AA 274A: Principles of Robot Autonomy I
Problem Set X

Name: Li Quan Khoo
SUID: lqkhoo (06154100)

Problem 1: Trajectory Generation via Differential Flatness

(i) We are given initial and final conditions in terms of variables $\{x, y, V, \theta\}$. The equations are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \\ x(t_f) \\ \dot{x}(t_f) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ y(t_f) \\ \dot{y}(t_f) \end{bmatrix}$$

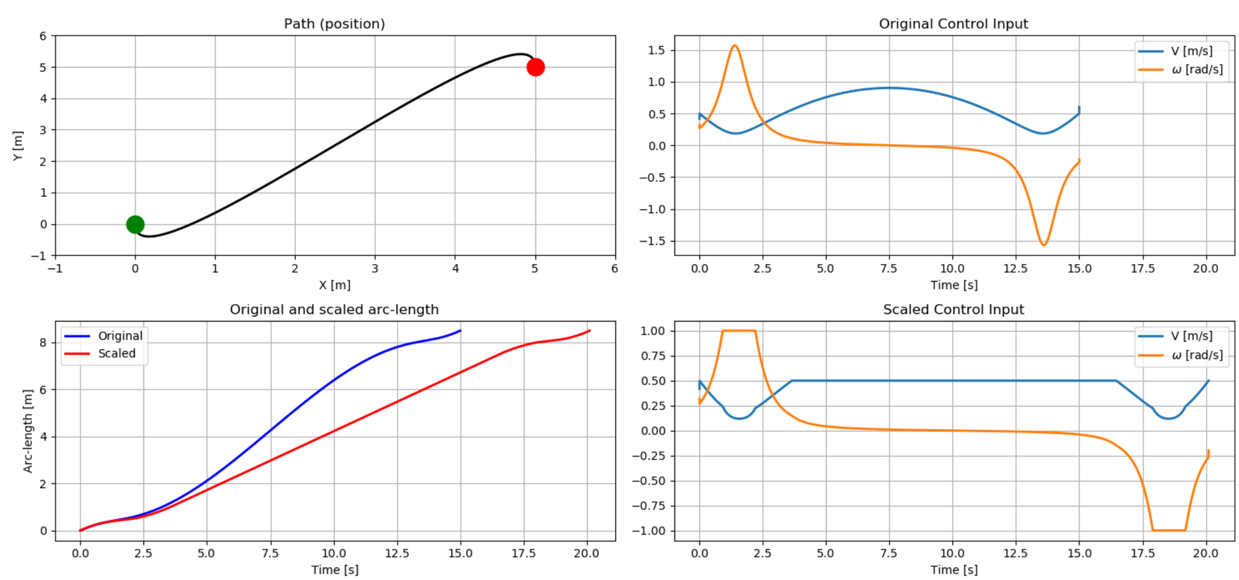
where $\dot{x}(t) = V\cos\theta$ and $\dot{y}(t) = V\sin\theta$ as given by the robot's kinematic model.

(ii) Since $\det(\boldsymbol{J}) = V$, $V > 0\ \forall t$ is a sufficient and necessary condition for the matrix $\boldsymbol{J}$ to be invertible.

(iii) (code)

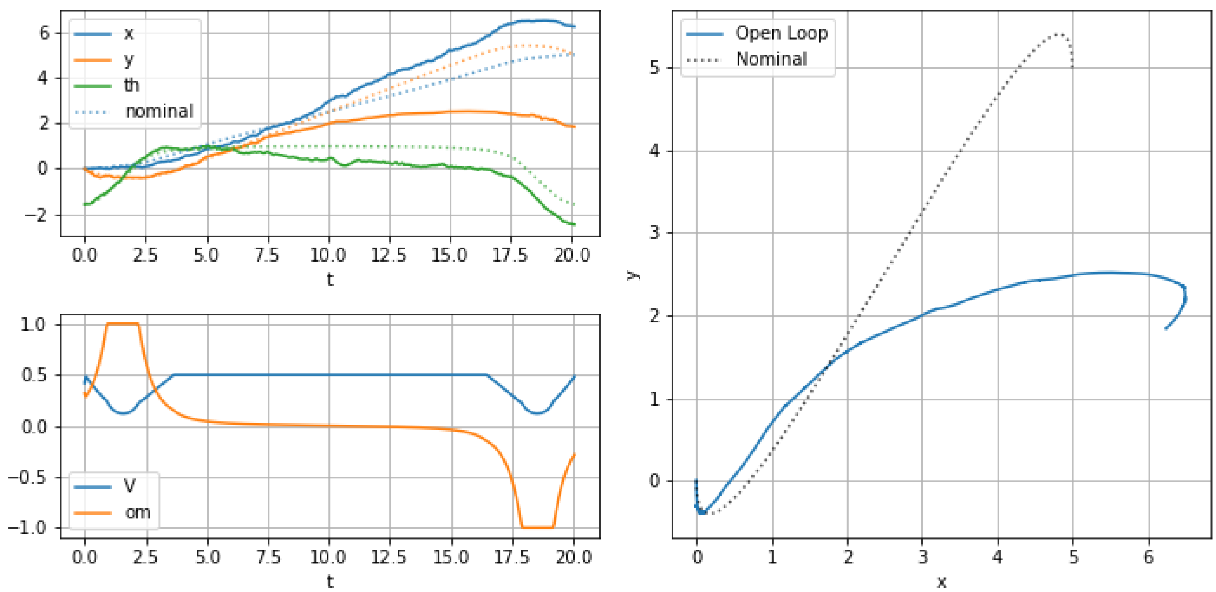
(iv) (code)

(v)



Trajectory of unicycle model in absence of noise. Initial and final conditions as given.

(vi)



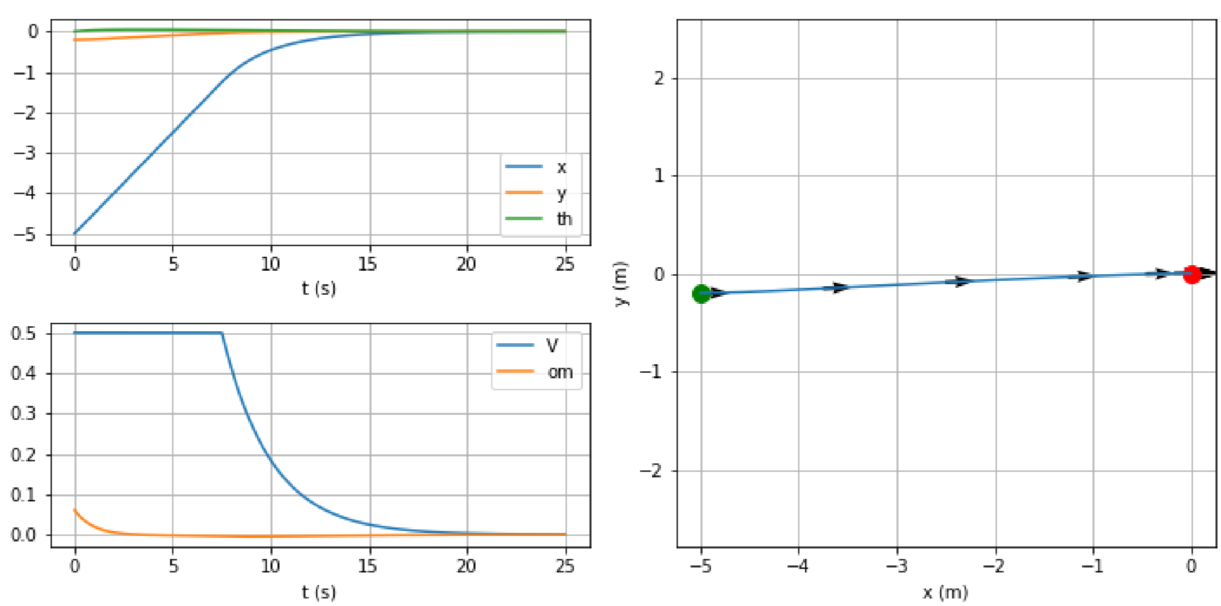
Trajectory of unicycle model where control vector $u_{\text{noisy}} = u + \epsilon$ where ϵ is simulated isotropic Gaussian noise.

Problem 2: Pose Stabilization

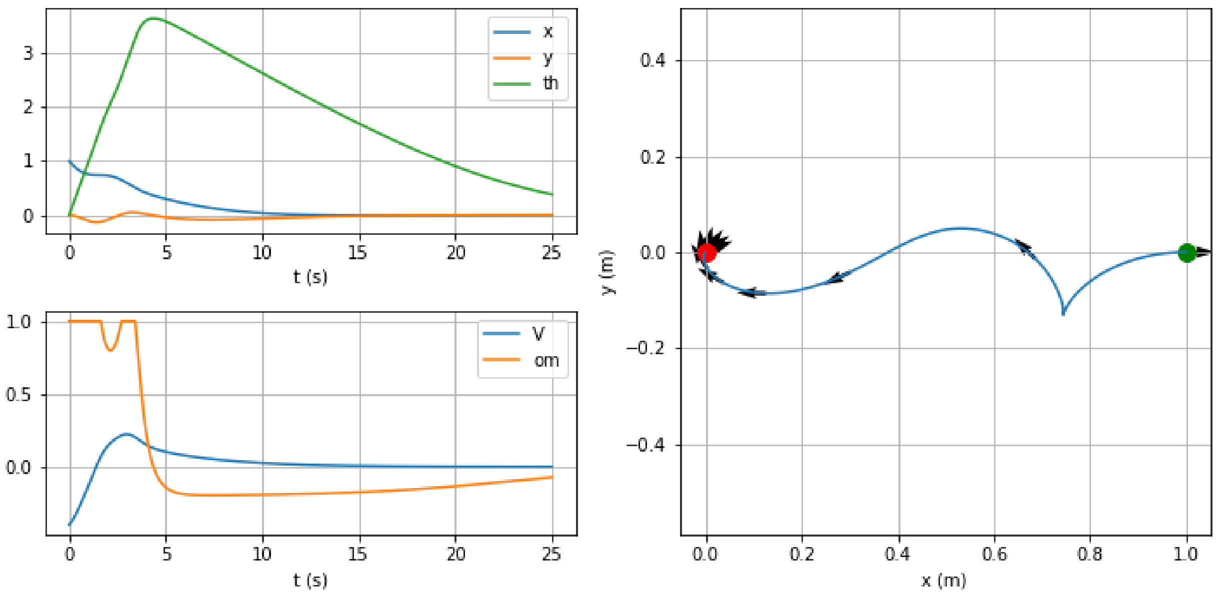
(i) (code)

(ii) (code)

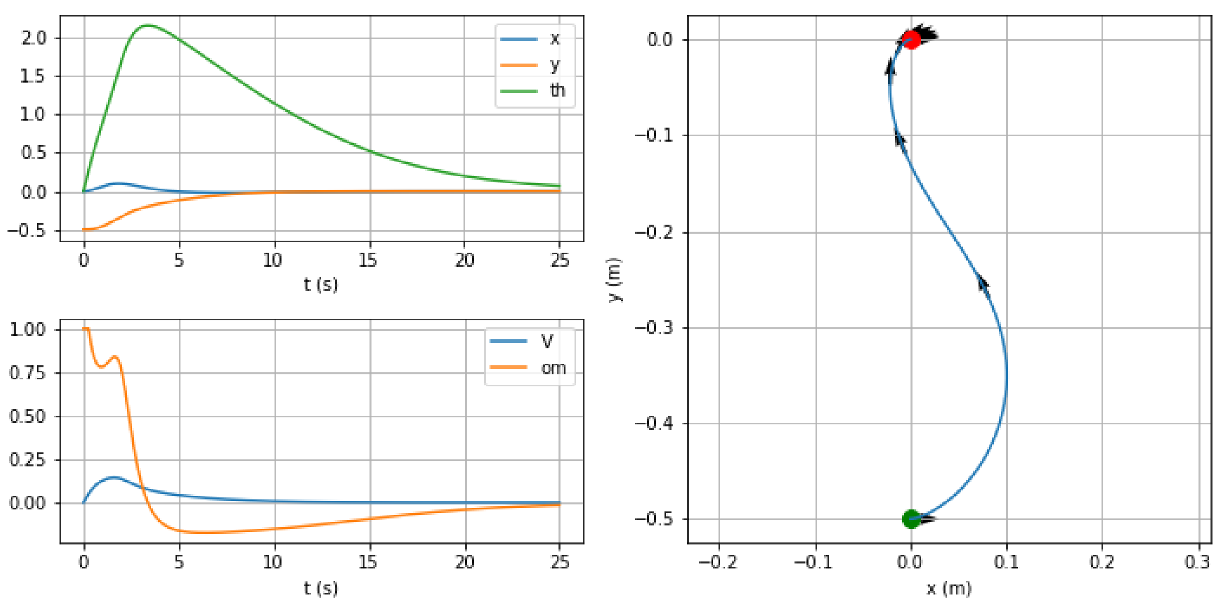
(iii)



Forward parking



Reverse parking



Parallel parking

Problem 3: Trajectory Tracking

(i) Starting from our extended unicycle model, the given kinematic equations are:

$$\begin{aligned} \dot{x}(t) &= V\cos(\theta) \\ \dot{y}(t) &= V\sin(\theta) \\ \dot{V}(t) &= a(t) \\ \dot{\theta}(t) &= \omega(t) \end{aligned} \tag{1}$$

$$\underbrace{\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix}}_{\ddot{\boldsymbol{z}}=\boldsymbol{z}^{(q+1)}} = \underbrace{\begin{bmatrix} \cos(\theta) & -V\sin(\theta) \\ \sin(\theta) & V\cos(\theta) \end{bmatrix}}_{:=\boldsymbol{J}} \begin{bmatrix} a \\ \omega \end{bmatrix} := \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{w}} \tag{2}$$

Equation (2) is in the form of a linear ODE $\boldsymbol{z}^{(q+1)} = \boldsymbol{w}$. For the unicycle model, q is known to be 1. We want to design a control law for the virtual input term $\boldsymbol{w}$.

Below is the exact linearization scheme in lecture 4. Where superscript (j) denotes the jth derivative w.r.t. t, subscript d denotes the 'desired' open-loop trajectory that we wish to track, we define ith component of the linear tracking error to be:

$$\begin{aligned} e_i(t) &:= z_i(t) - z_{i,d}(t) \\ e_i^{(q+1)}(t) &= z_i^{(q+1)}(t) - z_{i,d}^{(q+1)}(t) \\ &= w_i - w_{i,d} \\ \boldsymbol{e}(t) &= \boldsymbol{w}(t) - \boldsymbol{w}_d(t) \end{aligned} \tag{3}$$

, which is a second-order ODE for our system.

Following lecture 4 notes, we consider a closed-loop control law of the form:

$$w_i(t) = w_{i,d}(t) - \sum_{j=0}^{q} k_{i,j} e_i^{(j)}(t) \tag{4}$$

which results in closed-loop dynamics of the form:

$$\boldsymbol{z}^{(q+1)} = \boldsymbol{w}_d - \sum_{j=0}^{q} K_j \boldsymbol{e}^{(j)} \tag{5}$$

, where K_j is a diagonal matrix containing elements $k_{i,j}$. Since $\boldsymbol{z}_d^{(q+1)} = \boldsymbol{w}_d$, we finally get:

$$\boldsymbol{e}^{(q+1)} + \sum_{j=0}^{q} K_j \boldsymbol{e}^{(j)} = 0 \tag{6}$$

For our system, (6) becomes:

$$\begin{aligned} &(\ddot{x} - \ddot{x}_d) + k_{dx}(\dot{x} - \dot{x}) \quad + k_{px}(x_d - x) = 0 \\ &\underbrace{(\ddot{y} - \ddot{y}_d)}_{\ddot{e}} + k_{dy}\underbrace{(\dot{y} - \dot{y})}_{\dot{e}} \quad + k_{py}\underbrace{(y_d - y)}_{e} = 0 \end{aligned} \tag{7}$$

This is a 2nd order ODE in $\boldsymbol{e}$ in standard form. We know that for the system to be stable, the real parts of both roots of its characteristic equation must be > 0. We also know that the system is critically-damped when the roots are equal. This tells us what the values of K should be.

Rearranging (7), since $\ddot{x} = u_1$ and $\ddot{y} = u_2$ from (2), we get:

$$\begin{aligned} u_1 &= \ddot{x}_d \quad + k_{px}(x_d - x) + k_{dx}(\dot{x} - \dot{x}) = 0 \\ u_2 &= \ddot{y}_d \quad + k_{py}(y_d - y) + k_{dy}(\dot{y} - \dot{y}) = 0 \end{aligned} \tag{8}$$

Finally, to recover our real controls $\{a, \omega\}$, we invert J in (2) and solve for:

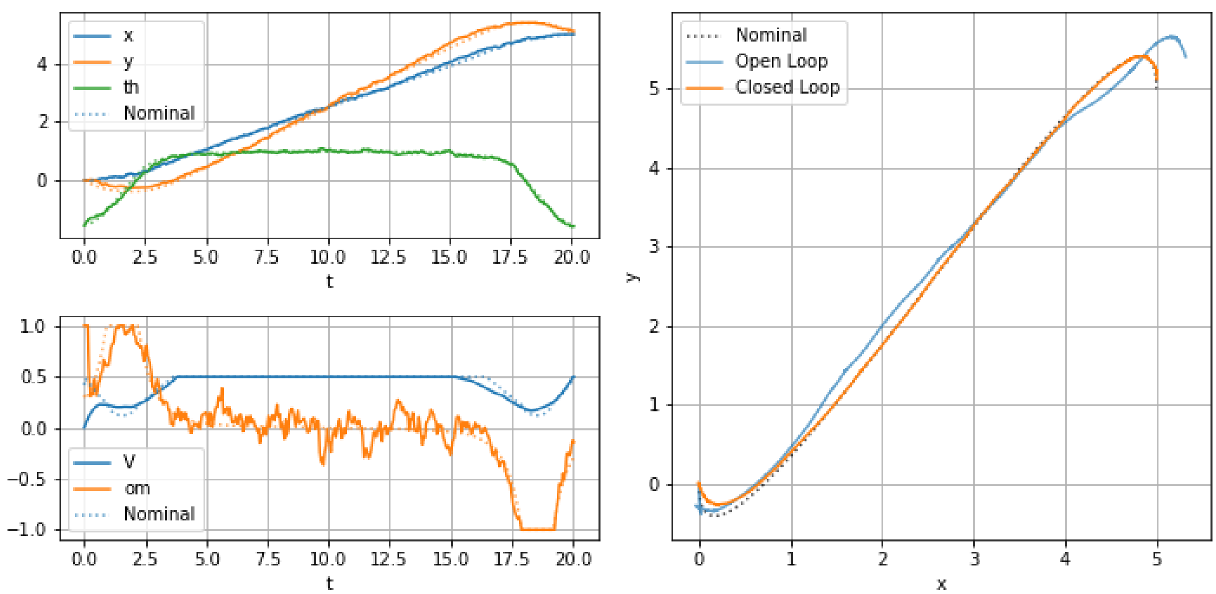
$$\begin{aligned} \begin{bmatrix} a \\ w \end{bmatrix} &= J^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \ , \ V > 0 \\ \begin{bmatrix} a \\ w \end{bmatrix} &= \frac{1}{V} \begin{bmatrix} V\cos(\theta) & V\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \tag{9}$$

To recover V as the problem requested,

$$\begin{aligned} V(T) &= \int_{t_0}^{T} a(t)\ dt \\ V_t &= V_{t-1} + a_t \Delta t \end{aligned} \tag{10}$$

(ii) (code)

(iii)



Trajectory of unicycle under two-part tracking controller.

Extra Problem: Optimal Control and Trajectory Optimization

(i) To minimize:

$$J = \int_0^{t_f} \underbrace{[\lambda + V^2 + \omega^2]}_{g} \, dt \;, \quad \lambda \text{ constant} \tag{11}$$

Subject to:

$$\begin{aligned} \dot{x} &= V\cos\theta \\ \dot{y} &= V\sin\theta \\ \dot{\theta} &= \omega \end{aligned} \tag{12}$$

Boundary conditions:

$$\begin{array}{ccc} x(0) = 0 & y(0) = 0 & \theta(0) = -\frac{\pi}{2} \\ x(t_f) = 5 & y(t_f) = 5 & \theta(t_f) = -\frac{\pi}{2} \end{array} \tag{13}$$

We follow the procedure outlined in lecture 5.

Let:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \;,\; \boldsymbol{u} = \begin{bmatrix} V \\ \omega \end{bmatrix} \;,\; \boldsymbol{f} = \begin{bmatrix} V\cos\theta \\ V\sin\theta \\ \omega \end{bmatrix} \tag{14}$$

1. First, we need the Hamiltonian for the system:

$$\begin{aligned} \mathcal{H} &= g + \boldsymbol{p}^{\mathsf{T}}\boldsymbol{f} \\ \mathcal{H} &= \lambda + V^2 + \omega^2 + p_1 V\cos\theta + p_2 V\sin\theta + p_3\omega \end{aligned} \tag{15}$$

2. Next, we find the NOCs, which are defined by:

$$\begin{aligned} \dot{\boldsymbol{x}}^* &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \\ \dot{\boldsymbol{p}}^* &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} \\ 0 &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{u}} \end{aligned} \tag{16}$$

For our system, these correspond to $2n+m$ equations. To solve for t_f, we additionally define a dummy state variable r such that $r = t_f$ and $\dot{r} = 0$.

The NOCs for our system are:

$$\begin{array}{lll} \dot{x}^* = V\cos\theta & \dot{p}_1^* = 0 & 0 = \frac{\partial \mathcal{H}}{\partial V} \\ \dot{y}^* = V\sin\theta & \dot{p}_2^* = 0 & 0 = \frac{\partial \mathcal{H}}{\partial \omega} \\ \dot{\theta}^* = \omega & \dot{p}_3^* = p_1 V\sin(\theta) - p_2 V\cos(\theta) & \\ \dot{r}^* = 0 & & \end{array} \tag{17}$$

Since

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \omega} = 2\omega + p_3 &= 0 \\ \omega &= -\frac{p_3}{2} \end{aligned} \tag{18}$$

3. We need to have $2n+1$ boundary conditions to solve for the $2n$ differential equations plus 1 free variable t_f. With x_f fixed and t_f free, these are governed by:

$$\begin{aligned} x^*(t_0) &= x_0 \\ x^*(t_f) &= x_f \\ \mathcal{H} + \frac{\partial h}{\partial t} &= 0 \end{aligned} \tag{19}$$

In our case, $h = 0$ as we don't have initial cost. These correspond to:

$$\begin{gathered} \begin{aligned} \dot{x}^*(0) &= 0 & \dot{x}^*(t_f) &= 5 \\ \dot{y}^*(0) &= 0 & \dot{y}^*(t_f) &= 5 \\ \dot{\theta}^*(0) &= -\tfrac{\pi}{2} & \dot{\theta}^*(t_f) &= -\tfrac{\pi}{2} \end{aligned} \\ \lambda + V^2 + \omega^2 + p_1 V \cos\theta + p_2 V \sin\theta + p_3 \omega = 0 \end{gathered} \tag{20}$$

4. Now we need to rescale for time. Let τ be our new time scale such that:

$$\begin{aligned} \tau &= \frac{t}{t_f}\ ,\ \tau \in [0,1] \\ \frac{d}{d\tau} &= \frac{dt}{d\tau} \cdot \frac{d}{dt} = t_f \frac{d}{dt} \end{aligned} \tag{21}$$

Denoting $\frac{d}{d\tau}$ as $[.]'$, the NOCs become:

$$\begin{aligned} x'^* &= rV\cos\theta & p_1'^* &= 0 & 0 &= \tfrac{\partial \mathcal{H}}{\partial V} \\ y'^* &= rV\sin\theta & p_2'^* &= 0 & 0 &= \tfrac{\partial \mathcal{H}}{\partial \omega} \\ \theta'^* &= r\omega & p_4'^* &= rp_1 V\sin(\theta) - rp_2 V\cos(\theta) \\ r'^* &= 0 \end{aligned} \tag{22}$$

The boundary conditions do not have t terms, so the expressions remain the same.

(ii) (code)

(iii) TODO

(iv) TODO

(v) TODO