

# Differentiation Under the Integral Sign

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We often need to differentiate under the integral sign:

$$\frac{d}{dt} \int_X f(x, t) d\mu(x) = \int_X \partial_t f(x, t) d\mu(x),$$

which allows us to exchange the order of derivative and integral sign.

The following is a rigorous statement of the result using Fubini's theorem.

**Theorem 1.** Let  $(X, \mu)$  be a  $\sigma$ -finite measure space and  $I = [a, b] \subset \mathbb{R}$  a compact interval. Let  $f : X \times I \rightarrow \mathbb{R}$  be a measurable function satisfying:

1. *Absolute Continuity:* For  $\mu$ -a.e.  $x \in X$ , the map  $t \mapsto f(x, t)$  is absolutely continuous on  $I$ .
2. *Integrable Function:* There exists  $t_0 \in I$  such that  $f(\cdot, t_0)$  is integrable on  $X$ :

$$\int_X |f(x, t_0)| d\mu(x) < \infty.$$

3. *Integrable Derivative:* The partial derivative  $\partial_t f$  is integrable on  $X \times I$ :

$$\int_X \int_I |\partial_t f(x, t)| dt d\mu(x) < \infty.$$

Then the function  $F(t) := \int_X f(x, t) d\mu(x)$  is absolutely continuous on  $I$ , and for almost every  $t \in I$ :

$$F'(t) = \int_X \partial_t f(x, t) d\mu(x).$$

*Proof.* 1. *Integral representation of  $f$ .* Since  $t \mapsto f(x, t)$  is absolutely continuous for almost every  $x$  (Assumption 1), the Fundamental Theorem of Calculus implies that for any  $t \in I$ :

$$f(x, t) = f(x, t_0) + \int_{t_0}^t \partial_t f(x, s) ds \quad \text{for } \mu\text{-a.e. } x.$$

2. *Finiteness of  $F(t)$ .* To ensure  $F(t)$  is well-defined, we check the integrability of  $f(\cdot, t)$ . By the triangle inequality:

$$|f(x, t)| \leq |f(x, t_0)| + \left| \int_{t_0}^t \partial_t f(x, s) ds \right| \leq |f(x, t_0)| + \int_I |\partial_t f(x, s)| ds.$$

Integrating over  $X$ :

$$\int_X |f(x, t)| d\mu(x) \leq \int_X |f(x, t_0)| d\mu(x) + \int_X \int_I |\partial_t f(x, s)| ds d\mu(x).$$

Thus,  $f(\cdot, t) \in L^1(\mu)$  for all  $t \in I$ , so  $F(t)$  is finite everywhere.

*3. Application of Fubini's theorem.* Consider the difference  $F(t) - F(t_0) = \int_X \left( \int_{t_0}^t \partial_t f(x, s) ds \right) d\mu(x)$ . Let  $h(x, s) := \partial_t f(x, s) \cdot \mathbf{1}_{[t_0, t]}(s)$ . Since  $|\partial_t f| \in L^1(X \times I)$ , Fubini's Theorem applies:

$$\int_X \int_{t_0}^t \partial_t f(x, s) ds d\mu(x) = \int_{t_0}^t \left( \int_X \partial_t f(x, s) d\mu(x) \right) ds.$$

*4. Conclusion.* Let  $G(s) := \int_X \partial_t f(x, s) d\mu(x)$ . Then  $F(t) = F(t_0) + \int_{t_0}^t G(s) ds$ . By fundamental theorem of calculus for Lebesgue integrals,  $F$  is absolutely continuous and  $F'(t) = G(t)$  for a.e.  $t$ .  $\square$

**Remark 1.** A common sufficient condition (stronger than Assumption 3) is to start from the difference quotient  $\frac{f(x, t+\varepsilon) - f(x, t)}{\varepsilon}$  and justify exchanging  $\lim_{\varepsilon \rightarrow 0}$  and  $\int_X (\cdot) d\mu$  by dominated convergence. This typically requires an *integrable envelope* that dominates the difference quotients (or, equivalently, dominates  $|\partial_t f(x, t)|$  uniformly for  $t$  near some  $t_0$ , i.e. there exists  $C \in L^1(X)$  and a neighborhood  $U$  of  $t_0$  such that  $|\partial_t f(x, t)| \leq C(x)$  for all  $t \in U$  and  $\mu$ -a.e.  $x$ ), which is more restrictive than assuming  $\partial_t f \in L^1(X \times I)$  above.

Conceptually, this approach applies the Fundamental Theorem of Calculus in the  $t$ -variable and then reduces the interchange-of-derivative question to commuting two integrals, which can be handled by Fubini's theorem.

See <https://planetmath.org/differentiationundertheintegralssign>.

**Remark 2.** (*Differentiating under the integral sign with a moving domain.*) If the integration region depends on a parameter, say  $\Omega(t) \subset \mathbb{R}^d$ , then one has an extra boundary term. For instance, assume  $\partial\Omega(t)$  is smooth and evolves with a velocity field  $v(\cdot, t)$ , in the sense that each boundary point  $x(t) \in \partial\Omega(t)$  moves according to  $\dot{x}(t) = v(x(t), t)$ . Let  $n(x, t)$  denote the outward unit normal to  $\partial\Omega(t)$  at  $x \in \partial\Omega(t)$ , so the normal speed is  $v(x, t) \cdot n(x, t)$ . Then

$$\frac{d}{dt} \int_{\Omega(t)} f(x, t) dx = \int_{\Omega(t)} \partial_t f(x, t) dx + \int_{\partial\Omega(t)} f(x, t) v(x, t) \cdot n(x, t) dS(x).$$

This is known as the *Reynolds transport theorem* (a Leibniz rule for moving domains).

## 1 Divergence Theorem on $\mathbb{R}^d$

We use this to prove the divergence theorem on  $\mathbb{R}^d$ :

$$\int_{\mathbb{R}^d} \nabla f(x) dx = 0, \quad \forall f \in W^{1,1}(\mathbb{R}^d),$$

where  $W^{1,1}(\mathbb{R}^d)$  is the Sobolev space of weakly differentiable functions with  $f \in L^1(\mathbb{R}^d)$  and  $\nabla f \in L^1(\mathbb{R}^d; \mathbb{R}^d)$ .

We first establish the result for functions that satisfy the *Absolute Continuity on Lines* (ACL) property.

**Theorem 2** (Divergence Theorem for ACL Functions). Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a measurable function such that:

1.  $f \in L^1(\mathbb{R}^d)$ .
2. For each  $i \in \{1, \dots, d\}$ ,  $f$  is absolutely continuous on almost every line parallel to the  $i$ -th coordinate axis.
3. The classical partial derivatives  $\partial_i f$  (which exist a.e. by condition 2) are in  $L^1(\mathbb{R}^d)$ .

Then for each  $i \in \{1, \dots, d\}$ , we have  $\int_{\mathbb{R}^d} \partial_i f(x) dx = 0$ .

*Proof.* Fix  $i \in \{1, \dots, d\}$ . By the translation invariance of the Lebesgue measure, the integral of  $f$  is invariant under shifts in the  $e_i$  direction:

$$F(h) := \int_{\mathbb{R}^d} f(x + h\mathbf{e}_i) dx = \int_{\mathbb{R}^d} f(y) dy = \text{const.}$$

We check the conditions to differentiate  $F(h)$  under the integral sign:

- *AC*:  $h \mapsto f(x + h\mathbf{e}_i)$  is AC for a.e.  $x$  by assumption.
- *Integrability*:  $f \in L^1$  ensures  $F(h)$  is finite.
- *Derivative*: The partial derivative  $\partial_h f(x + h\mathbf{e}_i) = \partial_i f(x + h\mathbf{e}_i)$  is integrable over  $\mathbb{R}^d \times I$  by Tonelli's theorem:  $\int_I \int_{\mathbb{R}^d} |\partial_i f(x + s\mathbf{e}_i)| dx ds = |I| \|\partial_i f\|_1 < \infty$ .

Applying the differentiation theorem:

$$0 = F'(h) = \int_{\mathbb{R}^d} \partial_i f(x + h\mathbf{e}_i) dx = \int_{\mathbb{R}^d} \partial_i f(x) dx,$$

where we used the translation invariance on  $\partial_i f$ . □

Now we connect this result to the Sobolev space  $W^{1,1}(\mathbb{R}^d)$  using the standard characterization of Sobolev functions.

**Theorem 3** (ACL Characterization of  $W^{1,1}$ ). A function  $f$  belongs to  $W^{1,1}(\mathbb{R}^d)$  if and only if there exists a function  $\hat{f}$  such that  $\hat{f} = f$  almost everywhere,  $\hat{f}$  is ACL, and the classical partial derivatives  $\partial_i \hat{f}$  belong to  $L^1(\mathbb{R}^d)$  for all  $i = 1, \dots, d$ . In this case,  $\partial_i \hat{f}$  coincides with the weak derivative  $\partial_i f$  almost everywhere.

**Theorem 4** (Divergence Theorem for  $W^{1,1}$ ). If  $f \in W^{1,1}(\mathbb{R}^d)$ , then  $\int_{\mathbb{R}^d} \nabla f(x) dx = 0$ .

*Proof.* Let  $\hat{f}$  be as in the ACL characterization above. Since  $f \in W^{1,1}(\mathbb{R}^d)$ ,  $\hat{f}$  satisfies all three conditions of the ACL Divergence Theorem:  $\hat{f} \in L^1$ ,  $\hat{f}$  is ACL, and  $\partial_i \hat{f} = D_i f \in L^1$ . Thus,

$$\int_{\mathbb{R}^d} \partial_i f(x) dx = \int_{\mathbb{R}^d} \partial_i \hat{f}(x) dx = 0.$$

The result for the gradient follows by applying this component-wise.  $\square$