0.1 Matrix Multiplication

In the previous section an appeal to your sense of organization was made in the discussion of matrix addition, matrix subtraction and scalar multiplication. Each operation was done component-wise, something people find rather natural. Devoid of context, however, there is nothing natural or intuitive about matrix multiplication

If you can master the product of a **row matrix** (a $1 \times n$ matrix) with a **column matrix** (an $m \times 1$ matrix), you can master the product of any two matrices. The following example illustrates the process.

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 32 \end{bmatrix}$$

Given a row matrix R and a column matrix C with the same number of entries, say n, their product is the sum of the products of corresponding entries. That is,

$$RC = \begin{bmatrix} r_{1,1}c_{1,1} + r_{1,2}c_{2,1} + \cdots + r_{1,n}c_{n,1} \end{bmatrix}.$$

The first entry of R (reading from left to right) corresponds with the first entry of C (reading from top to bottom). The second entry of R corresponds with the second entry of C. And so on. As with addition, multiplication is an operator, so the product of two matrices is a matrix. In this case, a 1×1 matrix. If R and C have differing length the product RC is undefined.

For the product of other matrices, this row times column calculation is repeated for each entry of the product. If A and B are matrices, then the product $P = AB$ is calculated by setting $p_{i,j}$ equal to the lone entry of $A_{i,:}B_{:,j}$ (where this makes sense). That is, the i,j-entry of P is the single entry of the product of the i^{th} row of A by the j^{th} column of B (again, where this makes sense). Several conclusions can be drawn from this description.

- The rows of A and the columns of B must all have the same length (number of entries). Otherwise the product of a row of A by a column of B, $A_{i,:}B_{:,j}$ is undefined.
- P has the same number of rows as A (P and A have the same height).
- P has the same number of columns as B (P and B have the same width).

These observations lead to a graphical organization technique for multiplication. Writing B to the right of A and just below leaves a space above B and to the right of A that's exactly the right size for the product AB. Plus, the row needed for calculating $p_{i,j}$ is directly left and the column needed for calculating $p_{i,j}$ is directly below. See figure 1. The size of P is determined by the number of rows in A and the number of columns in B.

Figure 1: $p_{2,4} = a_{2,1}b_{1,4} + a_{2,2}b_{2,4}$

Two entries
in each row

$P = AB$

$$\begin{bmatrix} \cdot & \cdot \\ a_{2,1} & a_{2,2} \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & p_{2,4} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

A

$$\begin{bmatrix} \cdot & \cdot & \cdot & b_{1,4} & \cdot \\ \cdot & \cdot & \cdot & b_{2,4} & \cdot \end{bmatrix}$$

Two entries
in each column.

B