaning that does not readily transer to objects such as matrices and functions (vectors in vector spaces other than $\mathbb{R}^2$ or $\mathbb{R}^3$). What would it mean for two polynomials to be perpendicular, for example? Is there some geometric way to visualize this? Likely you are coming up short on an answer. No matter. If we can come up with an inner product on the vector space of polynomials, then orthogonality of two vectors (polynomials) is defined by the requirement that their inner product be zero.

Can you verify that $\langle , \rangle : \mathbb{P}_2(\mathbb{R}) \times \mathbb{P}_2(\mathbb{R}) \to \mathbb{R}$,

$$\langle p_0 + p_1 x + p_2 x^2, q_0 + q_1 x + q_2 x^2 \rangle = \frac{2}{5} p_2 q_2 + \frac{2}{3} p_0 q_2 + \frac{2}{3} p_1 q_1 + \frac{2}{3} p_2 q_0 + 2 p_0 q_0 \tag{4.6.1}$$

is an inner product? If you have taken calculus, this is equivalent to $\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx$. Answer without using calculus on page 165.

Key Concepts

inner product an operator $\langle , \rangle : V \times V \to \mathbb{R}$ on a real vector space V such that

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u}$ in V
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$
3. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v}$ in V
4. $\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V
5. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v}$ in V and all scalars c

inner product space a vector space endowed with an inner product.

norm extension of the idea of magnitude in $\mathbb{R}^2$ or $\mathbb{R}^3$ to vectors in any inner product space. The norm of a vector $\mathbf{v}$ is denoted by $||\mathbf{v}||$ and is calculated as $||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$.

distance extension of the idea of distance in $\mathbb{R}^2$ or $\mathbb{R}^3$ to vectors in any inner product space. The distance between two vectors $\mathbf{u}$ and $\mathbf{v}$ is denoted by $d(\mathbf{u}, \mathbf{v})$ and is calculated as $d(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$.

orthogonal extension of the idea of perpendicular in $\mathbb{R}^2$ or $\mathbb{R}^3$ to vectors in any inner product space. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Exercises

1. Verify that the operator is an inner product.
2. Find the distance between the vectors.
3. Find the norm of the vector
4. Are the vectors orthogonal?
5. Explore quadratic forms (???), inner products of the form $\mathbf{v}^T A^T A\mathbf{v}$ [$= (A\mathbf{v})^T A\mathbf{v}$].
6. Justify the claim.

   (a) For any vector $\mathbf{v}$ in an inner product space, $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{0} \rangle = 0$.

Answers

dot product identities Let $\mathbf{u} = u_1, u_2, \ldots, u_n$, $\mathbf{v} = v_1, v_2, \ldots, v_n$, $\mathbf{w} = w_1, w_2, \ldots, w_n$ be arbitrary elements of $\mathbb{R}^n$. Then

$$\begin{aligned}
(\mathbf{u} + \mathbf{w}) \cdot \mathbf{v} &= (u_1, u_2, \ldots, u_n \;+\; w_1, w_2, \ldots, w_n) \;\cdot\; v_1, v_2, \ldots, v_n \\
&= u_1 + w_1, u_2 + w_2, \ldots, u_n + w_n \;\cdot\; v_1, v_2, \ldots, v_n \\
&= (u_1 + w_1)v_1 + (u_2 + w_2)v_2 + \cdots + (u_n + w_n)v_n \\
&= u_1v_1 + w_1v_1 + u_2v_2 + w_2v_2 + \cdots + u_nv_n + w_nv_n \\
&= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (w_1v_1 + w_2v_2 + \cdots + w_nv_n) \\
&= \mathbf{u} \cdot \mathbf{v} + \mathbf{w} \cdot \mathbf{v}
\end{aligned}$$

and

$$\begin{aligned}
(c\mathbf{u}) \cdot \mathbf{v} &= (c \;\times\; u_1, u_2, \ldots, u_n) \;\cdot\; v_1, v_2, \ldots, v_n \\
&= (cu_1, cu_2, \ldots, cu_n) \;\cdot\; v_1, v_2, \ldots, v_n \\
&= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n \\
&= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n) \\
&= c(\mathbf{u} \cdot \mathbf{v}).
\end{aligned}$$

dot product zero Suppose $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ for some vector $\mathbf{u} = u_1, u_2, \ldots, u_n$ in $\mathbb{R}^n$. That is, $u_1^2 + u_2^2 + \cdots + u_n^2 = 0$. Since this is a sum of nonnegative real numbers that add to zero, each term must itself be zero: $u_1 = u_2 = \cdots = u_n = 0$. Hence $\mathbf{u} = \mathbf{0}$. Now suppose $\mathbf{u} = \mathbf{0}$. Then $u_1 = u_2 = \cdots = u_n = 0$ and $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2 = 0 + 0 + \cdots + 0 = 0$.

inner product on polynomials Given polynomials $p(x) = p_0 + p_1x + p_2x^2$, $q(x) = q_0 + q_1x + q_2x^2$, and $r(x) = r_0 + r_1x + r_2x^2$ in $\mathbb{P}_2(\mathbb{R})$ and the operator

$$\langle p(x), q(x) \rangle = \frac{2}{5}p_2q_2 + \frac{2}{3}p_0q_2 + \frac{2}{3}p_1q_1 + \frac{2}{3}p_2q_0 + 2p_0q_0,$$

the most challenging part of the verification is the need for some fancy algebra to show properties 1 and 2. The expression $\langle p, p \rangle$ is manipulated into a sum of squares for this purpose.

1. For any p in $\mathbb{P}_2(\mathbb{R})$, $\langle p,p\rangle = \frac{2}{5}p_2^2+\frac{4}{3}p_0p_2+\frac{2}{3}p_1^2+2p_0^2 = \frac{1}{15}\left[6p_2^2+20p_0p_2+10p_1^2+30p_0^2\right] = \frac{1}{15}\left[(2p_2+5p_0)^2+2p_2^2+5p_0^2+10p_1^2\right]$ which is a sum of squares and therefore greater than or equal to 0.
2. $\langle p,p\rangle = \frac{1}{15}\left[(2p_2+5p_0)^2+2p_2^2+5p_0^2+10p_1^2\right]$ equals 0 if and only if $p_0=p_1=p_2=0$ since the square of each appears as a term in the sum. And $p_0=p_1=p_2=0$ if and only if $p(x)=0$ (that is, $p=\mathbf{0}$).
3. For any p,q in $\mathbb{P}_2(\mathbb{R})$,

$$\begin{aligned}
\langle p,q\rangle &= \frac{2}{5}p_2q_2+\frac{2}{3}p_0q_2+\frac{2}{3}p_1q_1+\frac{2}{3}p_2q_0+2p_0q_0\\
&= \frac{2}{5}q_2p_2+\frac{2}{3}q_2p_0+\frac{2}{3}q_1p_1+\frac{2}{3}q_0p_2+2q_0p_0\\
&= \frac{2}{5}q_2p_2+\frac{2}{3}q_0p_2+\frac{2}{3}q_1p_1+\frac{2}{3}q_2p_0+2q_0p_0\\
&= \langle q,p\rangle
\end{aligned}$$

4. For any p,q,r in $\mathbb{P}_2(\mathbb{R})$,

$$\begin{aligned}
\langle p+q,r\rangle &= \left\langle\left(p_0+p_1x+p_2x^2\right)+\left(q_0+q_1x+q_2x^2\right)\ ,\ r_0+r_1x+r_2x^2\right\rangle\\
&= \left\langle(p_0+q_0)+(p_1+q_1)\,x+(p_2+q_2)\,x^2\ ,\ r_0+r_1x+r_2x^2\right\rangle\\
&= \frac{2}{5}\,(p_2+q_2)\,r_2+\frac{2}{3}\,(p_0+q_0)\,r_2+\frac{2}{3}\,(p_1+q_1)\,r_1+\frac{2}{3}\,(p_2+q_2)\,r_0+2\,(p_0+q_0)\,r_0\\
&= \frac{2}{5}p_2r_2+\frac{2}{5}q_2r_2+\frac{2}{3}p_0r_2+\frac{2}{3}q_0r_2+\frac{2}{3}p_1r_1+\frac{2}{3}q_1r_1\\
&\quad+\frac{2}{3}p_2r_0+\frac{2}{3}q_2r_0+2p_0r_0+2q_0r_0\\
&= \frac{2}{5}p_2r_2+\frac{2}{3}p_0r_2+\frac{2}{3}p_1r_1+\frac{2}{3}p_2r_0+2p_0r_0\\
&\quad+\frac{2}{5}q_2r_2+\frac{2}{3}q_0r_2+\frac{2}{3}q_1r_1+\frac{2}{3}q_2r_0+2q_0r_0\\
&= \langle p,r\rangle+\langle q,r\rangle
\end{aligned}$$

5. For any p,q in $\mathbb{P}_2(\mathbb{R})$ and scalar c,

$$\begin{aligned}
\langle cp,q\rangle &= \left\langle c\left(p_0+p_1x+p_2x^2\right)\ ,\ q_0+q_1x+q_2x^2\right\rangle\\
&= \left\langle cp_0+cp_1x+cp_2x^2\ ,\ q_0+q_1x+q_2x^2\right\rangle\\
&= \frac{2}{5}cp_2q_2+\frac{2}{3}cp_0q_2+\frac{2}{3}cp_1q_1+\frac{2}{3}cp_2q_0+2cp_0q_0\\
&= c\left(\frac{2}{5}p_2q_2+\frac{2}{3}p_0q_2+\frac{2}{3}p_1q_1+\frac{2}{3}p_2q_0+2p_0q_0\right)\\
&= c\langle p,q\rangle
\end{aligned}$$

Thus $\langle p,q\rangle = \frac{2}{5}p_2q_2+\frac{2}{3}p_0q_2+\frac{2}{3}p_1q_1+\frac{2}{3}p_2q_0+2p_0q_0$ satisfies the five properties of an inner product (and is therefore an inner product on $\mathbb{P}_2(\mathbb{R})$.

Chapter 5

Exploring Vector Spaces and Inner Product Spaces

5.1 Solution Spaces []

Given a coefficient matrix M and a particular vector $\mathbf{b}$, we can use row reduction to determine whether a solution of $M\mathbf{v} = \mathbf{b}$ exists and find it if it does, but we have no way to tell ahead of time whether a solution will exist. Nor do we have an efficient way of finding all the solutions when there are more than one. We can apply the lessons of sections to address these shortcomings and generally better understand solution sets of linear systems.

Definitions

Let M be an $m \times n$ matrix and define $N = \{\mathbf{v} \in \mathbb{R}^n : M\mathbf{v} = \mathbf{0}\}$ (the solutions set of $M\mathbf{v} = \mathbf{0}$) and $C = \{M\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$ (the set of all linear combinations of the columns of M). Then C and N are vector spaces. PROOF: Since C is the collection of all linear combinations of the columns of M, $C = \text{span}\{M_{:,1}, M_{:,2}, \ldots, M_{:,n}\}$. Being the span of a set of vectors, C is a vector space (see "span is a subspace" on page 138). C is called the **column space** of M. Since N is a subset of $\mathbb{R}^n$, we need check three things: (1) $\mathbf{0} \in N$ since $M\mathbf{0} = \mathbf{0}$. (2) if $\mathbf{u}$ and $\mathbf{v}$ in N and c scalar then $M(\mathbf{u} + c\mathbf{v}) = M\mathbf{u} + M(c\mathbf{v}) = \mathbf{0} + c(M\mathbf{v}) = c\mathbf{0} = \mathbf{0}$ so $\mathbf{u} + c\mathbf{v}$ is in N. N is called the **null space** of M. The dimension of the column space of M is called the **rank** of M. The dimension of the null space of M is called the **nullity** of M. For any eigenvalue λ of M, the null space of $M - \lambda I$ is called the **eigenspace** of M corresponding to λ.

Implications

Row operations were defined to maintain the solution sets of linear systems. Solutions of row reduced linear systems are solutions of the original linear system. Using the matrix form for a linear system, this means given a particular vector $\mathbf{b}$, $M\mathbf{v} = \mathbf{b}$ has the exact same solution set as $(EM)\mathbf{v} = \mathbf{b}$ for any elementary matrix E. In particular this means the null space of M (the solution set of $M\mathbf{v} = \mathbf{0}$) and the null space of EM (the solution set of $(EM)\mathbf{v} = \mathbf{0}$) are equal for any elementary matrix E. **Row operations do not affect the null space of a matrix**. Stated another way, $\mathbf{v}$ is in the null space of M if and only if $\mathbf{v}$ is in the null space of EM. To rigrously prove this claim, let M be an $m \times n$ matrix and E be an $n \times n$ elementary matrix. Then $\mathbf{v}$ in the null space of M implies $M\mathbf{v} = \mathbf{0}$. Hence $(EM)\mathbf{v} = E(M\mathbf{v}) = E\mathbf{0} = \mathbf{0}$, so $\mathbf{v}$ is in null space of EM. [This establishes that if $\mathbf{v}$ is in the null space of M then $\mathbf{v}$ is in the null space of

EM.] Furthermore, $\mathbf{v}$ in the null space of EM implies $(EM)\mathbf{v} = \mathbf{0}$. Hence $E(M\mathbf{v}) = \mathbf{0}$ so $M\mathbf{v} = E^{-1}\mathbf{0} = \mathbf{0}$ and $\mathbf{v}$ is in the null space of M. [This establishes that if $\mathbf{v}$ is in the null space of EM then $\mathbf{v}$ is in the null space of M.] We have now established that the following two statements are equivalent for any $m \times n$ matrix M.

1. $\mathbf{v}$ is in the null space of M.
2. $\mathbf{v}$ is in the null space of EM.

To further this implication, this means a certain set of columns of M are linearly dependent if and only if the same set of columns of EM are linearly dependent. This follows since "a certain set of columns of M are linearly dependent" means there is a linear combination of those columns that sums to $\mathbf{0}$. To be rigorous, if M is an $m \times n$ matrix, E is an $n \times n$ elementary matrix, and $\{c_1, c_2, \ldots, c_k\}$ is a set of indices, then the following are equivalent.

1. $\{M_{:,c_1}, M_{:,c_2}, \ldots, M_{:,c_k}\}$ is linearly dependent.
2. $\{(EM)_{:,c_1}, (EM)_{:,c_2}, \ldots, (EM)_{:,c_k}\}$ is linearly dependent.

Assuming 1 holds, there are scalars $a_1, a_2, \ldots, a_k$, not all zero, such that $a_1 M_{:,c_1} + a_2 M_{:,c_2} + \cdots + a_k M_{:,c_k} = \mathbf{0}$. It follows that

$$\begin{aligned}
a_1(EM)_{:,c_1} + a_2(EM)_{:,c_2} + \cdots + a_k(EM)_{:,c_k} &= a_1\left(EM_{:,c_1}\right) + a_2\left(EM_{:,c_2}\right) + \cdots + a_k\left(EM_{:,c_k}\right) \\
&= E\left(a_1 M_{:,c_1}\right) + E\left(a_2 M_{:,c_2}\right) + \cdots + E\left(a_k M_{:,c_k}\right) \\
&= E\left(a_1 M_{:,c_1} + a_2 M_{:,c_2} + \cdots + a_k M_{:,c_k}\right) \\
&= E\mathbf{0} \\
&= \mathbf{0}
\end{aligned}$$

so $\{(EM)_{:,c_1}, (EM)_{:,c_2}, \ldots, (EM)_{:,c_k}\}$ is linearly dependent. Assuming 2 holds, 1 can be proven in a similar manner. Can you provide the details? Answer on page 172. Equivalently, a certain set of columns of M is linearly independent if and only if the same set of columns of EM are linearly independent. **Row operations do not affect the linear dependence relationships among the columns of a matrix**.

Crumpet 23: Uniqueness of Reduced Row Echelon Form

The fact that row operations do not affect the linear dependence relationships among the columns of a matrix lies at the heart of a proof that *the reduced row echelon form of a matrix exists and is unique*. The row reduction algorithm provides existence.

Suppose that an $m \times n$ matrix M has two reduced row echelon forms, A and B. The pivot columns of A and B are exactly those columns that are linearly independent of the columns to their left. This follows from the facts that (i) each pivot column of a matrix in reduced row echelon form is linearly independent of the columns on its left (it has a nonzero entry in a row where all the columns to its left have zeros); and (ii) each non-pivot column is linearly dependent on the columns to its left (it can be written as a linear combination of those columns). Because row operations do not alter the linear dependence relations among the columns of a matrix (and A and B are the results of series of row operations on M), the pivot columns of A must be the same as those of B. Since the pivot columns of a reduced row echelon form contain a 1 in the pivot position and zeros elsewhere, the pivot columns of A and B are in fact equal.

Suppose $A_{:,j}$ is a nonpivot column of A. If $A_{:,j}$ is to the left of all the pivot columns of A, it must be a column of zeros. In this case, the same is true of $B_{:,j}$ so $A_{:,j} = B_{:,j}$. Suppose $A_{:,j}$ is to the right of at least one pivot column of A. Then $A_{:,j}$ is a linear combination of the pivot columns to its left, which is to say $A_{:,j}$ is either a column of zeros or it is linear dependent on the pivot columns to its left. Since $B_{:,j}$ has the same linear dependence relation to the pivot columns on its left (so must be zero if $A_{:,j}$ is zero and must be the *same* linear combination of the pivot columns on its left if nonzero) and the pivot columns of A and B are equal, it follows that $B_{:,j} = A_{:,j}$. This completes the proof.

Bases for the null space and column space of a matrix

The very nature of the row reduction algorithm followed by writing solutions of the homogeneous equation $M\mathbf{v} = \mathbf{0}$ in parametric form provides a basis for the null space. Each free variable gives rise to one vector in the parametric vector form (see section 3.7). The collection of all the vectors from this form comprise a basis.

If $v_{f_1}, v_{f_2}, \ldots, v_{f_k}$ are the free variables of the linear system $M\mathbf{v} = \mathbf{0}$, then the row reduction algorithm leads to a solution of the form

$$\mathbf{v} = v_{f_1}\begin{bmatrix} c_{1,1} \\ c_{2,1} \\ \vdots \\ c_{n,1} \end{bmatrix} + v_{f_2}\begin{bmatrix} c_{1,2} \\ c_{2,2} \\ \vdots \\ c_{n,2} \end{bmatrix} + \cdots + v_{f_k}\begin{bmatrix} c_{1,k} \\ c_{2,k} \\ \vdots \\ c_{n,k} \end{bmatrix}.$$

This form constitutes all of the solutions of the homogeneous equation, so the columns of the matrix

$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,k} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,k} \end{bmatrix}$$

span the null space of M. The algorithm also provides that $c_{f_1,1} = 1$ while $c_{f_1,2} = c_{f_1,3} = \cdots = c_{f_1,k} = 0$. To reiterate, $C_{f_1,:}$ (row f_1 of C) has a 1 for its first entry and zeros elsewhere. The entries of $C_{f_2,:}$ are all zero except the second, the entries of row $C_{f_3,:}$ are all zero except the third, and so on. Consequently the columns of C are linearly independent. Hence the columns of C (the vectors of the parametric form of the solution) are a linearly independent spanning set—a basis—for the null space of M.

Suppose $v_{b_1}, v_{b_2}, \ldots, v_{b_\ell}$ are the basic variables for the linear system $M\mathbf{v} = \mathbf{b}$ and $b_1 < b_2 < \cdots < b_\ell$. We will argue that columns $M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}$ form a basis for the column space. To see that these columns are linearly independent, let R be the reduced row echelon form of M. Then $R_{:,b_j}$ (column b_j of R) cannot be written as a linear combination of columns $R_{:,b_1}, R_{:,b_2}, \ldots, R_{:,b_{j-1}}$ (the columns to the left of column b_j corresponding to basic variables) for any $j > 1$. This is enough to show that $R_{:,b_1}, R_{:,b_2}, \ldots, R_{:,b_\ell}$ are linearly independent. Can you provide the details? HINT: Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, $\mathbf{v}_1 \neq \mathbf{0}$ are linearly dependent if and only if there is a $k > 1$ such that $\mathbf{v}_k$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$. Answer on page 172. Because row operations do not affect the linear dependence relationships among the columns of a matrix, we have that $M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}$ are linearly independent.

To see that $M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}$ span the column space of M we will rely on the fact that if $\mathbf{v}$ is in the span of (the arbitrary set of vectors) $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, then $\operatorname{span} V = \operatorname{span}(V \cup \{\mathbf{v}\})$. Can you support this claim? Answer on page 173. To see why this is useful, note that if column f of R corresponds to a free variable, then $R_{:,f}$ is a linear combination of the columns of basic variables to the left of $R_{:,f}$, say $R_{:,b_1}, R_{:,b_2}, \ldots, R_{:,b_j}$ where $b_j < f < b_{j+1}$. This is because R_{b_1}, the leftmost column corresponding to a basic variable, has a 1 in its first entry and zeros elsewhere; R_{b_2} has a 1 in its second entry and zeros elsewhere; R_{b_3} has a 1 in its third entry and zeros elsewhere; and so on. By construction, $R_{:,f}$ cannot have a nonzero entry below row b_j (if it did, it would contain a leading entry and therefore not be the column of a free variable). In symbols,

$$\begin{bmatrix} R_{:,b_1} & R_{:,b_2} & \cdots & R_{:,b_j} & R_{:,f} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & R_{1,f} \\ 0 & 1 & \cdots & 0 & R_{2,f} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & R_{j,f} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Hence, $R_{:,f}$ is a linear combination of $R_{:,b_1}, R_{:,b_2}, \ldots, R_{:,b_\ell}$. Because row operations do not affect the linear dependence relationships among the columns of a matrix, we have that $M_{:,f}$ is a linear combination of $M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}$. In other words, $M_{:,f}$ is in the span of $M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}$. Repeatedly adding the columns of free variables (which are in the span of the columns of basic variables) to the set $\{M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}\}$ leads to the conclusion that

$$\begin{aligned} \operatorname{span}\{M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}\} &= \operatorname{span}\{M_{:,1}, M_{:,2}, \ldots, M_{:,n}\} \\ &= \text{column space of } M. \end{aligned}$$

Hence $\{M_{:,b_1}, M_{:,b_2}, \ldots, M_{:,b_\ell}\}$ is a linearly independent spanning set—a basis—for the column space of M

The preceding dicussion justifies two general statements about any $m \times n$ matrix M, the combination of which leads to one of the most fundamental theorems of linear agebra.

- a basis for the null space of M can be formed using one vector for each free variable of the linear system $M\mathbf{v} = \mathbf{b}$.
- a basis for the column space of M is formed from the columns of M corresponding to the basic variables of the linear system $M\mathbf{v} = \mathbf{b}$.

These statements mean the rank of M equals the number of basic variables of $M\mathbf{v} = \mathbf{b}$, and the nullity of M equals the number of free variables of M. Since $M\mathbf{v} = \mathbf{b}$ has n variables in total, we have the following theorem.

Theorem 12. [*Rank* and Nullity] *If M is an $m \times n$ matrix, then the rank of M and the nullity of M sum to n.*

General Solutions

Let $\mathbf{v}_p$ be a particular solution of $M\mathbf{v} = \mathbf{b}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for the null space of M. Now suppose $\mathbf{v}$ is any solution of $M\mathbf{v} = \mathbf{b}$. Then

$$\begin{aligned} M(\mathbf{v} - \mathbf{v}_p) &= M\mathbf{v} - M\mathbf{v}_p \\ &= \mathbf{b} - \mathbf{b} \\ &= \mathbf{0}. \end{aligned}$$

By definition, $\mathbf{v} - \mathbf{v}_p$ is in the null space of M. Because $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a basis for the null space of M, there are coefficients $a_1, a_2, \ldots, a_k$ such that $\mathbf{v} - \mathbf{v}_p = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$. Hence

$$\mathbf{v} = \mathbf{v}_p + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k.$$

On the other hand, if $\mathbf{v} = \mathbf{v}_p + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ for some coefficients $a_1, a_2, \ldots, a_k$ and particular solution $\mathbf{v}_p$, then

$$\begin{aligned} M\mathbf{v} &= M\left(\mathbf{v}_p + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k\right) \\ &= M\mathbf{v}_p + M\left(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k\right) \\ &= \mathbf{b} + \mathbf{0} \\ &= \mathbf{b}. \end{aligned}$$

To summarize, these comments justify the following theorem.

Theorem 13. [Characterization of Solutions of a Linear System] *For a consistent linear system $M\mathbf{v} = \mathbf{b}$, the solution set is*

$$\mathbf{v} = \mathbf{v}_p + a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

where $\mathbf{v}_p$ is any particular solution of $M\mathbf{v} = \mathbf{b}$ and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a basis for the null space of M.

Key Concepts

column space of an $m \times n$ matrix M is $\text{span}\{M_{:,1}, M_{:,2}, \ldots, M_{:,n}\} = \{M\mathbf{v} : \mathbf{v} \in \mathbb{R}^n\}$.

null space of an $m \times n$ matrix M is $\{\mathbf{v} \in \mathbb{R}^n : M\mathbf{v} = \mathbf{0}\}$, the solution set of $M\mathbf{v} = \mathbf{0}$.

eigenspace of M corresponding to λ is the null space of $M - \lambda I$.

vector spaces the column space of a matrix and the null space of a matrix are vector spaces.

column space basis the columns of M corresponding to basic variables form a basis for the column space of M.

null space basis the vectors in the parametric vector form of the solution of $M\mathbf{v} = \mathbf{0}$ form a basis for the null space of M.

characterization of solutions of a linear system see theorem 13.

row operations (i) do not affect the null space of a matrix, and (ii) do not affect the linear dependence relationships of the columns of a matrix.

Exercises

1. Is $\mathbf{b}$ in the column space of M?
2. Find a basis for the column space of M.
3. Find a basis for the null space of M.
4. M reduces to R.
   (a) Find a basis for the column space of M.
   (b) Find a basis for the null space of M.
   (c) Verify that $\mathbf{b}$ is in the column space of M.
   (d) Find the general solution of $M\mathbf{v} = \mathbf{b}$.
5. Is the solution set of $M\mathbf{v} = \mathbf{b}$, $\mathbf{b} \neq \mathbf{0}$ be a vector space?

6. Columns 2 and 5 of M form a basis for the column space of M. Use this information to help decide whether $M\mathbf{v} = \mathbf{b}$ (give M and $\mathbf{b}$) is consistent.

7. Let $M =$ (fill it in).

   (a) Solve $M\mathbf{v} = \mathbf{b}$ by inspection ($\mathbf{b}$ is just one of the columns of M).

   (b) Use the fact that $\{\ldots\}$ (fill in) is a basis for the null space of M to write down all the solutions of $M\mathbf{v} = \mathbf{b}$ in parametric vector form.

   (c) Write down three distinct solutions of $M\mathbf{v} = \mathbf{b}$ all different from the solution in (a).

8. Use the fact that row operations were used to reduce $A =$ to $B =$ to find a set of columns of A that are linearly independent.

9. Let $A =$ (has a parameter k in it), $B =$, and $\mathbf{v} =$.

   (a) Verify that $B\mathbf{v} = \mathbf{0}$.

   (b) Given that A and B are row equivalent, find k.

10. Eigenspace questions.

11. What is the rank of a matrix of all zeros?

12. Given an $m \times n$ matrix M and invertible $n \times n$ matrix P, show that the rank of MP equals the rank of M by the following argument. Let S_M be the column space of M and S_{MP} be the column space of MP.

    (a) Suppose $\mathbf{s}$ is in S_M, and show that $\mathbf{s}$ is in S_{MP}.

    (b) Suppose $\mathbf{s}$ is in S_{MP}, and show that $\mathbf{s}$ is in S_M.

    (c) Conclude that the rank of M equals the rank of MP.

13. Given an $m \times n$ matrix M and invertible $m \times m$ matrix Q, show that the rank of QM equals the rank of M by the following argument. Let N_M be the null space of M and N_{QM} be the null space of QM.

    (a) Suppose $\mathbf{v}$ is in N_M, and show that $\mathbf{v}$ is in N_{QM}.

    (b) Suppose $\mathbf{v}$ is in N_{QM}, and show that $\mathbf{v}$ is in N_M.

    (c) Conclude that the rank of M equals the rank of QM.

Answers

linearly dependent columns Assuming $\{(EM)_{:,c_1}, (EM)_{:,c_2}, \ldots, (EM)_{:,c_k}\}$ is linearly dependent for the set of indices $\{c_1, c_2, \ldots, c_k\}$., there are scalars $a_1, a_2, \ldots, a_k$, not all zero, such that $a_1(EM)_{:,c_1} + a_2(EM)_{:,c_2} + \cdots + a_k(EM)_{:,c_k} = \mathbf{0}$. It follows that

$$\begin{aligned}
a_1 M_{:,c_1} + a_2 M_{:,c_2} + \cdots + a_k M_{:,c_k} &= (E^{-1}E)\left(a_1 M_{:,c_1} + a_2 M_{:,c_2} + \cdots + a_k M_{:,c_k}\right) \\
&= E^{-1}\left(E\left(a_1 M_{:,c_1} + a_2 M_{:,c_2} + \cdots + a_k M_{:,c_k}\right)\right) \\
&= E^{-1}\left(E\left(a_1 M_{:,c_1}\right) + E\left(a_2 M_{:,c_2}\right) + \cdots + E\left(a_k M_{:,c_k}\right)\right) \\
&= E^{-1}\left(a_1\left(EM_{:,c_1}\right) + a_2\left(EM_{:,c_2}\right) + \cdots + a_k\left(EM_{:,c_k}\right)\right) \\
&= E^{-1}\left(a_1(EM)_{:,c_1} + a_2(EM)_{:,c_2} + \cdots + a_k(EM)_{:,c_k}\right) \\
&= E^{-1}\mathbf{0} \\
&= \mathbf{0}
\end{aligned}$$

so $\{M_{:,c_1}, M_{:,c_2}, \ldots, M_{:,c_k}\}$ is linearly dependent.

linear dependence part 2 Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, $\mathbf{v}_1 \neq \mathbf{0}$ are linearly dependent if and only if there is a $k > 1$ such that $\mathbf{v}_k$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$. Supposing $\mathbf{v}_k$ can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{k-1}$, we have immediately that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are linearly dependent. Now suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, $\mathbf{v}_1 \neq \mathbf{0}$ are linearly dependent. Then there exists a linear combination

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p = \mathbf{0}.$$

It must be that at least one of $a_2, a_3, \ldots, a_p$ is nonzero since $\mathbf{v}_1 \neq \mathbf{0}$. Set $k = \max\{i : a_i \neq 0\}$. Then $k > 1$, $a_k \neq 0$, and $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$, so

$$\mathbf{v}_k = -\frac{a_1}{a_k}\mathbf{v}_1 - \frac{a_2}{a_k}\mathbf{v}_2 - \cdots - \frac{a_{k-1}}{a_k}\mathbf{v}_{k-1}.$$

span Show that if $\mathbf{v}$ is in the span of (the arbitrary set of vectors) $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, then $\operatorname{span} V = \operatorname{span}(V \cup \{\mathbf{v}\})$. First, $\operatorname{span} V \subseteq \operatorname{span}(V \cup \{\mathbf{v}\})$ always since every linear combination of the vectors in V is also a linear combination of vectors in $V \cup \{\mathbf{v}\}$ (with the coefficient of $\mathbf{v}$ equal 0, for example). It remains to show that $\operatorname{span}(V \cup \{\mathbf{v}\}) \subseteq \operatorname{span} V$. In other words, if $\mathbf{w} \in \operatorname{span}(V \cup \{\mathbf{v}\})$ then $\mathbf{w} \in \operatorname{span} V$. To that end, suppose $\mathbf{w} \in \operatorname{span}(V \cup \{\mathbf{v}\})$ and (a) write $\mathbf{w}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p, \mathbf{v}$; and (b) write $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ (which is possible since $\mathbf{v}$ is in the span of V):

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p + a\mathbf{v}\\ \mathbf{v} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p.\end{aligned}$$

By substitution,

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p + a(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_p\mathbf{v}_p)\\ &= (a_1 + ab_1)\mathbf{v}_1 + (a_2 + ab_2)\mathbf{v}_2 + \cdots + (a_p + ab_p)\mathbf{v}_p\end{aligned}$$

and therefore $\mathbf{w} \in \operatorname{span} V$.

5.2 Coordinate Vectors []

In section 4.2 it was noted that given a basis $\mathcal{B}$ for a vector space V, each vector in V has a unique (exactly one) representation as a linear combination of the vectors in $\mathcal{B}$. In other words, there is a one-to-one correspondence between elements of V and linear combinations of vectors in $\mathcal{B}$. Every linear combination of vectors in $\mathcal{B}$ corresponds to exactly one vector in V, and every vector in V corresponds to exactly one linear combination of vectors in $\mathcal{B}$. To provide an analogy, linear combinations of vectors in $\mathcal{B}$ serve as unique identifiers for the vectors of V much the same way social security numbers serve as unique identifiers for citizens of the United States (and other countries with a social security system. The column matrix formed from the coefficients of such a linear combination is called the **coordinate vector** with respect to $\mathcal{B}$.

As pointed out in section 4.2,

$$\mathcal{E} = \{I_{:,1}, I_{:,2}, \ldots, I_{:,n}\}$$

is a basis for the vector space $\mathcal{M}_{n\times 1}(\mathbb{R})$ (the collection of all $n \times 1$ matrices with real coefficients), also known as $\mathbb{R}^n$[1]. In fact, $n \times 1$ matrices as we have been using them all along are coordinate vectors with respect to this basis (called the standard basis of $\mathbb{R}^n$).

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 I_{:,1} + v_2 I_{:,2} + \cdots + v_n I_{:,n}$$

(and this is the only such linear combination) so $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is the coordinate vector of $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with respect to $\mathcal{E} = \{I_{:,1}, I_{:,2}, \ldots, I_{:,n}\}$.

In the context of coordinate vectors, the vectors of a basis must be ordered, and the entries in the coordinate vector must correspond to this ordering. The first entry of the coordinate vector corresponds with the first vector in the basis. The second entry of the coordinate vector corresponds with the second vector in the basis. And so on. This correspondence is required to maintain the uniqueness of representation. Therefore different orderings of the same basis may provide different coordinate vectors for a single given vector. The basis $\mathcal{B} = \{I_{:,2}, I_{:,1}, \ldots, I_{:,n}\}$ is different from the standard basis $\mathcal{E}$ (even though, as sets, $\mathcal{B}$ and $\mathcal{E}$ are equal). Of course, it is still true that

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 I_{:,1} + v_2 I_{:,2} + \cdots + v_n I_{:,n}$$

but this means $\begin{bmatrix} v_2 \\ v_1 \\ \vdots \\ v_n \end{bmatrix}$ is the coordinate vector of $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ with respect to $\mathcal{B} = \{I_{:,2}, I_{:,1}, \ldots, I_{:,n}\}$. To keep the various coordinate vectors straight, the coordinate vector with respect to a basis $\mathcal{B}$ will be subscripted

[1] Technically, $\mathcal{M}_{n\times 1}(\mathbb{R})$ and $\mathbb{R}^n$ are isomorphic (see section 4.5)

with a $\mathcal{B}$ as in

$$\begin{bmatrix} v_2 \\ v_1 \\ \vdots \\ v_n \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Given that there is essentially no difference between a vector and its representation with respect to the standard basis—

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{\mathcal{E}}$$

for example—all vectors in the remainder of this section will be subscripted to help avoid confusion. In general, $[\mathbf{v}]_{\mathcal{B}}$ means the coordinate vector of $\mathbf{v}$ with respect to the basis $\mathcal{B}$. In this vein we have

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} v_2 \\ v_1 \\ \vdots \\ v_n \end{bmatrix}_{\mathcal{B}}.$$

Letting $C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{E}} \right\}$, can you verify that

$$\begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}_{C} ?$$

Answer on page 177.

Notice that

$$\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}_{C} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} + 7\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{E}} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix} \right)_{\mathcal{E}}$$

and

$$\begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix}_{\mathcal{B}} = 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} + 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} - 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix} \right)_{\mathcal{E}}.$$

In general, if $\mathcal{B}$ is a basis of $\mathbb{R}^n$ with its vectors written with respect to the standard basis, and $[\mathcal{B}]_{\mathcal{E}}$ is the matrix whose columns are the vectors of $\mathcal{B}$, respecting order, then

$$[\mathbf{v}]_{\mathcal{E}} = [\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}}. \tag{5.2.1}$$

But there is nothing terribly special about the standard basis beyond the fact that it is the most familiar. If the basis $\mathcal{B}$ is written with respect to the basis C (and $[\mathcal{B}]_{C}$ is the matrix whose columns are these vectors, respecting order), then

$$[\mathbf{v}]_{C} = [\mathcal{B}]_{C}\,[\mathbf{v}]_{\mathcal{B}}. \tag{5.2.2}$$

This can be verified by direct calculation. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ and write the vectors of $\mathcal{B}$ with respect to C:

$$\begin{aligned}
\mathbf{b}_1 &= M_{1,1}\mathbf{c}_1 + M_{2,1}\mathbf{c}_2 + \cdots + M_{n,1}\mathbf{c}_n \\
\mathbf{b}_2 &= M_{1,2}\mathbf{c}_1 + M_{2,2}\mathbf{c}_2 + \cdots + M_{n,2}\mathbf{c}_n \\
&\vdots \\
\mathbf{b}_n &= M_{1,n}\mathbf{c}_1 + M_{2,n}\mathbf{c}_2 + \cdots + M_{n,n}\mathbf{c}_n
\end{aligned} \tag{5.2.3}$$

and $\mathbf{v}$ with respect to $\mathcal{B}$:

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n. \tag{5.2.4}$$

Then

$$[\mathcal{B}]_C = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,n} \end{bmatrix}$$

and

$$\begin{aligned}
[\mathcal{B}]_C\,[\mathbf{v}]_{\mathcal{B}} &= v_1 \begin{bmatrix} M_{1,1} \\ M_{2,1} \\ \vdots \\ M_{n,1} \end{bmatrix} + v_2 \begin{bmatrix} M_{1,2} \\ M_{2,2} \\ \vdots \\ M_{n,2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} M_{1,n} \\ M_{2,n} \\ \vdots \\ M_{n,n} \end{bmatrix} \\
&= \begin{bmatrix} v_1 M_{1,1} + v_2 M_{1,2} + \cdots + v_n M_{1,n} \\ v_1 M_{2,1} + v_2 M_{2,2} + \cdots + v_n M_{2,n} \\ \vdots \\ v_1 M_{n,1} + v_2 M_{n,2} + \cdots + v_n M_{n,n} \end{bmatrix}.
\end{aligned} \tag{5.2.5}$$

On the other hand, direct substitution of (5.2.3) into (5.2.4) yields

$$\begin{aligned}
\mathbf{v} &= v_1\left(M_{1,1}\mathbf{c}_1 + M_{2,1}\mathbf{c}_2 + \cdots + M_{n,1}\mathbf{c}_n\right) \\
&\quad + v_2\left(M_{1,2}\mathbf{c}_1 + M_{2,2}\mathbf{c}_2 + \cdots + M_{n,2}\mathbf{c}_n\right) \\
&\quad \vdots \\
&\quad + v_n\left(M_{1,n}\mathbf{c}_1 + M_{2,n}\mathbf{c}_2 + \cdots + M_{n,n}\mathbf{c}_n\right) \\
&= \left(v_1 M_{1,1} + v_2 M_{1,2} + \cdots + v_n M_{1,n}\right)\mathbf{c}_1 \\
&\quad + \left(v_1 M_{2,1} + v_2 M_{2,2} + \cdots + v_n M_{2,n}\right)\mathbf{c}_2 \\
&\quad \vdots \\
&\quad + \left(v_1 M_{n,1} + v_2 M_{n,2} + \cdots + v_n M_{n,n}\right)\mathbf{c}_n
\end{aligned}$$

which verifies that (5.2.5) is $[\mathbf{v}]_C$. Equation (5.2.2) is one formula for a so-called **change of basis**. It gives a formula for changing the basis with respect to which $\mathbf{v}$ is written from $\mathcal{B}$ to C.

Being a basis of $\mathbb{R}^n$, $\mathcal{B}$ contains n linearly independent vectors with n entries each, so $[\mathcal{B}]_C$ is invertible for any basis $\mathcal{B}$ written with respect to any basis C (how it is written is actually immaterial as far as linear independence is concerned). In particular, if both bases $\mathcal{B}$ and C are written with respect to the standard basis and $\mathbf{v}$ is an arbitrary vector, we have $[\mathbf{v}]_{\mathcal{E}} = [\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}}$ and $[\mathbf{v}]_{\mathcal{E}} = [C]_{\mathcal{E}}\,[\mathbf{v}]_C$, so

$$[\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}} = [C]_{\mathcal{E}}\,[\mathbf{v}]_C$$

and we can conclude

$$[C]_{\mathcal{E}}^{-1}\,[\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{C} \quad \text{and} \quad [\mathbf{v}]_{\mathcal{B}} = [\mathcal{B}]_{\mathcal{E}}^{-1}\,[C]_{\mathcal{E}}\,[\mathbf{v}]_{C}\,.$$

Comparing this to (5.2.2) it must be that $[C]_{\mathcal{E}}^{-1}\,[\mathcal{B}]_{\mathcal{E}} = [\mathcal{B}]_{C}$ or

$$[\mathbf{v}]_{C} = [C]_{\mathcal{E}}^{-1}\,[\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}}\,. \tag{5.2.6}$$

In retrospect, this should not be surprising. Multiplying $[\mathbf{v}]_{\mathcal{B}}$ by $[\mathcal{B}]_{\mathcal{E}}$ gives $[\mathbf{v}]_{\mathcal{E}}$ (see equation (5.2.1)). This same equation tells us that multiplying $[\mathbf{v}]_{\mathcal{E}}$ by $[C]_{\mathcal{E}}^{-1}$ gives $[\mathbf{v}]_{C}$. Diagrammatically

$$[\mathbf{v}]_{\mathcal{B}} \xrightarrow{\text{times } [\mathcal{B}]_{\mathcal{E}}} [\mathbf{v}]_{\mathcal{E}} \xrightarrow{\text{times } [C]_{\mathcal{E}}^{-1}} [\mathbf{v}]_{C}\,,$$

which is just another way to write (5.2.6).

Key Concepts

coordinate vector of $\mathbf{v}$ with respect to basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is the vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}_{\mathcal{B}}$$

where $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n$.

change of basis given bases $\mathcal{B}$ and C of a vector space V and $\mathbf{v} \in V$,

$$[\mathbf{v}]_{C} = [\mathcal{B}]_{C}\,[\mathbf{v}]_{\mathcal{B}} = [C]_{\mathcal{E}}^{-1}\,[\mathcal{B}]_{\mathcal{E}}\,[\mathbf{v}]_{\mathcal{B}}\,.$$

Exercises

1. I

Answers

equivalent coordinate vectors Remember, any vector subscripted with an $\mathcal{E}$ is the same as the vectors we have been writing all along without subscripts, so they should be treated as if the subscript were not there.

$$\begin{bmatrix} 2 \\ 7 \\ -3 \end{bmatrix}_{C} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} + 7\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 2+7-3 \\ 7-3 \\ -3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}_{\mathcal{E}}$$

$$\begin{bmatrix} 4 \\ 6 \\ -3 \end{bmatrix}_{\mathcal{B}} = 4I_{:,2} + 6I_{:,1} - 3I_{:,3} = 4\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{E}} + 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{E}} - 3\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 6 \\ 4 \\ -3 \end{bmatrix}_{\mathcal{E}}.$$

5.3 Orthogonalization []

Determining the linear combination of basis vectors that sums to a given vector is big business in engineering and the sciences. This problem is generally aided by careful choice of basis vectors (see Legendre Polynomials, Fourier Series, and Finite Element Methods, for example). But what makes one set of basis vectors more amenable than another? To get some idea, suppose we have an arbitrary basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of an inner product space V and an arbitrary vector $\mathbf{v}$ in V. The problem stated mathematically is to find coefficients $a_1, a_2, \ldots, a_n$ such that

$$\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n. \tag{5.3.1}$$

If $V = \mathbb{R}^n$, this is the vector form of a linear system. It can be solved by row reduction. If $V \neq \mathbb{R}^n$, it is less obvious how to solve. Using coordinate vectors (see section 5.2) is one way to turn (5.3.1) into a linear system, but in neither case does the generated linear system shed light on simplifying the process. Since V is an inner product space, perhaps the inner product can be used to illuminate instead.

To start, can you show that if $\langle \mathbf{w}, \mathbf{b}_1\rangle = \langle \mathbf{w}, \mathbf{b}_2\rangle = \cdots = \langle \mathbf{w}, \mathbf{b}_n\rangle = 0$ for any vector $\mathbf{w} \in V$, then $\mathbf{w} = \mathbf{0}$? Answer on page 183. Only the zero vector is orthogonal to (has inner product zero with) every vector in a basis.

Taking the inner product with each basis vector on both sides of (5.3.1) produces the following linear system.

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{b}_1\rangle &= \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_1\rangle \\
\langle \mathbf{v}, \mathbf{b}_2\rangle &= \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_2\rangle \\
&\vdots \\
\langle \mathbf{v}, \mathbf{b}_n\rangle &= \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_n\rangle
\end{aligned} \tag{5.3.2}$$

and by inner product space properties 4 and 5,

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{b}_1\rangle &= a_1\langle \mathbf{b}_1, \mathbf{b}_1\rangle + a_2\langle \mathbf{b}_2, \mathbf{b}_1\rangle + \cdots + a_n\langle \mathbf{b}_n, \mathbf{b}_1\rangle \\
\langle \mathbf{v}, \mathbf{b}_2\rangle &= a_1\langle \mathbf{b}_1, \mathbf{b}_2\rangle + a_2\langle \mathbf{b}_2, \mathbf{b}_2\rangle + \cdots + a_n\langle \mathbf{b}_n, \mathbf{b}_2\rangle \\
&\vdots \\
\langle \mathbf{v}, \mathbf{b}_n\rangle &= a_1\langle \mathbf{b}_1, \mathbf{b}_n\rangle + a_2\langle \mathbf{b}_2, \mathbf{b}_n\rangle + \cdots + a_n\langle \mathbf{b}_n, \mathbf{b}_n\rangle
\end{aligned}$$

and in matrix format

$$\begin{bmatrix}
\langle \mathbf{b}_1, \mathbf{b}_1\rangle & \langle \mathbf{b}_2, \mathbf{b}_1\rangle & \cdots & \langle \mathbf{b}_n, \mathbf{b}_1\rangle \\
\langle \mathbf{b}_1, \mathbf{b}_2\rangle & \langle \mathbf{b}_2, \mathbf{b}_2\rangle & \cdots & \langle \mathbf{b}_n, \mathbf{b}_2\rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle \mathbf{b}_1, \mathbf{b}_n\rangle & \langle \mathbf{b}_2, \mathbf{b}_n\rangle & \cdots & \langle \mathbf{b}_n, \mathbf{b}_n\rangle
\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}
=
\begin{bmatrix} \langle \mathbf{v}, \mathbf{b}_1\rangle \\ \langle \mathbf{v}, \mathbf{b}_2\rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{b}_n\rangle \end{bmatrix}. \tag{5.3.3}$$

No matter the vector and no matter the basis, this problem is reduced to solving this linear system of equations, a problem that we have studied extensively!

If that were all there were to it, though, this method would hardly be better than using coordinate vectors. Either way, we can turn (5.3.1) into a linear system. This section began by suggesting that careful choice of basis would help. Since the process of row reduction involves producing zeros above and below the pivots, starting with some zeros in the coefficient matrix of (5.3.3) would be beneficial. For example, if $\langle \mathbf{b}_1, \mathbf{b}_2\rangle$ were zero, it would put zeros in the 1, 2-entry and the 2, 1-entry. More generally, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$ were zero, it would put zeros in the i, j-entry and the j, i-entry. The more orthogonal pairs of basis vectors (zero inner products between basis vectors) the better. Thinking greedily, if $\langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0$ for all pairs $i, j, i \neq j$, the system reads

$$\begin{bmatrix} \langle \mathbf{b}_1, \mathbf{b}_1 \rangle & 0 & \cdots & 0 \\ 0 & \langle \mathbf{b}_2, \mathbf{b}_2 \rangle & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \langle \mathbf{b}_n, \mathbf{b}_n \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle \mathbf{v}, \mathbf{b}_1 \rangle \\ \langle \mathbf{v}, \mathbf{b}_2 \rangle \\ \vdots \\ \langle \mathbf{v}, \mathbf{b}_n \rangle \end{bmatrix} \tag{5.3.4}$$

and has solution $a_1 = \frac{\langle \mathbf{v}, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle}$, $a_2 = \frac{\langle \mathbf{v}, \mathbf{b}_2 \rangle}{\langle \mathbf{b}_2, \mathbf{b}_2 \rangle}, \ldots, a_n = \frac{\langle \mathbf{v}, \mathbf{b}_n \rangle}{\langle \mathbf{b}_n, \mathbf{b}_n \rangle}$. Better yet, if $\langle \mathbf{b}_1, \mathbf{b}_1 \rangle = \langle \mathbf{b}_2, \mathbf{b}_2 \rangle = \cdots = \langle \mathbf{b}_n, \mathbf{b}_n \rangle = 1$, the solution is simply $a_1 = \langle \mathbf{v}, \mathbf{b}_1 \rangle$, $a_2 = \langle \mathbf{v}, \mathbf{b}_2 \rangle, \ldots, a_n = \langle \mathbf{v}, \mathbf{b}_n \rangle$—the coefficients are just the inner products of $\mathbf{v}$ with each of the basis vectors.

The question then turns to (a) establishing bases within which pairs of vectors are orthogonal, and (b) possibly ensuring all their norms (inner products $\langle \mathbf{b}_i, \mathbf{b}_i \rangle$) are one. The prototypical example of such a basis is the standard basis $\mathcal{E} = \{I_{:,1}, I_{:,2}, \ldots, I_{:,n}\}$ with inner product $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$ (the dot product, after which inner products were modeled). Can you verify that the inner product of every pair of vectors in $\mathcal{E}$ is zero and that the norm of each vector in $\mathcal{E}$ is one? Answer on page 183. The standard basis $\mathcal{E} = \{1, t, t^2\}$ of $\mathbb{P}_2(\mathbb{R})$ (see page 142) with inner product (4.6.1) does not have these properties. Can you identify at least one violation? Answer on page 183.

The SageMath output below demonstrates a process for taking any basis of $\mathbb{R}^3$ and modifying it so that all pairs of vectors are orthogonal, a process called orthogonalization.

```
Basis B:
------------
(5, -5, 0)
(7, -1, 9)
(-8, 3, 3)

Orthogonal Basis C:
--------------------
w1 = (5, -5, 0)
w2 = (7, -1, 9) - (4, -4, 0) = (3, 3, 9)
w3 = (-8, 3, 3) - (-11/2, 11/2, 0) - (4/11, 4/11, 12/11) = (-63/22, -63/22, 21/11)
```

Follow this SageCell link to generate random examples. Can you verify that all pairs of the vectors

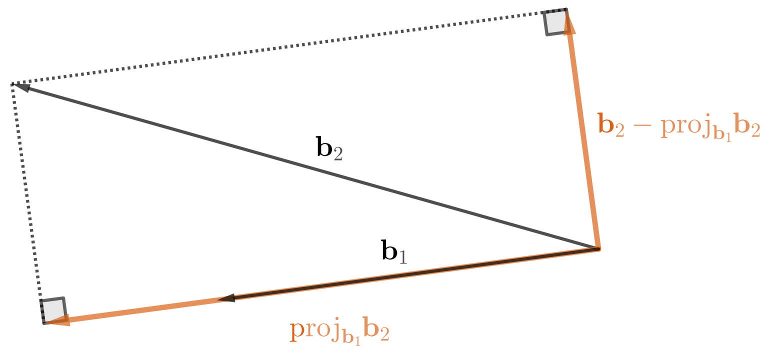
$$\begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} -\frac{63}{22} \\ -\frac{63}{22} \\ \frac{21}{11} \end{bmatrix}$$

from the SageMath snapshot are orthogonal? Answer on page 183. The first vector of the orthogonal basis is taken as the first vector of the original basis. The second vector of the orthogonal basis is taken as the second vector of the original basis minus a particular vector. The third vector of the orthogonal basis is taken as the third vector of the original basis minus two particular vectors. But what vectors ought to be subtracted? A clever observation will answer the question.

Given any nonzero vectors $\mathbf{b}_1$ and $\mathbf{b}_2$,

$$\begin{aligned} \left\langle \mathbf{b}_1, \mathbf{b}_2 - \frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \mathbf{b}_1 \right\rangle &= \langle \mathbf{b}_1, \mathbf{b}_2 \rangle - \left\langle \mathbf{b}_1, \frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \mathbf{b}_1 \right\rangle \\ &= \langle \mathbf{b}_1, \mathbf{b}_2 \rangle - \frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle} \langle \mathbf{b}_1, \mathbf{b}_1 \rangle \\ &= \langle \mathbf{b}_1, \mathbf{b}_2 \rangle - \langle \mathbf{b}_2, \mathbf{b}_1 \rangle \\ &= 0 \end{aligned} \tag{5.3.5}$$

so $\mathbf{b}_1$ and $\mathbf{b}_2 - \frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle}\mathbf{b}_1$ are always orthogonal (even if $\mathbf{b}_1$ and $\mathbf{b}_2$ are not). And this applies in any inner product space, not just $\mathbb{R}^n$. The term $\frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle}\mathbf{b}_1$ is called the component of $\mathbf{b}_2$ in the $\mathbf{b}_1$ direction or the (orthogonal) **projection** of $\mathbf{b}_2$ onto $\mathbf{b}_1$, and is often denoted $\text{proj}_{\mathbf{b}_1}\mathbf{b}_2$. Subtracting this term from $\mathbf{b}_2$ removes the component of $\mathbf{b}_2$ in the $\mathbf{b}_1$ direction, leaving only the component of $\mathbf{b}_2$ orthogonal to $\mathbf{b}_1$ (not in the direction of $\mathbf{b}_1$). In $\mathbb{R}^2$, this is seen geometrically in the following diagram.



A set of three nonzero vectors, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, can be orthogonalized by extending the process to $\mathbf{b}_3$. Its components in the directions of both $\mathbf{b}_1$ and $\mathbf{b}_2 - \frac{\langle \mathbf{b}_2, \mathbf{b}_1 \rangle}{\langle \mathbf{b}_1, \mathbf{b}_1 \rangle}\mathbf{b}_1$ will need to be subtracted. Letting $\mathbf{w}_1 = \mathbf{b}_1$ and $\mathbf{w}_2 = \mathbf{b}_2 - \text{proj}_{\mathbf{w}_1}\mathbf{b}_2$, $\mathbf{w}_3 = \mathbf{b}_3 - \text{proj}_{\mathbf{w}_1}\mathbf{b}_3 - \text{proj}_{\mathbf{w}_2}\mathbf{b}_3$. For larger sets, the process continues recursively.

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{b}_1 \\ \mathbf{w}_j &= \mathbf{b}_j - \text{proj}_{\mathbf{w}_1}\mathbf{b}_j - \text{proj}_{\mathbf{w}_2}\mathbf{b}_j - \cdots - \text{proj}_{\mathbf{w}_{j-1}}\mathbf{b}_j, \quad j = 2, 3, \ldots, n. \end{aligned} \tag{5.3.6}$$

The process as described by (5.3.6) is called **orthogonalization**, or Gram-Schmidt orthogonalization.

A couple details have thus far been overlooked. For one, formula (5.3.6) only works if the denominators $\langle \mathbf{w}_1, \mathbf{w}_1 \rangle, \langle \mathbf{w}_2, \mathbf{w}_2 \rangle, \ldots, \langle \mathbf{w}_{n-1}, \mathbf{w}_{n-1} \rangle$ of the projections are all nonzero. That is, the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$ are nonzero. Can you provide an argument that $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ being linearly independent assures $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{n-1}$ are nonzero? HINT: Show that if $\mathbf{w}_j = \mathbf{0}$ for some j, then $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is linearly dependent. Answer on page 184.

For two, the discussion at the beginning of the section establishes that (5.3.1) implies (5.3.3), but what we have been relying on is the converse, that (5.3.3) implies (5.3.1). Following our steps backward, the same properties of inner product spaces that gave us that (5.3.2) implies (5.3.3) work in reverse, giving us (5.3.3) implies (5.3.2). However, the implication from (5.3.2) to (5.3.1) is not as straightforward. Assuming (5.3.2) we need to show that $\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n$. Moving everything to the lefthand side in (5.3.2),

$$\begin{aligned} \langle \mathbf{v}, \mathbf{b}_1 \rangle - \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_1 \rangle &= 0 \\ \langle \mathbf{v}, \mathbf{b}_2 \rangle - \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_2 \rangle &= 0 \\ &\vdots \\ \langle \mathbf{v}, \mathbf{b}_n \rangle - \langle a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n, \mathbf{b}_n \rangle &= 0 \end{aligned}$$

which implies

$$\begin{aligned} \langle \mathbf{v} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n), \mathbf{b}_1 \rangle &= 0 \\ \langle \mathbf{v} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n), \mathbf{b}_2 \rangle &= 0 \\ &\vdots \\ \langle \mathbf{v} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n), \mathbf{b}_n \rangle &= 0 \end{aligned}$$

so $\mathbf{v} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n)$ is orthogonal to each basis vector. As shown in "zero inner products" on page 183, this means $\mathbf{v} - (a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n) = \mathbf{0}$ and therefore $\mathbf{v} = a_1\mathbf{b}_1 + a_2\mathbf{b}_2 + \cdots + a_n\mathbf{b}_n$.

For three, the span of the orthogonalized vectors is the same as the span of the original vectors. This is a particularly important feature of the process if you are orthogonalizing the basis of a subspace. Even more is true, though: $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\} = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$ for all $j = 1, 2, \ldots, n$. Let $W_j = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$, making W_j a vector space with dimension j. Each $\mathbf{w}_j$ is by construction a linear combination of $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$ and therefore in W_j. Of course $W_1 = \operatorname{span}\{\mathbf{b}_1\} \subset W_2 = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2\} \subset \cdots \subset W_j = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$ so $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j$ are all in W_j. If we knew that $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j$ were linearly independent, we would have j linearly independent vectors in a j-dimensional vector space, making $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\}$ a basis for W_j. Being a basis, $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\}$ would equal W_j and we would be done.

Direct computation shows that a set of nonzero vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in which every pair of distinct vectors is orthogonal is linearly independent. Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$ for scalars $c_1, c_2, \ldots, c_p$. Then[2]

$$\begin{aligned}
0 = \langle \mathbf{0}, \mathbf{v}_1 \rangle &= \left\langle (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p), \mathbf{v}_1 \right\rangle \\
&= \langle c_1\mathbf{v}_1, \mathbf{v}_1 \rangle + \langle c_2\mathbf{v}_2, \mathbf{v}_1 \rangle + \cdots + \left\langle c_p\mathbf{v}_p, \mathbf{v}_1 \right\rangle \\
&= c_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_1 \rangle + \cdots + c_p \left\langle \mathbf{v}_p, \mathbf{v}_1 \right\rangle \\
&= c_1 \langle \mathbf{v}_1, \mathbf{v}_1 \rangle
\end{aligned}$$

since $\left\langle \mathbf{v}_i, \mathbf{v}_j \right\rangle = 0$ whenever $i \neq j$ (every pair of distinct vectors is orthogonal). But $\mathbf{v}_1$ is a nonzero vector, so $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle \neq 0$ (inner product property 2). Hence c_1 must be zero. Similarly $c_2, c_3, \ldots, c_p$ must also be zero.

It remains to show that every pair of distinct vectors in $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\}$ is orthogonal. Every pair of distinct vectors in $\{\mathbf{w}_1\}$ is orthogonal (vacuously since there are no distinct paris in the set). By (5.3.5), $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal so every pair of distinct vectors in $\{\mathbf{w}_1, \mathbf{w}_2\}$ is orthogonal. Now suppose every pair of distinct vectors in $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_{j-1}\}$ is orthogonal for some $j \geq 3$. If $1 \leq i < j$, then

$$\begin{aligned}
\left\langle \mathbf{w}_i, \mathbf{w}_j \right\rangle &= \left\langle \mathbf{w}_i, \mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_1}\mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_2}\mathbf{b}_j - \cdots - \operatorname{proj}_{\mathbf{w}_{j-1}}\mathbf{b}_j \right\rangle \\
&= \left\langle \mathbf{w}_i, \mathbf{b}_j - \frac{\langle \mathbf{b}_j, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{b}_j, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 - \cdots - \frac{\langle \mathbf{b}_j, \mathbf{w}_{j-1} \rangle}{\langle \mathbf{w}_{j-1}, \mathbf{w}_{j-1} \rangle}\mathbf{w}_{j-1} \right\rangle \\
&= \left\langle \mathbf{w}_i, \mathbf{b}_j \right\rangle - \left\langle \mathbf{w}_i, \frac{\langle \mathbf{b}_j, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 \right\rangle - \left\langle \mathbf{w}_i, \frac{\langle \mathbf{b}_j, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 \right\rangle - \cdots - \left\langle \mathbf{w}_i, \frac{\langle \mathbf{b}_j, \mathbf{w}_{j-1} \rangle}{\langle \mathbf{w}_{j-1}, \mathbf{w}_{j-1} \rangle}\mathbf{w}_{j-1} \right\rangle \\
&= \left\langle \mathbf{w}_i, \mathbf{b}_j \right\rangle - \frac{\langle \mathbf{b}_j, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \langle \mathbf{w}_i, \mathbf{w}_1 \rangle - \frac{\langle \mathbf{b}_j, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \langle \mathbf{w}_i, \mathbf{w}_2 \rangle - \cdots - \frac{\langle \mathbf{b}_j, \mathbf{w}_{j-1} \rangle}{\langle \mathbf{w}_{j-1}, \mathbf{w}_{j-1} \rangle} \left\langle \mathbf{w}_i, \mathbf{w}_{j-1} \right\rangle \\
&= \left\langle \mathbf{w}_i, \mathbf{b}_j \right\rangle - \frac{\langle \mathbf{b}_j, \mathbf{w}_i \rangle}{\langle \mathbf{w}_i, \mathbf{w}_i \rangle} \langle \mathbf{w}_i, \mathbf{w}_i \rangle \\
&= \left\langle \mathbf{w}_i, \mathbf{b}_j \right\rangle - \left\langle \mathbf{w}_i, \mathbf{b}_j \right\rangle \\
&= 0
\end{aligned}$$

since $\langle \mathbf{w}_i, \mathbf{w}_k \rangle = 0$ whenever $k \neq i$. By induction, every pair of distinct vectors in $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\}$ is orthogonal.

A set of vectors in which every pair of distinct vectors is orthogonal is called an **orthogonal set**. A vector with norm 1 is called a **unit vector**. If each vector in an orthogonal set is a unit vector, it is called

[2]see exercise 6a in section 4.6.

an **orthonormal set**. If all the vectors in an orthogonal set are scaled to have norm 1 (normalized), the orthogonal set becomes an orthonormal set with the same span.

Returning to the original question of writing vectors as linear combinations of basis elements, we see that if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is an orthogonal basis, then (5.3.3) reduces to

$$a_i = \frac{\langle \mathbf{v}, \mathbf{b}_i \rangle}{\langle \mathbf{b}_i, \mathbf{b}_i \rangle}, \quad i = 1, 2, \ldots, n.$$

In words, writing a vector as a linear combination of orthogonal basis vectors amounts to projecting the vector onto each of the basis vectors. As a formula, if $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is an orthogonal basis and $\mathbf{v}$ is an arbitrary vector in an inner product space, then

$$\mathbf{v} = \left(\operatorname{proj}_{\mathbf{b}_1} \mathbf{v}\right) + \left(\operatorname{proj}_{\mathbf{b}_2} \mathbf{v}\right) + \cdots + \left(\operatorname{proj}_{\mathbf{b}_n} \mathbf{v}\right).$$

Key Concepts

unit vector vector with norm 1.

orthogonal set set in which every pair of distinct vectors is orthogonal.

orthonormal set orthogonal set of unit vectors.

(orthogonal) projection (of one vector onto another)$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}$.

orthogonal basis basis that is an orthogonal set. If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is an orthogonal basis and $\mathbf{v}$ is an arbitrary vector in an inner product space, then

$$\mathbf{v} = \left(\operatorname{proj}_{\mathbf{b}_1} \mathbf{v}\right) + \left(\operatorname{proj}_{\mathbf{b}_2} \mathbf{v}\right) + \cdots + \left(\operatorname{proj}_{\mathbf{b}_n} \mathbf{v}\right).$$

normalize scale a vector to meet a certain criterion. Often this means scaling so the norm is one.

(Gram-Schmidt) orthogonalzation given a linearly independent set $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, the set $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ defined by

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{b}_1 \\ \mathbf{w}_j &= \mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_1} \mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_2} \mathbf{b}_j - \cdots - \operatorname{proj}_{\mathbf{w}_{j-1}} \mathbf{b}_j, \quad j = 2, 3, \ldots, n. \end{aligned}$$

has the following properties for $j = 1, 2, \ldots, n$.

1. $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\}$ is an orthogonal set.
2. $\operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_j\} = \operatorname{span}\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$.

orthogonality to a basis The only vector orthogonal to every vector of a basis is $\mathbf{0}$.

orthogonal sets and linear independence an orthogonal set of nonzero vectors is linearly independent.

Exercises

1. P

Answers

orthogonal pairs The three possible dot products are all zero:

$$\begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} = 5(3) - 5(3) + 0(9) = 0$$

$$\begin{bmatrix} 5 \\ -5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -\frac{63}{22} \\ -\frac{63}{22} \\ \frac{21}{11} \end{bmatrix} = -5\left(\frac{63}{22}\right) + 5\left(\frac{63}{22}\right) + 0\left(\frac{21}{11}\right) = 0$$

$$\begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} -\frac{63}{22} \\ -\frac{63}{22} \\ \frac{21}{11} \end{bmatrix} = -3\left(\frac{63}{22}\right) - 3\left(\frac{63}{22}\right) + 9\left(\frac{21}{11}\right) = -\frac{189}{22} + \frac{189}{22} = 0$$

zero inner products Let $\mathbf{w} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$. Then $\langle \mathbf{w}, \mathbf{b}_1\rangle = \langle \mathbf{w}, \mathbf{b}_2\rangle = \cdots = \langle \mathbf{w}, \mathbf{b}_n\rangle = 0$ means $c_1\langle \mathbf{w}, \mathbf{b}_1\rangle = c_2\langle \mathbf{w}, \mathbf{b}_2\rangle = \cdots = c_n\langle \mathbf{w}, \mathbf{b}_n\rangle = 0$. By properties 3, 4 and 5 of inner product spaces (see page 164), $\langle \mathbf{w}, c_1\mathbf{b}_1\rangle = \langle \mathbf{w}, c_2\mathbf{b}_2\rangle = \cdots = \langle \mathbf{w}, c_n\mathbf{b}_n\rangle = 0$, so $\langle \mathbf{w}, c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n\rangle = 0$. But $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{w}$ so $\langle \mathbf{w}, \mathbf{w}\rangle = 0$ and by property 2, $\mathbf{w} = \mathbf{0}$.

standard basis inner products For pairs of basis vectors, $i < j$,

$$\begin{aligned}\langle I_{:,i}, I_{:,j}\rangle &= I_{:,i} \cdot I_{:,j} \\ &= 0\cdot 0 + \cdots + 0\cdot 0 + \overbrace{1\cdot 0}^{i^{th}\text{ term}} + 0\cdot 0 + \cdots + 0\cdot 0 + \overbrace{0\cdot 1}^{j^{th}\text{ term}} + 0\cdot 0 + \cdots + 0\cdot 0 \\ &= 0.\end{aligned}$$

For $i > j$, the same computation holds by symmetry (property 3 of inner product spaces). For the norms of basis vectors,

$$\begin{aligned}\|I_{:,i}\| &= \sqrt{I_{:,i} \cdot I_{:,i}} \\ &= \sqrt{0\cdot 0 + \cdots + 0\cdot 0 + \overbrace{1\cdot 1}^{i^{th}\text{ term}} + 0\cdot 0 + \cdots + 0\cdot 0} \\ &= 1.\end{aligned}$$

$\mathbb{P}_2(\mathbb{R})$ **violation** $\frac{2}{5}p_2q_2 + \frac{2}{3}p_0q_2 + \frac{2}{3}p_1q_1 + \frac{2}{3}p_2q_0 + 2p_0q_0$ Violations are easy to come by. The inner products $\langle 1, t\rangle$ and $\langle t, t^2\rangle$ are both zero, but $\langle 1, t^2\rangle$ is not:

$$\begin{aligned}\langle 1, t^2\rangle &= \langle 1(1) + 0(t) + 0(t^2),\ 0(1) + 0(t) + 1(t^2)\rangle \\ &= \frac{2}{5}(0\cdot 1) + \frac{2}{3}(1\cdot 1) + \frac{2}{3}(0\cdot 0) + \frac{2}{3}(0\cdot 0) + 2(1\cdot 0) \\ &= \frac{2}{3}.\end{aligned}$$

None of the norms are one:[3]

$$\begin{aligned}
\|1\| &= \sqrt{\langle 1, 1\rangle} = \sqrt{\frac{2}{5}(0\cdot 0) + \frac{2}{3}(1\cdot 0) + \frac{2}{3}(0\cdot 0) + \frac{2}{3}(0\cdot 1) + 2(1\cdot 1)} = \sqrt{2} \\
\|t\| &= \sqrt{\langle t, t\rangle} = \sqrt{\frac{2}{5}(0\cdot 0) + \frac{2}{3}(0\cdot 0) + \frac{2}{3}(1\cdot 1) + \frac{2}{3}(0\cdot 0) + 2(0\cdot 0)} = \sqrt{\frac{2}{3}} \\
\|t^2\| &= \sqrt{\langle t^2, t^2\rangle} = \sqrt{\frac{2}{5}(1, 1) + \frac{2}{3}(0\cdot 1) + \frac{2}{3}(0\cdot 0) + \frac{2}{3}(1\cdot 0) + 2(0\cdot 0)} = \sqrt{\frac{2}{5}}.
\end{aligned}$$

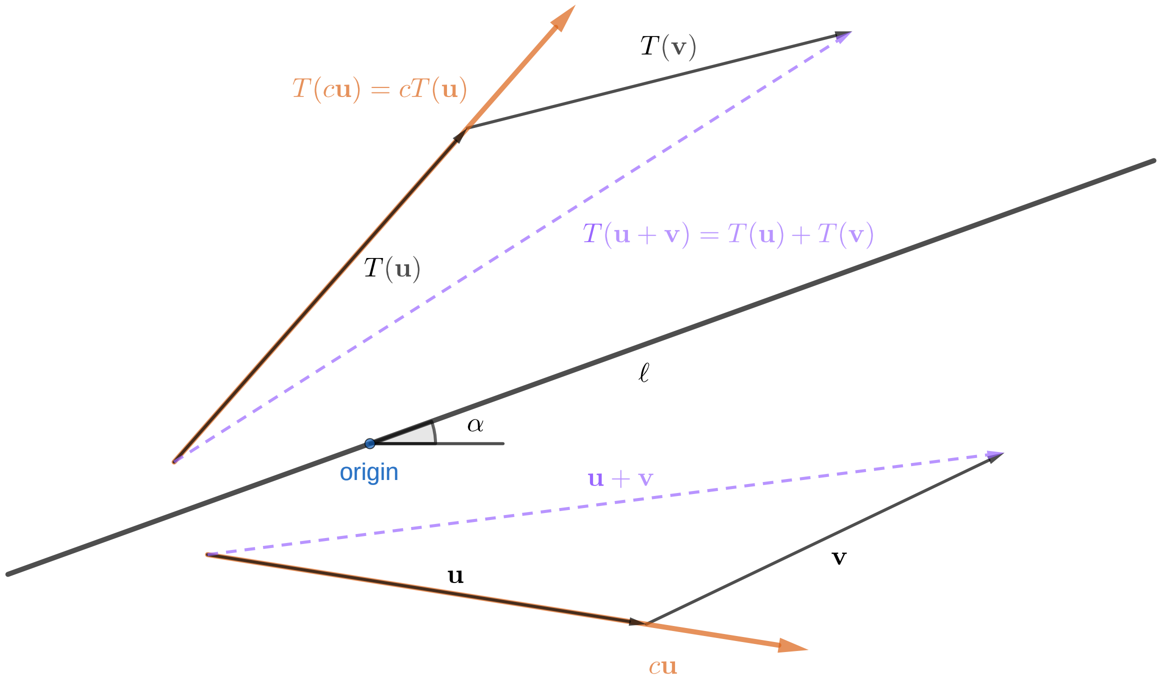
orthogonalized vectors are nonzero Suppose $\mathbf{w}_j = \mathbf{0}$ for some j. Since each $\mathbf{w}_i$ is a linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_i$

$$\begin{aligned}
\mathbf{w}_j &= \mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_1}\mathbf{b}_j - \operatorname{proj}_{\mathbf{w}_2}\mathbf{b}_j - \cdots - \operatorname{proj}_{\mathbf{w}_{j-1}}\mathbf{b}_j \\
&= \mathbf{b}_j + \text{some linear combination of } \mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{j-1}
\end{aligned}$$

giving a nontrivial linear combination of $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j$ that sums to $\mathbf{0}$. This implies $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_j\}$, and therefore $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$, is linearly dependent.

[3] Since $1^2 = 1$, it is just as well to see that the norm squared is not 1 as in $\|1\|^2 = \langle 1, 1\rangle = \frac{2}{5}(0\cdot 0) + \frac{2}{3}(1\cdot 0) + \frac{2}{3}(0\cdot 0) + \frac{2}{3}(0\cdot 1) + 2(1\cdot 1) = 2$.

Figure 5.4.1: Reflection about the line ℓ is linear



5.4 Similarity and Diagonalization []

Figure 5.4.1 illustrates that reflection about an arbitrary line through the origin $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation. As we saw in section 4.4, there must therefore be a 2×2 matrix M such that $T(\mathbf{x}) = M\mathbf{x}$. In the same section, we also learned that the columns of M must satisfy $M_{:j} = T(I_{:,j}),\ j = 1, 2$, so if we knew the images of the standard basis, we would know M.

There is another way. Imagine rotating the plane by angle $-\alpha$ about the origin, then reflecting about the x-axis, then rotating (back) by angle α about the origin. The line ℓ would first map onto the x-axis, all vectors/points/sets in the plane would then be reflected about this image, and then the line and the reflected images would be rotated back so that ℓ would be back where it started and all vectors/points/sets and their reflections would maintain their relative positions across ℓ. In the end, images would be reflected about line ℓ.

This composition is easy enough to write down. Using the standard matrix for reflection about the x-axis given in the discussion of section 4.4 and the standard matrix for counterclockwise rotation about the origin (from exercise 1 of section 4.4),

$$\begin{aligned} M &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \end{aligned} \tag{5.4.1}$$

which can of course be calculated to get

$$\begin{aligned} M &= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 2\cos\alpha\sin\alpha \\ 2\cos\alpha\sin\alpha & -(\cos^2\alpha - \sin^2\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}. \end{aligned} \tag{5.4.2}$$

Interestingly this form reveals that T can also be described by reflection about the x-axis followed by counterclockwise rotation by 2α about the origin. Can you see why? Answer on page 191. Standard matrices for scaling in the direction of any line and shearing along any line can be derived similarly.

Considering M in form (5.4.1) leads to a deeper perspective. As seen in equation (5.2.1), left-multiplying by an invertible matrix can always be interpreted as a change of basis. For example, left-multiplication by the matrix

$$P = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix},$$

the leftmost matrix in (5.4.1), can be interpreted as changing from the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}, \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right\}$$

to the standard basis,

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Note that the rightmost matrix of (5.4.1) is P^{-1}. Can you verify this? Answer on page 191. Accordingly, the rightmost matrix of (5.4.1) can be interpreted as changing from the standard basis to basis $\mathcal{B}$. Altogether then, left-multiplication by M represents a change from the standard basis to basis $\mathcal{B}$, then reflection about the first basis vector in $\mathcal{B}$ (which lies along line ℓ), followed by a change of basis from $\mathcal{B}$ to the standard basis. The transformation starts with coordinates relative the the standard basis and ends with coordinates relative to the standard basis.

Taking this new perspective allows us to understand general transformations such as

$$S : \mathbb{R}^2 \to \mathbb{R}^2,\ S(\mathbf{u}) = \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} \mathbf{u}$$

geometrically by writing the matrix as a product PAP^{-1} where the action of A is easily understood. In particular, matrices A of the form

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

(diagonal matrices) are easily understood geometrically. They can be writen as the product

$$\begin{bmatrix} 1 & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix},$$

which according to section 4.4 is the standard matrix of scaling by a factor of α in the direction of the first basis vector followed by scaling by a factor of β in the direction of the second basis vector. If α is negative, it incorporates a reflection about the second basis vector, and if β is negative it incorporates a reflection about the first basis vector.

Generalizing, we have the standard matrix M of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and we want to find a matrix P and a diagonal matrix D such that $M = PDP^{-1}$. Right multiplying both sides by P we require $MP = PD$. The left side of this equation can be written as

$$\begin{bmatrix} MP_{:,1} & MP_{:,2} & \cdots & MP_{:,n} \end{bmatrix}$$

and the right side can be written as

$$\begin{bmatrix} D_{1,1}P_{:,1} & D_{2,2}P_{:,2} & \cdots & D_{n,n}P_{:,n} \end{bmatrix}.$$

Equating columns of the two sides, we get

$$\begin{aligned} MP_{:,1} &= D_{1,1}P_{:,1} \\ MP_{:,2} &= D_{2,2}P_{:,2} \\ &\vdots \\ MP_{:,n} &= D_{n,n}P_{:,n}. \end{aligned}$$

The columns of P are eigenvectors of M and the entries of D are the corresponding eigenvalues!

In the particular case of S, the eigenvalues are -5 and 5 with corresponding eigenvectors $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ respectively. Can you provide the calculation? Answer on page 191. It folows that

$$\begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

and we see that S has the effect of reflection about the second vector of the basis $C = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$ plus expansion by a factor of 5. This is because multiplication by

$$\begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}$$

has the effect of reflection about the second basis vector (rooted at the origin) plus expansion by a factor of 5. As with reflection relative to the standard basis, the reflection about the second basis vector occurs *parallel* to the first vector. Because the standard basis vectors meet at a right angle, reflection is done along lines perpendicular to the axes. If the basis vectors meet at a different angle, reflection is done along lines meeting at that same angle.

This analysis can be verified by plotting a couple of points and their images under S. For example,

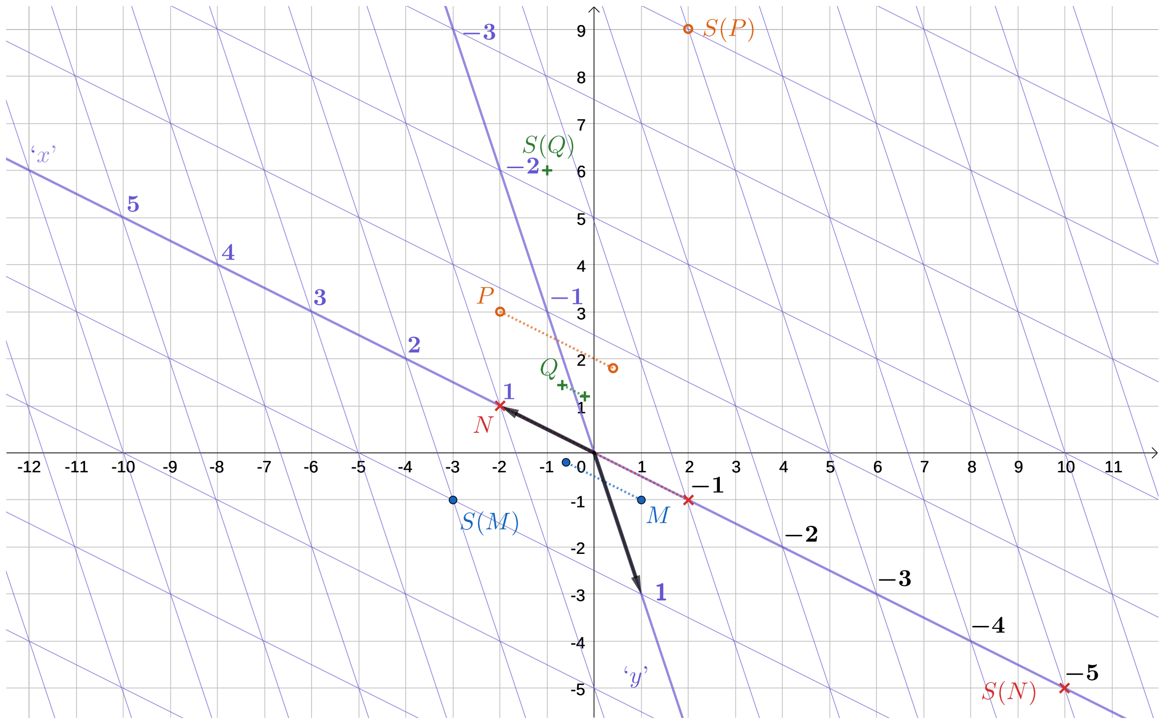
$$S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix};\ S\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 10 \\ -5 \end{bmatrix};\ S\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 9 \end{bmatrix};\ S\left(\begin{bmatrix} -\frac{17}{25} \\ \frac{36}{25} \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \tag{5.4.3}$$

Figure 5.4.2 shows the geometry of S with respect to both the standard basis and the basis C. The purple grid shows coordinates with respect to C. The $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ direction is the positive "x" axis (first basis vector) and the $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ direction is the positive "y" axis (second basis vector) in these coordinates. Notice the point $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ has coordinates $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with respect to the purple grid. In other words,

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_C$$

They are the same point in the plane! Coordinates with respect to C make it easy to trace its action under S. It is first reflected about the second basis vector, making its coordinates $\begin{bmatrix} -1 \\ 0 \end{bmatrix}_C$ (with respect to the purple grid) and then expanded by a factor of 5, making its cordinates $\begin{bmatrix} -5 \\ 0 \end{bmatrix}_C$ (with respect to the purple

Figure 5.4.2: Visualizing the action of S



grid). Other points can be understood similarly. The coordinates of point P are $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ with respect to the standard basis, but $\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}_C$ with respect to C (marked as P in figure 5.4.2). Can you verify this? Answer on page 192. Reflection across the second basis vector gives it coordinates $\begin{bmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}_C$ (shown in figure 5.4.2 connected to P by an orange dashed line segment crossing the "y" axis parallel to the "x" axis) and then expanding by a factor of 5 gives it coordinates $\begin{bmatrix} -3 \\ -4 \end{bmatrix}_C$. This point is marked as $S(P)$ in figure 5.4.2. What are the coordinates of $S(P)$ with respect to the standard basis? Answer on page 192.

Similarity

Separating the matrix calculations of the preceding discussion from their geometric interpretation, we have been examining matrices A and B related by the equation

$$B = P^{-1}AP,$$

a relation known as **similarity**. Indeed, matrices A and B are called **similar** if $B = P^{-1}AP$ (or equivalently $PBP^{-1} = A$). Consistent with the name, matrices related this way share a number of similar features.

Theorem 14. *If matrices A and B are similar, then A and B have the same (i) determinant, (ii) eigenvalues, and (iii) rank.*

(i) By theorem 8 and the fact that $\det P^{-1} = \frac{1}{\det P}$ (section 3.7), $\det B = \det(P^{-1}AP) = \det P^{-1} \cdot \det A \cdot \det P = \frac{1}{\det P} \cdot \det A \cdot \det P = \det A$.

(ii) If $\lambda, \mathbf{v}$ is an eigenpair of B then

$$\begin{aligned} B\mathbf{v} = \lambda\mathbf{v} &\Rightarrow (P^{-1}AP)\mathbf{v} = \lambda\mathbf{v} \\ &\Rightarrow P^{-1}A(P\mathbf{v}) = \lambda\mathbf{v} \\ &\Rightarrow A(P\mathbf{v}) = \lambda(P\mathbf{v}) \end{aligned}$$

so λ is an eigenvalue of A. Similarly, if $\lambda, \mathbf{w}$ is an eigenpair of A then

$$\begin{aligned} A\mathbf{w} = \lambda\mathbf{w} &\Rightarrow (PBP^{-1})\mathbf{w} = \lambda\mathbf{w} \\ &\Rightarrow PB(P^{-1}\mathbf{w}) = \lambda\mathbf{w} \\ &\Rightarrow B(P^{-1}\mathbf{w}) = \lambda(P^{-1}\mathbf{w}) \end{aligned}$$

so λ is an eigenvalue of B.

(iii) By exercises 12 and 13 of section 5.1, neither right multiplying nor left multiplying a matrix by an invertible matrix changes its rank, so rank of A, which equals the rank of PBP^{-1}, equals the rank of B.

Other similar features of similar matrices are explored in the exercises.

Another important feature of similar matrices is that powers of similar matrices are similar. That is, if A and B are square and similar, then A^k and B^k are similar, where the k^{th} power of M is defined by

$$M^k = \overbrace{M \cdot M \cdots M}^{k \text{ times}}$$

(analogous to powers of real numbers). To see that this is true, write $A = PBP^{-1}$ and compute

$$\begin{aligned} A^k &= (PBP^{-1})^k \\ &= \overbrace{PBP^{-1} \cdot PBP^{-1} \cdots PBP^{-1}}^{k \text{ times}} \\ &= P\overbrace{B \cdot B \cdots B}^{k \text{ times}}P^{-1} \\ &= PB^kP^{-1} \end{aligned}$$

This is particularly useful when A is similar to a diagonal matrix. In this case, $A = PDP^{-1}$ for a diagonal matrix D and

$$\begin{aligned} A^k &= PD^kP^{-1} \\ &= P\begin{bmatrix} D_{1,1} & 0 & \cdots & 0 \\ 0 & D_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n,n} \end{bmatrix}^k P^{-1} \\ &= P\begin{bmatrix} D_{1,1}^k & 0 & \cdots & 0 \\ 0 & D_{2,2}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & D_{n,n}^k \end{bmatrix} P^{-1} \end{aligned}$$

so the difficulty in raising A to any power is commensurate with the difficulty of raising D to that power. Earlier we found that

$$\begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

so

$$\begin{aligned} \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix}^5 &= \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -5 & 0 \\ 0 & 5 \end{bmatrix}^5 \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -3125 & 0 \\ 0 & 3125 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix} \\ &= \begin{bmatrix} -4375 & -2500 \\ 3750 & 4375 \end{bmatrix}. \end{aligned}$$

While that may not be the most pleasant computation, it certainly beats computing

$$\begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix}^5 = \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} -7 & -4 \\ 6 & 7 \end{bmatrix}$$

directly. This property of similar matrices is at the heart of the power method for estimating eigenvalues (section), which is at the heart of Markov chain problems (section).

When a matrix M is similar to a diagonal matrix D, we say that M is **diagonalizable** and that the matrix P of $P^{-1}MP = D$ **diagonalizes** M. We saw earlier that if $P^{-1}MP = D$ then the columns of P are eigenvectors. We shall now add the observation that the eigenvectors (columns of P) must be linearly independent, a requirement for P to be invertible. On the other hand, if M is an $n \times n$ matrix admitting n linearly independent eigenvectors, then M is diagonalizable and P, a matrix whose columns are n linearly independent eigenvectors of M, diagonalizes M:

$$\begin{aligned} P^{-1}MP &= P^{-1} \begin{bmatrix} \lambda_1 P_{:,1} & \lambda_2 P_{:,2} & \cdots & \lambda_n P_{:,n} \end{bmatrix} \\ &= P^{-1}P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \end{aligned}$$

which is diagonal. Altogether, an $n \times n$ matrix M is diagonalizable if and only if M has n linearly independent eigenvectors.

Key Concepts

similar matrices matrices A and B are similar if there is an invertible matrix P such that $A = P^{-1}BP$ (equivalently $PAP^-1 = B$ or $PA = BP$).

similarity Matrices that are similar have similarity. Matrices that have similarity have the same (i) determinant, (ii) eigenvalues, and (iii) rank.

diagonalizable a matrix that is similar to a diagonal matrix.

diagonalizability an $n \times n$ matrix M is diagonalizable if and only if M has n linearly independent eigenvectors. Such M is diagonalized by any matrix whose columns are n linearly independent eigenvectors.

powers of matrices The k^{th} power of matrix M is deifined by

$$M^k = \overbrace{M \cdot M \cdots M}^{k \text{ times}}.$$

similarity and powers The k^{th} powers of similar matrices are similar.

geometry of diagonalizable matrices If M is a diagonalizable 2×2, respectively 3×3, matrix, its action on the plane, respectively space, can be understood as a scaling (and possibly reflecting) transformation relative to a basis of eigenvectors.

Exercises

1. Justify the claim.

   (a) Similar matrices have the same trace (sum of the entries on the main diagonal).

   (b) Similar matrices have equal eigenspace dimensions.

Answers

reflection about ℓ The standard matrix for reflection about the x-axis followed by counterclockwise rotation by 2α about the origin is

$$\begin{bmatrix} \cos(2\alpha) & -\sin(2\alpha) \\ \sin(2\alpha) & \cos(2\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = M.$$

inverse of P It is easiest to verify that the product of the two matrices is the identity:

$$\begin{aligned} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} &= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

eigenvalues and eigenvectors The characteristic equation is

$$\begin{aligned} \begin{vmatrix} -7-\lambda & -4 \\ 6 & 7-\lambda \end{vmatrix} &= (-7-\lambda)(7-\lambda) - (-4)(6) \\ &= -49 + \lambda^2 + 24 \\ &= \lambda^2 - 25 = 0 \end{aligned}$$

so the eigenvalues are -5 and 5. Eigenvectors are found by reducing the matrices

$$\begin{bmatrix} -2 & -4 \\ 6 & 12 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -12 & -4 \\ 6 & 2 \end{bmatrix}$$

to

$$\begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -12 & -4 \\ 0 & 0 \end{bmatrix},$$

which gives eigenvectors of the forms $x = -2y$ for $\lambda = -5$ and $y = -3x$ for $\lambda = 5$. Choosing eigenvectgors with small integer entries, we take eigenpairs

$$-5, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{and} \quad 5, \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

coordinates with respect to C The change of coordinates matrix from the standard basis to C is $\begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$, so point P has coordinates

$$\begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}_{C}.$$

coordinates with respect to C As can be seen from figure 5.4.2, the coordinates of $S(P)$ are $\begin{bmatrix} 2 \\ 9 \end{bmatrix}_{\mathcal{E}}$, consistent with (5.4.3), which shows $S\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

Part III

Applications

Chapter 6

Mathematical Applications

6.1 LU Factorization []

Computers, as the name would suggest, are famously adept at computation. Handling fractions, square roots, irrational constants, and $1 + 2$ are all the same to a computer. Ask a human to execute five, six, or twenty numerical operations to solve a single problem and they might think you're simply asking too much. Ask a computer to do the same and you'll have your answer in just a few milliseconds. Humans and computers bring very different skills to task.

Humans provide the algorithms and instructions for the computer and the computer grinds out the computations with amazing speed. The speed and accuracy of computer calculations make the otherwise impractical practical. Linear systems with a hundred, thousand, or even a hundred thousand equations are within the practical limits of computers. But at that size, efficiency plays a major role in practicality. An efficient algorithm may be able to handle a computation in a few seconds while an inefficient one may take a few days for the same result.

The efficiency of an algorithm is calculated by comparing the number of computations needed with common functions such as n^2 or 3^n where n is some measure of the "size" of the problem. This way, we have a good idea how the time it takes to solve a problem grows as the size grows. In solving linear systems n represents the number of equations to be solved. The row reduction algorithm of section 2.2, also known as Gaussian elimination, requires $\frac{n^3}{3} + n^2 - \frac{n}{3}$ multiplications/divisions and $\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$ additions/subtractions ([3] section 6.1) to execute. Table 6.1 lists the number of arithmetic operations required to execute reduction to reduced row echelon form for a general linear system with n equations in

Table 6.1: Arithmetic operations required for row reduction

equations, n	$\times/\div$	$+/-$	total	$\frac{2}{3}n^3$
2	6	3	9	$5 + \frac{1}{3}$
3	17	11	28	18
4	36	26	62	$42 + \frac{2}{3}$
9	321	276	597	486
51	46,801	45,475	92,276	88,434
102	364,106	358,853	722,959	707,472
501	42,168,001	42,042,250	84,210,251	83,834,334
2,001	2,674,672,001	2,672,669,000	5,347,341,001	5,341,337,334
10,002	333,633,410,006	333,583,385,003	667,216,795,009	667,066,746,672

n unknowns. For a human with a handheld calculator, systems with 2 or 3 variables are doable. Systems with 4 variables would be tedious in general, but practical with a few strategically placed zeros (which can dramatically reduce the number of computations required). However, even 9 equations in 9 unknowns would be a bit much for a person.

You might well ask whether systems of 500 equations in 500 unknowns are practical even for a computer. Their solution by Gaussian elimination requires over 83 million computations! Even if the computer does one computation every microsecond (millionth of a second), solving a system of 500 equations in 500 unknowns will take about 83 seconds. Depending on how quickly results are needed, this may or may not be practical. The same computer would take over $5,300$ seconds (about an hour and a half) to solve a system of $2,000$ equations in $2,000$ unknowns and over $666,000$ seconds (over 7 and a half days!) to solve a system of $10,000$ equations in $10,000$ unknowns. Of course faster computers or clusters of computers could be put to the task to speed up the computation, but no matter how much computing power is supplied, there will always be a size outside its practical limits.

A better option is to streamline the algorithm. The numerical "speed" of Gaussian elimination is approximately proportional to n^3. The rightmost column of table 6.1 illustrates this point. The number of computations can be well approximated by $\frac{2}{3}n^3$ for large n. In the parlance of numerical analysis, one would say the algorithm executes in $O(n^3)$ time, read "big-oh of n^3 time". The implication is doubling the size of the problem multiplies the time it takes to execute by 8 and more generally increasing the size by a factor of k increases its execution time by a factor of k^3.

If the algorithm executed in, say, $O(n^2)$ time it would reduce the number of computations needed to solve large problems by several orders of magnitude. For example, an algorithm that required approximately $4n^2$ computations would need "only" about $1,000,000$ computations to solve a system of 500 equations in 500 unknowns (compare this to the 83 million for Gausian elimination); about $16,000,000$ for a system of $2,000$ equations in $2,000$ unknowns (compare this to the over 53 bilion for Gaussian elimination); and about 400 million for a system of $10,000$ equations in $10,000$ unknowns (reducing the estimated time of 7.5 days to about 6 minutes!).

Now imagine you have to solve the system multiple times for multiple sets of constants, but the same coefficient matrix. 7.5 days is preferable to 75 days for solving the system with ten different sets of constants. This is the approximate effect and purpose of using LU factorization—solving large systems for multiple sets of constants. The factorization itself takes about the same effort as Gaussian elimination, but subsequent solutions take $O(n^2)$ time.

The efficiency gain is due to turning the general problem into a special case that is much quicker to solve. As noted earlier, asking a human with a handheld calculator to solve a system of 4 equations in 4 unknowns borders on the impractical as it requires as many as 62 individual arithmetic operations. But asking a human to solve a system of 4 equations in 4 unknowns where the coefficient matrix is upper triangular is well within reason. You might have a go at solving the system

$$\begin{bmatrix} -14 & 0 & -10 & 12 \\ 0 & 9 & -8 & 4 \\ 0 & 0 & -15 & -6 \\ 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -11 \\ -2 \\ 6 \\ 15 \end{bmatrix}$$

to see that it takes 26 arithmetic operations—still not terribly appealing but much better than the 62 for Gaussian elimination of a general 4-equation, 4-variable system. A similar reduction is achieved when the coefficient matrix is lower triangular. These observations are at the heart of the LU factorization (upper-lower factorization) algorithm. It requires many fewer computations to solve two linear systems, one with an upper triangular coefficient matrix and the other with a lower triangular coefficient matrix, than it does to solve a single general linear system. In general, solving a system with either upper triangular or lower triangular coefficient matrix can be done in $O(n^2)$ time.

If we could factor a general coefficient matrix M into the product of a lower triangular and upper triangular matrix, we could achieve such efficiency. Solving $M\mathbf{v} = \mathbf{b}$ directly by Gaussian elimination requires $O(n^3)$ operations while solving $LU\mathbf{v} = \mathbf{b}$ "twice"—first to find $U\mathbf{v}$ by solving $L\mathbf{w} = \mathbf{b}$ and second to find $\mathbf{v}$ by solving $U\mathbf{v} = \mathbf{w}$—requires only $O(n^2)$ operations.

The process for factoring M into LU essentially amounts to saving your progress in executing Gaussian elimination at the point where M has first reached echelon form. As we did several times in sections 3.6 and 3.7, we will rely on the fact that row reduction can be expressed by multiplication by (invertible) elementary matrices. If we record the row operations (elementary matrices) that reduce M to echelon form, we have

$$E_p \cdots E_2 E_1 M = U$$

where U (the echelon from of M) is upper triangular. Assuming no row swaps have been done, the product $E_p \cdots E_2 E_1$ is lower triangular since all row replacements are done by adding a multiple of one row to a row below it (and row scaling does not affect the locations of zeros). By the same reasoning the inverse of $E_p \cdots E_2 E_1$ is lower triangular, so setting $L^{-1} = E_p \cdots E_2 E_1$ we have

$$M = (E_p \cdots E_2 E_1)^{-1} U = LU.$$

It is not realistic to expect that reduction can be done without row swaps, however. If, for example, the $1,1$-entry of M is zero and there is at least one nonzero entry below it, there is no choice. The first operation of row reduction requires a row swap. This means M cannot always be factored into a lower triangular times upper triangular product, and allowing row swaps is necessary. In the end, a factorization that includes row swaps does not factor M into a product LU. Instead, it factors PM into a product LU where P is a permutation matrix, a matrix that holds the same rows as the identity matrix but in a possibly different order. Including P in the computation this way does not add any arithmetic operations to the algorithm. It adds only swapping of values, a computer operation that is so fast as to be insignificant compared to arithmetic operations. Allowing row swaps in an LU decomposition is known as LU decomposition with **partial pivoting**.

An Example

To illustrate the method, we will factor (factorize, or decompose)

$$M = \begin{bmatrix} 18 & -35 & -4 & -56 \\ -14 & 21 & 4 & 42 \\ 6 & -7 & -1 & -23 \\ 6 & -9 & -2 & -18 \end{bmatrix}.$$

operations	result
$M_{1,:} \leftrightarrow M_{4,:}$	$\begin{bmatrix} 6 & -9 & -2 & -18 \\ -14 & 21 & 4 & 42 \\ 6 & -7 & -1 & -23 \\ 18 & -35 & -4 & -56 \end{bmatrix}$
$M_{1,:} \to \frac{1}{3} M_{1,:}$	$\begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ -14 & 21 & 4 & 42 \\ 6 & -7 & -1 & -23 \\ 18 & -35 & -4 & -56 \end{bmatrix}$

$$\begin{array}{rl} M_{2,:} & \to 7M_{1,:} + M_{2,:} \\ M_{3,:} & \to -3M_{1,:} + M_{3,:} \\ M_{4,:} & \to -9M_{1,:} + M_{4,:} \end{array} \qquad \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 2 & 1 & -5 \\ 0 & -8 & 2 & -2 \end{bmatrix}$$

$$M_{2,:} \leftrightarrow M_{3,:} \qquad \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & -8 & 2 & -2 \end{bmatrix}$$

$$M_{4,:} \to 4M_{2,:} + M_{4,:} \qquad \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 6 & -22 \end{bmatrix}$$

$$M_{4,:} \to 9M_{3,:} + M_{4,:} \qquad \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -22 \end{bmatrix}$$

We have reached echelon form, so we have identified U (the echelon form itself):

$$U = \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -22 \end{bmatrix}$$

and the product of the 8 elementary matrices corresponding to the 8 row operations form $P^{-1}L$. Apply the inverse of each row operation, in reverse order, starting with the identity matrix:

$$\begin{array}{ll} \text{operation} & \text{inverse} \\ M_{4,:} \to 9M_{3,:} + M_{4,:} & M_{4,:} \to -9M_{3,:} + M_{4,:} \\ M_{4,:} \to 4M_{2,:} + M_{4,:} & M_{4,:} \to -4M_{2,:} + M_{4,:} \\ M_{2,:} \leftrightarrow M_{3,:} & M_{2,:} \leftrightarrow M_{3,:} \\ M_{4,:} \to -9M_{1,:} + M_{4,:} & M_{4,:} \to 9M_{1,:} + M_{4,:} \\ M_{3,:} \to -3M_{1,:} + M_{3,:} & M_{3,:} \to 3M_{1,:} + M_{3,:} \\ M_{2,:} \to 7M_{1,:} + M_{2,:} & M_{2,:} \to -7M_{1,:} + M_{2,:} \\ M_{1,:} \to (1/3)M_{1,:} & M_{1,:} \to 3M_{1,:} \\ M_{1,:} \leftrightarrow M_{4,:} & M_{1,:} \leftrightarrow M_{4,:} \end{array}$$

In full detail:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -4 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 9 & -4 & -9 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 9 & -4 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 9 & -4 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 9 & -4 & -9 & 1 \end{bmatrix} \to \begin{bmatrix} 9 & -4 & -9 & 1 \\ -7 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} = P^{-1}L$$

At this point, we have

$$M = \begin{bmatrix} 9 & -4 & -9 & 1 \\ -7 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -22 \end{bmatrix},$$

which, as expected, is not lower triangular times upper triangular. The final step is to permute the rows of M. The permutation matrix P can be constructed by applying only the row swaps to the identity matrix, in the same order in which they were applied during row reduction. In this case, that means $M_{1,:} \leftrightarrow M_{4,:}$ and then $M_{2,:} \leftrightarrow M_{3,:}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P.$$

In this case the order of the swaps does not matter, but when an index is repeated within the set of swaps, the order will matter. Finally, we have

$$PM = LU = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -7 & 0 & 1 & 0 \\ 9 & -4 & -9 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & -\frac{2}{3} & -6 \\ 0 & 2 & 1 & -5 \\ 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & -22 \end{bmatrix}$$

or $M = P^{-1}LU$. Can you verify this?

Therefore the system $M\mathbf{v} = \mathbf{b}$ is equivalent to $P^{-1}LU\mathbf{v} = \mathbf{b}$. Solving amounts to first applying P to $\mathbf{b}$, permuting its entries, yielding $LU\mathbf{v} = P\mathbf{b}$; second, solving $L\mathbf{w} = P\mathbf{b}$, which yields $\mathbf{w} = L^{-1}P\mathbf{b}$; and third, solving $U\mathbf{v} = \mathbf{w}$, which yields $\mathbf{v}$ such that $U\mathbf{v} = L^{-1}P\mathbf{b}$.

When M is invertible, U will be invertible, and we will have $\mathbf{v} = U^{-1}L^{-1}P\mathbf{b} = M^{-1}\mathbf{b}$. But the method works even when M is noninvertible—whenever $U\mathbf{v} = \mathbf{w}$ is consistent, its solutions will also be solutions of $M\mathbf{v} = \mathbf{b}$ (and whenever $U\mathbf{v} = \mathbf{w}$ is inconsistent $M\mathbf{v} = \mathbf{b}$ will also be inconsistent).

Key Concepts

factorizaton factoring a matrix into a product of matrices.

decomposition factorization.

LU **factorization** factoring a matrix into a product of a lower triangular matrix (L) by an upper triangular matrix (U).

partial pivoting allowing row swaps in LU factorization. In this case, the algorithm factorizes PM into LU for some permutation matrix P.

permutation matrix a matrix containing the same rows as an identity matrix but in a possibly different order. Such a matrix will have exactly one 1 in each row and each column and zeros elsewhere.

LU **advantage** solving a system in $O(n^3)$ time, subsequently solving systems with the same coefficient matrix but different constants in $O(n^2)$ time.

Exercises

1. I

Answers

e

6.2 The Power Method []

A great deal of numerical methods are concerned with approximating numbers when those numbers cannot be computed exactly. The roots of the modest-looking function $f(x) = x^5 - x - 1$ is an example. It is a result of Galois theory that there are unsolvable fifth degree polynomials, and f is one such quintic. However, being a fifth degree polynomial with rational coefficients, it must have at least one real root. We just can't write down an exact expression for it. On the other hand, there are a host of numerical methods that can approximate the root to any desired precision. For example, if we set $x_0 = 1$ and let $x_{i+1} = x_i - \frac{f(x_i)}{5x_i^4-1}$, we can generate a sequence of approximations. $x_1 = x_0 - \frac{f(x_0)}{5x_0^4-1} = 1 - \frac{f(1)}{5(1)^4-1} = \frac{5}{4}$ and $x_2 = x_1 - \frac{f(x_1)}{5x_1^4-1} \approx 1.17846$. Rounded to five decimal places, the sequence $x_0, x_1, \ldots$ begins

$$1,\ 1.25,\ 1.17846,\ 1.16754,\ 1.16730$$

and 1.16730 is a root of f accurate to 5 decimal places. You may recognize this as Newton's method. There is a formula that is applied to some starting value, the output of which is then input to the formula, the output of which is then input to the formula, and so on, each time feeding the output of the formula into the formula as input. As the iterations proceed, the output gets closer and closer (we hope!) to the desired quantity. Such methods are called iterative methods, and there are a number of them for linear algebra problems too.

Let

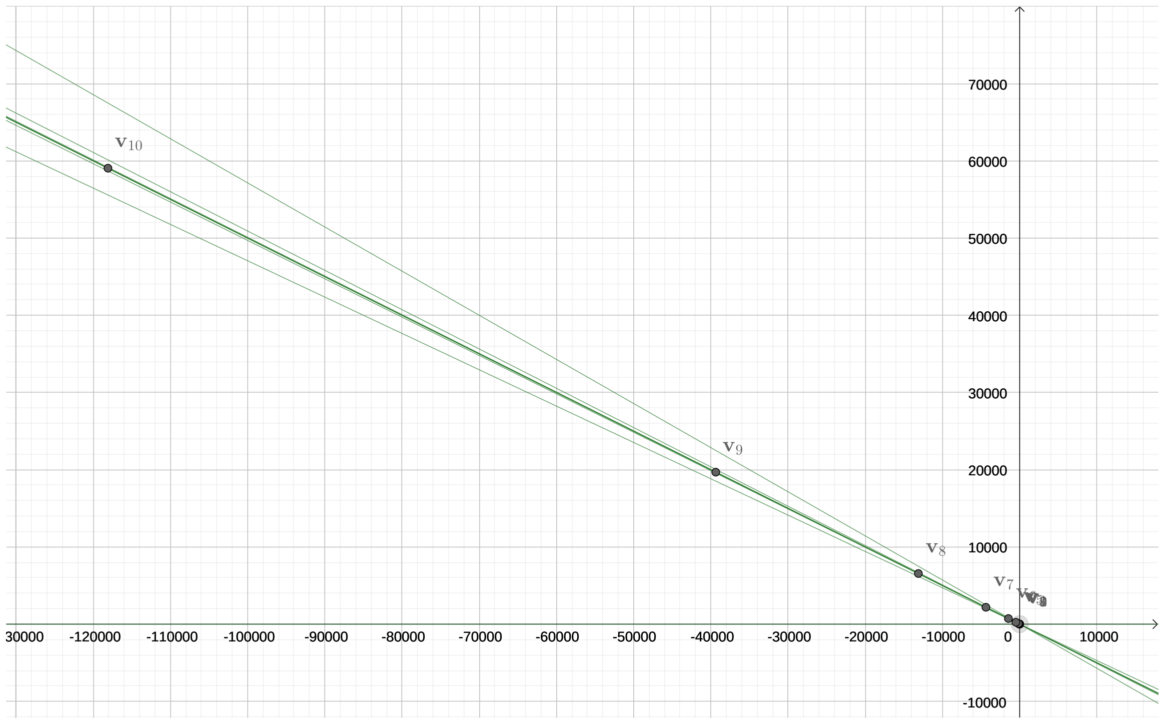
$$M = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix},$$

$\mathbf{v}_0 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T$ and $\mathbf{v}_{i+1} = M\mathbf{v}_i$. Then

$$\begin{aligned}
\mathbf{v}_1 &= M\mathbf{v}_0 = \begin{bmatrix} -7 & 4 \end{bmatrix}^T \\
\mathbf{v}_2 &= M\mathbf{v}_1 = \begin{bmatrix} -17 & 8 \end{bmatrix}^T \\
\mathbf{v}_3 &= M\mathbf{v}_2 = \begin{bmatrix} -55 & 28 \end{bmatrix}^T \\
\mathbf{v}_4 &= M\mathbf{v}_3 = \begin{bmatrix} -161 & 80 \end{bmatrix}^T \\
\mathbf{v}_5 &= M\mathbf{v}_4 = \begin{bmatrix} -487 & 244 \end{bmatrix}^T \\
\mathbf{v}_6 &= M\mathbf{v}_5 = \begin{bmatrix} -1457 & 728 \end{bmatrix}^T \\
\mathbf{v}_7 &= M\mathbf{v}_6 = \begin{bmatrix} -4375 & 2188 \end{bmatrix}^T \\
\mathbf{v}_8 &= M\mathbf{v}_7 = \begin{bmatrix} -13121 & 6560 \end{bmatrix}^T \\
\mathbf{v}_9 &= M\mathbf{v}_8 = \begin{bmatrix} -39367 & 19684 \end{bmatrix}^T \\
\mathbf{v}_{10} &= M\mathbf{v}_9 = \begin{bmatrix} -118097 & 59048 \end{bmatrix}^T
\end{aligned}$$

Though it is likely not apparent, something remarkable is happening here. Each vector is closer than the last to a vector of interest. Plotting the vectors helps reveal the phenomenon. Figure 6.2.1 shows eleven lines, one in the direction of each $\mathbf{v}_i$. (In the parlance of linear algebra, the lines are the 1-dimensional vector spaces generated by the individual $\mathbf{v}_i$.) The head of each $\mathbf{v}_i$ is marked with a point, but $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_6$ are nearly on top of one another. Nonetheless, the figure ilustrates what is happening. The slopes of the lines are converging. The last three lines, $\{r\mathbf{v}_8 : r \in \mathbb{R}\}$, $\{r\mathbf{v}_9 : r \in \mathbb{R}\}$, and $\{r\mathbf{v}_{10} : r \in \mathbb{R}\}$ all seem to more

Figure 6.2.1: Calculating $\mathbf{v}_{i+1} = M\mathbf{v}_i$ shows a sort of convergence



or less lie on the line $y = -\frac{1}{2}x$. That is, they essentially lie in the direction of $\begin{bmatrix} -2 & 1 \end{bmatrix}^T$ (or $\begin{bmatrix} 2 & -1 \end{bmatrix}^T$ if you prefer). Further iteration will reveal more of the same. What if $\mathbf{v}_0$ were different, though? Can you calculate a similar sequence of vectors starting with a different $\mathbf{v}_0$? Do the vectors of your sequence approach the direction $\begin{bmatrix} -2 & 1 \end{bmatrix}^T$ too? Answers on page 205.

Figure 6.2.1 nicely illustrates the convergence geometrically, but we ought be able to detect it algebraically as well. Let $\hat{\mathbf{v}}_i = \frac{1}{(\mathbf{v}_i)_{2,1}}\mathbf{v}_i$, $i = 1, 2, \ldots, 10$. That is, let $\hat{\mathbf{v}}_i$ be $\mathbf{v}_i$ scaled by the reciprocal of its second entry. Then

$$
\begin{aligned}
\hat{\mathbf{v}}_1 &= \mathbf{v}_1/(\mathbf{v}_1)_{2,1} = \begin{bmatrix} -1.75 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_2 &= \mathbf{v}_2/(\mathbf{v}_2)_{2,1} = \begin{bmatrix} -2.125 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_3 &= \mathbf{v}_3/(\mathbf{v}_3)_{2,1} \approx \begin{bmatrix} -1.96429 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_4 &= \mathbf{v}_4/(\mathbf{v}_4)_{2,1} = \begin{bmatrix} -2.0125 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_5 &= \mathbf{v}_5/(\mathbf{v}_5)_{2,1} \approx \begin{bmatrix} -1.99590 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_6 &= \mathbf{v}_6/(\mathbf{v}_6)_{2,1} \approx \begin{bmatrix} -2.00137 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_7 &= \mathbf{v}_7/(\mathbf{v}_7)_{2,1} \approx \begin{bmatrix} -1.99954 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_8 &= \mathbf{v}_8/(\mathbf{v}_8)_{2,1} \approx \begin{bmatrix} -2.00015 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_9 &= \mathbf{v}_9/(\mathbf{v}_9)_{2,1} \approx \begin{bmatrix} -1.99995 & 1 \end{bmatrix}^T \\
\hat{\mathbf{v}}_{10} &= \mathbf{v}_{10}/(\mathbf{v}_{10})_{2,1} \approx \begin{bmatrix} -2.00002 & 1 \end{bmatrix}^T
\end{aligned}
$$

Being a scalar multiple, the vector $\hat{\mathbf{v}}_i$ points in the same direction as $\mathbf{v}_i$. The list of $\hat{\mathbf{v}}_i$ numerically demonstrates that the $\hat{\mathbf{v}}_i$, and therefore the $\mathbf{v}_i$, are pointing closer and closer to the $\begin{bmatrix} -2 & 1 \end{bmatrix}^T$ direction. Interestingly,

$$M \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \end{bmatrix},$$

or in short $M \begin{bmatrix} -2 & 1 \end{bmatrix}^T = 3 \begin{bmatrix} -2 & 1 \end{bmatrix}^T$, so $3, \begin{bmatrix} -2 & 1 \end{bmatrix}^T$ is an eigenpair of M!

Sit back and think about this for a moment. We started with a seemingly arbitrary matrix and a (pretty shabby) approximation of one of its eigenvectors. We then proceeded to multiply this vector (and its products) by M, the result of which was a very good approximation of an eigenvector of M. With a computer at hand to do the calculation, this is a whole lot easier than solving a characteristic equation. More importantly, though, as noted earlier, general polynomials of degree five or higher cannot be solved exactly. This means eigenpairs for square matrices of size 5×5 and up canonot generally be found exactly. Numerical methods must be used!

This approach of iteratively multiplying some vector by the matrix whose eigenpairs are desired, known as the **power method**, seems to have potential, but the example should leave you with lots of questions. For example,

Does this always work?
Why does it work?
If it doesn't always work, when does it work?
Are there other methods we can try?
Can we say how many iterations are needed to get a good approximation?
Which eigenpair is found when it works?
Does it matter what vector is chosen for $\mathbf{v}_0$?
Will the method always produce the same eigenvector?
What about other eigenvectors?

The computation in crumpet 24 answers several questions, partially answers others, and leaves some open. For example, the method works when

(i) M is diagonalizable, and

(ii) one of the eigenvalues is **dominant** in the sense that its magnitude is larger than all the others, and

(iii) the dominant eigenvalue has a 1-dimensional eigenspace, and

(iv) $\mathbf{v}_0$ is chosen appropriately.

That's not to say it won't work in other instances. As mathematicians say, these are sufficient conditions, not necessary conditions. Other answers include

1. The method only determines the eigenvector corresponding to the dominant eigenvalue.

2. The accuracy of the approximation is proportional to the largest ratio $\left|\frac{\lambda_j}{\lambda_i}\right|^k$, $i \neq j$.

The exercises explore a number of these points, and describes a modification of the power method that can be used to find any eigenpair.

On a practical note, implementation of the method will include scaling $\mathbf{v}_k$ with each iteration. As the example shows, the norm of $\mathbf{v}_k$ can grow very large very quickly. Crumpet 1 reveals that $\mathbf{v}_k$ will grow (or decay) exponentially. Since it is only the direction of $\mathbf{v}_k$ that matters, scaling does not affect the success of the algorithm. Typically, $\mathbf{v}_k$ will be scaled by $\max\{|(\mathbf{v}_k)_{j,1}| : j = 1, 2, \ldots, n\}$ so that the magnitude of the greatest (or least) entry of $\mathbf{v}_k$ is one.

Crumpet 24: The Power Method

Suppose M is a diagonalizable $n \times n$ matrix and P is an $n \times n$ matrix whose columns are linearly independent eigenvectors of M (see section 5.4), and let $D = P^{-1}MP$. Further suppose that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of M are such that $|\lambda_j| > |\lambda_i|$ for all $i \neq j$ and the eigenspace of λ_j has dimension one. Pick an arbitrary vector $\mathbf{v}_0$ in $\mathbb{R}^n$ and let $\mathbf{w} = P^{-1}\mathbf{v}_0$. Finally, define $\mathbf{v}_k = M\mathbf{v}_{k-1}$ for $k > 0$ and Z as the $n \times n$ matrix with zeros everywhere except the j, j-entry where it has a one. Then for large enough k,

$$\begin{aligned}
\mathbf{v}_k = M^k\mathbf{v}_0 &= PD^kP^{-1}\mathbf{v}_0 \\
&= PD^k\mathbf{w} \\
&= P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}^k \mathbf{w} \\
&= P\begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \mathbf{w} \\
&= \lambda_j^k P\begin{bmatrix} \frac{\lambda_1^k}{\lambda_j^k} & 0 & \cdots & 0 \\ 0 & \frac{\lambda_2^k}{\lambda_j^k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda_n^k}{\lambda_j^k} \end{bmatrix} \mathbf{w} \\
&\approx \lambda_j^k PZ\mathbf{w} \\
&= \lambda_j^k \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & P_{:,j} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathbf{w} \\
&= \lambda_j^k \mathbf{w}_{j,1} P_{:,j}.
\end{aligned}$$

But $P_{:,j}$ is an eigenvector of M corresponding to λ_j, so $\mathbf{v}_k$ is approximately an eigenvector of M corresponding to λ_j as long as $\mathbf{w}_{j,1} \neq 0$.

Key Concepts

dominant eigenvalue an eigenvalue with larger magnitude than all other eigenvalues of a given matrix.

power method iteration of the recurrence relation $\mathbf{v}_k = M\mathbf{v}_{k-1}$ for some initial vector $\mathbf{v}_0$. Under certain conditions the sequence $\mathbf{v}_0, \mathbf{v}_1, \ldots$ will converge to an eigenvector of M.

Exercises

1. Find the dominant eigenvalue.

2. Does the matrix have a dominant eigenvalue?

3. Here are the first 6 terms of $\mathbf{v}_k = M\mathbf{v}_{k-1}$. Estimate an eigenpair.

4. Here are the first 6 terms of $\mathbf{v}_k = \frac{M\mathbf{v}_{k-1}}{||M\mathbf{v}_{k-1}||_\infty}$. Esti-

mate an eigenpair.

5. Find $\mathbf{v}_{10}$ and $M\mathbf{v}_{10}$ of the power method. Does it seem that the method will converge? You may use the code at .

6. Redo question 3 find $\mathbf{v}_{100}$ and $M\mathbf{v}_{100}$ instead. Does it seem that the method will converge? You may use the code at . Compare your answer to your answer in question 3.

Dominant eigenvalue has multi-dimensional eigenspace

7. The dominant eigenvalue of M has a 2-dimensional eigenspace (algebraic multiplicity = geometric multiplicity = 2). Apply the power method anyway. Does it work?

Matrix not diagonalizable

8. M is not diagonalizable but has a dominant eigenvalue with multiplicity 1 (1-dimensional eigenspace). Apply the power method anyway. Does it work?

Failure

9. Dominant eigenvalue with algebraic mult greater than geom multiplicity (dim of nullspace of $M - \lambda I$). Compute $M^{\text{high power}}$ and look at columns. Is there an eigenvector?

10. Two eigenvalues with the same modulus/absolute value (alg mult 1 each). Run the method anyway. Do you get a linear combination of the two eigenvectors?

The inverse power method

11. Show that if $\lambda, \mathbf{v}$ is an eigenpair of M and M is invertible, then $\frac{1}{\lambda}, \mathbf{v}$ is an eigenpair of M^{-1}.

12. Show that if $\lambda, \mathbf{v}$ is an eigenpair of M then $\lambda - \alpha, \mathbf{v}$ is an eigenpair of $M - \alpha I$.

13. Combine 11 and 12 to show that if $\lambda, \mathbf{v}$ is an eigenpair of M and α is not an eigenvalue of M, then $\frac{1}{\lambda - \alpha}, \mathbf{v}$ is an eigenpair of $(M - \alpha I)^{-1}$.

14. 2 is a good estimate of one of the non-dominant eigenvalues of M. Apply the power method to $(M - 2I)^{-1}$ to find the eigenvalue and an associated eigenvector.

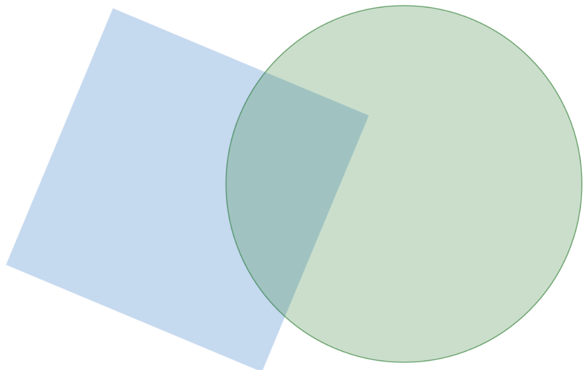
Answers

different $\mathbf{v}_0$ Setting $\mathbf{v}_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, for example, leads to the sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{10}$

$$\begin{bmatrix} 15 & -9 \end{bmatrix}^T, \begin{bmatrix} 33 & -15 \end{bmatrix}^T, \begin{bmatrix} 111 & -57 \end{bmatrix}^T, \begin{bmatrix} 321 & -159 \end{bmatrix}^T, \begin{bmatrix} 975 & -489 \end{bmatrix}^T,$$
$$\begin{bmatrix} 2913 & -1455 \end{bmatrix}^T, \begin{bmatrix} 8751 & -4377 \end{bmatrix}^T, \begin{bmatrix} 26241 & -13119 \end{bmatrix}^T,$$
$$\begin{bmatrix} 78735 & -39369 \end{bmatrix}^T, \begin{bmatrix} 236193 & -118095 \end{bmatrix}^T.$$

Again the second entries are approximately $-\frac{1}{2}$ times the first, and getting closer the further we go in the sequence. For example, $\frac{-118095}{236193} \approx -0.499994$. Unless you chose $\mathbf{v}_0$ in the direction of $\begin{bmatrix} 1 & -1 \end{bmatrix}^T$, you should have noticed the same thing for your sequence.

Figure 6.3.1: What is the area of the overlapping region?



6.3 Geometry: Determinants, Eigenvalues, and Area []

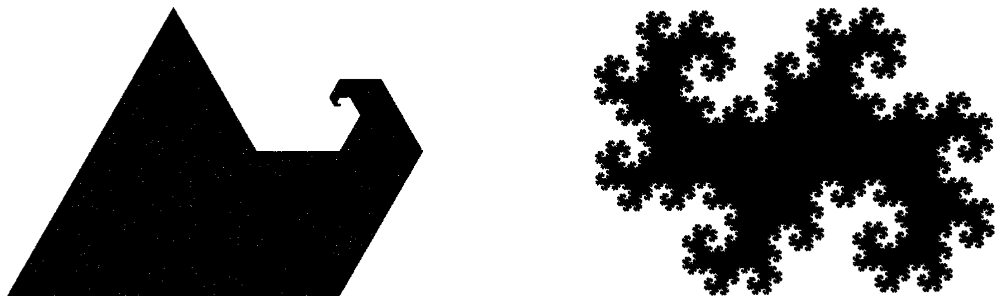
Intuitively, we might think of area as the amount of paint needed in painting a particular shape. The more paint needed, the larger its area, and the larger its area, the more paint needed. To have some sense of what is meant by the area of an object, this intuition is good enough. Larger shapes have larger area while smaller shapes have smaller area, and the area of a shape is some measure of this size.

Calculating the areas of a shapes (assigning numbers to areas) is another story. We certainly are not going to require that to find the area of an object it needs to be painted and the amount of paint used measured. What paint should be used, by whom, and what instrument should do the measuring? This process would be so imprecise it would lead to a single shape having many areas (depending on the paint, the painter, and the measuring device), a rather undesirable situation. The area of a shape should be uniquely determined. A single shape has but a single size, and so it must have but a single measure of its size.

A number of area formulas are presented in grammar school. The area of a rectangle is its length times width. The area of a triangle is one half its base times height. The area of a circle is π times the square of its radius. Trapezoids, parallelograms, regular polygons, and unions of such shapes have calculable areas. Doesn't that pretty much cover it? As you might have imagined by now, the answer is no. For a simple example, take an arbitrary nonempty overlap between a square and circle where neither is the circle contained within the square nor is the square contained within the circle. See figure 6.3.1. Calculus provides a method for calculating its area and hints at the complexity of the general question. By slicing the shape into smaller and smaller approximating rectangles and adding up the areas of those rectangles, the area can be approximated more and more accurately. The limit of these areas as the widths of the approximating rectangles aproaches zero is the area of the overlap. If you've taken calculus, that probably reminds you of integration, and it should! If you have not taken calculus, that probably sounds rather confusing and complicated, and it should! That is really the point. It is not an easy matter to calculate area, even of shapes that are easy to draw.

To stretch the point just a bit further, consider the shapes in figure 6.3.2. The figure on the left is the snail of Solomon Golomb[9] and features an infinitely spiraling appendage. The figure on the right is referred to as a twin dragon as it is the union of a pair of dragon curves. Neither of these figures can be drawn with perfect precision since each has infinitely small detail. The twin dragon is an example of a self-similar fractal with nonzero area. Its boundary (perimeter) is infinitely long and infinitely intricate. The more one magnifies the boundary, the more detail is revealed. While the snail can be formed by a union of infinitely many nonoverlapping triangles in a straightforward way, making its area calculable, the twin dragon cannot. Even applying calculus to the problem of finding the area of the twin dragon is

Figure 6.3.2: What are the areas of these shapes?



not a straightforward matter. Does it even have a calculable area? What does having a calculable area mean? Are there sets whose areas are not calculable? These questions can be followed deep into measure theory, a branch of analysis far outside the reaches of this textbook.

With the very definition of area up in the air,

Crumpet 25: A Definition of Area

The area of a bounded region of the plane, a shape S, can be defined as follows. Let R be a polygonal region containing S, and let $\mathcal{P}_R$ be a primitive partition of R (a finite set of parallelograms and triangles whose interiors do not overlap and whose union is R). Define the norm of a partition, denoted $||\mathcal{P}_R||$, as the maximum of the areas of the primitives in $\mathcal{P}_R$. Then

$$\operatorname{area}(S) = \lim_{||\mathcal{P}_R|| \to 0} \sum_{\substack{p \in \mathcal{P}_R \\ p \subseteq S}} \operatorname{area}(p)$$

whenever such limit exists.

it hardly makes practical sense to expect to prove the ways linear transformations affect the areas of general shapes. The following discussion is inherently incomplete this way. Certain claims regarding area will necessarily remain unproven.

Areas and determinants

In general, the image of a set S is defined as the set of images of all the points in S. That is, if S is a subset of A and $T : A \to B$, then the image of S under T is defined by $T(S) = \{T(s) : s \in S\}$. This definition typical in all of mathematics, not just linear algebra, and applies no matter the sets A and B.

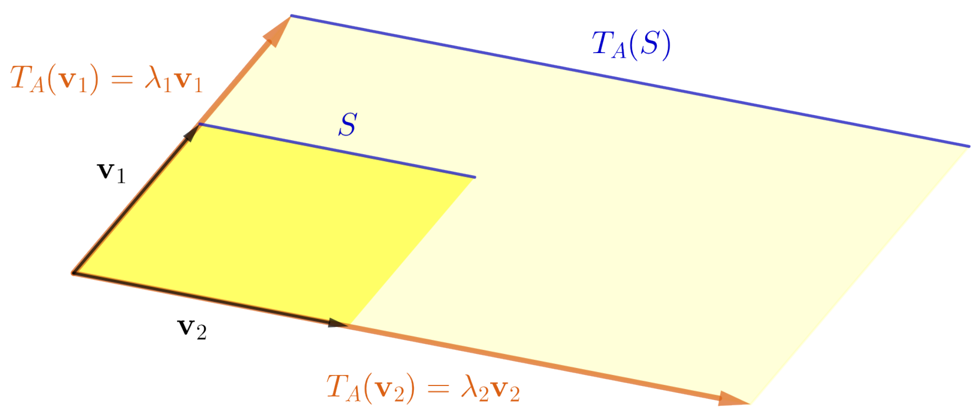
To understand how the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$, $T_A(\mathbf{v}) = A\mathbf{v}$ affects areas, it is convenient to write A is a product of elementary matrices, $A = E_p \cdots E_2 E_1$, as we have done before assuming A is invertible (page 120). Since $T_A(S) = (T_{E_p} \circ \cdots \circ T_{E_2} \circ T_{E_1})(S)$, if we can understand how linear transformations associated with elementary matrices affect area, we have a chance of understanding how general linear transformations affect area. If E is a row swap matrix, then T_E is a reflection about the line $y = x$, so in this case $\operatorname{area}(T_E(S)) = \operatorname{area}(S)$. Reflections do not change areas. If E is a row replace matrix, then T_E is a shear transformation, and it is a known result of calculus that shear transformations do not affect area, so again $\operatorname{area}(T_E(S)) = \operatorname{area}(S)$. If E is a row scale matrix, then T_E scales shapes either horizontally or vertically—not both!—by a factor of s, so $\text{area}(T_E(S)) = |s| \cdot \text{area}(S)$. In every case, $\text{area}(T_E(S)) = |\det E| \cdot \text{area}(S)$. Remember, the determinant of a row swap matrix is -1, the determinant of a row replace matrix is 1 and the determinant of a row scale matrix with scale factor s is s. Therefore,

$$\begin{aligned}
\text{area}\,(T_A(S)) &= \text{area}\left((T_{E_p} \circ \cdots \circ T_{E_2} \circ T_{E_1})(S)\right) \\
&= \text{area}\left(T_{E_p}\left(\cdots (T_{E_2}\left(T_{E_1}(S)\right)) \cdots\right)\right) \\
&= |\det E_p| \cdots |\det E_2| \cdot |\det E_1| \cdot \text{area}(S) \\
&= |\det A| \cdot \text{area}(S).
\end{aligned}$$

If A is noninvertible, then one of the columns of A is a multiple of the other, so any linear combination of the columns is also a multiple of that column. Therefore, the image of any vector, which is a linear combination of the columns of A, is a multiple of that column. Thus the image of every vector lies on the line determined by that column, giving the image of any shape area zero. The entire image is contained within a line. Of course, $|\det A| = 0$, so again we have $\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$.

Areas and eigenvalues

Let A be a 2×2 matrix with linearly independent eigenpairs $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$. Then $T_A(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1$ and $T_A(\mathbf{v}_2) = \lambda_2 \mathbf{v}_2$. In fact, if we let $S = \{\mathbf{v}_1 + \alpha \mathbf{v}_2 : 0 \le \alpha \le 1\}$, the line segment from $\mathbf{v}_1$ to $\mathbf{v}_1 + \mathbf{v}_2$, then $T_A(S) = \{T_A(\mathbf{v}_1 + \alpha \mathbf{v}_2) : 0 \le \alpha \le 1\} = \{\lambda_1 \mathbf{v}_1 + \alpha \lambda_2 \mathbf{v}_2 : 0 \le \alpha \le 1\}$ is the line segment from $T_A(\mathbf{v}_1)$ to $T_A(\mathbf{v}_2)$. Further analysis shows that the image of the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the parallelogram determined by $T_A(\mathbf{v}_1)$ and $T_A(\mathbf{v}_2)$. Can you supply this analysis? Answer on page 212.



Letting P be the parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$, we see that T_A scales P in the $\mathbf{v}_1$ direction by a factor of λ_1 and in the $\mathbf{v}_2$ direction by factor λ_2. Therefore, the area of $T_A(P)$ equals $|\lambda_1 \lambda_2|$ times the area of P. Since we have been arguing that linear transformations scale the areas of all shapes the same way, we have generally that $\text{area}(T_A(S)) = |\lambda_1 \lambda_2| \cdot \text{area}(S)$ for any shape whose area is measurable. With respect to the eigenvectors of A, T_A is a simple scaling.

Now we have that

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

and

$$\text{area}(T_A(S)) = |\lambda_1 \lambda_2| \cdot \text{area}(S).$$

It must be, then, that $|\det A| = |\lambda_1 \lambda_2|$. This is true for any 2×2 matrix including noninvertible ones, but the statement can be made much stronger, as in the following theorem.

Theorem 15. [Determinant and the Product of Eigenvalues] *If A is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are its n (possibly complex) eigenvalues, then*

$$\det A = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n.$$

Some but not all parts of the justification of this theorem are straightforward. For example, if A is upper triangular, then the conclusion follows quickly. As we have seen, $\det A$ is the product of the entries on the main diagonal. That is, $\det A = \prod_{i=1}^{n} A_{i,i}$. The characteristic equation

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} A_{1,1} - \lambda & \star & \cdots & \star \\ 0 & A_{2,2} - \lambda & \cdots & \star \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{n,n} - \lambda \end{bmatrix}\right) \\ &= (A_{1,1} - \lambda)(A_{2,2} - \lambda) \cdots (A_{n,n} - \lambda) \end{aligned}$$

has solutions $A_{1,1}, A_{2,2}, \ldots, A_{n,n}$, so the eigenvalues of A are the entries on the main diagonal of A. Hence $\prod_{i=1}^{n} A_{i,i} = \prod_{i=1}^{n} \lambda_i$ completing the proof for upper triangular matrices.

If A is any matrix, the conclusion follows from two facts.

1. The determinant and eigenvalues of $P^{-1}AP$ are the same as the determinant and eigenvalues of A for any invertible $n \times n$ matrix P.

2. For any $n \times n$ matrix A, there is an $n \times n$ matrix P such that $P^{-1}AP$ is upper triangular.

Assuming these facts true for a moment, if $U = P^{-1}AP$, then $\det A = \det U$ and the eigenvalues of A are the eigenvalues of U by fact 1. Now if P is that special matrix such that U is upper triangular, as guaranteed to exist by fact 2, then the determinant of U (which equals the determinant of A) and the product of the eigenvalues of U (which equals the product of the eigenvalues of A) are both $\prod_{i=1}^{n} U_{i,i}$ and therefore equal. This concludes the proof of the theorem for general matrices.

Justification of fact 2 takes a bit of work (see crumpet 26), but fact 1 can be justified in reasonably short order. To show that the eigenvalues of $P^{-1}AP$ are the same as the eigenvalues of A is a matter of a calculation. For any value λ,

$$\begin{aligned} \det(P^{-1}AP - \lambda I) &= \det(P^{-1}AP - P^{-1}(\lambda I)P) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1}) \cdot \det(A - \lambda I) \cdot \det P \\ &= \det(A - \lambda I) \end{aligned}$$

so the characteristic equations of $P^{-1}AP$ and A are equal making the eigenvalues of $P^{-1}AP$ and A equal. Can you show that $\det(P^{-1}AP) = \det A$? Answer on page 212.

Crumpet 26: Triangularization

For a square matrix M, $P^{-1}MP$ is a triangularization of M whenever $P^{-1}MP$ is upper triangular. We wish to show that there is a triangularization of any $n \times n$ matrix. Triangularization of a 1×1 matrix is simple enough since all 1×1 matrices are upper triangular. Choose, $P = \begin{bmatrix} 1 \end{bmatrix}$ for example. Proceeding by induction, assume a triangularization exists for every $(k-1)\times(k-1)$ matrix for some $k \geq 1$, and let M be a particular but arbitrary $k \times k$ matrix. Take any eigenpair $\lambda, \mathbf{v}$ of M and find vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}$ such that $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}\}$ is linearly independent. This set can always be found since $\mathbf{v}$ must have at least one nonzero entry ($\mathbf{0}$ is not a permissible eigenvector). Assuming $\mathbf{v}_{i,1} \neq 0$, we may take $\{\mathbf{v}, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{n-1}\} = \{\mathbf{v}\} \cup \{I_{:,j} : j \neq i\}$. Setting $Q = \begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix}$, Q is invertible (its columns are linearly independent), and

$$\begin{aligned} Q^{-1}AQ &= Q^{-1}A\begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} Q^{-1}A\mathbf{v} & Q^{-1}A\mathbf{u}_1 & Q^{-1}A\mathbf{u}_2 & \cdots & Q^{-1}A\mathbf{u}_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda Q^{-1}\mathbf{v} & Q^{-1}A\mathbf{u}_1 & Q^{-1}A\mathbf{u}_2 & \cdots & Q^{-1}A\mathbf{u}_{n-1} \end{bmatrix}. \end{aligned}$$

While we cannot say much about $Q^{-1}A\mathbf{u}_j$ for any j, we can say $\lambda Q^{-1}\mathbf{v} = \lambda I_{:,1}$ because $Q^{-1}Q = Q^{-1}\begin{bmatrix} \mathbf{v} & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{n-1} \end{bmatrix} = I$. Q^{-1} times the first column of Q must be the first column of I. Hence we have

$$Q^{-1}AQ = \begin{bmatrix} \lambda & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ \vdots & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \end{bmatrix}.$$

By the inductive hypothesis, there is a triangularization of $(Q^{-1}AQ)_{\setminus 1,1}$. Let R be such that $R^{-1}(Q^{-1}AQ)_{\setminus 1,1}R$ is upper triangular, and set $\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$. Then $\hat{Q}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix}$ and

$$\hat{Q}^{-1}(Q^{-1}AQ)\hat{Q} = \begin{bmatrix} 1 & 0 \\ 0 & R^{-1} \end{bmatrix}\begin{bmatrix} \lambda & \star \\ 0 & (Q^{-1}AQ)_{\setminus 1,1} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}$$

is upper triangular. Hence $(Q\hat{Q})^{-1}A(Q\hat{Q})$ is a triangularization of A. This result suffices for our purposes, but the result can be strengthened to specify that $(Q\hat{Q})$ have a certain property, a so-called Schur decomposition.

Hence we have two ways to measure the effect of a linear transformation on the plane. In rough terms, a linear transformation expands or compresses areas by a factor equal to the absolute value of the determinant (which is equal to the absolute value of the product of the eigenvalues) of its standard matrix. More precisely a linear transformation expands or compresses areas in the direction of each eigenvector by a factor equal to the absolute value of the associated eigenvalue.

Determinants, eigenvalues, and volumes

The analysis of elementary 3×3 matrices follows much along the same lines as the analysis of 2×2 matrices in section 4.4. Vectors in $\mathbb{R}^3$ can be imagined as arrows or points just as they are in $\mathbb{R}^2$. Images of cubes in $\mathbb{R}^3$ under transformations associated with elementary matrices analogous to the images of the coffee cup in $\mathbb{R}^2$ can be derived. They will also be a collection of reflections, shears, and scalings. Rotation in $\mathbb{R}^3$ can be accomplished by a compostion of scalings and shears just as in $\mathbb{R}^2$. Noninvertible 3×3 matrices can be described by compositions of elementary matrices and projections as well. Hence theorem 11 can be proved for operators on $\mathbb{R}^3$. Generally, if the 2's in this section are replaced by 3's and the word area is replaced by the word volume, the discourse still applies with only minor additional modification. In particular, for 3×3 matrices M with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and three-dimensional regions of space, R,

$$\begin{aligned}\text{volume}(T_M(R)) &= |\det M| \cdot \text{volume}(R) \\ &= |\lambda_1\lambda_2\lambda_3| \cdot \text{volume}(R)\end{aligned}$$

and the concluding paragraph in the discussion of transformations of the plane might be rephrased for transformations of space as follows.

We have two ways to measure the effect of a linear transformation on space, $\mathbb{R}^3$. In rough terms, a linear transformation expands or compresses volumes by a factor equal to the absolute value of the determinant (which is equal to the absolute value of the product of the eigenvalues) of its standard matrix. More precisely a linear transformation expands or compresses volumes in the direction of each eigenvector by a factor equal to the absolute value of the associated eigenvalue.

Crumpet 27: Hyperspace

The main results of this section and the previous are stated and hold for $\mathbb{R}^n$, giving an enterprising individual a basis to extend the ideas of area and volume to dimensions higher than 3! The notion of a hypercube (in hyperspace) is exactly this enterprise.

Key Concepts

set image For any transformation (map or function) $f : A \to B$ and subset S of A,

$$f(S) = \{f(s) : s \in S\}$$

determinant and area For any linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ and any subset S of $\mathbb{R}^2$ with measurable area,

$$\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$$

determinant and volume For any linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ and any subset S of $\mathbb{R}^3$ with measurable volume,

$$\text{volume}(T_A(S)) = |\det A| \cdot \text{volume}(S)$$

determinant and eigenvalues The determinant of any square matrix is the product of its eigenvalues.

triangularization For any square matrix M there is an invertible matrix P such that $P^{-1}MP$ is upper triangular.

Exercises

1. Prove that any square matrix M can be factored as PUP^{-1} for some invertible matrix P and upper triangular matrix U.
2. Use the fact that $\text{area}(T_A(S)) = |\det A| \cdot \text{area}(S)$ to justify the claim that the area of the parallelogram

determined by the columns of a 2×2 matrix A is $\det A$.

3. Use the fact that $\text{volume}(T_A(S)) = |\det A| \cdot \text{volume}(S)$ to justify the claim that the volume of the parallelepiped determined by the columns of a 3×3 matrix A is $\det A$.

Answers

further analysis The parallelogram determined by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the set $S = \{\beta\mathbf{v}_1 + \alpha\mathbf{v}_2 : 0 \leq \alpha, \beta \leq 1\}$ so its image is

$$\begin{aligned} T_A(S) &= T_A(\{\beta\mathbf{v}_1 + \alpha\mathbf{v}_2 : 0 \leq \alpha, \beta \leq 1\}) \\ &= \{T_A(\beta\mathbf{v}_1 + \alpha\mathbf{v}_2) : 0 \leq \alpha, \beta \leq 1\} \\ &= \{\beta T_A(\mathbf{v}_1) + \alpha T_A(\mathbf{v}_2) : 0 \leq \alpha, \beta \leq 1\} \\ &= \{\beta\lambda_1\mathbf{v}_1 + \alpha\lambda_2\mathbf{v}_2 : 0 \leq \alpha, \beta \leq 1\} \end{aligned}$$

which is the paralelogram determined by $T_A(\mathbf{v}_1)$ and $T_A(\mathbf{v}_2)$.

equal determinants

$$\det(P^{-1}AP) = \det(P^{-1}) \cdot \det A \cdot \det P = \frac{1}{\det P} \cdot \det A \cdot \det P = \det A$$

6.4 Approximation []

From the very beginning of our discussion of linear systems, we acknowledged that there were systems with no solution (see section 2.1 exercise 4). This was a familiar state of affairs as you undoubtedly have seen equations like $x^2 + 1 = 0$, $\sin\theta = 2$, and $\frac{1}{2+e^t} = 3$, all of which have "no solution". Full disclosure, what was meant by "no solution" was *no real number solution*. All three equations have complex number solutions. $\sqrt{-1}$, $\sin^{-1}(2)$, and $\ln\left(-\frac{5}{3}\right)$ are perfectly well defined complex numbers, and are, respectively, solutions of the three equations. It's possible you studied complex numbers enough to know this already, but it's also possible this comes as a revelation. No worries either way.

Linear systems with no solution are different. When we say they have no solution, they have no integer solution, no rational number solution, no real number solution, and no complex number solution. They simply have no solution. What more is there to say?

The linear equation

$$54x + 30y = 17 \tag{6.4.1}$$

has *no integer solution*. This can be seen by factoring a 6 from the left-hand side:

$$6(9x + 5y) = 17$$

showing that the left side is, for any integers x and y, a multiple of 6 while the right side is not. The best we can hope for are integers x and y that make $54x + 30y$ close to 17. To say it another way, we can look for integers x and y so that

$$|(54x + 30y) - 17|$$

(the distance between $54x+30y$ and 17) is small. In fact, if we could find a minimum of this quantity, that would mean something. Among all the pairs of integers x and y, this pair (or these pairs) make $54x + 30y$ as close to 17 as possible. Can you find the minimum possible value of $|(54x + 30y) - 17|$ for integers x and y? Answer on page 218.

$54(-34)+30(78) = 18$, so $(\hat{x}, \hat{y}) = (-34, 78)$ is a **best approximation** of an integer solution of (6.4.1). The pair $(-34, 78)$ does not solve (6.4.1), but it makes its two sides as close as possible (using integers). That is,

$$|(54\hat{x} + 30\hat{y}) - 17| \le |(54x + 30y) - 17|$$

for all integer pairs (x, y). We could use calculus to seek best real number approximations of $x^2 + 1 = 0$, $\sin\theta = 2$, and $\frac{1}{2+e^t} = 3$, but the point is made. Even when an equation has no solution, it may have a best approximation.

Using this discussion as a model for linear systems with no solution, we ask whether

$$M\mathbf{v} = \mathbf{b}$$

has a best approximation when inconsistent. That is, can we find $\hat{\mathbf{v}}$ such that

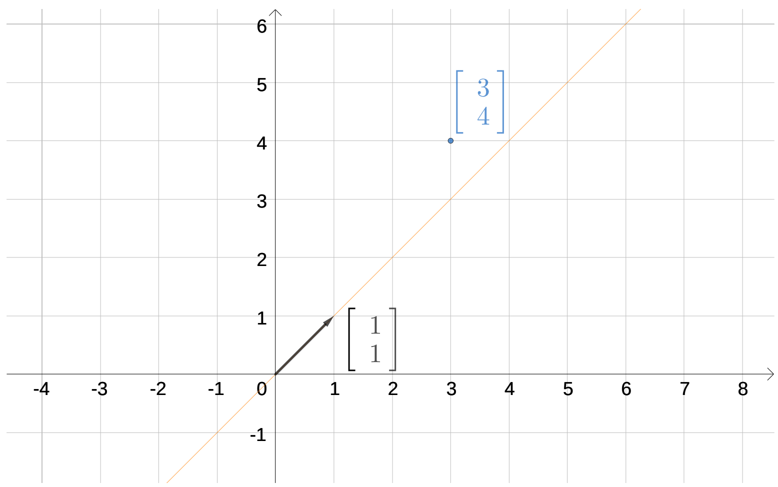
$$\|M\hat{\mathbf{v}} - \mathbf{b}\| < \|M\mathbf{v} - \mathbf{b}\|$$

for all $\mathbf{v} \neq \hat{\mathbf{v}}$? For example,

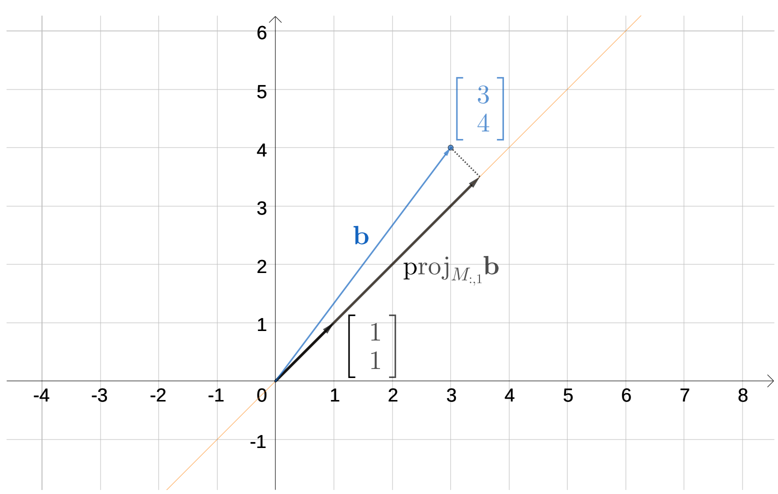
$$\begin{bmatrix} 1 & -3 \\ 4 & 9 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix} \tag{6.4.2}$$

is inconsistent. Can you show this? Answer on page 218. To say it another way, $\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$ is not in the column space of $\begin{bmatrix} 1 & -3 \\ 4 & 9 \\ -9 & 6 \end{bmatrix}$, hinting at how to find a best approximation—look inside the column space of $\begin{bmatrix} 1 & -3 \\ 4 & 9 \\ -9 & 6 \end{bmatrix}$ for a vector that is as close to $\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$ as possible.

In fact, we have done this to some extent already! $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is not in the column space of $M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ since $\mathbf{b}$ is not a multiple (linear combination) of $M_{:,1}$. Nonetheless there is a multiple (linear combination) of $M_{:,1}$ that is closest to $\mathbf{b}$. Geometrically, this means there is a point on the line determined by $M_{:,1}$ closest to $\mathbf{b}$. This situation is diagrammed here.



We know the shortest distance between a point and a line is measured perpendicularly. The point on the line where this shortest distance occurs marks exactly the (orthogonal) projection of $\mathbf{b}$ onto $M_{:,1}$, as diagrammed here.



Helpful to this discussion is to then see orthogonal projection as projecting a vector onto a subspace (rather than another vector). The line determined by $M_{:,1}$ is a subspace of $\mathbb{R}^2$ as it is the span of $M_{:,1}$. In three dimensions, there is a point in a plane nearest any point/vector not in the plane. That point occurs exactly at the projection of the vector onto the plane (and that plane is a subspace of $\mathbb{R}^3$). Again, projection is best viewed as projecting a vector onto a subspace.

With this in mind, we have to address the questions of (i) how to project a vector onto a subspace and (ii) whether that projection is always the nearest point/vector within the subspace. A lot of this work has already been done, but there is a bit more to do now.

First, let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ be an orthogonal basis for a subspace W of an inner product space V. Then the **orthogonal projection** of any $\mathbf{v}$ in V onto W, denoted $\text{proj}_W\mathbf{v}$, is defined by

$$\text{proj}_W\mathbf{v} = \text{proj}_{\mathbf{b}_1}\mathbf{v} + \text{proj}_{\mathbf{b}_2}\mathbf{v} + \cdots + \text{proj}_{\mathbf{b}_p}\mathbf{v}.$$

Because each projection is a multiple of one of the basis elements, this is a linear combination of the basis elements and therefore lies in W. Next, we will need some terminology.

If W is a subspace of an inner product space V and $\mathbf{v}$ is orthogonal to every vector in W, then we say $\mathbf{v}$ is **orthogonal** to W. The set of all vectors in V orthogonal to W is called the **orthogonal complement** of W and is denoted $W^\perp$ (read "W perp"). Can you show that $W^\perp$ is a subspace of V? Answer on page 219.

Just as $\mathbf{v}$ and $\mathbf{v} - \text{proj}_{\mathbf{w}}\mathbf{v}$ are orthogonal for any vectors $\mathbf{v}$ and $\mathbf{w}$ of an inner product space V (see section 5.3), we can now show that $\mathbf{v}$ and $\mathbf{v} - \text{proj}_W\mathbf{v}$ are orthogonal for any vector $\mathbf{v}$ and subspace W of an inner product space V. Letting $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ be an *orthogonal basis* of W,

$$\begin{aligned}
\langle \mathbf{v} - \text{proj}_W\mathbf{v}, \mathbf{b}_j\rangle &= \left\langle \mathbf{v} - \left(\text{proj}_{\mathbf{b}_1}\mathbf{v} + \text{proj}_{\mathbf{b}_p}\mathbf{v} + \cdots + \text{proj}_{\mathbf{b}_p}\mathbf{v}\right), \mathbf{b}_j\right\rangle \\
&= \left\langle \mathbf{v}, \mathbf{b}_j\right\rangle - \left\langle \text{proj}_{\mathbf{b}_1}\mathbf{v}, \mathbf{b}_j\right\rangle - \left\langle \text{proj}_{\mathbf{b}_p}\mathbf{v}, \mathbf{b}_j\right\rangle - \cdots - \left\langle \text{proj}_{\mathbf{b}_p}\mathbf{v}, \mathbf{b}_j\right\rangle \\
&= \left\langle \mathbf{v}, \mathbf{b}_j\right\rangle - \left\langle \text{proj}_{\mathbf{b}_j}\mathbf{v}, \mathbf{b}_j\right\rangle \\
&= \left\langle \mathbf{v}, \mathbf{b}_j\right\rangle - \left\langle \frac{\langle \mathbf{v}, \mathbf{b}_j\rangle}{\langle \mathbf{b}_j, \mathbf{b}_j\rangle}\mathbf{b}_j, \mathbf{b}_j\right\rangle \\
&= \left\langle \mathbf{v}, \mathbf{b}_j\right\rangle - \frac{\langle \mathbf{v}, \mathbf{b}_j\rangle}{\langle \mathbf{b}_j, \mathbf{b}_j\rangle}\left\langle \mathbf{b}_j, \mathbf{b}_j\right\rangle \\
&= 0
\end{aligned}$$

for each $j = 1, 2, \ldots, p$, so $\mathbf{v} - \text{proj}_W\mathbf{v}$ is orthogonal to every element of a basis of W. This is enough to show that $\mathbf{v} - \text{proj}_W\mathbf{v}$ is orthogonal to every vector in W and therefore $\mathbf{v} - \text{proj}_W\mathbf{v}$ is in $W^\perp$. Can you provide the details? Answer on page 219. This leads to the following theorem.

Theorem 16. [Orthogonal Decomposition] *Let W be a subspace of an inner product space V. Each $\mathbf{v} \in V$ can be written uniquely as a sum*

$$\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$$

where $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$.

Existence: we have just shown that $\mathbf{v} - \text{proj}_W\mathbf{v}$ is in $W^\perp$. Since $\text{proj}_W\mathbf{v}$ is in W and

$$\mathbf{v} = \text{proj}_W\mathbf{v} + (\mathbf{v} - \text{proj}_W\mathbf{v})$$

we have existence.

Uniqueness: suppose $\mathbf{v} = \hat{\mathbf{w}} + \hat{\mathbf{w}}^\perp$ for some (possibly other) $\hat{\mathbf{w}}$ in W and $\hat{\mathbf{w}}^\perp$ in $W^\perp$. Then, of course, $\mathbf{w} + \mathbf{w}^\perp = \hat{\mathbf{w}} + \hat{\mathbf{w}}^\perp$ so $\mathbf{w} - \hat{\mathbf{w}} = \hat{\mathbf{w}}^\perp - \mathbf{w}^\perp$. Noting that $\mathbf{w} - \hat{\mathbf{w}}$ is in W and $\hat{\mathbf{w}}^\perp - \mathbf{w}^\perp$ is in $W^\perp$, we have $\langle \mathbf{w} - \hat{\mathbf{w}}, \hat{\mathbf{w}}^\perp - \mathbf{w}^\perp\rangle = 0$. Setting $\mathbf{x} = \mathbf{w} - \hat{\mathbf{w}} = \hat{\mathbf{w}}^\perp - \mathbf{w}^\perp$, this means

$$\langle \mathbf{x}, \mathbf{x}\rangle = \langle \mathbf{w} - \hat{\mathbf{w}}, \hat{\mathbf{w}}^\perp - \mathbf{w}^\perp\rangle = 0,$$

so $\mathbf{x} = \mathbf{0}$ and therefore $\mathbf{w} = \hat{\mathbf{w}}$ and $\hat{\mathbf{w}}^\perp = \mathbf{w}^\perp$.

Corollary 17. $\mathbf{v} = proj_W\mathbf{v} + (\mathbf{v} - proj_W\mathbf{v})$ *is the unique decomposition of* $\mathbf{v}$ *into the sum of two vectors, one in* W *and one in* $W^\perp$.

Finally, we are ready to answer our original question. In the form of a theorem, we have the following.

Theorem 18. [Best Approximation] *If* W *is a subspace of an inner product space* V *and* $\mathbf{v}$ *is in* V*, then* $\mathbf{w} = proj_W\mathbf{v}$ *is the closest point to* $\mathbf{v}$ *in* W.

Justification of theorem 18 relies on a generalization of the Pythagorean theorem. Can you prove that if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (vectors of an inner product space), then $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$? Answer on page 219. Now let $\hat{\mathbf{w}}$ be any vector in W, $\hat{\mathbf{w}} \neq \mathbf{w}$. Since $\mathbf{v} - \mathbf{w}$ is in $W^\perp$ and $\mathbf{w} - \hat{\mathbf{w}}$ is in W, they are orthogonal, and the Pythagorean applies. But $(\mathbf{v} - \mathbf{w}) + (\mathbf{w} - \hat{\mathbf{w}}) = \mathbf{v} - \hat{\mathbf{w}}$ so

$$||\mathbf{v} - \hat{\mathbf{w}}||^2 = ||\mathbf{v} - \mathbf{w}||^2 + ||\mathbf{w} - \hat{\mathbf{w}}||^2 .$$

Since $\hat{\mathbf{w}} \neq \mathbf{w}$, $||\mathbf{w} - \hat{\mathbf{w}}||^2 > 0$ and therfore $||\mathbf{v} - \hat{\mathbf{w}}||^2 > ||\mathbf{v} - \mathbf{w}||^2$. In other words, $\mathbf{w}$ is the closest point to $\mathbf{v}$ in W.

Corollary 19. [Best Approximation for a Linear System] *Given any* $m \times n$ *matrix* M *and vector* $\mathbf{b}$ *in* $\mathbb{R}^m$, $\hat{\mathbf{v}} = proj_W\mathbf{b}$ *is the best approximation to a solution of* $M\mathbf{v} = \mathbf{b}$*, where* W *is the column space of* M.

Can you use theorem 18 to prove theorem 19? Answer on page 219. Finally, we can return to (6.4.2) and provide an answer. We need to project

$$\mathbf{b} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

onto the column space of

$$M = \begin{bmatrix} 1 & -3 \\ 4 & 9 \\ -9 & 6 \end{bmatrix}.$$

This requires an orthogonal basis of the column space of M. Using Gram-Schmidt orthogonalization, let

$$\mathbf{w}_1 = M_{:,2} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$$

and

$$\mathbf{w}_2 = M_{:,1} - \mathrm{proj}_{\mathbf{w}_1} M_{:,1} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix} - \frac{\langle M_{:,1}, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -9 \end{bmatrix} - \frac{-1}{6} \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix}.$$

Taking (scalar multiples of $\mathbf{w}_1$ and $\mathbf{w}_2$)

$$\{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 11 \\ -16 \end{bmatrix} \right\}$$

as the orthogonal basis of the column space of M, the projection of $\mathbf{b}$ onto this subspace is

$$\begin{aligned}\text{proj}_{\mathbf{b}_1}\mathbf{b} + \text{proj}_{\mathbf{b}_2}\mathbf{b} &= \frac{\langle \mathbf{b}, \mathbf{b}_1\rangle}{\langle \mathbf{b}_1, \mathbf{b}_1\rangle}\mathbf{b}_1 + \frac{\langle \mathbf{b}, \mathbf{b}_2\rangle}{\langle \mathbf{b}_2, \mathbf{b}_2\rangle}\mathbf{b}_2 = \frac{3}{2}\mathbf{b}_1 - \frac{7}{54}\mathbf{b}_2 \\ &= \frac{1}{27}\begin{bmatrix} -44 \\ 83 \\ 137 \end{bmatrix}.\end{aligned}$$

Hence

$$\frac{1}{27}\begin{bmatrix} -44 \\ 83 \\ 137 \end{bmatrix} \approx \begin{bmatrix} -1.63 \\ 3.07 \\ 5.07 \end{bmatrix}$$

is the closest vector to

$$\begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

in the column space of M. The distance between these two vectors happens to be

$$\begin{aligned}\left\|\frac{1}{27}\begin{bmatrix} -44 \\ 83 \\ 137 \end{bmatrix} - \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}\right\| = \frac{1}{27}\left\|\begin{bmatrix} -44 \\ 83 \\ 137 \end{bmatrix} - \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}\right\| &= \frac{1}{27}\left\|\begin{bmatrix} -52 \\ 78 \\ 130 \end{bmatrix}\right\| \\ &= \frac{26}{27}\sqrt{38} \approx 5.936\end{aligned}$$

and there is no vector in the column space of M closer to $\mathbf{b}$. See figure 6.4.1.

Key Concepts

best approximation of $M\mathbf{v} = \mathbf{b}$ is a vector $\hat{\mathbf{v}}$ such that

$$||M\hat{\mathbf{v}} - \mathbf{b}|| < ||M\mathbf{v} - \mathbf{b}||$$

for all $\mathbf{v} \neq \hat{\mathbf{v}}$. ($M$ is an $m \times n$ matrix, $\mathbf{v}, \hat{\mathbf{v}}$ are in $\mathbb{R}^n$ and $\mathbf{b}$ is in $\mathbb{R}^m$.)

best approximation theorem see theorem 18 and corollary 19.

orthogonal a vector $\mathbf{v}$ is orthogonal to a subspace W if $\mathbf{v}$ is orthogonal to every vector in W. $\mathbf{v}$ is orthogonal to W if and only if $\mathbf{v}$ is orthogonal to every vector in a basis of W.

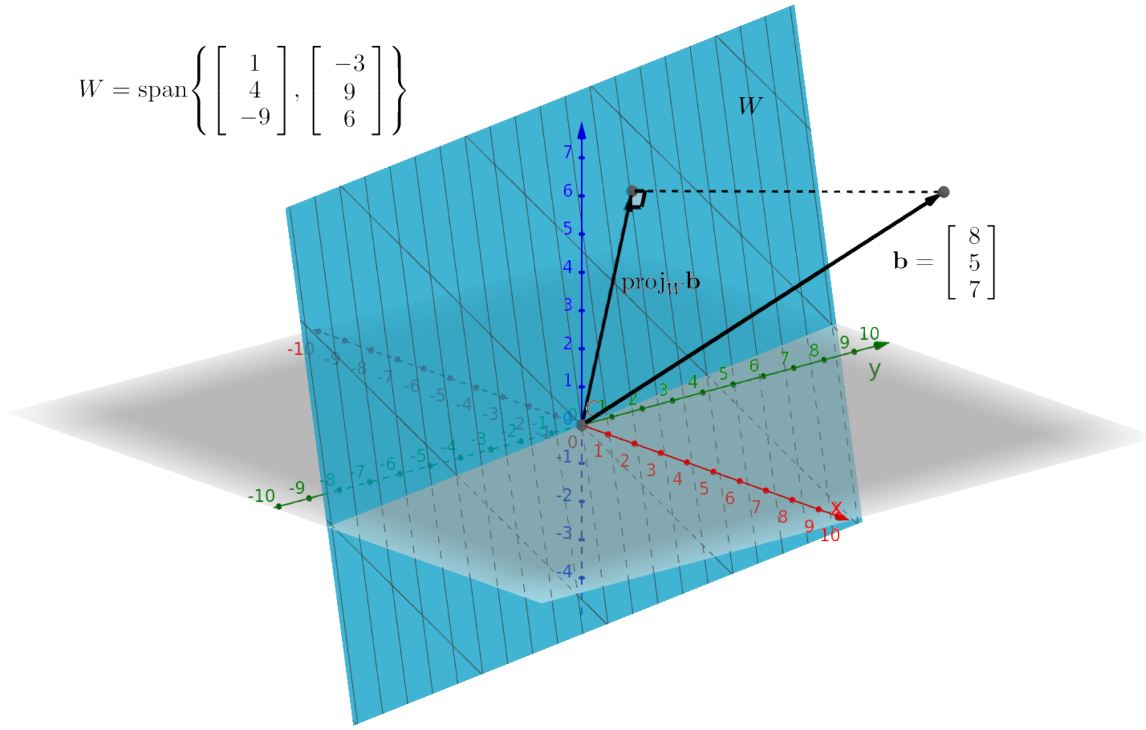
(orthogonal) projection of a vector $\mathbf{v}$ onto a subspace W, denoted $\text{proj}_W\mathbf{v}$, is defined by

$$\text{proj}_W\mathbf{v} = \text{proj}_{\mathbf{b}_1}\mathbf{v} + \text{proj}_{\mathbf{b}_2}\mathbf{v} + \cdots + \text{proj}_{\mathbf{b}_p}\mathbf{v}.$$

orthogonal complement of a subspace W is the set of all vectors orthogonal to W, denoted $W^\perp$. For any vector $\mathbf{v}$, $\mathbf{v} - \text{proj}_W\mathbf{v}$ is in $W^\perp$.

orthogonal decomposition writing $\mathbf{v}$ as a sum $\mathbf{w} + \mathbf{w}^\perp$ where $\mathbf{w}$ is in W and $\mathbf{w}^\perp$ is in $W^\perp$. See theorem 16.

Figure 6.4.1: Best Approximation of a Solution of $\begin{bmatrix} 1 & -3 \\ 4 & 9 \\ -9 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$



Exercises

1. P

Answers

minimum integer solution We know that $|(54x + 30y) - 17|$ cannot equal zero, so the best we can hope for is to find integers x and y so that $|(54x + 30y) - 17| = 1$. That is, $(54x + 30y) - 17 = 1$ or $(54x + 30y) - 17 = -1$. Adding 17 to both sides of these equations, we seek integer solutions of $54x + 30y = 18$ or $54x + 30y = 16$. Since 16 is not a multiple of 6, there is no hope of finding integer solutions of $54x+30y = 16$. Since 18 is a multiple of 6, perhaps there are integer solutions of $54x+30y = 18$. Dividing both sides of the equation by 6, $9x+5y = 3$ or $9x = 3-5y$. As long as $3-5y$ is a multiple of 9, we will have a solution. For example, $y = -3$ makes $3-5y = 18$ (and $x = 2$ makes $9x = 18$), so one solution is $(x, y) = (2, -3)$. Sure enough, $|(54\cdot2+30\cdot-3)-17| = |108-90-17| = 1$. There are others.

inconsistent system The augmented matrix for the system reduces as follows.

$$\left[\begin{array}{cc|c} 1 & -3 & 8 \\ 4 & 9 & 5 \\ -9 & 6 & 7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 8 \\ 0 & 21 & -27 \\ 0 & -21 & 79 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & 8 \\ 0 & 21 & -27 \\ 0 & 0 & 52 \end{array}\right]$$

An echelon form has a pivot in the rightmost column (the third row represents the equation $0 = 52$), so the system is inconsistent.

$W^\perp$ is a subspace Let $\mathbf{u}$ and $\mathbf{v}$ be in $W^\perp$ and c be scalar. (By definition, $\langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$ for all $\mathbf{w}$ in W). Then for any $\mathbf{w}$ in W,

$$\langle \mathbf{0}, \mathbf{w} \rangle = \langle 0\mathbf{w}, \mathbf{w} \rangle = 0\langle \mathbf{w}, \mathbf{w} \rangle = 0$$

and

$$\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle = 0$$

and

$$\langle c\mathbf{u}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle = c \cdot 0 = 0$$

so $\mathbf{0}$, $\mathbf{u} + \mathbf{v}$, and $c\mathbf{u}$ are all in $W^\perp$. Since $W^\perp$ is a subset of V, this is sufficient to show that $W^\perp$ is a subspace.

$\mathbf{v} - \mathbf{proj}_W\mathbf{v}$ is in $W^\perp$ Suppose $\mathbf{v}$ is orthogonal to every element of *any* basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_p\}$ of a subspace W. Then for any scalars $c_1, c_2, \ldots, c_p$,

$$\begin{aligned}
\langle \mathbf{v}, c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p \rangle &= \langle \mathbf{v}, c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p \rangle \\
&= \langle \mathbf{v}, c_1\mathbf{b}_1 \rangle + \langle \mathbf{v}, c_2\mathbf{b}_2 \rangle + \cdots + \langle \mathbf{v}, c_p\mathbf{b}_p \rangle \\
&= c_1\langle \mathbf{v}, \mathbf{b}_1 \rangle + c_2\langle \mathbf{v}, \mathbf{b}_2 \rangle + \cdots + c_p\langle \mathbf{v}, \mathbf{b}_p \rangle \\
&= 0.
\end{aligned}$$

Since every vector in W has the form $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_p$, this shows $\mathbf{v}$ is orthogonal to every vector in W and therefore is in $W^\perp$.

REMARK: Note that if $\mathbf{v}$ is in $W^\perp$, then $\mathbf{v}$ is orthogonal to every element of *any* basis $\mathcal{B}$ (since $\mathbf{v}$ is orthogonal to *every* vector in W—including basis vectors). Altogether we have that $\mathbf{v}$ is in $W^\perp$ if and only if $\mathbf{v}$ is orthogonal to every element of a basis of W.

Pythagorean theorem Because $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ and therefore

$$\begin{aligned}
||\mathbf{u} + \mathbf{v}||^2 &= \langle \mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + 2\langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{v} \rangle \\
&= ||\mathbf{u}||^2 + ||\mathbf{v}||^2 .
\end{aligned}$$

theorem 18 implies corollary 19 Corollary 19 is the special case of theorem 18 where $V = \mathbb{R}^m$ and $\mathbf{v} = \mathbf{b}$ (and W is the column space of M).

Chapter 7

Applications in Other Disciplines

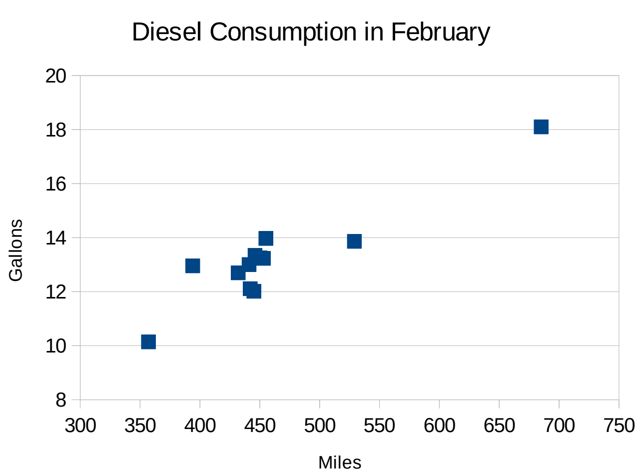
7.1 Linear Regression []

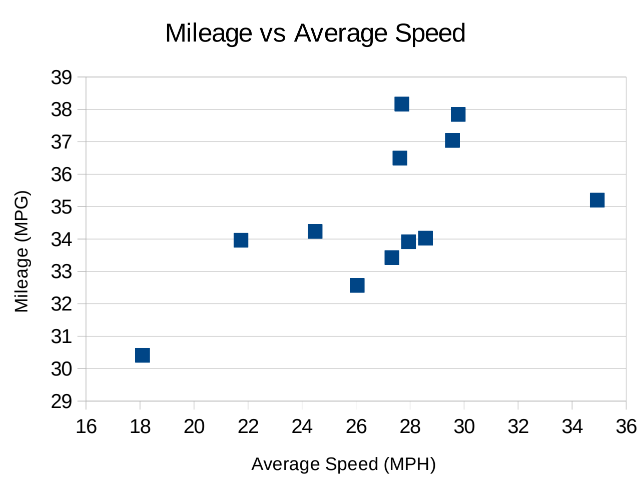
Perhaps the most ubiquitous application of linear algebra outside the boundaries of mathematics is linear regression, used to test hypotheses and produce models of phenomena in innumerable fields including meteorology, criminology, economics, materials science, archaeology, enginering, and psychology.[20, 7, 15, 5, 1, 22, 2] Anywhere two or more quantities are suspected of correlation, regression analysis can be performed. In its simplest form, two quantities are suspected of having a linear relationship. Data are collected on the two quantities, and a model (linear function) predicting one quantity based on the other is produced and analyzed.

For example, it is well known that the distance a gas or diesel powered vehicle is driven (in miles, for example) is more or less directly proportional to the volume of fuel (in gallons, for example) consumed. Also understood is that highway driving generally uses less fuel per mile than city driving. This is why statistics on new cars will include both a highway and a city mileage estimate. The graphs in figure 7.1.1 were produced from the February driving data for a 2010 VW Jetta Sportwagen TDI in the chart below. Only February data are considered because it is also known that ambient temperature affects a combustion engine's efficiency. This car was driven in New England, where the average temperature in February is around 30°F, 0°C.

The graphs confirm the claims that driving longer distances requires more fuel and that highway

Figure 7.1.1: Graphs of Diesel Data





driving uses fuel more efficiently than city driving (as average speed increases so does mileage). When trends like these are observed, linear regression provides a way to quantify the relationship between the variables in the form of a function. This function can then be used to predict one quantity from the other.

Fill-up Date	Elapsed Miles	Gallons	Average Speed	Price per Gallon
02/08/12	450	13.25	21.739	4.36
02/23/12	685	18.101	29.783	4.40
02/17/13	394	12.956	18.098	4.36
02/01/14	445	12.014	29.568	4.38
02/16/14	432	12.696	28.571	4.60
02/26/14	529	13.861	27.696	4.46
02/06/15	453	13.233	24.486	3.25
02/22/15	357	10.142	34.932	3.00
02/12/16	442	12.11	27.625	2.30
02/16/17	455	13.971	26.045	2.68
02/02/18	446	13.343	27.328	3.10
02/20/18	441	13.003	27.947	3.15

Let's say the owner of this vehicle is planning a trip from New Haven, CT to Augusta, ME (approximately 600-miles round trip) next February and is interested in how much fuel will be used. Perhaps the simplest way to estimate is to sum the elapsed miles, sum the gallons, and divide. This gives an average of about 0.0286996 gallons per mile. A 600-mile trip at this rate of consumption would require $0.0286996 \cdot 600 = 17.21976$ gallons of diesel.

For this application, that is probably good enough. However, we can do (slightly) better using linear regression. We know that fuel consumed is (roughly) directly proportional to miles driven, so they are (approximately) related by a function of the form $y = kx$. Either variable can represent either quantity, but since we are interested in predicting fuel required given distance driven, we are looking for a function of the form $f(x) = kx$ where x = distance driven and $f(x)$ is the fuel required. The estimate above produces the model $f(x) = 0.0286996x$, but what is the best value for k?

It depends on how you define "best value", but one reasonable definition is to minimize the sum of the squared errors, where an error, also known as a residual, is the difference between an observed response and the modeled response to the same input. For example, 12.956 is the observed response to the input 394 (observed on 02/17/13). The modeled response, using $f(x) = 0.0286996x$, is $f(394) = 0.0286996 \cdot 394 \approx 11.308$ gallons, however. The error, or residual, for this observation is therefore $11.308 - 12.956$ or -1.648 gallons. The squared error is $(-1.648)^2 \approx 2.716$. A similar squared error can be calculated for each observation. The sum of the squared errors is, accurate to 5 decimal places, 9.30835.

As a linear algebra problem, finding the best value of k in this sense amounts to finding the best approximation (see theorem 19) of $M\mathbf{v} = \mathbf{b}$ where

$$M = \begin{bmatrix} 450 \\ 685 \\ 394 \\ \vdots \\ 441 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} k \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13.25 \\ 18.101 \\ 12.956 \\ \vdots \\ 13.003 \end{bmatrix}$$

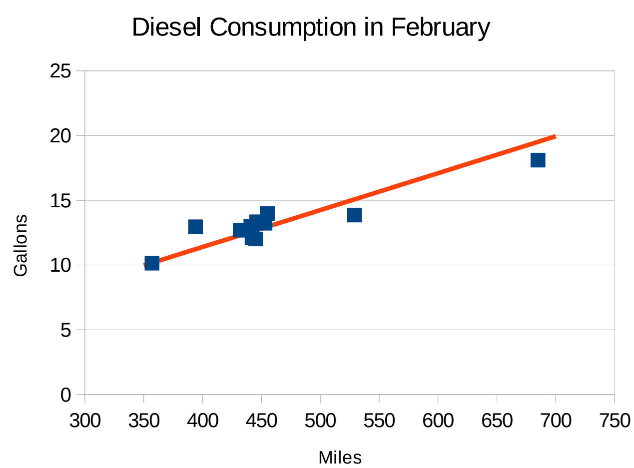
since

$$\begin{aligned} \|M\mathbf{v} - \mathbf{b}\| &= \left\| \begin{matrix} 450k - 13.25 \\ 685k - 18.101 \\ 394k - 12.956 \\ \vdots \\ 441k - 13.003 \end{matrix} \right\| \\ &= \sqrt{(450k - 13.25)^2 + \cdots + (441k - 13.003)^2}, \end{aligned}$$

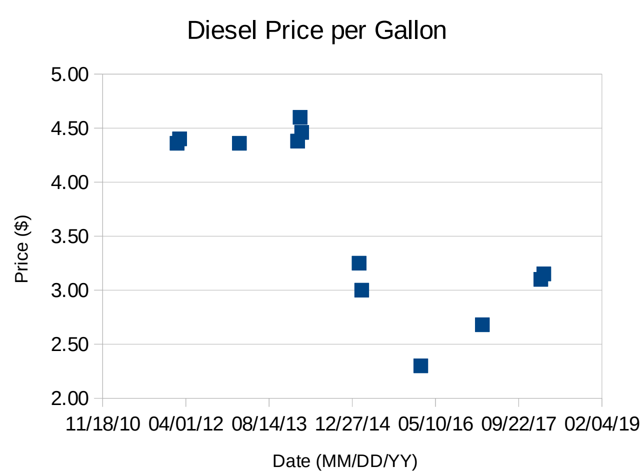
the square root of the sum of the squared errors. And 19 tells us the best approximation is the projection of $\mathbf{b}$ onto the column space of M. Letting W be the column space of M,

$$\begin{aligned} \text{proj}_W \mathbf{b} = \text{proj}_{M_{:,1}} \mathbf{b} = \frac{\mathbf{b} \cdot M_{:,1}}{M_{:,1} \cdot M_{:,1}} M_{:,1} &= \frac{74639.689}{2619895} \\ &\approx 0.0284896 M_{:,1} \end{aligned}$$

so the best value of k is 0.0284896 (giving about 17.09 gallons for a 600-mile trip). The sum of the squared errors for the model $f(x) = 0.0284896x$ is $\left\| M \left[\; 0.0284896 \; \right] - \mathbf{b} \right\|^2 \approx 9.19278$—slightly lower than the 9.30835 we got from the model $f(x) = 0.0286996x$. Plotting the model on the same axes as the data illustrates the closeness of fit.



The driving data provide other opportunities for linear regression. Plotting the price of diesel over time produces the following graph.



This graph shows an overall downward trend in price over the six year span of the data. Linear regression can be used to capture this overall trend. If we are interested in an average decresase in price over this time span, we could find the best fit model of the form $p(t) = p_0 + rt$, a linear model whose slope estimates the average annual drop in price.

As a linear algebra problem, we wish to find the best approximation to $M\mathbf{v} = \mathbf{b}$ where M holds the inputs, $\mathbf{b}$ holds the responses, and $\mathbf{v}$ holds the unknown parameters p_0 and r. In this case, the input variable is time, which we will measure in days since 1 February 2012:

$$M = \begin{bmatrix} 1 & 7 \\ 1 & 22 \\ 1 & 382 \\ \vdots & \vdots \\ 1 & 2211 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} p_0 \\ r \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4.36 \\ 4.40 \\ 4.36 \\ \vdots \\ 3.15 \end{bmatrix},$$

which has best approximation

$$\begin{bmatrix} p_0 \\ r \end{bmatrix} \approx \begin{bmatrix} 4.55591498045673 \\ -0.000845069933663 \end{bmatrix}.$$

Since we measured time in days, r represents the average change in price per day, not year. To get an annual change, we multiply r by 365 to get -0.308, an average decrease of approximately 31 cents per year.

The graph of diesel price over time does not indicate a steady decline, however. While the overall trend is downward, there is a fluctuation as well. A more accurate model of the actual price over this time period would come from a model that caputures this fluctuation. For example, a model of the form $f(t) = p_0 + rt + \alpha\sin(\omega t) + \beta\cos(\omega t)$ might provide reasonable results since it includes a linear portion $(p_0 + rt)$ to capture the overall decrease and periodic portion $(\alpha\sin(\omega t) + \beta\cos(\omega t))$ to capture the fluctuation. Linear regression only approximates parameters that very linearly with respect to the response, though. Since ω and f are not linearly related, the value $\omega = \frac{2\pi}{5\times 365}$ will be assumed. This value of ω gives the sine and cosine functions a 5-year period.

As a linear algebra problem, we wish to find the best approximation to $M\mathbf{v} = \mathbf{b}$ where M holds the inputs, $\mathbf{b}$ holds the responses, and $\mathbf{v}$ holds the unknown parameters. Again, time will be measured in days since 1 February 2012.

$$M = \begin{array}{c} \begin{array}{cccc} & t & \sin(\omega t) & \cos(\omega t) \end{array} \\ \begin{bmatrix} 1 & 7 & 0.0172134 & 0.999852 \\ 1 & 22 & 0.0540754 & 0.998537 \\ 1 & 382 & 0.807206 & 0.590269 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 2211 & -0.748605 & 0.663016 \end{bmatrix} \end{array}, \quad \mathbf{v} = \begin{bmatrix} p_0 \\ r \\ \alpha \\ \beta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4.36 \\ 4.40 \\ 4.36 \\ \vdots \\ 3.15 \end{bmatrix},$$

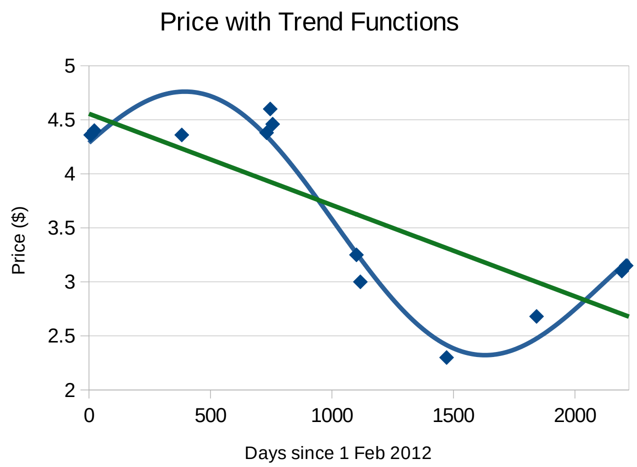
which has best approximation

$$\begin{bmatrix} p_0 \\ r \\ \alpha \\ \beta \end{bmatrix} \approx \begin{bmatrix} 4.47274711624545 \\ -0.000907047402863 \\ 0.71642901672004 \\ -0.22510779212064 \end{bmatrix}.$$

The sums of the squared errors for the two models

$$
\begin{aligned}
p(x) &= 4.55591 - 845070(10)^{-4}t \\
f(x) &= 4.47275 - 907047(10)^{-4}t + 0.716429\sin(\omega t) - 0.225108\cos(\omega t)
\end{aligned}
$$

are 3.02939 and 0.460556, respectively. The graphs of the two models superimposed on the data clearly illustrate how much closer f comes to predicting the observed data.



For a final linear regression on the February driving data, we return to the thought that mileage is affected by driving speed. One model that incorporates this fact is $g(x, s) = (k + \beta s)x = kx + \beta sx$, where x is distance (as before) and s is average speed. The number of gallons consumed per mile, $(k + \beta s)$, varies with average speed s and $g(0, s) = 0$. This model is slightly different from the ones we have derived so far. Here, we have two input variables, making this a multiple linear regression, or multilinear regression. The principle is the same, however. We wish to find the best approximation of $M\mathbf{v} = \mathbf{b}$ where M holds the inputs, $\mathbf{b}$ holds the responses, and $\mathbf{v}$ holds the unknown parameters. In this case,

$$
\begin{array}{cc}
 & \begin{array}{cc} x & sx \end{array} \\
M = & \begin{bmatrix} 450 & 9782.55 \\ 685 & 20401.355 \\ 394 & 7130.612 \\ \vdots & \vdots \\ 441 & 12324.627 \end{bmatrix}
\end{array}, \quad \mathbf{v} = \begin{bmatrix} k \\ \beta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13.25 \\ 18.101 \\ 12.956 \\ \vdots \\ 13.003 \end{bmatrix},
$$

which has best approximation

$$
\begin{bmatrix} k \\ \beta \end{bmatrix} \approx \begin{bmatrix} 0.0379147381262610 \\ -0.000346513745672586 \end{bmatrix}
$$

and sum of squared errors 5.21664. Compare this to the 9.30835 we found without considering average driving speed.

Given that the hypothetical trip from New Haven to Augusta is to be driven mostly on the highway, we can approximate the required fuel by, for example,

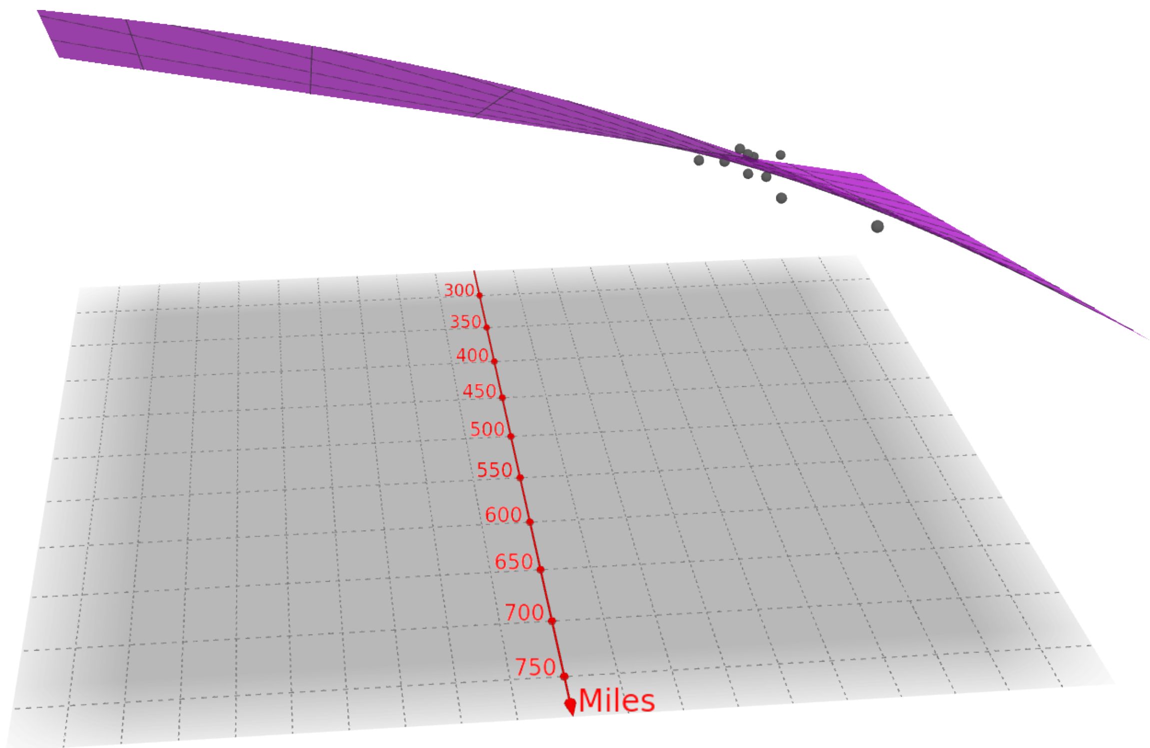
$$
\begin{aligned}
g(600, 45) &= (0.0379147 - 0.000346514(45))600 \\
&\approx 13.39,
\end{aligned}
$$

significantly different from the original estimates of over 17 gallons. The 13.39 gallon estimate should be met with some skepticism, however. It uses an average speed of 45 miles per hour while the highest average speed for which we have data is about 35 miles per hour. There is no evidence that the model applies to an average speed of 45 miles per hour. More data and possibly a revision to the model should be considered before using an average speed of 45. On the other hand, hypothesizing that it is reasonable to expect the car's efficiency to be better at an average speed of 45 miles per hour than it is at an average speed of 35 miles per hour, we can use the model with an average speed of 35 to get an (expected) overestimate of the required volume of fuel.

$$\begin{aligned} g(600, 35) &= (0.0379147 - 0.000346514(35))600 \\ &\approx 15.47 \end{aligned}$$

is still considerably less than 17—and likely an overestimate.

A linear regression model with two input parameters is, geometrically, a regression surface. A plot of $g(x, s)$ with the twelve data points is shown below.



Normal Equations

As presented in section 6.4, the calculation of a best approximation involves projecting onto the column space of a coefficient matrix, requiring an orthogonal basis for the column space. While (Gram-Schmidt) orthogonalization can be applied to find such a basis, the process is computationally intensive and more detrimental to the results, error prone. In practice, the normal equations

$$M^T M\mathbf{v} = M^T\mathbf{b}$$

are solved instead. It is known that $\mathbf{v}$ is a solution of the normal equations if and only if $\mathbf{v}$ is a best approximation to a solution of $M\mathbf{v} = \mathbf{b}$.

Crumpet 28: Normal Equations

Letting W be the column space of M, an $m \times n$ matrix, the following statements are equivalent.

1. $\hat{\mathbf{v}}$ is a best approximation of $M\mathbf{v} = \mathbf{b}$.
2. $M\hat{\mathbf{v}}$ is the closest point to $\mathbf{b}$ in the column space of M.
3. $M\hat{\mathbf{v}} = \text{proj}_W \mathbf{b}$.
4. $\mathbf{b} - M\hat{\mathbf{v}}$ is in $W^\perp$.
5. $(\mathbf{b} - M\hat{\mathbf{v}})^T M_{:,j} = 0$ for all $j = 1, 2, \ldots, n$.
6. $M^T M\hat{\mathbf{v}} = M^T\mathbf{b}$.

$1 \Leftrightarrow 2$ by definition of best approximation. $2 \Leftrightarrow 3$ by theorem 19. $3 \Rightarrow 4$ by the fact that $(\mathbf{b} - \text{proj}_W\mathbf{b})$ is in $W^\perp$. $4 \Rightarrow 3$ by the facts that (i) $\mathbf{b} = M\hat{\mathbf{v}} + (\mathbf{b} - M\hat{\mathbf{v}})$; (ii) $M\mathbf{v}$ is in W; (iii) $\mathbf{b} - M\hat{\mathbf{v}}$ is in $W^\perp$; and (iv) corollary 17. $4 \Leftrightarrow 5$ by definition of $W^\perp$ and the fact that each column of M is in W. $5 \Leftrightarrow 6$ by matrix algebra: for each $j = 1, 2, \ldots, n$, $(\mathbf{b} - M\hat{\mathbf{v}})^T M_{:,j} = 0 \Leftrightarrow M_{:,j}^T(\mathbf{b} - M\hat{\mathbf{v}}) = 0 \Leftrightarrow M_{:,j}^T\mathbf{b} - M_{:,j}^T M\hat{\mathbf{v}} = 0 \Leftrightarrow M_{:,j}^T\mathbf{b} = M_{:,j}^T M\hat{\mathbf{v}}$, this last equality being true for all j if and only if $M^T M\hat{\mathbf{v}} = M^T\mathbf{b}$.

Because the set of best approximations of $M\mathbf{v} = \mathbf{b}$ equals precisely the solution set of $M^T M\hat{\mathbf{v}} = M^T\mathbf{b}$, the linear system $M\mathbf{v} = \mathbf{b}$ has a unique best approximation for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if $M^T M\hat{\mathbf{v}} = M^T\mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$. By theorem 7 $M^T M\hat{\mathbf{v}} = M^T\mathbf{b}$ has a unique solution for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if $M^T M\hat{\mathbf{v}} = \mathbf{0}$ has only the trivial solution if and only if $M^T M$ is invertible. Hence the linear system $M\mathbf{v} = \mathbf{b}$ has a unique best approximation for each $\mathbf{b}$ in $\mathbb{R}^m$ if and only if $M^T M$ is invertible.

Solving the normal equations amounts to solving a linear system of p equations in p variables where p is the number of parameters (not the number of data points). The normal equations represent a relatively small system with known, dependable solution techniques.

For example, the model $p(t) = p_0 + rt$, which came from a best approximation with

$$M = \begin{bmatrix} 1 & 7 \\ 1 & 22 \\ 1 & 382 \\ \vdots & \vdots \\ 1 & 2211 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} p_0 \\ r \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4.36 \\ 4.40 \\ 4.36 \\ \vdots \\ 3.15 \end{bmatrix}$$

can be solved by first computing

$$M^T M = \begin{bmatrix} 12 & 12580 \\ 12580 & 19526278 \end{bmatrix} \quad \text{and} \quad M^T\mathbf{b} = \begin{bmatrix} 44.04 \\ 40812.34 \end{bmatrix}$$

and then solving

$$\begin{bmatrix} 12 & 12580 \\ 12580 & 19526278 \end{bmatrix} \begin{bmatrix} p_0 \\ r \end{bmatrix} = \begin{bmatrix} 44.04 \\ 40812.34 \end{bmatrix}.$$

Can you provide this solution? Answer on page 228.

Key Concepts

least squares solution a best approximation $\hat{\mathbf{v}}$ of $M\mathbf{v} = \mathbf{b}$ having sum of squared errors $||M\hat{\mathbf{v}} - \mathbf{b}||$.

sum of squared errors given observations (X_i, y_i), $i = 1, 2, \ldots, N$, and model $y = f(X)$, $(f(X_1) - y_1)^2 + (f(X_2) - y_2)^2 + \cdots + (f(X_N) - y_N)^2$.

linear regression given a model of the form $f(X) = \beta_1 f_1(X) + \beta_2 f_2(X) + \cdots + \beta_p f_p(X)$ and observations (X_i, y_i), $i = 1, 2, \ldots, N$, linear regression refers to finding a best approximation of

$$\begin{bmatrix} f_1(X_1) & f_2(X_1) & \cdots & f_p(X_1) \\ f_1(X_2) & f_2(X_2) & \cdots & f_p(X_2) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(X_N) & f_2(X_N) & \cdots & f_p(X_N) \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}.$$

normal equations $M^T M\mathbf{v} = M^T\mathbf{b}$, whose solutions coincide precisely with best approximations of $M\mathbf{v} = \mathbf{b}$.

multiple linear regression linear regression with more than one (multiple) input variables.

multilinear regression another name for multiple linear regression.

Exercises

1. I

Answers

linear system solution The solution can be reached by row reduction or the inverse method. Since the coefficient matrix is 2×2 and it is easy enough to compute a 2×2 inverse, that is probably the easiest route:

$$\begin{aligned} \begin{bmatrix} 12 & 12580 \\ 12580 & 19526278 \end{bmatrix}^{-1} &= \frac{1}{12(19526278) - 12580^2} \begin{bmatrix} 19526278 & -12580 \\ -12580 & 12 \end{bmatrix} \\ &= \begin{bmatrix} \frac{19526278}{76058936} & -\frac{12580}{76058936} \\ -\frac{12580}{76058936} & \frac{12}{76058936} \end{bmatrix} \approx \begin{bmatrix} 0.256726 & -1.65398(10)^{-4} \\ -1.65398(10)^{-4} & 1.57772(10)^{-7} \end{bmatrix} \end{aligned}$$

so

$$\begin{aligned} \mathbf{v} = \begin{bmatrix} p_0 \\ r \end{bmatrix} &\approx \begin{bmatrix} 0.256726 & -1.65398(10)^{-4} \\ -1.65398(10)^{-4} & 1.57772(10)^{-7} \end{bmatrix} \begin{bmatrix} 44.04 \\ 40812.34 \end{bmatrix} \\ &\approx \begin{bmatrix} 4.556 \\ -8.451(10)^{-4} \end{bmatrix}. \end{aligned}$$

7.2 Markov Chains []

From https://www.capitalbikeshare.com/about

> Capital Bikeshare is metro DC's bikeshare system, with more than 4,300 bikes available at 500 stations across six jurisdictions: Washington, DC; Arlington, VA; Alexandria, VA; Montgomery County, MD; Prince George's County, MD; Fairfax County, VA; and the City of Falls Church, VA. Capital Bikeshare provides residents and visitors with a convenient, fun and affordable transportation option for getting from Point A to Point B.
>
> Capital Bikeshare, like other bikeshare systems, consists of a fleet of specially designed, sturdy and durable bikes that are locked into a network of docking stations throughout the region. The bikes can be unlocked from any station and returned to any station in the system, making them ideal for one-way trips. People use bikeshare to commute to work or school, run errands, get to appointments or social engagements and more.
>
> Capital Bikeshare is available for use 24 hours a day, 7 days a week, 365 days a year. Riders have access to a bike at any station across the system.

Capital Bikeshare makes their trip data available to the general public free of charge. The data includes (i) duration of trip, (ii) start date and time, (iii) end date and time, (iv) starting station name and number, (v) ending station name and number, and more.[1]

The following chart was processed from real Capital Bikeshare data for the year 2018. It shows the total number of rides that started and ended in the section of Alexandria containing bike stations 31041 through 31048. The locations of the stations are to the right of the chart. The total number of rides accounted for is $10,364$.

		From								
		31041	**31042**	**31043**	**31044**	**31045**	**31046**	**31047**	**31048**	
To	**31041**	664	163	77	99	103	55	265	256	**Prince St & Union St**
	31042	124	519	152	159	161	101	658	710	**Market Square / King St & Royal St**
	31043	66	206	102	87	58	129	611	18	**Saint Asaph St & Pendleton St**
	31044	56	121	55	156	32	73	114	501	**King St & Patrick St**
	31045	41	128	41	22	70	121	64	187	**Commerce St & Fayette St**
	31046	20	97	98	41	76	96	70	11	**Henry St & Pendleton St**
	31047	172	561	568	88	64	180	182	28	**Braddock Rd Metro**
	31048	78	215	17	197	41	13	33	93	**King St Metro South**
	Total:	**1221**	**2010**	**1110**	**849**	**605**	**768**	**1997**	**1804**	**10364**

From these data, linear algebra can be applied to estimate the distribution of bicycles among the stations. The method begins by dividing each column of the chart by the total number of rides in the column. The first column is divided by 1221, the second column by 2010, and so on, resulting in a table whose columns all sum to 1. Accurate to five decimal places, this normalized chart is collected in the matrix M:

[1] See https://www.capitalbikeshare.com/system-data.

$$M = \begin{bmatrix} 0.54382 & 0.08109 & 0.06937 & 0.11661 & 0.17025 & 0.07161 & 0.13270 & 0.14191 \\ 0.10156 & 0.25821 & 0.13694 & 0.18728 & 0.26612 & 0.13151 & 0.32949 & 0.39357 \\ 0.05405 & 0.10249 & 0.09189 & 0.10247 & 0.09587 & 0.16797 & 0.30596 & 0.00998 \\ 0.04586 & 0.06020 & 0.04955 & 0.18375 & 0.05289 & 0.09505 & 0.05709 & 0.27772 \\ 0.03358 & 0.06368 & 0.03694 & 0.02591 & 0.11570 & 0.15755 & 0.03205 & 0.10366 \\ 0.01638 & 0.04826 & 0.08829 & 0.04829 & 0.12562 & 0.12500 & 0.03505 & 0.00610 \\ 0.14087 & 0.27910 & 0.51171 & 0.10365 & 0.10579 & 0.23438 & 0.09114 & 0.01552 \\ 0.06388 & 0.10697 & 0.01532 & 0.23204 & 0.06777 & 0.01693 & 0.01652 & 0.05155 \end{bmatrix}$$

M thereby represents the percentage of rides starting at the station represented by the column that end at the station represented by the row. For example, the 0.06937 in the first row, third column means that 6.937% of rides starting at station number 31043 ended at station number 31041. Empirically speaking, about 7% of the bikes at station number 31043 are destined for station number 31041.

Multiplying $M_{2,1}M_{1,5}$ then represents the percentage of bikes at the station of column 5 (31045) that are destined for the station of column 2 (31042) after being dropped at the station of column 1 (31041). Similarly, $M_{2,2}M_{2,5}$ represents the percentage of bikes at station 31045 that are destined for station 31042 via station 31042 (the second ride is from station 31042 back to station 31042); $M_{2,3}M_{3,5}$ represents the percentage of bikes at station 31045 that are destined for station 31042 via station 31043; and so on. The sum $M_{2,1}M_{1,5} + M_{2,2}M_{2,5} + \cdots + M_{8,2}M_{8,5}$ therefore represents the total percentage of bikes at station 31045 that will end up at station 31042 *after two rides*. Notice that's just a row-column product (row 2 times column 5), which is the $2,5$-entry of M^2.

Generalizing, the i, j-entry of

$$M^2 = \begin{bmatrix} 0.34772 & 0.14568 & 0.14367 & 0.16169 & 0.17933 & 0.14162 & 0.14917 & 0.16962 \\ 0.18009 & 0.25757 & 0.26034 & 0.24737 & 0.21787 & 0.22429 & 0.20081 & 0.22320 \\ 0.09918 & 0.14711 & 0.20639 & 0.09837 & 0.11594 & 0.15050 & 0.11192 & 0.09361 \\ 0.07128 & 0.08900 & 0.06889 & 0.13178 & 0.08122 & 0.07528 & 0.06639 & 0.10299 \\ 0.04552 & 0.05952 & 0.05190 & 0.06237 & 0.07117 & 0.06664 & 0.05208 & 0.05619 \\ 0.03239 & 0.05021 & 0.05196 & 0.04321 & 0.06101 & 0.07067 & 0.05955 & 0.05025 \\ 0.15859 & 0.18732 & 0.17162 & 0.16729 & 0.20517 & 0.21018 & 0.29330 & 0.17834 \\ 0.06525 & 0.06360 & 0.04523 & 0.08794 & 0.06829 & 0.06081 & 0.06678 & 0.12580 \end{bmatrix}$$

holds the percentage of bikes starting at station $3104j$ that end up at station $3104i$ after two rides. Likewise, the i, j-entry of M^k holds the percentage of bikes starting at station $3104j$ that end up at station $3104i$ after k rides. Multiplying M^2 by itself,

$$M^4 = \begin{bmatrix} 0.22039 & 0.18022 & 0.17861 & 0.18487 & 0.18748 & 0.17934 & 0.18079 & 0.18673 \\ 0.21605 & 0.22893 & 0.23122 & 0.22754 & 0.22432 & 0.22723 & 0.22265 & 0.22529 \\ 0.12247 & 0.13283 & 0.13920 & 0.12638 & 0.12854 & 0.13309 & 0.12804 & 0.12492 \\ 0.08042 & 0.08277 & 0.08089 & 0.08616 & 0.08189 & 0.08129 & 0.08003 & 0.08512 \\ 0.05346 & 0.05606 & 0.05568 & 0.05638 & 0.05587 & 0.05612 & 0.05539 & 0.05599 \\ 0.04633 & 0.05067 & 0.05076 & 0.04970 & 0.05058 & 0.05155 & 0.05181 & 0.04994 \\ 0.19212 & 0.20053 & 0.19847 & 0.19753 & 0.20252 & 0.20392 & 0.21272 & 0.19884 \\ 0.06877 & 0.06799 & 0.06518 & 0.07144 & 0.06880 & 0.06746 & 0.06857 & 0.07318 \end{bmatrix}$$

and then M^4 by itself,

$$M^8 = \begin{bmatrix} 0.19015 & 0.18855 & 0.18844 & 0.18879 & 0.18885 & 0.18851 & 0.18857 & 0.18887 \\ 0.22448 & 0.22495 & 0.22501 & 0.22487 & 0.22483 & 0.22494 & 0.22486 & 0.22484 \\ 0.12913 & 0.12955 & 0.12965 & 0.12942 & 0.12944 & 0.12956 & 0.12947 & 0.12938 \\ 0.08181 & 0.08186 & 0.08185 & 0.08189 & 0.08185 & 0.08185 & 0.08183 & 0.08189 \\ 0.05532 & 0.05542 & 0.05542 & 0.05541 & 0.05540 & 0.05542 & 0.05541 & 0.05541 \\ 0.04985 & 0.05004 & 0.05005 & 0.05001 & 0.05001 & 0.05005 & 0.05005 & 0.05000 \\ 0.20067 & 0.20111 & 0.20109 & 0.20102 & 0.20108 & 0.20116 & 0.20127 & 0.20103 \\ 0.06858 & 0.06853 & 0.06849 & 0.06858 & 0.06855 & 0.06852 & 0.06854 & 0.06859 \end{bmatrix}$$

and so on,

$$M^{16} = \begin{bmatrix} 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 \\ 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 \\ 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 \\ 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 \\ 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 \\ 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 \\ 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 \\ 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 \end{bmatrix}$$

and

$$M^{32} = \begin{bmatrix} 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 & 0.18890 \\ 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 & 0.22483 \\ 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 & 0.12944 \\ 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 & 0.08185 \\ 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 & 0.05540 \\ 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 & 0.05000 \\ 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 & 0.20104 \\ 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 & 0.06855 \end{bmatrix}.$$

Notice that (i) accurate to five decimal places, $M^{16} = M^{32}$, and (ii) the columns of M^{16} are all the same! Higher powers of M will be no different.

This means that after 16 rides each, about 18.89% of bikes from station 31041 will end up at station 31041. 18.89% of bikes from station 31042 will end up at station 31041. 18.89% of bikes from station 31043 will end up at station 31041. Altogether, then, about 18.89% of all the bikes in the neighborhood will end up at station 31041. Likewise, about 22.48% of all the bikes in the neighborhood will end up at station 2, 12.94% at station 3, and so on. No matter how the bikes are initially distributed, they will end up distributed this way after a short time, and stay that way (so long as the empirical percentages hold).

Crumpet 29: Why it Works

Suppose M is a positive column-stochastic matrix. The Gershgorin circle theorem ensures that M^T (and therefore M) has no eigenvalue with magnitude greater than one and its only possible eigenvalue with magnitude equal to one is 1. Letting $\mathbf{1}$ be the column vector whose entries are all 1, note that $M^T\mathbf{1} = \mathbf{1}$ (this is equivalent to saying the rows of M^T sum to 1) so 1 is indeed an eigenvalue of M^T.

Hence M has dominant eigenvalue 1. Since $M^T - I$ is real, its null space admits a basis of real vectors. Suppose $\mathbf{w} \neq k\mathbf{1}$ is a real nonzero vector in the null space of $M^T - I$, and assume $\mathbf{w}$ has at least one positive entry (if it does not, multiply it by -1). Now set

$$\alpha_{max} = \max\{\alpha : \mathbf{1} - \alpha\mathbf{w} \text{ is nonnegative}\}$$

(the set is nonempty since it contains 0 and closed since the limit of a nonnegative sequence is nonnegative, so it has a maximum). Now $\mathbf{u} = \mathbf{1} - \alpha_{max}\mathbf{w}$ has at least one zero entry (if it does not, then α_{max} is not maximal). But $\mathbf{u}$ is then a nonnegative eigenvector (of the positive matrix M) and hence must be positive, contradicting that it has at least one zero entry. Thus no such $\mathbf{w}$ exists, and the eigenspace of 1 is one-dimensional. A dominant eigenvalue with a one-dimensional eigenspace is exactly what is needed for the power method to work (section 6.2). In this case, the matrix M is stochastic, so instead of computing $\mathbf{v}_k = M^k\mathbf{v}_0$, we simply calculate M^k. The entries of M^k will never tend to 0 or infintity since powers of stochastic matrices are stochastic, so we do not have to worry about scaling after each iteration. We can then multiply any nonzero $\mathbf{v}_0$ by M^k to find the approximation $\mathbf{v}_k$ of the dominant eigenvector.

Vocabulary

- A **positive matrix** is one whose entries are all positive.
- A **nonnegative matrix** is one who entries are all nonnegative.
- A **stochastic matrix** is a nonnegative matrix whose columns each sum to 1.

Lemmas

In each of the following lemmas, M is an $n \times n$ matrix.

- If M is positive and $\mathbf{w}$ is a nonnegative eigenvector, then $\mathbf{w}$ is positive.

  Proof. Since eigenvectors are nonzero and $\mathbf{w}$ is nonnegative, $\mathbf{w}$ has a positive entry, say its i^{th}. If follows that

  $$(M\mathbf{w})_{j,1} = \sum_{k=1}^{n} M_{j,k}\mathbf{w}_{k,1} \geq M_{j,i}\mathbf{w}_{i,1} > 0$$

  for each $j = 1, 2, \ldots, n$. □

- The eigenvalues of M and M^T are the same.

  Proof. Since the determinants of a matrix and its transpose are equal (section 3.5), for any scalar λ,

  $$\det(M - \lambda I) = \det(M - \lambda I)^T = \det(M^T - (\lambda I)^T) = \det(M^T - \lambda I).$$

  Hence M and M^T have the same characteristic equation and therefore the same eigenvalues. □

- If the entries of M are real numbers and λ is a real number, then the null space of $M - \lambda I$ admits a basis of vectors with real entries.

  Proof. Since the null space of any matrix can be found through row reduction, and row reducing a real matrix does not requre complex numbers, a real basis of $M - \lambda I$ (which has real entries) exists. □

- Gershgorin circle theorem: Every (possibly complex) eigenvalue of M lies in at least one disk with center $M_{i,i}$ and radius $r_i = \sum_{j \neq i} |M_{i,j}|$, $i = 1, 2, \ldots, n$.

  Proof. Suppose $\lambda, \mathbf{w}$ is an eigenpair of M and let $\mathbf{w}_{i,1}$ be the entry of $\mathbf{w}$ with greatest magnitude. Because $M\mathbf{w} = \lambda\mathbf{w}$, we have for each $i = 1, 2, \ldots, n$, $\lambda x_{i,1} = \sum_{j=1}^{n} M_{i,j} x_{j,1}$. Hence

  $$(\lambda - M_{i,i})x_{i,1} = \lambda x_{i,1} - M_{i,i}x_{i,1} = \sum_{j \neq i} M_{i,j}x_{j,1}$$

  from which it follows

  $$\left|\lambda - M_{i,i}\right| = \frac{1}{|x_{i,1}|}\left|\sum_{j \neq i} M_{i,j}x_{j,1}\right| \leq \sum_{j \neq i} |M_{i,j}| \cdot \left|\frac{x_{j,1}}{x_{i,1}}\right| \leq \sum_{j \neq i} |M_{i,j}|.$$

  □

- Powers of stochastic matrices are stochastic.

  Proof. Let S be a stochastic matrix. By definition, $\mathbf{1}^T S = \mathbf{1}^T$ (the sum of each column of S is one). By induction, assume S^k is stochastic for some $k \geq 1$. Then

  $$\mathbf{1}^T S^{k+1} = \left(\mathbf{1}^T S^k\right) S = \mathbf{1}^T S = \mathbf{1}^T$$

  so the columns of S^{k+1} sum to one. Of course powers of nonnegative matrices are nonnegative, so S^{k+1} is stochastic. □

Suppose Capital Bikeshare supplies each station with the same number of bicycles. That information can be recorded in a column vector with length eight and each entry equal to $\frac{1}{8}$. After one month, we assume that the empirical data on transitions from one station to another is reasonably accurate, so the distribution of bicycles will be approximately

$$M \begin{bmatrix} \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{8} \end{bmatrix} \approx \begin{bmatrix} 0.166 \\ 0.226 \\ 0.116 \\ 0.103 \\ 0.071 \\ 0.062 \\ 0.185 \\ 0.071 \end{bmatrix}.$$

Multiplying the distribution vector by the **transition matrix** M gives the new distribution of bicycles among the stations. After another month, the distribution of bicycles will be approximately

$$M\begin{bmatrix}0.166\\0.226\\0.116\\0.103\\0.071\\0.062\\0.185\\0.071\end{bmatrix}\approx M^2\begin{bmatrix}\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\end{bmatrix}\approx\begin{bmatrix}0.180\\0.226\\0.128\\0.086\\0.058\\0.052\\0.196\\0.073\end{bmatrix}$$

and after four months,

$$M^2\begin{bmatrix}0.180\\0.226\\0.128\\0.086\\0.058\\0.052\\0.196\\0.073\end{bmatrix}\approx M^4\begin{bmatrix}\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\end{bmatrix}\approx\begin{bmatrix}0.187\\0.225\\0.129\\0.082\\0.056\\0.050\\0.201\\0.069\end{bmatrix}$$

and after eight months,

$$M^4\begin{bmatrix}0.187\\0.225\\0.129\\0.082\\0.056\\0.050\\0.201\\0.069\end{bmatrix}\approx M^8\begin{bmatrix}\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\\\frac{1}{8}\end{bmatrix}\approx\begin{bmatrix}0.189\\0.225\\0.129\\0.082\\0.055\\0.050\\0.201\\0.069\end{bmatrix}$$

which is, accurate to three decimal places, equal to the columns of M^{16} (and M^{32} for that matter). The distribution of bicycles does not change much after the first two months.

More importantly, the distribution we see after 8 months will be the eventual distribution of bicycles no matter the initial distribution! Given any initial distribution of bicycles, $\begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix}^T$ (where $\sum_{i=1}^{8} w_i = 1$ and each w_i is nonnegative),

$$M^{16}\begin{bmatrix}w_1\\w_2\\w_3\\w_4\\w_5\\w_6\\w_7\\w_8\end{bmatrix}\approx\begin{bmatrix}0.18890\sum_{i=1}^{8} w_i\\0.22483\sum_{i=1}^{8} w_i\\0.12944\sum_{i=1}^{8} w_i\\0.08185\sum_{i=1}^{8} w_i\\0.05540\sum_{i=1}^{8} w_i\\0.05000\sum_{i=1}^{8} w_i\\0.20104\sum_{i=1}^{8} w_i\\0.06855\sum_{i=1}^{8} w_i\end{bmatrix}=\sum_{i=1}^{8} w_i\begin{bmatrix}0.18890\\0.22483\\0.12944\\0.08185\\0.05540\\0.05000\\0.20104\\0.06855\end{bmatrix}=\begin{bmatrix}0.18890\\0.22483\\0.12944\\0.08185\\0.05540\\0.05000\\0.20104\\0.06855\end{bmatrix}.$$

This means the distribution of bicycles in the long-run is given by this vector, $\mathbf{v} = M_{i,1}$, the eigenvector of M corresponding with eigenvalue 1. This distribution is called the steady-state distribution because if reached, it never deviates: $M\mathbf{v} = \mathbf{v}$. We would expect to see the bicycles distributed among the stations in these proportions after adequate time.

Formalities

The sequence of bicycle distributions in the Capital Bikeshare scenario is an example of a discrete-time **Markov chain**. Any Markov chain necessarily consists of a set of **states** (stations in our example), a set of probabilities that some object (bicycle in our example) will transition from one state to any other depending only on its current state after one time step, and an initial distribution among the possible states. The matrix M, where $M_{i,j}$ is the probability of transitioning from state j to state i, is the **transition matrix**, and each distribution $\mathbf{v}_n$ other than $\mathbf{v}_0$ statisfies $\mathbf{v}_n = M\mathbf{v}_{n-1}$.

Other situations that can be modeled by Markov chains include

1. board games whose movement is determined by the roll of a die, such as Snakes and Ladders (the spaces on the board are the states, the game piece is the object, and the roll of the die provides the transition probabilities);

2. sentence construction, as used by computer auto-completion (the states are the words of a specific language, the object is the reader's focus, and the probability of one word following another in a sentence provides the transition probabilites);

3. a closed economy—one where a set of commodities is produced and consumed by the same group (the states are the sectors of the economy, the commodities are the objects, and transitioning is interpreted as consumption);

4. the weather (the states are weather conditions such as sunny, cloudy, and rainy, the object is the weather, and the transition probabilities are the conditional probabilities that one weather condition will follow another on, say, the next day);

5. gambling (the states are the amounts of money the gambler could have, the object is the gambler, and the likelihoods of winning or losing certain amounts of money provide the transition probabilities);

6. a single server queue (the states are the possible sizes of the queue, the object is the queue, and the likelihoods of increasing or decreasing the size of the queue by certain amounts provide the transition probabilities).

Key Concepts

transition matrix a matrix M where $M_{i,j}$ is the probability of transitioning from state j to state i in one time step. The entries of M are nonnegative and each column of M sums to 1.[2]

Markov chain a sequence of distributions arising from an initial distribution $\mathbf{v}_0$ and the recurrence $\mathbf{v}_n = M\mathbf{v}_{n-1}$, $n > 0$ for some transition matrix M.[3]

state one of the possible conditions of the object associated with a Markov chain.

steady-state distribution a distribution $\mathbf{v}$ such that $M\mathbf{v} = \mathbf{v}$ (by definition an eigenvector corresponding to eigenvalue 1).

properties of a transition matrix If M is a transition matrix,

[2] A square matrix whose entries are nonnegative and whose columns each sum to 1 is also called a **stochastic matrix** (whether it models state transition probabilities or not).

[3] In a more general setting, the transition matrix may change with time, and would then be replaced by M_n.

1. each column of M sums to one;
2. each entry of M is nonnegative;
3. 1 is one of its eigenvalues;
4. none of its eigenvalues has magnitude greater than one.

If, additionally, all the entries of some power, M^k, of M are all positive,

1. 1 is a dominant eigenvalue;
2. the eigenspace of 1 is one-dimensional;
3. each column of M^k approaches the same eigenvector, that corresponding with the eigenvalue 1, the steady-state vector.

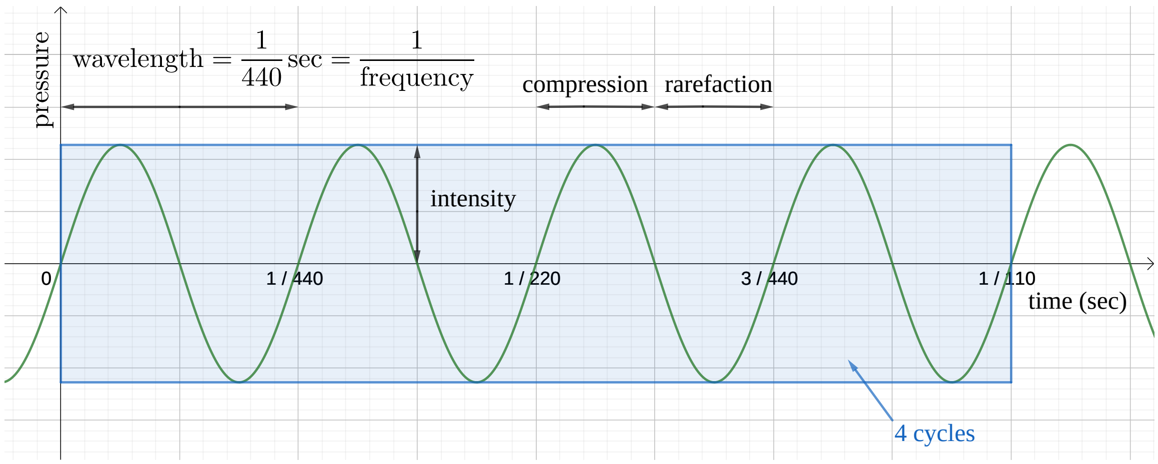
Exercises

1. I

Answers

l

Figure 7.3.1: Mathematical Model of a 440 Hz Simple Harmonic Sound Wave



7.3 Fourier Series []

Sound is the perception of pressure variation. Tuning forks, speakers, musical instruments, voice boxes, whistles, and anything else that makes sound must somehow cause varying pressure. One common way to create pressure variation is through physical vibration. Tuning forks, speakers, the strings of stringed instruments, and vocal cords all use this technique. Their vibrations cause alternating moments of compression (increased pressure) and rarefaction (reduced pressure) in the air.

The greater the difference in high and low pressures, the louder the sound. Sometimes the pressure difference (volume or intensity) is so great, our whole bodies vibrate. Thunder, subwoofers, fireworks, and helicopters can do this, but for most sounds it is only our eardrums that perceive the pressure variation.

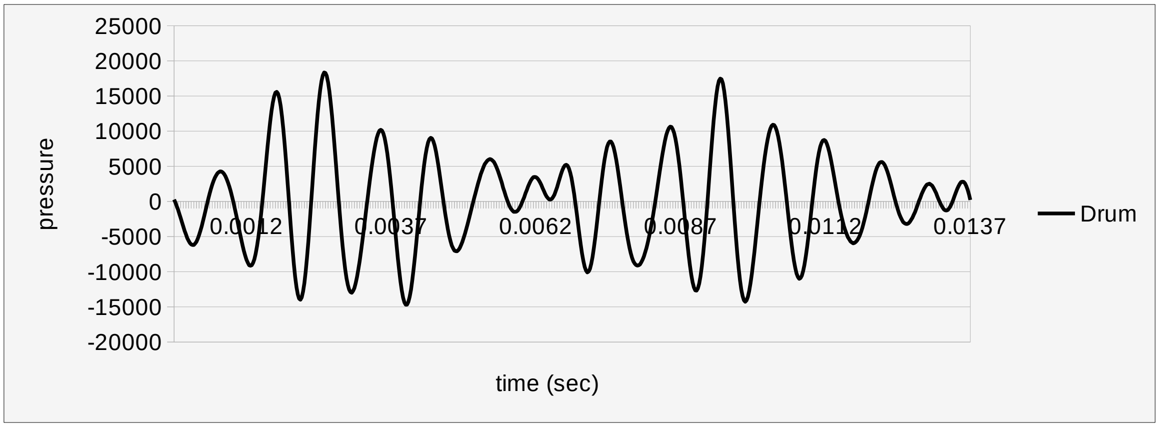
The faster the pressure alternates between high and low, the higher the pitch. Middle C, for example, is the result of pressure varying from neutral to high to low and back to neutral approximately 261.6 times per second. Each variation through neutral, high, low and back to neutral is one cycle, so we also say that middle C has a frequency of 261.6 cycles per second. One Hertz, abreviated Hz, equals one cycle per second, so we also say middle C has a frequency of 261.6 Hz. The highest note on a piano, B_8, has a frequency of about 7902.1 Hz and the lowest note on a piano has a frequency of about 16.35 Hz.

The human ear is capable of perceiving frequencies between about 20 Hz and 20,000 Hz. Air pressure can certainly alternate between high and low slower than 20 cycles per second and faster than 20,000 cycles per second. Our ears just won't pick up those vibrations. Dog whistles ideally emit sound between 23,000 Hz and 54,000 Hz[4], above the range of human hearing but within the range of canine hearing. Elephants were the first land animals to be observed to produce sound below the range of human hearing[18], creating calls with frequencies as low as 14 Hz.

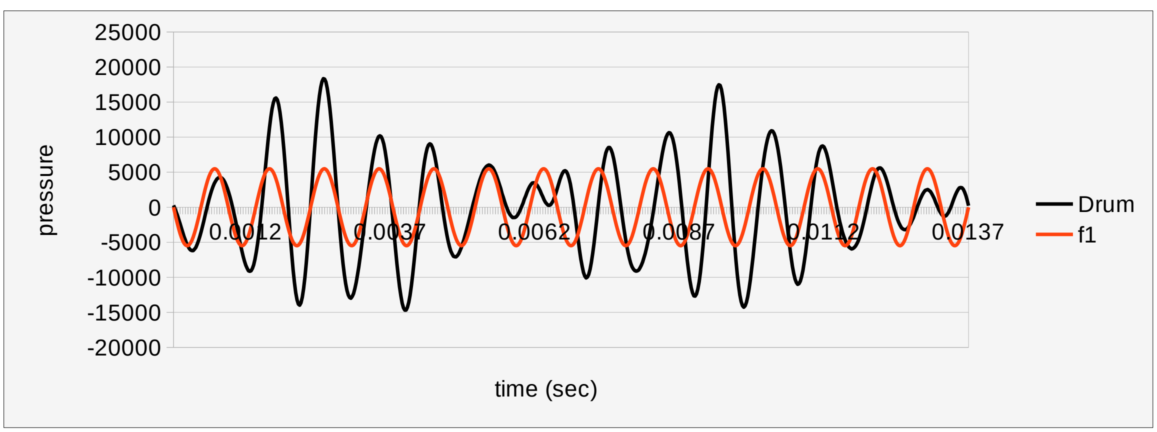
A sound wave, therefore can be modeled by a record of the pressure it causes on a receiver such as a microphone or eardrum over time. The simplest type of sound waves are simple harmonic vibrations– one intensity and one frequency, shifting from high to low pressure smoothly, like a sine curve. Until the advent of electronics, the closest approximation to a simple harmonic vibration was the sound of a tuning fork. Over the course of a second or so, neither the frequency nor the intensity of a vibrating tuning fork changes appreciably, and the vibrations are sinusoidal. The graph of a 440 Hz sine wave is a mathematical model of this sound. See figure 7.3.1.

Naturally produced sounds are not so neat. Even a tuning fork does not produce sound in a perfect sine wave. Its intensity decreases continuously as it rings, and air particles do not compress and rarefy

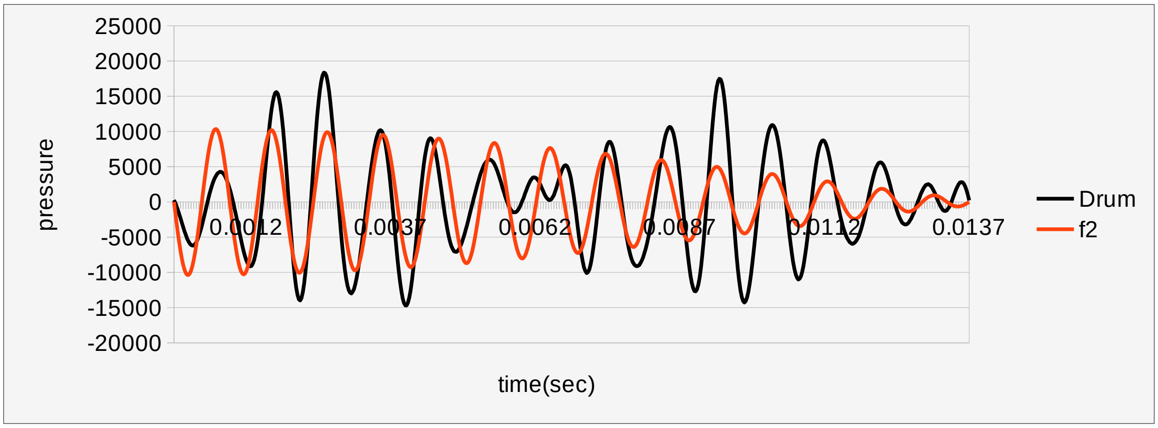
Figure 7.3.2: Two cycles of the sound of a drum.



in a perfectly sinusoidal pattern. The matter is even more complex for musical instruments. Even on a stringed instrument where a string vibrates at a steady frequency, the body picks up the vibration and imparts its own signature frequencies to the sound. Wind and percussion instruments are the same. The richness of their sound comes from a variable intensity patchwork of many frequencies. The graph of two cycles of an actual recording of a drum are shown in figure 7.3.2. The wave is clearly not sinusoidal, featuring 8 peaks and 8 valleys per cycle. In a sense, the best sinusoidal approximation of this sound wave is shown below as $f1(t) = -5498.21 \sin\left(29 \cdot \frac{8820}{121}\pi t\right)$, a frequency of $\frac{1}{2} \cdot 29 \cdot \frac{8820}{121} \approx 1056.9$ Hz.



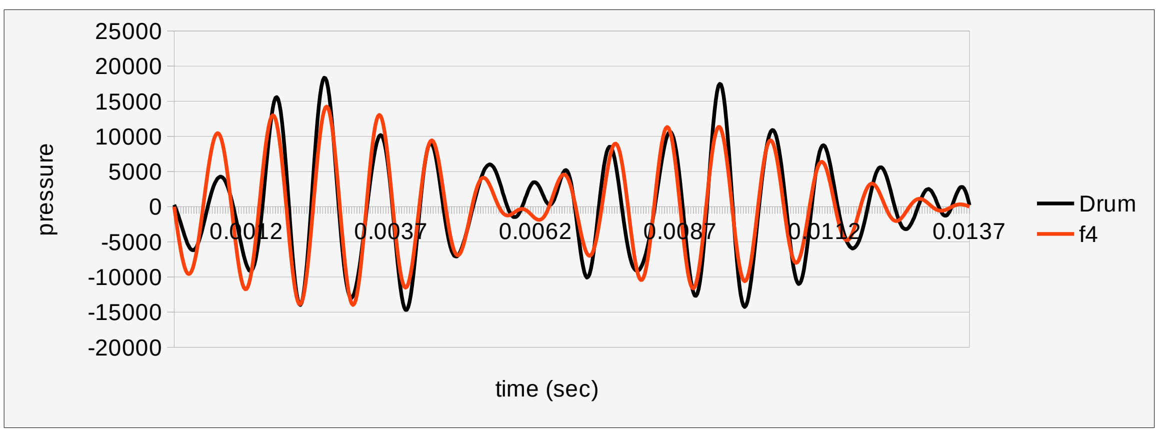
As expected, this sine wave does a poor job of approximating the drum wave. But we can do better. Allowing the sum of two sine waves, we can improve the approximation considerably, shown below as $f2(t) = -5498.21 \sin\left(29 \cdot \frac{8820}{121}\pi t\right) - 4891.4 \sin\left(28 \cdot \frac{8820}{121}\pi t\right)$, a combination of frequencies 1056.9 Hz and $\frac{1}{2} \cdot 28 \cdot \frac{8820}{121} \approx 1020.5$ Hz.



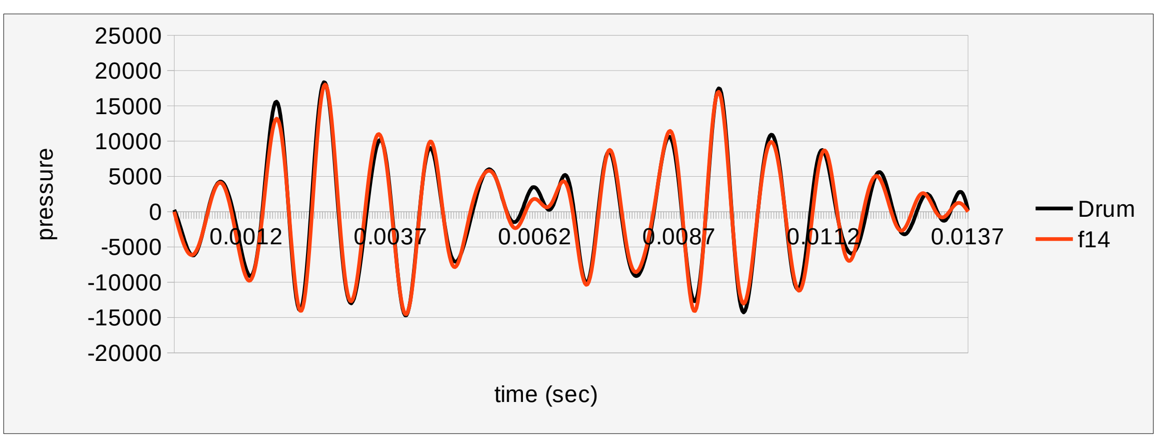
The approximation now peters out as does the drum wave, and the peaks and valleys match better. Allowing a combination of four sinusoidal waves,

$$f4(t) = -5498.21\sin\left(29\cdot\frac{8820}{121}\pi t\right) - 4891.4\sin\left(28\cdot\frac{8820}{121}\pi t\right) + 4394.8\sin\left(33\cdot\frac{8820}{121}\pi t\right) - 3469.0\sin\left(31\cdot\frac{8820}{121}\pi t\right)$$

the approximation continues to improve, as seen here.



The more sinusoidal waves allowed, the better the approximation. The differences largely disappear with the allowance of 14 sinusoidal waves:



In order of greatest to least intensity, the frequencies represented in $f14$ are 1056.9, 1020.5, 1202.7, 1129.8, 984.0, 1166.3, 291.6, 1640.1, 1093.4, 1457.9, 1567.2, 1275.6, 328.0, and 1239.2 Hz. For the brief moment represented in the graph $\left(\frac{121}{8820} \approx 0.01372 \text{ sec}\right)$, this means the sound of the drum was dominated by these 14 frequencies.

Similarly examining a full 2 seconds of the audio reveals that the overall dominant frequencies of this particular drum sound, in order from most to least dominant are 290, 1177, and 1027 Hz. The note played was likely D_4, equivalent to the D string on a violin or viola played open.

But how do we know which frequencies and intensities to use? In principle, the answer is simple. Theorem 18 of section 6.4 provides the answer. Orthogonal projection of the function gives the right intensities. All we need are an appropriate inner product space and a basis for some subspace. Except in extreme cases, pressure waves, and therefore sound waves, vary continuously, so it makes sense to consider the vector space of continuous functions over some interval. In particular, let $V_{[0,L]} = \{f(x) : f \text{ is continuous over } [0, L]\}$. In an analysis class, this space would likely be denoted $C[0, L]$. $C[0, L] = V_{[0,L]}$ is the set of all functions which are continuous on the closed interval $[0, L]$. Much like the justification that $\mathbb{P}_n(\mathbb{R})$ is a vector space from section 4.1, it is a short exercise to note that $V_{[0,L]}$ is a vector space (exercise 1).

Much like the formulation of the inner product $\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$ on $\mathbb{P}_2(\mathbb{R})$ from section 4.6, $\langle f, g\rangle = \int_0^L f(x)g(x)\,dx$ defines an inner product on $V_{[0,L]}$. Can you justify it? Answer on page 247.

Vectors in $V_{[0,L]}$ are continuous functions, so a basis for any subspace will have to be a collection of functions which are continuous on $[0, L]$. Collections of trigonometric functions provide convenient bases because for all $m = 1, 2, \ldots$ and all $n = 1, 2, \ldots$

$$\left\langle \cos\left(\frac{m\pi}{L}t\right), \cos\left(\frac{n\pi}{L}t\right)\right\rangle = 0 \tag{7.3.1}$$

and

$$\left\langle \sin\left(\frac{m\pi}{L}t\right), \sin\left(\frac{n\pi}{L}t\right)\right\rangle = 0 \tag{7.3.2}$$

whenever $m \neq n$. That is, for any distinct positive integers m and n, $\cos\left(\frac{m\pi}{L}t\right)$ and $\cos\left(\frac{n\pi}{L}t\right)$ are orthogonal, as are $\sin\left(\frac{m\pi}{L}t\right)$ and $\sin\left(\frac{n\pi}{L}t\right)$. Inner products (7.3.1) and (7.3.2) can be verified with the help of two trigonometric identities. Recall that $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$, so

$$\frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right] = \cos\alpha\cos\beta \tag{7.3.3}$$

and

$$\frac{1}{2}\left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right] = \sin\alpha\sin\beta. \tag{7.3.4}$$

Can you use (7.3.4) to show (7.3.2)? Answer on page 247. Additionally, when $m = 0$, the function $\cos\left(\frac{m\pi}{L}t\right) = 1$ for all t and provides another function orthogonal to all $\cos\left(\frac{m\pi}{L}t\right)$, $m > 0$. And now we have our candidates for a vector space, $V_{[0,L]}$, and basis elements, $\sin\left(\frac{m\pi}{L}t\right)$ with $m > 0$ or $\cos\left(\frac{m\pi}{L}t\right)$ with $m \geq 0$.

When the function you are trying to approximate takes the value zero at both endpoints of the interval of interest, as is the case for the sound waves we have looked at, it makes best sense to use sine functions for a basis. Every sine function of the form $\sin\left(\frac{m\pi}{L}t\right)$ takes the value zero at both $t = 0$ and $t = L$ so the approximation is exact at the endpoints no matter how many basis elements are used. If the function were nonzero at either endpoint, it would make more sense to use cosine functions for a basis as this allows approximation of the nonzero endpoint(s).

Crumpet 30: A theorem of Fejér

[Fejer, 1900] *Let f be a continuous function on $[-\pi, \pi]$ for which $f(-\pi) = f(\pi)$. Then the sequence of Cesàro means of the partial sums of the Fourier series for f converges uniformly to f on $[-\pi, \pi]$.*[21] This theorem applies to any continuous function on $[0, \pi]$ by extending it as an even function over $[-\pi, \pi]$ (in which case the sine terms all have zero Fourier coefficients, yielding a cosine series) or, when $f(0) = f(\pi) = 0$, as an odd function over $[-\pi, \pi]$ (in which case the cosine terms all have zero Fourier coefficients, yielding a sine series). If f additionally has a piecewise continuous first derivative, then the sequence of partial sums of the Fourier series for f converge uniformly to f on $[-\pi, \pi]$. For many physical applications, such as the sound waves discussed here, the functions we are trying to approximate have continuous first derivatives and therefore their extensions have piecewise continuous first derivatives, and the theorem applies. By proper scaling, these results can be modified to apply to domains such as $[-L, L]$ or $[0, L]$.

The upshot of results like this is that there exist bases of trigonometric functions for subspaces containing vectors (linear combinations of trigonometric functions) arbitrarily close to a given vector (function). In other words, certain functions can be approximated with arbitrary precision using sums of sines and cosines.

Given a continuous function, we choose a set of sine functions or a set of cosine functions as basis for a subspace and we project onto the subspace to find the best approximation. For example, consider approximating $f(x) = x(x-1)(x-2)$ over the interval $[0, 2]$. How closely can we approximate f by vectors in the spans of

1. $\mathcal{B}_1 = \left\{1, \cos\left(\frac{\pi}{2}t\right), \cos(\pi t)\right\}$ (from the family of cosine functions with $m = 0, 1, 2$)?

   Answer: The best approximation is $\mathbf{v} = \mathrm{proj}_{\mathrm{span}\{\mathcal{B}_1\}} f$:

$$\frac{\langle f, 1\rangle}{\langle 1, 1\rangle}1 + \frac{\left\langle f, \cos\left(\frac{\pi}{2}t\right)\right\rangle}{\left\langle \cos\left(\frac{\pi}{2}t\right), \cos\left(\frac{\pi}{2}t\right)\right\rangle}\cos\left(\frac{\pi}{2}t\right) + \frac{\langle f, \cos(\pi t)\rangle}{\langle \cos(\pi t), \cos(\pi t)\rangle}\cos(\pi t).$$

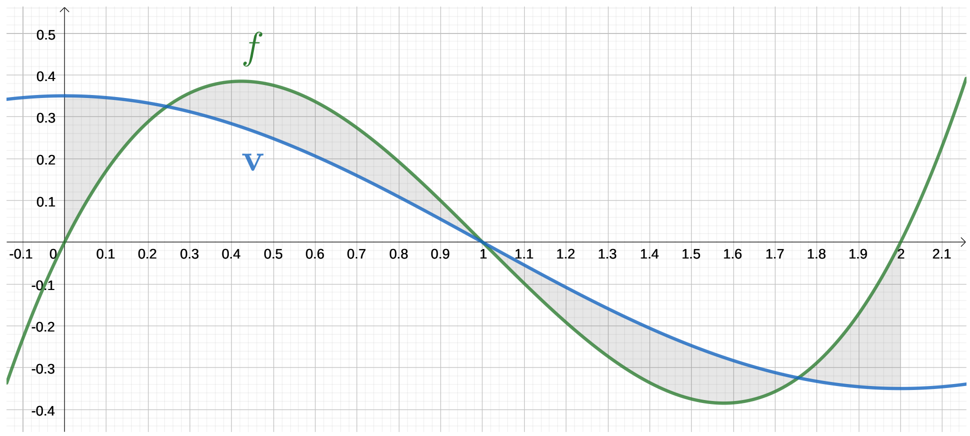
   Using a computer algebra system to help with the integration,

$$\mathbf{v} = -16\frac{\pi^2 - 12}{\pi^4}\cos\left(\frac{\pi}{2}t\right).$$

   Due to symmetry, the first and third terms are zero. The distance between $\mathbf{v}$ and f (see section 4.6) is

$$d(f, \mathbf{v}) = \|f - \mathbf{v}\| = \sqrt{\langle f - \mathbf{v}, f - \mathbf{v}\rangle} \approx 0.0299.$$

   The graphs of f and $\mathbf{v}$ are shown below with the area between the two shaded over the interval $[0, 2]$. Since the distance between f and $\mathbf{v}$ involves integrating $(f - \mathbf{v})^2$, the shaded area does not represent the distance between the vectors exactly, but it helps give a visual sense of this distance.



2. $\mathcal{B}_2 = \left\{\sin\left(\frac{\pi}{2}t\right), \sin(\pi t), \sin\left(\frac{3\pi}{2}t\right)\right\}$ (from the family of sine functions with $m = 1, 2, 3$)?

Answer: The best approximation is $\mathbf{w} = \text{proj}_{\text{span}\{\mathcal{B}_2\}} f$:

$$\frac{\left\langle f, \sin\left(\frac{\pi}{2}t\right)\right\rangle}{\left\langle \sin\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right)\right\rangle} \sin\left(\frac{\pi}{2}t\right) + \frac{\langle f, \sin(\pi t)\rangle}{\langle \sin(\pi t), \sin(\pi t)\rangle} \sin(\pi t) + \frac{\left\langle f, \sin\left(\frac{3\pi}{2}t\right)\right\rangle}{\left\langle \sin\left(\frac{3\pi}{2}t\right), \sin\left(\frac{3\pi}{2}t\right)\right\rangle} \sin\left(\frac{3\pi}{2}t\right).$$

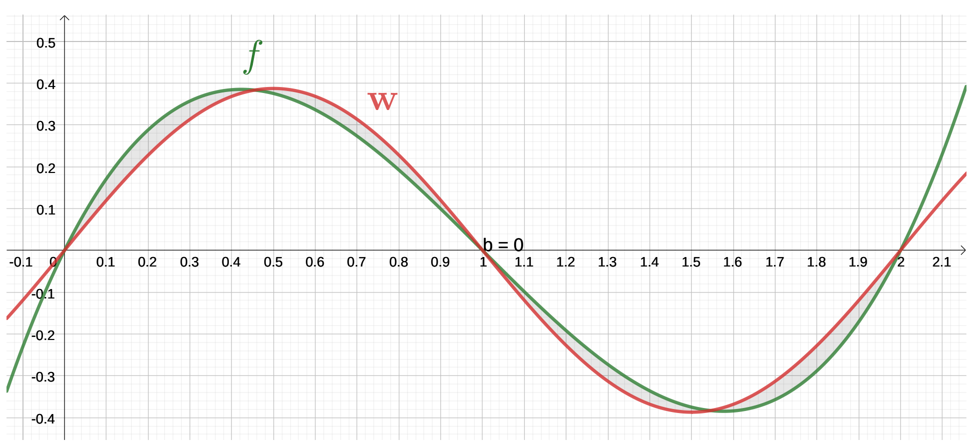
Using a computer algebra system to help with the integration,

$$\mathbf{w} = \frac{12}{\pi^3} \sin(\pi x).$$

Due to symmetry the first and third terms are zero again. The distance between $\mathbf{w}$ and f is

$$d(f, \mathbf{w}) = \sqrt{\langle f - \mathbf{w}, f - \mathbf{w}\rangle} \approx 0.0026.$$

The graphs of f and $\mathbf{w}$ are shown below with the area between the two shaded.



3. $\mathcal{B}_3 = \left\{\sin\left(\frac{m\pi}{2}t\right) : m = 1, 2, \ldots, 10\right\}$?

Answer: The best approximation is $\mathbf{x} = \text{proj}_{\text{span}\{\mathcal{B}_3\}} f$:

$$\sum_{m=1}^{10} \frac{\left\langle f, \sin\left(\frac{m\pi}{2}t\right)\right\rangle}{\left\langle \sin\left(\frac{m\pi}{2}t\right), \sin\left(\frac{m\pi}{2}t\right)\right\rangle} \sin\left(\frac{m\pi}{2}t\right).$$

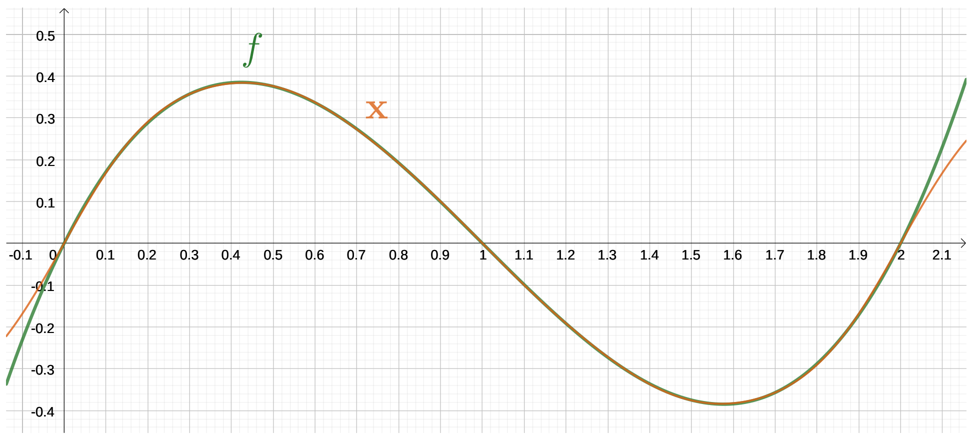
Using a computer algebra system to help with the integration,

$$\mathbf{x} = \frac{12}{125\pi^3}\sin(5\pi x) + \frac{3}{16\pi^3}\sin(4\pi x) + \frac{4}{9\pi^3}\sin(3\pi x) + \frac{3}{2\pi^3}\sin(2\pi x) + \frac{12}{\pi^3}\sin(\pi x).$$

Due to symmetry the odd terms are zero. The distance between $\mathbf{x}$ and f is

$$d(f, \mathbf{x}) \approx 0.00000572.$$

The graphs of f and $\mathbf{x}$ are shown below. The area between the vectors over the interval $[0, 2]$ is imperceptible as the distance calculation might suggest.



The choice of subspace, and therefore its basis, makes all the difference in how close the given function might be approximated.

In the case of the sound wave that opened this section, coefficients

$$\frac{\left\langle \text{Drum}, \sin\left(m \cdot \frac{8820}{121}\pi t\right)\right\rangle}{\left\langle \sin\left(m \cdot \frac{8820}{121}\pi t\right), \sin\left(m \cdot \frac{8820}{121}\pi t\right)\right\rangle}, \ m = 1, 2, \ldots, 120$$

were calculated and sorted from greatest magnitude to least. In decreasing order, the top 14 magnitudes were from the coefficients corresponding to $m = 29, 28, 33, 31, 27, 32, 8, 45, 30, 40, 43, 35, 9, 34$, so the set $\left\{\sin\left(m \cdot \frac{8820}{121}\pi t\right) : m = 29, 28, 33, 31, 27, 32, 8, 45, 30, 40, 43, 35, 9, 34\right\}$, and subsets, were chosen as bases to create the approximations. The frequency of any harmonic function, $\sin(\omega x)$ or $\cos(\omega x)$, is $\frac{\omega}{2\pi}$, so the frequencies of these basis elements are

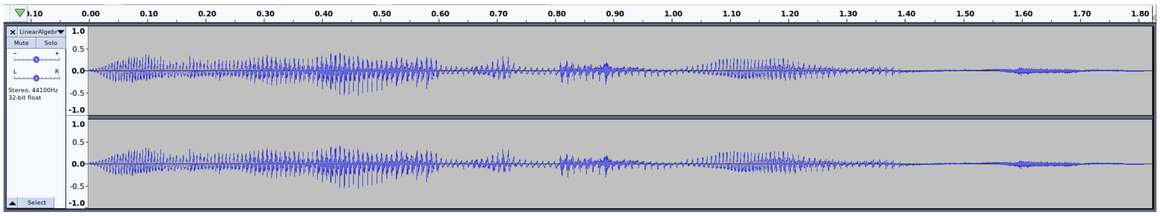
$$\begin{aligned}&\left\{m \cdot \frac{44100}{121} : m = 29, 28, 33, 31, 27, 32, 8, 45, 30, 40, 43, 35, 9, 34\right\}\\ &= \{1056.9, 1020.5, 1202.7, 1129.8, 984.0, 1166.3, 291.6, 1640.1,\\ &\quad 1093.4, 1457.9, 1567.2, 1275.6, 328.0, 1239.2\}\end{aligned}$$

as noted previously.

For a function f defined over the interval $[0, L]$, the (infinite) series

$$b_1 \sin\left(\frac{\pi}{L}x\right) + b_2 \sin\left(2 \cdot \frac{\pi}{L}x\right) + \cdots$$

Figure 7.3.3: Recording of "linear algebra rules" as displayed by Audacity



where

$$b_m = \frac{\left\langle f, \sin\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}{\left\langle \sin\left(m \cdot \frac{\pi}{L} t\right), \sin\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}$$

is called the **Fourier sine series** for f. The (infinite) series

$$a_0 + a_1 \cos\left(\frac{\pi}{L} x\right) + a_2 \cos\left(2 \cdot \frac{\pi}{L} x\right) + \cdots$$

where

$$a_m = \frac{\left\langle f, \cos\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}{\left\langle \cos\left(m \cdot \frac{\pi}{L} t\right), \cos\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}$$

is called the **Fourier cosine series** for f. The a_m and b_m are called **Fourier coefficients**, and the functions $\sin\left(m \cdot \frac{\pi}{L} x\right)$ and $\cos\left(m \cdot \frac{\pi}{L} x\right)$ are called the m^{th} **harmonics**. Most of our effort to now has concentrated on the sine series since we have been examining functions for which $f(0) = f(L) = 0$. In general though, any piecewise continuous functions can be approximated arbitrarily closely using a finite number of terms of either the Fourier sine series, as we have done, *or* the Fourier cosine series. The fit will simply be better with certain selections of harmonics.

A general Fourier series involves both sine and cosine functions and is defined over domains symmetric about zero. For any function f defined over the interval $[-L, L]$, the (infinite) series

$$a_0 + a_1 \cos\left(\frac{2\pi}{L} x\right) + b_1 \sin\left(\frac{2\pi}{L} x\right) + a_2 \cos\left(2 \cdot \frac{2\pi}{L} x\right) + b_2 \sin\left(2 \cdot \frac{2\pi}{L} x\right) + \cdots$$

where

$$a_m = \frac{\left\langle f, \cos\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}{\left\langle \cos\left(m \cdot \frac{2\pi}{L} t\right), \cos\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle} \quad \text{and} \quad b_m = \frac{\left\langle f, \sin\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}{\left\langle \sin\left(m \cdot \frac{2\pi}{L} t\right), \sin\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}$$

is called the **Fourier series** for f. Indeed, the term Fourier series, with no qualifiers, refers to this series, not the sine or cosine series. As before the a_m and b_m are called Fourier coefficients and the functions $\sin\left(m \cdot \frac{2\pi}{L} x\right)$ and $\cos\left(m \cdot \frac{2\pi}{L} x\right)$ are called the m^{th} harmonics.

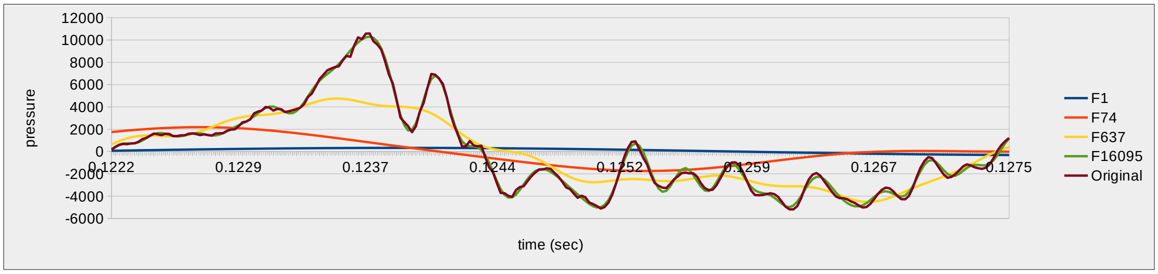
Whether any of these series converges to something useful is a deep and interesting topic of analysis. Generally, though, piecewise continuous functions can be approximated arbitrarily closely using a finite number of terms from any one of the Fourier series. Because sines and cosines are periodic, especially good approximations with small numbers of harmonics can often be found for functions f where $f(-L) = f(L)$ (for general Fourier series) or $f(0) = f(L)$ (for sine or cosine series).

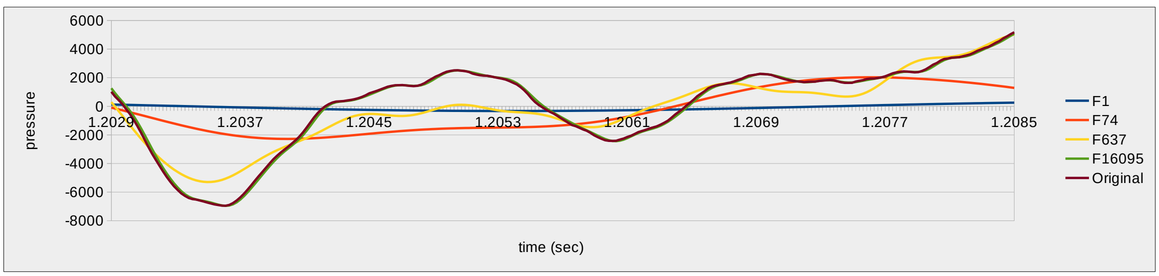
It is not only functions that show some regularity or symmetry that can be approximated by Fourier series. The real power of Fourier analysis is to suss out the most important frequencies when none are apparent. Circling back to sound waves, figure 7.3.3[4] shows the full 1.82 seconds of a voice saying

[4]Audacity: https://www.audacityteam.org/

"linear algebra rules". The sound wave shows no particular symmetry or regularity since the sound is constantly changing throughout.

The first 16,095 Fourier sine series coefficients were calculated. Using the dominant 1, 74, 637, and 16095 harmonics, best approximations were calculated. The following graphs of the "linear algebra rules" audio and its approximations over two separate time intervals illustrate how extending the basis improves the approximation. F1, F74, F637, and F16095 are the approximations using the dominant 1, 74, 637, and 16095 harmonics, respectively.





F16095 is barely visible beneath the original curve, suggesting that the approximation is very good (which should probably be expected having used so many terms). However, the distance between F16095 and the original wave is about 254.4, a number that may seem large. Distances are relative to the function being approximated, though. The norm of the original is 3493.5. $d(\text{f16095}, \text{original})$ is only about one fourteenth (3.7%) the norm (size) of the original—not bad. On the other hand, the distances between the original and F1, F74, and F637 are 3479.5, 3032.0, and 2084.8, respectively. As their distances are similar in magnitude to the norm of the original, it should be expected that they do a poor job approximating the original sound wave. This expectation is born out by the graphs.

In the end, though, the sound of the reproduction should be the judge of the quality of the approximation. The ancillary website contains all the data and playable sound files for the sounds mentioned in this section as well as several others, such as boiling water and birds chirping. The audio corresponding to F1 is a simple computer tone and does not resemble the original audio at all except that it catures the overall pitch. The sound of F74 is slightly better in that it oscillates, but in no way sounds like speech. The words can clearly be heard behind a noisy foreground in F637, but it is still a poor reproduction. Think Alexander Graham Bell and his first phone call. Finally, F16095 and the original audio are indistinguishable, at least to my ear. Have a listen!

Key Concepts

Fourier series for functions f defined over $[-L, L]$, the series

$$a_0 + a_1 \cos\left(\frac{2\pi}{L}x\right) + b_1 \sin\left(\frac{2\pi}{L}x\right) + a_2 \cos\left(2 \cdot \frac{2\pi}{L}x\right) + b_2 \sin\left(2 \cdot \frac{2\pi}{L}x\right) + \cdots$$

where

$$a_m = \frac{\left\langle f, \cos\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}{\left\langle \cos\left(m \cdot \frac{2\pi}{L} t\right), \cos\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle} \quad \text{and} \quad b_m = \frac{\left\langle f, \sin\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}{\left\langle \sin\left(m \cdot \frac{2\pi}{L} t\right), \sin\left(m \cdot \frac{2\pi}{L} t\right)\right\rangle}.$$

Fourier sine series for functions f defined over $[0, L]$, the series

$$b_1 \sin\left(\frac{\pi}{L} x\right) + b_2 \sin\left(2 \cdot \frac{\pi}{L} x\right) + \cdots$$

where

$$b_m = \frac{\left\langle f, \sin\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}{\left\langle \sin\left(m \cdot \frac{\pi}{L} t\right), \sin\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}.$$

Fourier cosine series for functions f defined over $[0, L]$, the series

$$a_0 + a_1 \cos\left(\frac{\pi}{L} x\right) + a_2 \cos\left(2 \cdot \frac{\pi}{L} x\right) + \cdots$$

where

$$a_m = \frac{\left\langle f, \cos\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}{\left\langle \cos\left(m \cdot \frac{\pi}{L} t\right), \cos\left(m \cdot \frac{\pi}{L} t\right)\right\rangle}.$$

Fourier coefficients the a_m and b_m of a Fourier series, Fourier sine series, or Fourier cosine series.

harmonics the functions $\cos\left(m \cdot \frac{2\pi}{L} t\right)$, $\sin\left(m \cdot \frac{2\pi}{L} x\right)$, $\cos\left(m \cdot \frac{\pi}{L} t\right)$, and $\sin\left(m \cdot \frac{\pi}{L} t\right)$ appearing in Fourier series are called m^{th} harmonics, or simply harmonics when not referring to any particular frequency.

approximations piecewise continuous functions can be approximated arbitrarily closely using a finite number of terms from any one of the Fourier series. Especially good approximations with small numbers of harmonics can often be found for smooth functions f where $f(-L) = f(L)$ (for general Fourier series) or $f(0) = f(L)$ (for sine or cosine series).

Fourier analysis the process of determining a large selection of Fourier coefficients with the purpose of identifying those with some particular characteristic.

Exercises

1. Argue that $V_{[0,L]}$ is a vector space.
2. Heat equation stuff.
3. Show (7.3.3).
4. Why is $m = 0$ not included in the family of harmonics for Fourier series?
5. Show that, in the inner product space $V_{[0,L]}$, $\left\langle \sin\left(m \cdot \frac{\pi}{L} t\right), \sin\left(m \cdot \frac{\pi}{L} t\right)\right\rangle = \frac{L}{2}$ for $m = 1, 2, \ldots$
6. Show that, in the inner product space $V_{[0,L]}$,

   (a) $\langle 1, 1\rangle = L$

   (b) $\left\langle \cos\left(m \cdot \frac{\pi}{L} t\right), \cos\left(m \cdot \frac{\pi}{L} t\right)\right\rangle = \frac{L}{2}$ for $m = 1, 2, \ldots$

Answers

inner product on $V_{[0,L]}$ The properties of an inner product are justified one by one below.

1. For any function f in $V_{[0,L]}$, $\langle f, f\rangle = \int_0^L f^2(x)\,dx \geq 0$ since $f^2(x) \geq 0$ for all x in $[0, L]$. In other words, $f^2(x)$ is nonnegative, so its definite integral is nonnegative.
2. Of course, if $f(x) = 0$ (that is, $f = \mathbf{0}$), then $\langle f, f\rangle = \int_0^L f^2(x)\,dx = \int_0^L 0\,dx = 0$. Now suppose $f \neq \mathbf{0}$. That is, there is some x_0 in $[0, L]$ for which $f(x_0) \neq 0$. Let $z = f^2(x_0) > 0$. Since f^2 is continuous, there is a δ such that whenever $|x - x_0| < \delta$, x in $[0, L]$, $|f^2(x) - f^2(x_0)| = |f^2(x) - z| < \frac{z}{2}$. This establishes an interval I of width at least δ within $[0, L]$ where $f^2(x) > \frac{z}{2}$ so $\langle f, f\rangle = \int_0^L f^2(x)\,dx \geq \int_I f^2(x)\,dx \geq \delta\frac{z}{2} > 0$. Hence if $f \neq \mathbf{0}$ then $\langle f, f\rangle \neq 0$, or contrapositively if $\langle f, f\rangle = 0$ then $f = \mathbf{0}$.
3. For any f, g in $V_{[0,L]}$, $\langle f, g\rangle = \int_0^L f(x)g(x)\,dx = \int_0^L g(x)f(x)\,dx = \langle g, f\rangle$ since multiplication is commutative.
4. For any f, g, h in $V_{[0,L]}$, $\langle f + g, h\rangle = \int_0^L (f(x) + g(x))h(x)\,dx = \int_0^L (f(x)h(x) + g(x)h(x))\;dx = \int_0^L f(x)h(x)\,dx + \int_0^L g(x)h(x)\,dx = \langle f, h\rangle + \langle g, h\rangle$ by the distributive property for real numbers and a standard result of calculus.
5. For any f, g in $V_{[0,L]}$ and any scalar c, $\langle cf, g\rangle = \int_0^L cf(x)g(x)\,dx = c\int_0^L f(x)g(x)\,dx = c\langle f, g\rangle$ by a standard result of calculus.

sine functions are orthogonal By (7.3.4),

$$\sin\left(\frac{m\pi}{L}t\right)\sin\left(\frac{n\pi}{L}t\right) = \frac{1}{2}\left[\cos\left((m-n)\frac{\pi}{L}t\right) - \cos\left((m+n)\frac{\pi}{L}t\right)\right]$$

so

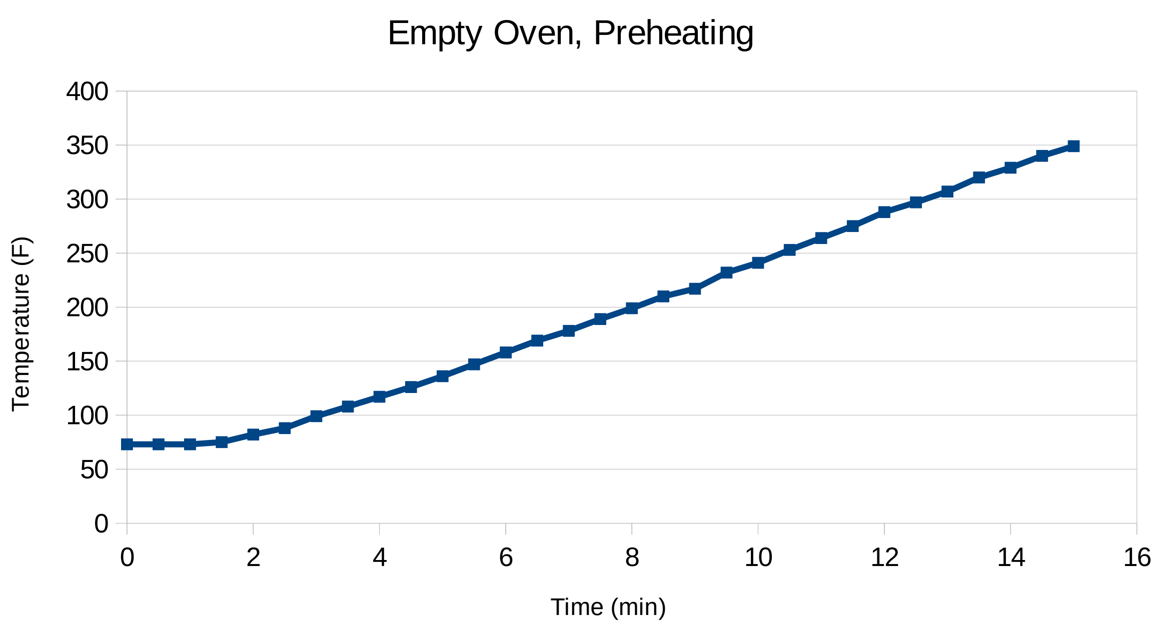
$$\begin{aligned}
\left\langle \sin\left(\frac{m\pi}{L}t\right), \sin\left(\frac{n\pi}{L}t\right)\right\rangle &= \int_0^L \sin\left(\frac{m\pi}{L}t\right)\sin\left(\frac{n\pi}{L}t\right)dt \\
&= \frac{1}{2}\int_0^L \left[\cos\left((m-n)\frac{\pi}{L}t\right) - \cos\left((m+n)\frac{\pi}{L}t\right)\right]dt \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left((m-n)\frac{\pi}{L}t\right) - \frac{L}{\pi(m+n)}\sin\left((m+n)\frac{\pi}{L}t\right)\right]_0^L \\
&= 0.
\end{aligned}$$

7.4 Discrete Dynamical Systems []

You have just finished chopping, slicing, mixing, blending, marinating, layering, and otherwise preparing your favorite dish. You are ready to place it in the oven when you realize it has not been preheated. Preheat now, or put your assembled dish in, start the oven and guess how long it will take to properly bake? Neither! Model the situation with a discrete dynamical system and know just how long to put it in a cold oven.

Next time you bake brownies, you might try this experiment. It requires an over thermometer with a probe that can be inserted into the brownie mix, preferably connected to a console that sits outside the oven, displaying the inside temperature. First, gather the preheating characteristics of the oven. Place the probe in the empty oven approximately where you will later put the brownies. Note the temperature of the probe (air inside the oven) and begin preheating the oven. Likely the recipe will require a 350°F or 175°C oven. Record the thermometer reading every 30 seconds until the oven reaches the target temperature.

Have a look at the graph of temperature versus time. If your oven is like mine, you will get a curve that looks something like this.[5]



Curiously there is essentially no heating during the first 90 seconds, after which the temperature increases in a very steadily linear fashion at about 21°F per minute. We will use this observation to model the temperature of the oven during preheating.

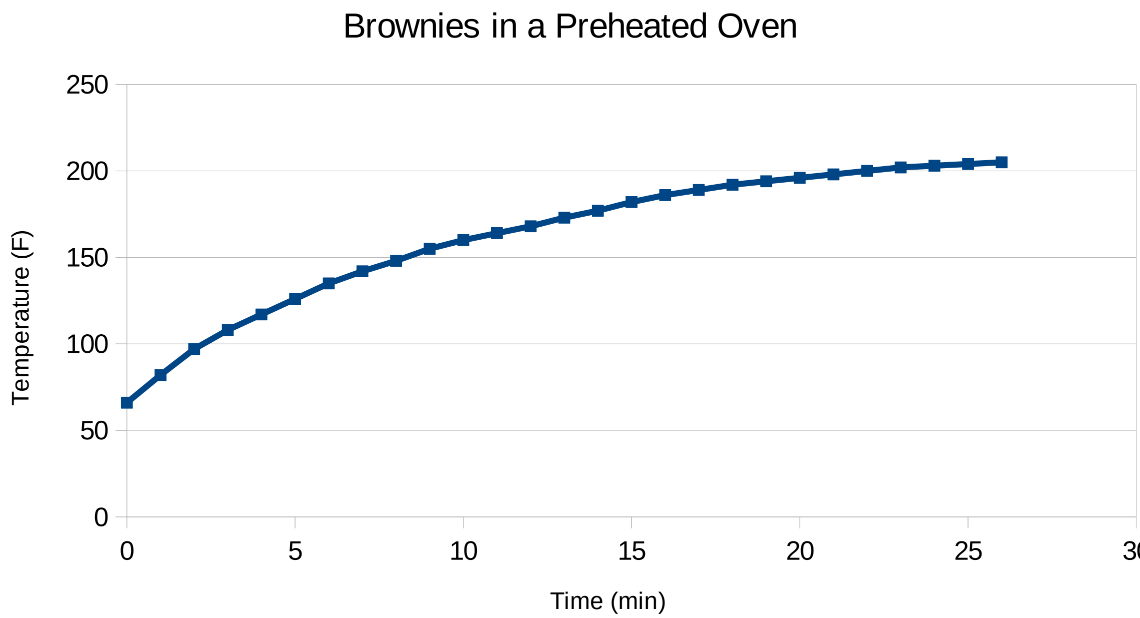
Remove the probe from the oven, let it cool, and let the oven return to its preheated temperature. When the probe has cooled to room temperature and the brownie mix has been poured into the brownie pan, insert the probe into the brownie mix. Record its internal temperature and put the brownies in the preheated oven. Record the thermometer reading every minute until the brownies are done. You will notice the heating is not linear.

Newton's law of cooling, which applies equally to heating, suggests that the change in temperature of a body is approximately proportional to the difference between the temperature of the body and the temperature of its surroundings, ambient temperature. As an equation,

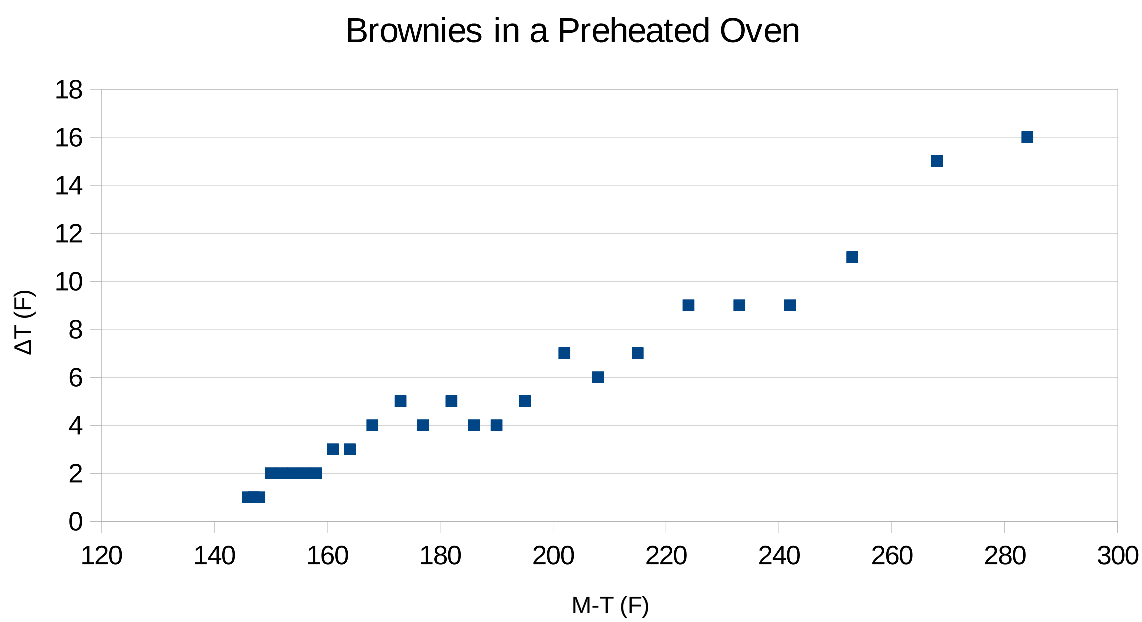
$$\Delta T \approx k(M - T).$$

[5] Data, graphs, and calculations for this entire discussion are available in a spreadsheet at the ancillary site.

If brownies obey this law, plotting the temperature over time will reveal a concave down graph. As the brownies' temperature increases, the difference between ambient (oven) temperature and brownie temperature, $(M-T)$, decreases. In turn, the change in temperature over a fixed amount of time, ΔT, will also decrease. This is, at least as a general characteristic, exactly what the data provide!



However, if the brownies truly follow Newton's law of cooling, a plot of $M-T$ versus ΔT will reveal a straight line passing through the origin, just as any two directly related variables will. Alas this is not what the data suggest.



The scatterplot looks quite linear, but clearly would not pass through the origin if extended to $M-T=0$. Unfortunately, this is a critical feature of the law. When there is no difference between the temperature of a body and its surroundings, the body will neither heat nor cool. Laying a glass of water on the counter for hours, days, or weeks, it will remain at room temperature for the duration.

Nonetheless, this is what the data are telling us, law or no law. We apply linear regression (section 7.1) to the data, deriving a model of the from $\Delta T = \alpha_0 + \alpha_1(M-T)$ that applies when $146°\text{F} \leq M-T \leq 284°\text{F}$.

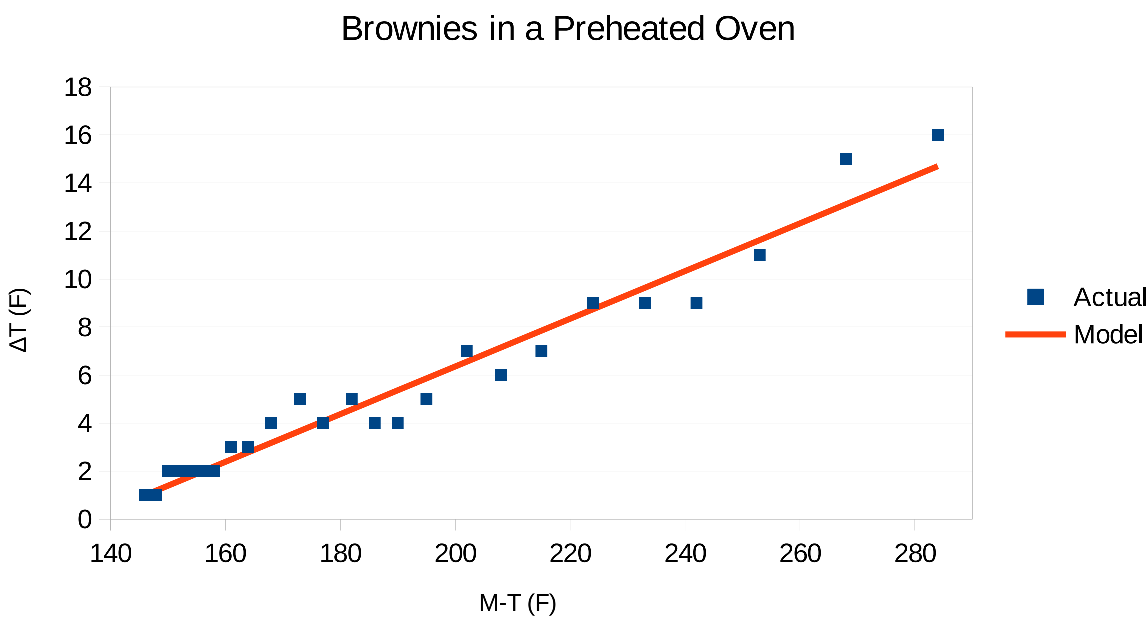
The normal equations are

$$\begin{bmatrix} 26 & 4936 \\ 4936 & 977404 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 139 \\ 30395 \end{bmatrix}$$

and have solution

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 26 & 4936 \\ 4936 & 977404 \end{bmatrix}^{-1} \begin{bmatrix} 139 \\ 30395 \end{bmatrix} = \begin{bmatrix} -13.516 \\ .099356 \end{bmatrix}.$$

Hence the temperature of the brownies can reasonably be modeled by $\Delta T = -13.516 + .099356(M - T)$. The graph here illustrates the reasonableness of this model.



Understanding that Newton's law of cooling only applies for "small" temperature differences we can only conclude that for our brownies the 350°F oven provides a temperature difference outside the range of "small". To understand the heating of brownies placed in a cold oven, though, we need a model for small temperature differences. Without any further data, we will assume that Newton's law of cooling applies at temperature differences less than 146°F (the smallest observed temperature difference). According to our model, at $M - T = 146$, we have $\Delta T = -13.516 + .099356(146) = 0.98998 \approx 1$. Since Newton's law implies that ΔT and $M - T$ are directly proportional, we arrive at the simple relation $\Delta T = \frac{1}{146}(M - T)$ for $0 \leq M - T \leq 146$.

Finally we are ready to run a simulation and find out just how long the brownies should bake starting in a cold oven. From the observation of oven temperature during preheating, we have

$$\Delta M = 21$$

90 seconds or more into heating (and $\Delta M = 0$ prior since the oven does not heat during the first 90 seconds). Substituting $\Delta T = T(t+1) - T(t)$ and $\Delta M = M(t+1) - M(t)$, the changes in temperature over the course of 1 minute, we have starting at 1.5 minutes,

$$\begin{bmatrix} M(t+1) \\ T(t+1) \end{bmatrix} = \begin{bmatrix} M(t) \\ T(t) \end{bmatrix} + \begin{bmatrix} 21 \\ \frac{1}{146}(M(t) - T(t)) \end{bmatrix}. \tag{7.4.1}$$

Given that the brownies began at 66°F and the oven began at 72°F, we have

$$\begin{aligned}
\begin{bmatrix} M(1.5) \\ T(1.5) \end{bmatrix} &= \begin{bmatrix} 72 \\ 66 \end{bmatrix}, \\
\begin{bmatrix} M(2.5) \\ T(2.5) \end{bmatrix} &= \begin{bmatrix} M(1.5) \\ T(1.5) \end{bmatrix} + \begin{bmatrix} 21 \\ \frac{1}{146}(M(1.5) - T(1.5)) \end{bmatrix} = \begin{bmatrix} 72 \\ 66 \end{bmatrix} + \begin{bmatrix} 21 \\ \frac{1}{146}(72 - 66) \end{bmatrix} = \begin{bmatrix} 93 \\ 66.041 \end{bmatrix}, \\
\begin{bmatrix} M(3.5) \\ T(3.5) \end{bmatrix} &= \begin{bmatrix} M(2.5) \\ T(2.5) \end{bmatrix} + \begin{bmatrix} 21 \\ \frac{1}{146}(M(2.5) - T(2.5)) \end{bmatrix} = \begin{bmatrix} 114 \\ 66.226 \end{bmatrix}, \ldots
\end{aligned}$$

and more succinctly,

$$\begin{bmatrix} M(1.5) \\ T(1.5) \end{bmatrix}, \begin{bmatrix} M(2.5) \\ T(2.5) \end{bmatrix}, \begin{bmatrix} M(3.5) \\ T(3.5) \end{bmatrix}, \ldots =$$
$$\begin{bmatrix} 72 \\ 66 \end{bmatrix}, \begin{bmatrix} 93 \\ 66.041 \end{bmatrix}, \begin{bmatrix} 114 \\ 66.226 \end{bmatrix}, \begin{bmatrix} 135 \\ 66.553 \end{bmatrix}, \begin{bmatrix} 156 \\ 67.022 \end{bmatrix}, \begin{bmatrix} 177 \\ 67.631 \end{bmatrix}, \begin{bmatrix} 198 \\ 68.380 \end{bmatrix}, \begin{bmatrix} 219 \\ 69.268 \end{bmatrix}, \ldots$$

bringing us to 8.5 minutes. At this point, $M - T = 219 - 69.268 = 149.73$, which exceeds 146. To continue, we need to start using $\Delta T = -13.516 + .099356(M - T)$, or $T(t + 1) = T(t) - 13.516 + .099356(M(t) - T(t))$ for the change in brownie temperature. In other words, we now have

$$\begin{bmatrix} M(t+1) \\ T(t+1) \end{bmatrix} = \begin{bmatrix} M(t) \\ T(t) \end{bmatrix} + \begin{bmatrix} 21 \\ -13.516 + .099356(M(t) - T(t)) \end{bmatrix}. \tag{7.4.2}$$

So

$$\begin{bmatrix} M(9.5) \\ T(9.5) \end{bmatrix} = \begin{bmatrix} 219 \\ 69.268 \end{bmatrix} + \begin{bmatrix} 21 \\ -13.516 + .099356(149.73) \end{bmatrix} = \begin{bmatrix} 240 \\ 70.629 \end{bmatrix}$$

and so on,

$$\begin{bmatrix} M(10.5) \\ T(10.5) \end{bmatrix}, \begin{bmatrix} M(11.5) \\ T(11.5) \end{bmatrix}, \begin{bmatrix} M(12.5) \\ T(12.5) \end{bmatrix}, \ldots =$$
$$\begin{bmatrix} 261 \\ 73.941 \end{bmatrix}, \begin{bmatrix} 282 \\ 79.010 \end{bmatrix}, \begin{bmatrix} 303 \\ 85.662 \end{bmatrix}, \begin{bmatrix} 324 \\ 93.740 \end{bmatrix}, \begin{bmatrix} 345 \\ 103.10 \end{bmatrix}, \ldots$$

at which point we reach another milestone. The oven temperature does not jump another 21°F at this point. It will only increase another 5°F, so is essentially up to working temperature. The first 14.5 minutes of baking brings the oven to $\sim$ 350°F and the brownies to $\sim$ 103°F, a state that the brownies baking in a preheated oven reached in about 2.5 minutes. To summarize, brownies in a cold oven took 14.5 minutes to get to the same point (oven temperature 350°F, brownie temperature 103°F) the brownies reached in a preheated oven in only 2.5 minutes. From here out it is safe to assume the baking will proceed similarly. Therefore it takes 12 more minutes to bake brownies starting in a cold oven than it does starting in a preheated oven. We simply add 12 minutes to the baking time and proceed. Presumably this applies to any baking done at 350°F. The first 14.5 minutes of baking starting with a cold oven are equivalent to only 2.5 minutes of baking starting with a preheated oven.

This is neither an engineering nor math modeling class, yet the majority of the discussion to now has been about the modeling process. Not to worry, the reader will not be asked to create their own models. Instead, focus on the results of the modeling process, equations (7.4.1) and (7.4.2). These are discrete dynamical systems. The rest of the discussion has hopefuly grabbed your attention and motivated study. In case not, I should mention discrete dynamical systems are used to model phenomena in biology, medicine, physics, economics, engineering, and a host of other areas. Chances are, if you are studying linear algebra, discrete dynamical systems appear in your field of study.

A **first order homogeneous discrete dynamical system** is an equation

$$\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k), \tag{7.4.3}$$

which paired with an **initial condition**

$$\mathbf{x}_0 = \mathbf{v} \tag{7.4.4}$$

defines a sequence $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$. (7.4.3) is an example of a **recurrence** or **recurrence relation** and determines all of the terms of the sequence except $\mathbf{x}_0$, which must be supplied separately. For purpose of our study of linear algebra, $\mathbf{x}_k$ are in $\mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary function.

Equation (7.4.1) can be rewritten in terms of this definition by setting $\mathbf{x}_k = \begin{bmatrix} M(k) \\ T(k) \end{bmatrix}$, from which it follows

$$\begin{aligned}
\mathbf{f}(\mathbf{x}_k) &= \begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \begin{bmatrix} 21 \\ \frac{1}{146}(M(k) - T(k)) \end{bmatrix} \\
&= \begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \frac{1}{146}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \frac{1}{146}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{146}\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \\
&= \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\begin{bmatrix} M(k) \\ T(k) \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \\
&= \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\mathbf{x}_k + \begin{bmatrix} 21 \\ 0 \end{bmatrix}.
\end{aligned}$$

This is an example of a **nonlinear discrete dynamical system** as the function $\mathbf{f}$ is not linear. More precisely in this case, the system is **affine**. It takes the form

$$\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{b}. \tag{7.4.5}$$

$\mathbf{f}$ is an affine map.

As noted in the definition, a discrete dynamical system determines a sequence. Our first order of business is to understand how so. For each initial condition, the sequence defined by a discrete dynamical system can be calculated term-by-term. The recurrence relation defines each term after the first. For example, the first few terms of the sequence defined by the system

$$\mathbf{x}_{k+1} = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix}\mathbf{x}_k + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \tag{7.4.6}$$

with initial condition

$$\mathbf{x}_0 = \begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix}$$

can be calculated as follows. According to (7.4.6),

$$\mathbf{x}_1 = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix}\mathbf{x}_0 + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix}\begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -4.4 \\ .5 \\ -1.8 \end{bmatrix}.$$

Also according to (7.4.6),

$$\mathbf{x}_2 = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \mathbf{x}_1 + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \begin{bmatrix} -4.4 \\ .5 \\ -1.8 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 5.52 \\ -7.35 \\ 6.74 \end{bmatrix}.$$

Similarly,

$$\mathbf{x}_3 = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \begin{bmatrix} 5.52 \\ -7.35 \\ 6.74 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -2.416 \\ -.035 \\ -.782 \end{bmatrix}$$

and

$$\mathbf{x}_4 = \begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \begin{bmatrix} -2.416 \\ -.035 \\ -.782 \end{bmatrix} + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 3.933 \\ -6.198 \\ 5.443 \end{bmatrix}$$

accurate to 3 decimal places. The first five terms of the sequence are $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$, which have just been calculated (using this SageMath SageCell) as

$$\begin{bmatrix} 8 \\ -5 \\ 6 \end{bmatrix}, \begin{bmatrix} -4.4 \\ .5 \\ -1.8 \end{bmatrix}, \begin{bmatrix} 5.52 \\ -7.35 \\ 6.74 \end{bmatrix}, \begin{bmatrix} -2.416 \\ -.035 \\ -.782 \end{bmatrix}, \begin{bmatrix} 3.933 \\ -6.198 \\ 5.443 \end{bmatrix}.$$

Further terms can be calculated similarly. The process of calculating the terms is called **iteration**. The terms themselves are called **iterates** or **iterations**, and the sequence is called the **orbit** of $\mathbf{x}_0$.

Given the dynamical system

$$\mathbf{x}_{k+1} = \frac{1}{146} \begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 21 \\ 0 \end{bmatrix} \tag{7.4.7}$$

from the brownie baking model, can you find the first 5 iterates in the orbit of

$$\mathbf{x}_0 = \begin{bmatrix} 72 \\ 66 \end{bmatrix}?$$

Answer on page 257.

As a final exercise in iteration, the first 5 iterates in the orbit of $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for the dynamical system

$$\mathbf{x}_{k+1} = \begin{bmatrix} \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{-\sqrt{6}+\sqrt{2}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \tag{7.4.8}$$

are (approximately)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1.293 \\ 3.224 \end{bmatrix}, \begin{bmatrix} -4.083 \\ 4.780 \end{bmatrix}, \begin{bmatrix} -7.182 \\ 5.560 \end{bmatrix}, \begin{bmatrix} -10.376 \\ 5.512 \end{bmatrix}.$$

Can you verify these terms (using SageMath)? Answer on page 258.

The first few iterates of an orbit are often not the ultimate goal. For many applications, the point of interest is the long run. How can the 1000^{th} through 2000^{th} or $1,000,000^{th}$ through $1,000,012^{th}$ iterations of an orbit be described in general terms? Such a desctiption is called the system's **long term behavior**.

In the case of (7.4.6),

$$\mathbf{x}_5 = \begin{bmatrix} -1.146 \\ -1.174 \\ 0.401 \end{bmatrix}, \ \mathbf{x}_{30} = \begin{bmatrix} 1.108 \\ -3.416 \\ 2.647 \end{bmatrix}, \ \mathbf{x}_{55} = \begin{bmatrix} 1.111 \\ -3.419 \\ 2.650 \end{bmatrix}, \text{ and } \mathbf{x}_{80} = \begin{bmatrix} 1.111 \\ -3.419 \\ 2.650 \end{bmatrix}$$

accurate to 3 decimal places. It takes a short while, but the terms reveal a pattern. All terms $\mathbf{x}_k$, $k \geq 55$ are, accurate to 3 decimal places, equal to $\begin{bmatrix} 1.111 \\ -3.419 \\ 2.650 \end{bmatrix}$. The iterates after the 54^{th} do not change much. When the iterates of a dynamical system settle down this way, we say that the orbit approaches a particular vector. But what vector, and can we predict it without computing large numbers of iterates?

By definition, $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$, so when a dynamical system settles on a particular vector, we have $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) \approx \mathbf{x}_k$. Therein lies the answer to the mystery.

The iterates are getting closer and closer to satisfying the equation $\mathbf{f}(\mathbf{x}_k) = \mathbf{x}_k$. Any solution of the equation $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$ is called a **fixed point** of $\mathbf{f}$, and if $\mathbf{x}_k$ were such a value, we would have $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k) = \mathbf{f}(\mathbf{x}^*) = \mathbf{x}^* = \mathbf{x}_k$. The sequence would be fixed forever more at the value $\mathbf{x}^*$.

In the case of (7.4.6) a fixed point of $\mathbf{f}$ satisfies

$$\begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \mathbf{x}^* + \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = \mathbf{x}^*,$$

an equation we can solve:

$$\begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} \mathbf{x}^* - \mathbf{x}^* = -\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$$

$$\left(\begin{bmatrix} -.15 & -1.3 & -1.95 \\ -.55 & 1.8 & 3.15 \\ .15 & -1.3 & -2.25 \end{bmatrix} - I \right) \mathbf{x}^* = -\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}.$$

By row reduction (using this SageMath SageCell), it turns out

$$\mathbf{x}^* = \frac{1}{117} \begin{bmatrix} 130 \\ -400 \\ 310 \end{bmatrix} \approx \begin{bmatrix} 1.111111111 \\ -3.418803419 \\ 2.649572650 \end{bmatrix}.$$

It seems clear enough the orbit is **approaching the fixed point**. When this happens, we say $\mathbf{x}^*$ is an attracting fixed point or, simply, an **attractor**.

Not all orbits of discrete dynamical systems approach a fixed point, however. The dynamical system

$$\mathbf{x}_{k+1} = \frac{1}{146} \begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} 21 \\ 0 \end{bmatrix}$$

with initial condition

$$\mathbf{x}_0 = \begin{bmatrix} 72 \\ 66 \end{bmatrix}$$

from the brownie baking model does not. This fact is clear by observing the behavior of the first entry of $\mathbf{x}_k$. It simply increases by 21 with each iteration. To be precise, $(\mathbf{x}_k)_{1,1} = 72 + 21k$, which tends to infinity as k grows. Therefore, $||\mathbf{x}_k||$ diverges to ∞ and we say the orbit **tends toward infinity**.

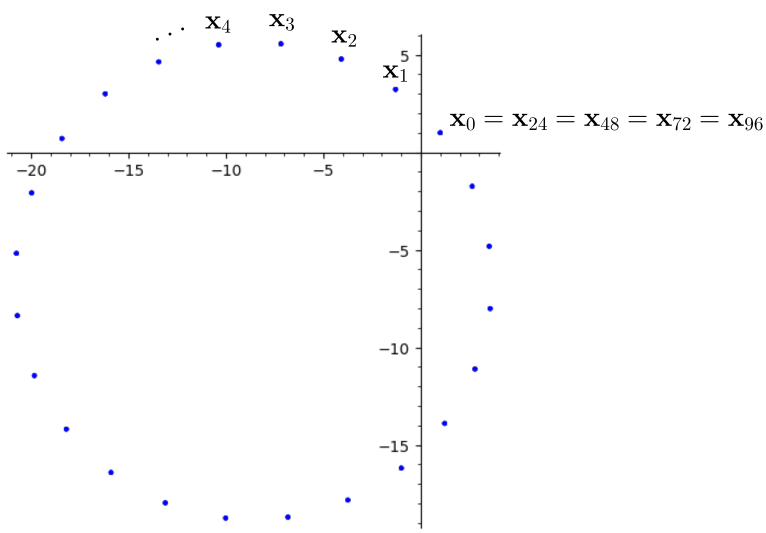
Finally, the orbit of $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for the dynamical system

$$\mathbf{x}_{k+1} = \begin{bmatrix} \frac{\sqrt{6}+\sqrt{2}}{4} & \frac{-\sqrt{6}+\sqrt{2}}{4} \\ \frac{\sqrt{6}-\sqrt{2}}{4} & \frac{\sqrt{6}+\sqrt{2}}{4} \end{bmatrix} \mathbf{x}_k + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

does not present a clear pattern even after 100 iterations. As computed in this SageMath SageCell, $\mathbf{x}_{98}$ through $\mathbf{x}_{100}$ are (again accurate to three decimal places),

$$\begin{bmatrix} -4.083 \\ 4.780 \end{bmatrix}, \begin{bmatrix} -7.182 \\ 5.560 \end{bmatrix}, \begin{bmatrix} -10.376 \\ 5.512 \end{bmatrix}.$$

Though it is likely not at all clear from this short list nor the SageMath output, the orbit exhibits a very simple pattern. To see it, a graph of the first 100 iterations:



It appears there are only 24 iterations, but that is because they repeat. As indicated, $\mathbf{x}_0 = \mathbf{x}_{24} = \mathbf{x}_{48} = \cdots$. Similarly, $\mathbf{x}_1 = \mathbf{x}_{25} = \mathbf{x}_{49} = \cdots$ and so on. As a result, $\mathbf{x}_{98}$ coincides with $\mathbf{x}_2$, $\mathbf{x}_{99}$ coincides with $\mathbf{x}_3$, and $\mathbf{x}_{100}$ coincides with $\mathbf{x}_4$. When a sequence of iterates repeats this way, we say the orbit is **periodic**.

As with the power method (section 6.2) and Markov chains (section 7.2), both of which can be framed as discrete dynamical systems, eigenvalues tell the story of long term behavior. For the example systems of this section, each of the form (7.4.5), eigenvalues of M and its spectral radius are listed in the chart.

System	Eigenvalues of M	Spectral Radius	Long term behavior
(7.4.6)	$-.8, -.3, .5$	.8	approaches fixed point
(7.4.7)	$1, \frac{145}{146}$	1	tends toward infinity
(7.4.8)	$\frac{\sqrt{6}+\sqrt{2}}{4} \pm i\frac{\sqrt{6}-\sqrt{2}}{4}$	1	periodic

The **spectral radius** of a square matrix is the maximum of the magnitudes (absolute values) of its eigenvalues. The magnitude of a complex number $a + ib$ is $\sqrt{a^2 + b^2}$, so

$$\begin{aligned} \left| \frac{\sqrt{6}+\sqrt{2}}{4} \pm i\frac{\sqrt{6}-\sqrt{2}}{4} \right| &= \sqrt{\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)^2 + \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2} \\ &= \sqrt{\frac{6+2\sqrt{12}+2}{16} + \frac{6-2\sqrt{12}+2}{16}} \\ &= \sqrt{\frac{6+2+6+2}{16}} \\ &= 1. \end{aligned}$$

Much like a geometric series, which converges when the ratio is less than one and diverges when the ratio is greater than one, an affine system will approach the fixed point when the spectral radius is less than one and will generally tend toward infinity when the spectral radius is greater than one. The analogy ends there. An affine dynamical system whose matrix has spectral radius one can exhibit several different behaviors: tendency toward infinity and periodicity as seen above, but also convergence to a fixed point, depending on the system and the initial condition.

Crumpet 31: Long Term Behavior

For an affine discrete dynamical system, $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{b}$, where 1 is not an eigenvalue of M, the system has a unique fixed point:

$$\begin{aligned} M\mathbf{x}^* + \mathbf{b} &= \mathbf{x}^* \\ M\mathbf{x}^* - \mathbf{x}^* &= -\mathbf{b} \\ (M - I)\mathbf{x}^* &= -\mathbf{b} \\ \mathbf{x}^* &= -(M - I)^{-1}\mathbf{b}. \end{aligned}$$

The inverse of $M - I$ exists because 1 is not an eigenvalue of M. Letting $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, which implies $\mathbf{x} = \mathbf{y} + \mathbf{x}^*$, and substituting into $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{b}$:

$$\begin{aligned} \mathbf{y}_{k+1} + \mathbf{x}^* &= M(\mathbf{y}_k + \mathbf{x}^*) + \mathbf{b} \\ &= M\mathbf{y}_k + (M\mathbf{x}^* + \mathbf{b}) \\ &= M\mathbf{y}_k + \mathbf{x}^*. \end{aligned}$$

So $\mathbf{y}_{k+1} = M\mathbf{y}_k$. The analysis of this linear dynamical system fully informs the behavior of the affine system. Assuming M is diagonalizable, we set $\mathbf{y}_0 = \mathbf{x}_0 - \mathbf{x}^*$ (by substitution) and write $\mathbf{y}_0$ in terms of a basis of eigenvectors, $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then

$$\mathbf{y}_0 = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n$$

for some scalars $c_1, c_2, \ldots, c_n$, and $\mathbf{y}_1 = M(c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n) = c_1\lambda_1\mathbf{w}_1 + c_2\lambda_2\mathbf{w}_2 + \cdots + c_n\lambda_n\mathbf{w}_n$, $\mathbf{y}_2 = M(c_1\lambda_1\mathbf{w}_1 + c_2\lambda_2\mathbf{w}_2 + \cdots + c_n\lambda_n\mathbf{w}_n) = c_1\lambda_1^2\mathbf{w}_1 + c_2\lambda_2^2\mathbf{w}_2 + \cdots + c_n\lambda_n^2\mathbf{w}_n$, and so on:

$$\mathbf{y}_k = c_1\lambda_1^k\mathbf{w}_1 + c_2\lambda_2^k\mathbf{w}_2 + \cdots + c_n\lambda_n^k\mathbf{w}_n.$$

This solution is dominated by the eigenvalue(s) with the greatest magnitude. If the dominant magnitude is less than one, $\mathbf{y}_k$ will tend toward zero and therefore $\mathbf{x}_k$ will tend toward the fixed point. If the dominant magnitude is greater than one, $\mathbf{y}_k$ will tend toward infinity as long as $c_j \neq 0$ for some j with $|\lambda_j| > 1$, in which case $\mathbf{x}_k$ will tend toward infinity.

Key Concepts

first order homogeneous discrete dynamical system an equation of the form $\mathbf{x}_{k+1} = \mathbf{f}(\mathbf{x}_k)$.

nonlinear discrete dynamical system a discrete dynamical system whose recurrence is nonlinear.

initial condition a value $\mathbf{v}$ for the first term of a dynamical system, usually given as $\mathbf{x}_0 = \mathbf{v}$.

recurrence a general term for the type of equation appearing in a discrete dynamical system.

recurrence relation a recurrence.

affine discrete dynamical system a dynamical system of the form $\mathbf{x}_{k+1} = M\mathbf{x}_k + \mathbf{b}$.

affine map a map $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ where $\mathbf{f}(\mathbf{x}) = M\mathbf{x} + \mathbf{b}$.

iteration the process of calculating the terms of the sequence determined by a discrete dynamical system.

iterates the terms of the sequence determined by a discrete dynamical system.

iterations iterates.

orbit the sequence determined by a discrete dynamical system—the solution of a dynamical system with initial condition.

long term behavior a qualitative description of the orbit of a dynamical system. The phrases "approaching the fixed point", "tending toward infinity", and "periodic" are often used. Another possibility for nonlinear dynamical systems is "chaotic".

fixed point a solution of the equation $\mathbf{f}(\mathbf{x}^*) = \mathbf{x}^*$.

attractor the fixed point of a dynamical system whose solutions tend toward it.

Useful results of search "real applications of discrete dynamical systems"

https://www.math.colostate.edu/~gerhard/MATH331/index.html
(Non-Newtonian fish heating)

https://www.researchgate.net/publication/344193230_Discrete_Dynamical_Systems_With_Applications_in_Biology
(Biology—parasites)

https://www.math.umd.edu/~tjp/131%2014.0%20Discrete%2

Exercises

1. A

Answers

brownie iterates Given that $\mathbf{x}_0 = \begin{bmatrix} 72 \\ 66 \end{bmatrix}$, accurate to three decimal places,

$$\mathbf{x}_1 = \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\begin{bmatrix} 72 \\ 66 \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 93 \\ 66.041 \end{bmatrix}$$
$$\mathbf{x}_2 = \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\begin{bmatrix} 93 \\ 66.041 \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 114 \\ 66.226 \end{bmatrix}$$
$$\mathbf{x}_3 = \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\begin{bmatrix} 114 \\ 66.226 \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 135 \\ 66.553 \end{bmatrix}$$
$$\mathbf{x}_4 = \frac{1}{146}\begin{bmatrix} 146 & 0 \\ 1 & 145 \end{bmatrix}\begin{bmatrix} 135 \\ 66.553 \end{bmatrix} + \begin{bmatrix} 21 \\ 0 \end{bmatrix} = \begin{bmatrix} 156 \\ 67.022 \end{bmatrix}$$

so the first five iterates of the orbit are

$$\begin{bmatrix} 72 \\ 66 \end{bmatrix}, \begin{bmatrix} 93 \\ 66.041 \end{bmatrix}, \begin{bmatrix} 114 \\ 66.226 \end{bmatrix}, \begin{bmatrix} 135 \\ 66.553 \end{bmatrix}, \begin{bmatrix} 156 \\ 67.022 \end{bmatrix}.$$

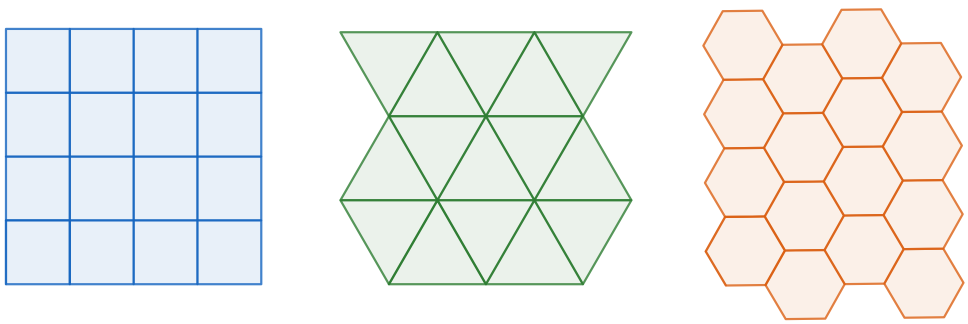
last iteration example Sample SageMath code that can be copied and pasted into a SageCell:

```
M=1/4*matrix(2,2,[sqrt(6.0)+sqrt(2.0), sqrt(2.0)-sqrt(6.0),
                   sqrt(6.0)-sqrt(2.0), sqrt(6.0)+sqrt(2.0)])
b=vector([-2.0,2.0])
x0=vector([1,1])
print("0 :",x0)
for i in range(1,5):
    x0=M*x0+b
    print(i,":",x0)
```

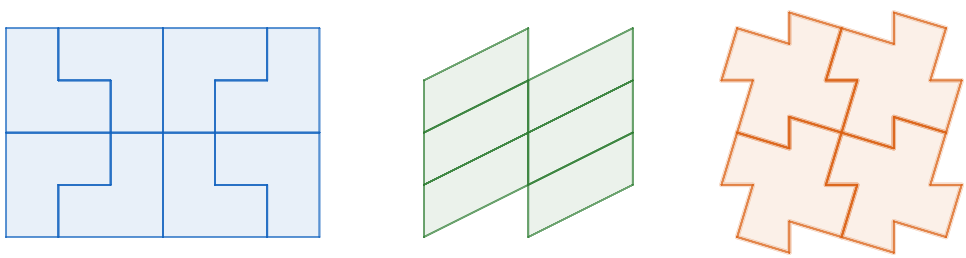
7.5 Rep-tiles []

The floors of kitchens, bathrooms, museums, and other spaces are more often than not tiled in some fashion. Flat ceramic or natural stone tiles are placed together in a regular pattern, sometimes more artistically than others. In any case, the tiles are laid flat, covering the whole floor without overlapping. Perhaps the most common tile shape for this purpose is the square. It is easy to design a covering of a floor using squares of the same size. Squares are flexible and simple this way. More fanciful floor coverings are made of rectangular, triangular, hexagonal, and octagonal tiles. Fitting these various shapes together to cover a floor can be an artistic endeavor, limited mostly by imagination.

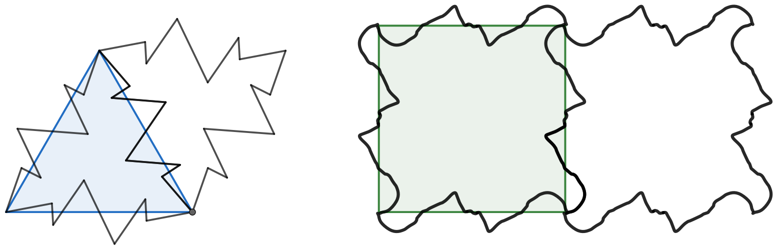
Tessellations are the mathematical equivalent of floor tiling. Any piecing together of a set of shapes, covering a flat surface without overlapping is called a tiling or tessellation. As with floors, perhaps the simplest way to tile, or tessellate, the plane is with squares. Any flat surface can be covered by fitting squares of the same size together at their corners, maintaining parallel sides, just like the pattern seen on graph paper. The same can be done with equilateral triangles and regular hexagons. These coverings are called regular tesselations, samples shown here. Each one can be extended indefinitely in any direction.



Irregular polygons tile the plane just as well. Parallelograms, pentagons, convex and concave, can all be fitted to tessellate, samples shown here.

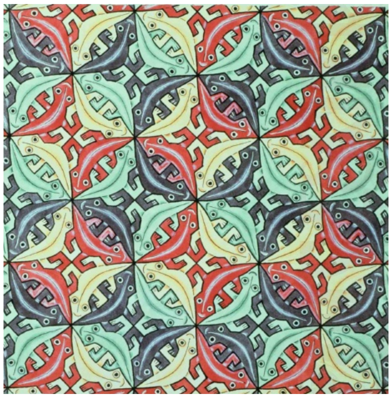


M.C. Escher famously made tessellation an artform. Tilings with irregularly shaped birds, fish, human figures, and other natural shapes appear in many of his most famous creations. See figure 7.5.1, for example. One way to create Escher-esque tessellations is to start with a regular tile and modify its perimeter in a symmetric way, preserving its tessellating nature. The third tessellation of the diagram above is created from squares where each side is replaced by a zig-zag. Among the infinite possibilities for tiles created this way are the two shown here.

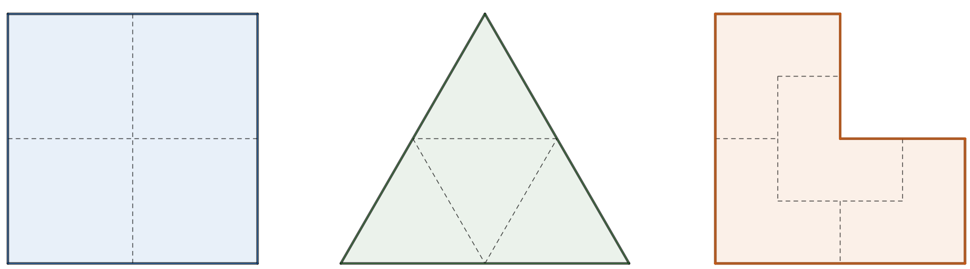


Each side of each tile is given 180 degree rotational symmetry. The tile created from the equilateral triangle is then given 120 degree rotational symmetry on the whole, and the tile created from the square is given 90 degree rotational symmetry on the whole.

Figure 7.5.1: M.C. Escher's System X(e)



Elaborate in a different way, some tiles are doubly tiling. Not only can they be fitted to tile the plane, but they can also be fitted to tile larger copies of themselves! These tiles are called **rep-tiles**, short for self-replicating tiles. Again, the square provides an immediate example. Four congruent squares fitted together at a corner, sides parallel to one another form a square with side length twice the original (and four times the area). Equilateral triangles, and in fact all triangles, are rep-tiles. Four congruent copies can be pieced together, three in the same orientation and the fourth rotated 180 degrees, to form a larger copy. Regular hexagons are not rep-tiles as no finite number of copies of a hexagon can be fitted together (as tiles, without overlap) to form a hexagon. However, there are irregular hexagons that are rep-tiles. One such hexagon has a symmetric L shape—three squares glued together in an ell. To see that they are self-replicating, the following diagram shows four of each shape fitted together to form a larger replica. We have already seen that these shapes tessellate the plane, so they are indeed rep-tiles.



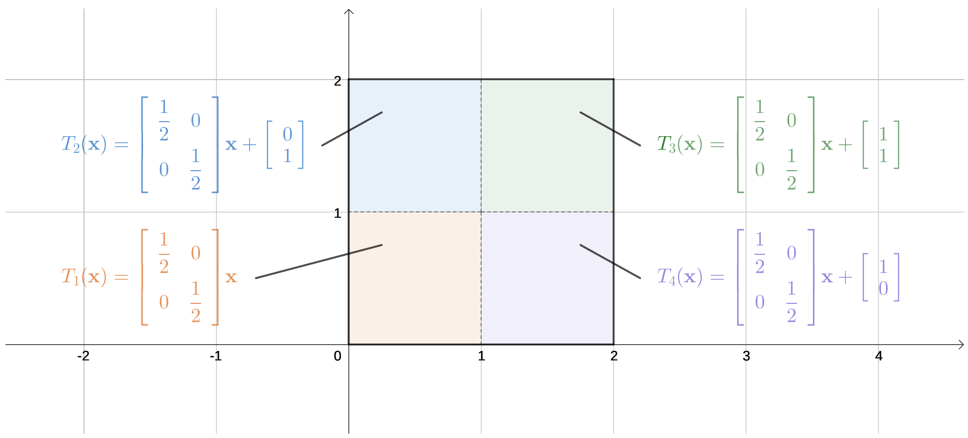
As these rep-tiles are polygons, rigorous mathematical descriptions of each are easily within reach. A square is described as a quadrilateral with four congruent sides and four congruent interior angles. With this description and the location of one pair of opposite vertices, exactly one square is determined and it can be sketched from this information. A triangle is a three-sided polygon. With this description and the locations of its vertices, exactly one triangle is determined and it can be sketched from this information. The L-shaped tile is harder to describe and harder still to provide sufficient information for drawing a particular such tile. Phrases like "three squares glued together" or "a square with one missing quadrant" are perhaps adequate, but general and flexible language for precisely describing general polygons is lacking. As we explore more and more rep-tiles, the language of plane geometry becomes less and less satisfying. Linear algebra provides a common language to describe all rep-tiles, polygons and non-polygons alike—transformations.

After building a larger replica of a rep-tile, one can switch perspecives and look at the construction as a dissection of the larger rep-tile. From this viewpoint, rep-tiles are plane figures that tessellate the plane and can be dissected into finitely many similar copies. All the figures of the previous diagram can be seen from this vantage. The large square is made up of four smaller squares. The large triangle is broken up into four similar triangles. The large L-shape is likewise divided into four smaller copies of itself.

Crumpet 32: Solomon Golomb

Solomon Golomb is credited with coining the term *rep-tile*, but his original paper[9] only uses the term “rep-k”, short for *replicating of order k*. To quote Golomb, a plane figure is called rep-k if “it can be dissected into k ‘replicas’, each congruent to the others and similar to the original”. Curiously, Martin Gardner[6] credits Golomb with laying the foundation for the study of rep-tiles and inventing the term in a series of private papers, all in his article appearing more than a year earlier than Golomb’s!

By imposing a set of axes on any of these figures, rigorous mathematical descriptions of the figures are available as sets of affine transformations—compositions of linear transformations and translations. Any placement of the axes—or from the perspective of the square, putting it anywhere in the plane—will do. We only need a frame of reference. For simplicity, if we arrange for opposite corners of the square to coincide with $(0,0)$ and $(2,2)$, the following diagram shows the transformations that determine the square.



Each transformation maps the whole square (the one with opposite corners $(0,0)$ and $(2,2)$) to one of the parts of its dissection. You should recognize the 2×2 matrices as scaling by a factor of $\frac{1}{2}$ in both horizontal and vertical directions. The addition of 2×1 vectors provide translations. In words, T_4 can be described as scaling by $\frac{1}{2}$ horizontally and vertically followed by translation (shifting) 1 unit horizontally. $T_4\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T_4\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, for example. More importantly, the image of the 2 by 2 square under T_4 is the purple square: contracting the 2 by 2 square by a factor of $\frac{1}{2}$ and then translating the contracted copy right 1 unit lands the image of the larger square (well, squarely) on top of the purple square—the bottom right square of the dissection. Letting S be the 2 by 2 square with opposite corners at $(0,0)$ and $(2,2)$, we thereby have $T_4(S)$ = the purple square. Similarly, $T_1(S)$ = the orange square (the bottom left square of the dissection); $T_2(S)$ = the blue square (the top left square of the dissection); and $T_3(S)$ = the green square (the top right square of the dissection). The union of the four images is the original square. In the form of an equation,

$$T_1(S) \cup T_2(S) \cup T_3(S) \cup T_4(S) = S. \tag{7.5.1}$$

A theorem of Hutchinson[13] asserts that S is the only compact set that satisfies (7.5.1) with the given transformations. In other words, T_1, T_2, T_3, T_4 determine S via equation (7.5.1). These four transformations provide a precise description of the square with opposite corners at $(0,0)$ and $(2,2)$. Incidentally, each transformation T_k is a **similitude**—a rigid transformation (rotation, reflection, translation, or composition thereof) composed with **dilation** (scaling by the same scale factor in all directions). Similitudes preserve shape but not necessarily size, exactly the type of transformation needed to map a shape onto one of the (similar) parts of its dissection.

More generally, let $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ be a set of similitudes in $\mathbb{R}^n$ with scale factors less than one, and define

$$\mathcal{H}_{\mathscr{C}}(A) = C_1(A) \cup C_2(A) \cup \cdots \cup C_p(A) \tag{7.5.2}$$

for any subset A of $\mathbb{R}^n$. **Hutchinson's theorem** concludes that there is exactly one compact set K in $\mathbb{R}^n$ such that $\mathcal{H}_{\mathscr{C}}(K) = K$. Moreover,

$$\lim_{k\to\infty} \mathcal{H}_{\mathscr{C}}^{\circ k}(A) = K \tag{7.5.3}$$

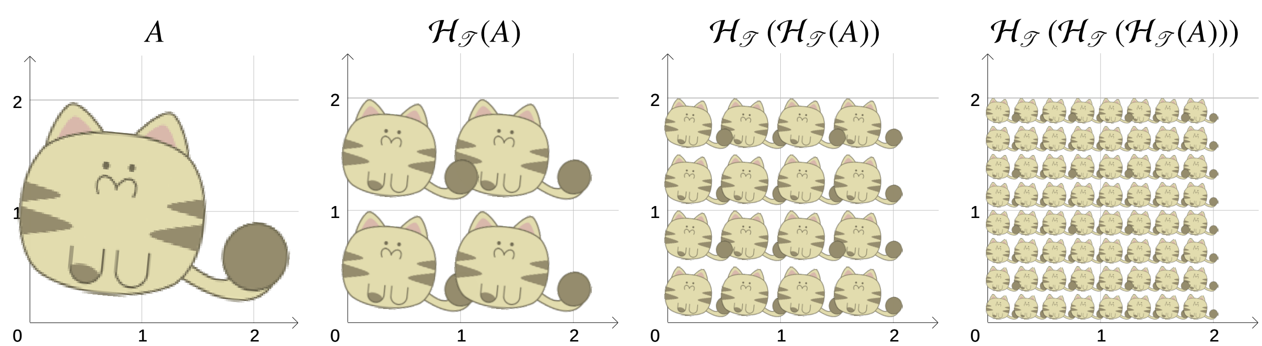
for any compact set A. Not only does the theorem assert the existence and uniqueness of the set K, it gives a way to construct it from the similitudes.

Crumpet 33: Hutchinson

The original theorem of Hutchinson and its proof lie along the fence between real analysis and topology. *Let $X = (X, d)$ be a complete metric space and $\mathscr{S} = \{S_1, \ldots, S_N\}$ be a finite set of contraction maps on X. Then there exists a unique closed bounded set K such that $K = \bigcup_{i=1}^{N} S_i K$. Furthermore, K is compact and is the closure of the set of fixed points $s_{i_1\cdots i_p}$ of finite compositions $S_{i_1} \circ \ldots \circ S_{i_p}$ of members of $\mathscr{S}$.*

For arbitary $A \subset X$ let $\mathscr{S}(A) = \bigcup_{i=1}^{N} S_i A$, $\mathscr{S}^P(A) = \mathscr{S}(\mathscr{S}^{P-1}(A))$. Then for closed bounded A, $\mathscr{S}^P(A) \to K$ in the Hausdorff metric.

Applying this theorem to the set $\mathscr{T} = \{T_1, T_2, T_3, T_4\}$, we do not have to know anything about the origin of the transformations T_k. All the work of dissecting the square, placing it in the plane, and deriving the transformations in $\mathscr{T}$ can be forgotten. All we need is a compact set A (and a lot of patience!) to recover the square. It is the limit of the sequence $\mathcal{H}_{\mathscr{T}}(A), \mathcal{H}_{\mathscr{T}}(\mathcal{H}_{\mathscr{T}}(A)), \mathcal{H}_{\mathscr{T}}(\mathcal{H}_{\mathscr{T}}(\mathcal{H}_{\mathscr{T}}(A))), \ldots$, the iteration of $\mathcal{H}_{\mathscr{T}}$ on *any* compact set A. The first few terms of this sequence are shown below, where A takes the shape of a kitty[6].



The following set of similitudes determines an L-shaped rep-tile within the square with opposite vertices

[6] Kitty image downloaded from https://openclipart.org/detail/292277/cute-cat.

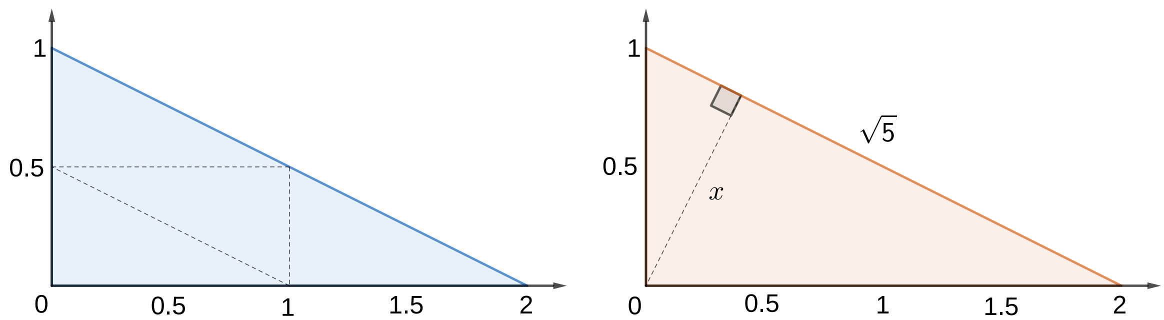
(0, 0) and (2, 2).

$$L_1(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\mathbf{x} \qquad L_2(\mathbf{x}) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$L_3(\mathbf{x}) = \begin{bmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad L_4(\mathbf{x}) = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Can you generate the L-shaped rep-tile by applying (7.5.3) to $\mathscr{C} = \{L_1, L_2, L_3, L_4\}$ (and some set A of your own creation)? Answer on page 265.

The following diagram illustrates three things. The dissection of a reptile is not unique (two different dissections are shown for the same right triangle). The number of parts in a dissection of a rep-tile is not always four. The parts of a dissection need not be congruent to one another (they only need be similar to the whole).



Can you find the similitudes associated with these dissections (note there will be four similitudes associated with the first dissection and two similitudes associated with the second)? Answer on page 265.

In the second dissection, the two scale factors are $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$. Not coincidentally $\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2 = 1$. If $\mathscr{C} = \{T_1, T_2, \ldots, T_p\}$ is the set of similitudes that determine a rep-tile R, then

$$T_1(R) \cup T_2(R) \cup \cdots \cup T_p(R) = R$$

so

$$\text{area}\,(T_1(R)) + \text{area}\,(T_2(R)) + \cdots + \text{area}\left(T_p(R)\right) = \text{area}(R). \tag{7.5.4}$$

But we know from section 6.3 that area $(T_k(R)) = |\det L_k| \cdot \text{area}(R)$ where L_k is the matrix of the linear part of similitude T_k. We also know from section 3.7 that the determinant of a product is the product of the determinants and from section 3.5 that multiplying a 2×2 matrix by scalar c multiplies its determinant by c^2. Combined with the facts that the determinants of reflections and rotations are -1 and 1 respectively, the determinant of any matrix of a similitude is s^2 where s is its scale factor. By extension, a similitude scales the area of a figure by a factor of s^2 as well. Applying this information to equation (7.5.4),

$$s_1^2\text{area}(R) + s_2^2\text{area}(R) + \cdots + s_p^2\text{area}(R) = \text{area}(R)$$

where s_k is the scale factor of similitude T_k.

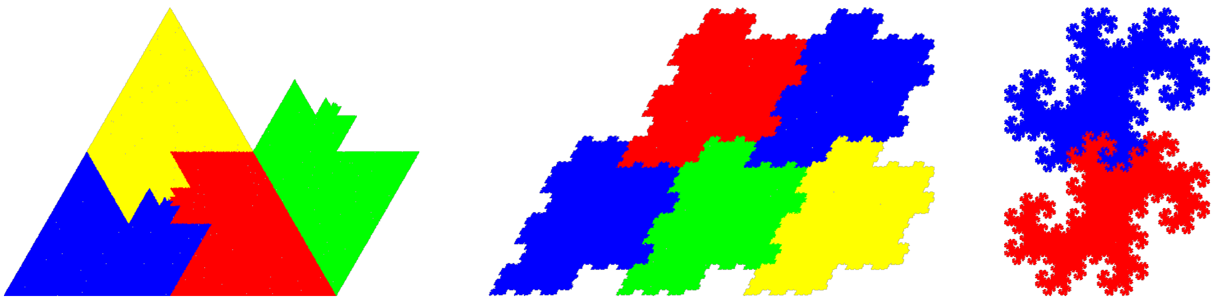
Hence

$$s_1^2 + s_2^2 + \cdots + s_p^2 = 1. \tag{7.5.5}$$

The square, the L-shape, and the triangle were all dissected into four parts, each of which was a $\frac{1}{2}$ scale replica of the whole. By equation (7.5.5) it must be that $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$, an equality that is not hard to verify. As a matter of vocabulary, the set of similitudes associated with a rep-tile is an **iterated function system**, or **IFS**. Hence, if $s_1, s_2, \ldots, s_p$ are the scale factors of the similitudes of the IFS of a rep-tile, then $s_1^2 + s_2^2 + \cdots + s_p^2 = 1$.

That was a lot of work to be able to describe these special polygon-shaped rep-tiles, and if rep-tiles did not come in more exotic shapes, it would probably not have been worth the effort. To justify the work, try to imagine how else you might describe these rep-tiles,

dissections of which are shown here.



Key Concepts

rep-tile a plane figure that tessellates the plane and can be dissected into finitely many similar copies. Equivalently, a plane figure that tiles the plane and tiles an enlarged replica of itself.

Hutchinson's theorem (a special case) Let $\mathscr{C} = \{C_1, C_2, \ldots, C_n\}$ be a set of similitudes in $\mathbb{R}^2$ with scale factors less than one, and define

$$\mathcal{H}_{\mathscr{C}}(A) = C_1(A) \cup C_2(A) \cup \cdots \cup C_n(A) \tag{7.5.6}$$

for any subset A of $\mathbb{R}^2$. Then there is exactly one compact set K in $\mathbb{R}^2$ such that $\mathcal{H}_{\mathscr{C}}(K) = K$. Moreover,

$$\lim_{k\to\infty} \mathcal{H}_{\mathscr{C}}^{\circ k}(A) = K \tag{7.5.7}$$

for any compact set A.

similitude a rigid transformation composed with a dilation. Similitudes preserve shape but not necessarily size

rigid transformation a rotation, reflection, or translation.

dilation a map of the form $T(\mathbf{x}) = r\mathbf{x}$ for some real number $r \geq 0$—scaling by the same factor in all directions.

compact set a subset S of $\mathbb{R}^n$ is compact if it is closed and bounded.

closed set a subset S of $\mathbb{R}^n$ is closed if the limit of every convergent sequence of points in S is also in S. Alternatively, S is closed if it contains all of its limit points.

bounded set a subset S of $\mathbb{R}^n$ is bounded if there exists a real number M such that $S \subseteq \{\mathbf{x} \text{ in } \mathbb{R}^n : ||\mathbf{x}|| < M\}$. Alternatively, S is bounded if it is contained within some ball centered at the origin.

iterated function system a set of contraction mappings.

contraction mapping a map $T : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction (mapping) if for every distinct pair of points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$ there exists a real number $s < 1$ such that

$$\frac{d(T(\mathbf{x}), T(\mathbf{y}))}{d(\mathbf{x}, \mathbf{y})} \leq s.$$

A contraction mapping scales down the distance between every pair of distinct points. A similitude with scale factor less than one is a contraction mapping.

IFS iterated function system.

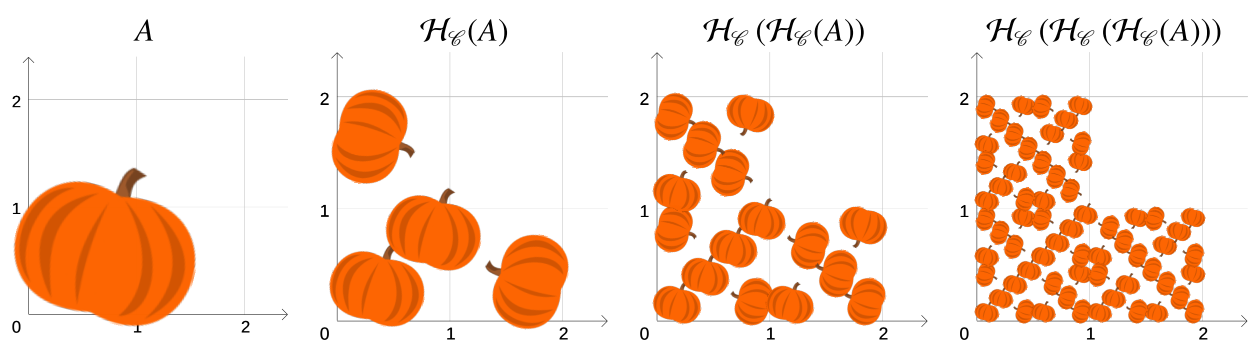
scale factors of the IFS of a rep-tile if $s_1, s_2, \ldots, s_p$ are the scale factors of the similitudes of the IFS of a rep-tile, then $s_1^2 + s_2^2 + \cdots + s_p^2 = 1$.

Exercises

1. Show that the $1, 2, \sqrt{5}$ triangle is rep-5 two different ways (pinwheel tiling and not).
2. Show that the 30-60-90 triangle is rep-3.
3. Trapezoids.
4. (i) Fixed points of affine transformations; and (ii) using them to help determine rep-tiles of an IFS.
5. Find IFS of dragon–put three line segments and a set of axes to help.

Answers

generating the L-shape Letting A take the shape of a pumpkin[7], the L-shape appears rather plainly after only three iterations:



dissecting the triangle For the first dissection, the four transformations mapping the whole triangle to the four parts are, in words,

1. scale by a factor of $\frac{1}{2}$,
2. scale by a factor of $\frac{1}{2}$ and then translate 1 unit right,

[7]Pumpkin image downloaded from https://openclipart.org/detail/86665/plain-pumpkin.

3. scale by a factor of $\frac{1}{2}$ and then translate $\frac{1}{2}$ unit up, and
4. scale by a factor of $\frac{1}{2}$, rotate (about the origin) by 180°, and then translate $\frac{1}{2}$ unit up and one unit right.

As affine transformations, the mappings are

$$\mathbf{x} \mapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x}; \qquad \mathbf{x} \mapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix};$$
$$\mathbf{x} \mapsto \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}; \qquad \mathbf{x} \mapsto \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}.$$

For the second dissection, remember all the triangles are similar, so corresponding parts are in proportion. In particular, the smallest triangle is a $\frac{1}{\sqrt{5}}$ scaled version of the whole and the remaining part is a $\frac{2}{\sqrt{5}}$ scaled version. Getting a little ahead of ourselves, transformations mapping the whole triangle to the two parts are, in words,

1. scale by a factor of $\frac{1}{\sqrt{5}}$, reflect about the y-axis, rotate by angle β (counterclockwise about the origin), then translate along line segment x; and
2. scale by a factor of $\frac{2}{\sqrt{5}}$, reflect about the x-axis, rotate by angle $-\theta$ (clockwise about the origin), then translate along line segment x.

To quantify the rotations and translations, we need to calculate x and the sines and cosines of β and θ. Using the Pythagorean theorem, $1^2 = x^2 + \left(\frac{x}{2}\right)^2$ so $x = \frac{2}{\sqrt{5}}$, and the coordinates of P are $(x\cos\beta, x\sin\beta)$. But $\cos\beta = \frac{1}{\sqrt{5}}$ and $\sin\beta = \frac{2}{\sqrt{5}}$. Finally $\cos\theta = \frac{2}{\sqrt{5}}$ and $\sin\theta = \frac{1}{\sqrt{5}}$, so the mappings are

$$\mathbf{x} \mapsto \frac{1}{\sqrt{5}} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}; \ \mathbf{x} \mapsto \frac{2}{\sqrt{5}} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}$$

which simplify as

$$\mathbf{x} \mapsto \begin{bmatrix} -\frac{1}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}; \qquad \mathbf{x} \mapsto \begin{bmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & -\frac{4}{5} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix}.$$

Solutions to Selected Exercises

Section 3.4

18: What makes any matrix M upper triangular is that $M_{k,\ell} = 0$ whenever $k > \ell$. That is, entries whose row number is greater than their column number are zero. After deleting the first row and some column of U, as shown below,

$$\begin{bmatrix} \boxed{\star} & \boxed{\star} & \boxed{\star} & \boxed{\star} & \boxed{\cdots} & \boxed{\star} \\ 0 & \star & \star & \boxed{\star} & \cdots & \star \\ 0 & 0 & \star & \boxed{\star} & \cdots & \star \\ 0 & 0 & 0 & \boxed{\star} & \cdots & \star \\ \vdots & \vdots & \vdots & \boxed{\vdots} & \ddots & \star \\ 0 & 0 & 0 & \boxed{0} & \cdots & \star \end{bmatrix}$$

there are two distinct regions of entries. Entries to the left of the deleted column have the same column index in $U_{\backslash i,j}$ as they do in U. Columns to the right of the deleted column have a column index one less than they do in U. All entries in $U_{\backslash 1,j}$ have a row index one less than they do in U. In symbols [and **the start of the proof**],

$$(U_{\backslash 1,j})_{k,\ell} = \begin{cases} U_{k+1,\ell} & \text{if } \ell < j \\ U_{k+1,\ell+1} & \text{if } \ell \geq j \end{cases}.$$

If $k > \ell$, then $k+1 > \ell$ and $k+1 > \ell+1$, so $U_{k+1,\ell} = U_{k+1,\ell+1} = 0$. Hence $(U_{\backslash 1,j})_{k,\ell} = 0$ whenever $k > \ell$.

19: As long as $j > 1$, $(U_{\backslash 1,j})_{1,1} = U_{2,1} = 0$.

20: ***[Commentary that is not strictly part of the proof wil be inserted in square brackets and bold italicized.]*** If U is a 1×1 matrix, it is upper triangular and $\det([U_{1,1}]) = U_{1,1}$. ***[This establishes part (i) of the proof. The claim is true for the particular value*** $n = 1$***.]*** Now assume that $\det U = U_{1,1}U_{2,2}\cdots U_{k,k}$ for some (arbitrary) value k greater than or equal to one and arbitrary upper triangular $k \times k$ matrix U. ***[That is, if*** U ***is an upper triangular*** $k \times k$ ***matrix and*** $k \geq 1$***, then the proposition is true. To complete the proof, we must use this information to prove that if*** U ***is a*** $(k+1)\times(k+1)$ ***upper triangular matrix then*** $\det U = U_{1,1}U_{2,2}\cdots U_{k+1,k+1}$***.]*** Additionally, suppose U is a $(k+1)\times(k+1)$ upper triangular matrix. By definition,

$$\det U = (-1)^{1+1}U_{1,1}\det U_{\backslash 1,1} + (-1)^{1+2}U_{1,2}\det U_{\backslash 1,2}\cdots + (-1)^{1+3}U_{1,3}\det U_{\backslash 1,3}.$$

Since $U_{\setminus 1,j}$ has a zero on its diagonal whenever $j > 1$, all terms except the first are zero. Therefore,

$$\det U = (-1)^{1+1} U_{1,1} \det U_{\setminus 1,1}. \tag{7.5.8}$$

Since $U_{\setminus 1,1}$ is a $k \times k$ matrix, its determinant is the product of the entries on its diagonal ***[this is the inductive hypothesis]***, so $\det U_{\setminus 1,1} = U_{2,2} U_{3,3} \cdots U_{k+1,k+1}$. Substituting this expresion into (7.5.8), we have $\det U = U_{1,1} U_{2,2} U_{3,3} \cdots U_{k+1,k+1}$, and the proof is complete.

Index

Bibliography

[1] C. Adam Berrey. Making absolute population estimates in the intermediate area using the area and density of ceramic sherd scatters: An application of regression analysis. *Journal of Archaeological Science*, 97:147–158.

[2] Jamie B. Barker, Matthew J. Slater, Geoff Pugh, Stephen D. Mellalieu, Paul J. McCarthy, Marc V. Jones, and Aidan Moran. The effectiveness of psychological skills training and behavioral interventions in sport using single-case designs: A meta regression analysis of the peer-reviewed studies. *Psychology of Sport and Exercise*, 51:101746, 2020.

[3] Richard L. Burden, J. Douglas Faires, and Annette M. Burden. *Numerical Analysis*. Cengage Learning, 10th edition, 2016.

[4] D. Caroline Coile and Margaret H. Bonham. *Why Do Dogs Like Balls?: More Than 200 Canine Quirks, Curiosities, and Conundrums Revealed*. Sterling Publishing Company, Inc, 2008.

[5] Ángela García, Ofélia Anjos, Carla Iglesias, Helena Pereira, Javier Martínez, and Javier Taboada. Prediction of mechanical strength of cork under compression using machine learning techniques. *Materials & Design*, 82:304–311, 2015.

[6] Martin Gardner. Mathematical games. *Scientific American*, 199(5):136–144, 1958.

[7] Tita G.E. and Radil S.M. Spatial regression models in criminology: Modeling social processes in the spatial weights matrix. 2010.

[8] Kurt Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme i. *Monatsh. f. Mathematik und Physik*, 38:173–198, 1931. https://doi.org/10.1007/BF01700692.

[9] Solomon W. Golomb. Replicating figures in the plane. *The Mathematical Gazette*, 48(366):403–412, Dec 1964.

[10] Roger Hart. *The Chinese Roots of Linear Algebra*. Johns Hopkins University Press, 2011.

[11] Lester S. Hill. Cryptography in an algebraic alphabet. *Mathematical Monthly*, 36(6):306–312, June-July 1929.

[12] William L. Hosch. Pascal's triangle, August 2009. Encyclopedia Britannica. `https://www.britannica.com/science/Pascals-triangle`. Accessed January 29, 2021.

[13] John E. Hutchinson. Fractals and self similarity. *Indiana University Mathematics Journal*, 30(5):713–747, 1981.

[14] Edmund Landau. *Foundations of Analysis: The Arithmetic of Whole, Rational, Irrational and Complex Numbers*. Chelsea Publishing Company, third edition, 1966.

[15] Mariusz Malinowski and Lidia Jabłońska-Porzuczek. Female activity and education levels in selected european union countries. *Research in Economics*, 74(2):153–173, 2020.

[16] Elizabeth S. Meckes and Mark W. Meckes. *Linear Algebra*. Cambridge University Press, 2018.

[17] BBC News Online. Technology | key computer coding creator dies, June 2004. http://news.bbc.co.uk/2/hi/technology/3838845.stm. Accessed January 27, 2021.

[18] K.B. Payne, W.R. Langbauer, and E.M. Thomas. Infrasonic calls of the asian elephant (*Elephas maximus*). *Behav Ecol Sociobiol*, 18:297–301, 1986.

[19] Ioseph Peano. *Arithmetices Principia: Nova Methodo Exposita*. Augustae Taurinorum: Fratres Bocca, 1889.

[20] Mark A. Rose. Predicting daily maximum temperatures using linear regression and geopotential thickness forecasts, unknown. National Weather Service. https://www.weather.gov/ohx/predictingdailymaxtemps. Accessed April 5, 2021.

[21] Brian S. Thomson, Judith B. Bruckner, and Andrew M. Bruckner. *Elementary Real Analysis*. ClassicalRealAnalysis.com, second edition edition, 2008.

[22] B. Wuyts, M. Maenhoudt, S. Libbrecht, Z.X. Gao, E. Osquiguil, V.V. Moshchalkov, and Y. Bruynseraede. Linear relation between the hall angle slope and the carrier density in yba2cu3ox films. *Physica C: Superconductivity*, 235-240:1369–1370, 1994.