AA 274A: Principles of Robot Autonomy I
Problem Set 4

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Problem 1: EKF Localization

(i) (code). This written part is not required by the question, but I'll setup the problem here anyway since this is way too much for comments in code, and the derivation is neither in the notes or slides.

Given: A unicycle model with generalized coordinates and instantaneous control vector:

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ \theta(t) \end{bmatrix} \quad , \quad \mathbf{u}(t) = \begin{bmatrix} V(t) \\ \omega(t) \end{bmatrix} \tag{1}$$

Given: Continuous unicycle model dynamics:

$$\begin{aligned} \dot{x}(t) &= V(t)\cos(\theta(t)) \\ \dot{y}(t) &= V(t)\sin(\theta(t)) \\ \dot{\theta}(t) &= \omega(t) \end{aligned} \tag{2}$$

For clarity, we denote the value of a variable at discrete time step using subscript t from now on.

To find: Discrete-time state transition model

$$\mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) \tag{3}$$

g can be interpreted as our belief of the state variables after taking control $\mathbf{u}$ from state $\mathbf{x}_{t-1}$. $\mathbf{x}_t$ is not directly observable due to uncertainty, but assuming g is well-behaved i.e. continuous etc., for small time steps Δt, we may rely on local similarity in order to approximate it. Let $\tilde{\mathbf{x}}_{t-1}$ and $\tilde{\mathbf{u}}_t$ be small perturbations about $\mathbf{x}_{t-1}$ and $\mathbf{u}_t$. We can use the (multivariate) Taylor series approximation up to first order terms:

$$\begin{aligned} \mathbf{x}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) \approx \tilde{\mathbf{x}}_t &= g(\tilde{\mathbf{x}}_{t-1}, \tilde{\mathbf{u}}_t) \\ &\approx g(\mathbf{x}_{t-1}, \mathbf{u}_t) + G_{x,t}(\mathbf{x}_{t-1}, \mathbf{u}_t) \cdot (\tilde{\mathbf{x}}_{t-1} - \mathbf{x}_{t-1}) + G_{u,t}(\mathbf{x}_{t-1}, \mathbf{u}_t) \cdot (\tilde{\mathbf{u}}_t - \mathbf{u}_t) \end{aligned} \tag{4}$$

where G_x and G_u are Jacobians.

We also assume a zero-order hold on $\mathbf{u}$, i.e. $\mathbf{u}$ is constant over some time period Δt. For small Δt this is a good approximation. In order to approximate $\mathbf{x}_t$, we find $\tilde{\mathbf{x}}_t$ by discretizing the continuous model using small Δt and by using the zero-order hold.

$$\begin{aligned} \mathbf{x_t} \approx \tilde{\mathbf{x}}_t &= \mathbf{x}_{t-1} + \Delta \mathbf{x} \\ &= \mathbf{x}_{t-1} + \int_0^{\Delta t} \dot{\mathbf{x}}_{t-1} d\tau \end{aligned} \tag{5}$$

Individually then,

$$\begin{aligned} \tilde{\theta}_t &= \theta_{t-1} + \int_0^{\Delta t} \omega_t \; d\tau \;, \quad \omega_t \text{ constant} \\ &= \theta_{t-1} + \omega_t \tau \Big|_0^{\Delta t} \\ &= \theta_{t-1} + \omega_t \Delta t \end{aligned} \tag{6}$$

$$\begin{aligned} \tilde{x}_t &= x_{t-1} + \int_0^{\Delta t} \dot{x}_{t-1} \; d\tau \\ &= x_{t-1} + \int_0^{\Delta t} V_t \cos(\theta_t) \; d\tau \;, \quad V_t \text{ constant} \\ &= x_{t-1} + V_t \int_0^{\Delta t} \cos(\theta_{t-1} + \omega_t \tau) \; d\tau \\ &= x_{t-1} + \frac{V_t}{\omega_t} \int_0^{\Delta t} \omega_t \cdot \cos(\theta_{t-1} + \omega_t \tau) \; d\tau \\ &= x_{t-1} + \frac{V_t}{\omega_t} \sin(\theta_{t-1} + \omega_t \tau) \Big|_0^{\Delta t} \\ &= x_{t-1} + \frac{V_t}{\omega_t} \Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1}) \Big] \end{aligned} \tag{7}$$

Likewise,

$$\begin{aligned} \tilde{y}_t &= y_{t-1} + \int_0^{\Delta t} \dot{y}_{t-1} \; d\tau \\ &= y_{t-1} + \int_0^{\Delta t} V_t \sin(\theta_t) \; d\tau \\ &= y_{t-1} + \frac{V_t}{\omega_t} \int_0^{\Delta t} \omega_t \cdot \sin(\theta_{t-1} + \omega_t \tau) \; d\tau \\ &= y_{t-1} - \frac{V_t}{\omega_t} \Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1}) \Big] \end{aligned} \tag{8}$$

The Jacobian G_x at time t is then

$$G_{x,t} = \frac{\partial \mathbf{x}_t}{\partial \mathbf{x}_t} \approx \frac{\partial \tilde{\mathbf{x}}_t}{\partial \tilde{\mathbf{x}}_t} = \begin{bmatrix} 1 & 0 & \frac{\partial \tilde{x}}{\partial \tilde{\theta}} \\ 0 & 1 & \frac{\partial \tilde{y}}{\partial \tilde{\theta}} \\ 0 & 0 & 1 \end{bmatrix}_t \tag{9}$$

where

$$\begin{aligned} \frac{\partial \tilde{x}_t}{\partial \tilde{\theta}_t} &= \frac{\partial}{\partial \tilde{\theta}_t} \left(x_{t-1} + \frac{V_t}{\omega_t} \Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1}) \Big] \right) \\ &= \frac{V_t}{\omega_t} \Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1}) \Big] \\ \frac{\partial \tilde{y}_t}{\partial \tilde{\theta}_t} &= \frac{\partial}{\partial \tilde{\theta}_t} \left(y_{t-1} - \frac{V_t}{\omega_t} \Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1}) \Big] \right) \\ &= \frac{V_t}{\omega_t} \Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1}) \Big] \end{aligned} \tag{10}$$

Likewise, the Jacobian G_u at time t is

$$G_{u,t} = \frac{\partial \mathbf{x}_t}{\partial \mathbf{u}_t} \approx \frac{\partial \tilde{\mathbf{x}}_t}{\partial \tilde{\mathbf{u}}_t} = \begin{bmatrix} \frac{\partial \tilde{x}}{\partial \tilde{V}} & \frac{\partial \tilde{x}}{\partial \tilde{\omega}} \\ \frac{\partial \tilde{y}}{\partial \tilde{V}} & \frac{\partial \tilde{y}}{\partial \tilde{\omega}} \\ \frac{\partial \tilde{\theta}}{\partial \tilde{V}} & \frac{\partial \tilde{\theta}}{\partial \tilde{\omega}} \end{bmatrix}_t \tag{11}$$

where

$$\begin{aligned}
\frac{\partial \tilde{x}_t}{\partial \tilde{V}_t} &= \frac{\partial}{\partial \tilde{V}_t}\left(x_{t-1} + \frac{V_t}{\omega_t}\Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\Big]\right) \\
&= \frac{1}{\omega_t}\Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\Big] \\
\frac{\partial \tilde{y}_t}{\partial \tilde{V}_t} &= -\frac{1}{\omega_t}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1})\Big] \\
\frac{\partial \tilde{\theta}_t}{\partial \tilde{V}_t} &= 0 \\
\frac{\partial \tilde{x}_t}{\partial \tilde{\omega}_t} &= \frac{\partial}{\partial \tilde{\omega}_t}\left(x_{t-1} + \frac{V_t}{\omega_t}\Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\Big]\right) \\
&= \frac{\partial}{\partial \tilde{\omega}_t}\left(\frac{V_t}{\omega_t}\sin(\theta_{t-1} + \omega_t \Delta t) - \frac{V_t}{\omega_t}\sin(\theta_{t-1})\right) \\
&= -\frac{V_t}{\omega_t^2}\sin(\theta_{t-1} + \omega_t \Delta t) + \frac{V_t}{\omega_t}\cos(\theta_{t-1} + \omega_t \Delta t) \cdot \Delta t + \frac{V_t}{\omega_t^2}\sin(\theta_{t-1}) \\
&= \frac{V_t}{\omega_t^2}\Big[\sin(\theta_{t-1}) - \sin(\theta_{t-1} + \omega_t \Delta t) + \omega_t \Delta t \cos(\theta_{t-1} + \omega_t \Delta t)\Big] \\
\frac{\partial \tilde{y}_t}{\partial \tilde{\omega}_t} &= \frac{\partial}{\partial \tilde{\omega}_t}\left(-\frac{V_t}{\omega_t}\cos(\theta_{t-1} + \omega_t \Delta t) + \frac{V_t}{\omega_t}\cos(\theta_{t-1})\right) \\
&= \frac{V_t}{\omega_t^2}\cos(\theta_{t-1} + \omega_t \Delta t) + \frac{V_t}{\omega_t}\sin(\theta_{t-1} + \omega_t \Delta t) \cdot \Delta t - \frac{V_t}{\omega_t^2}\cos(\theta_{t-1}) \\
&= \frac{V_t}{\omega_t^2}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1}) + \omega_t \Delta t \sin(\theta_{t-1} + \omega_t \Delta t)\Big] \\
\frac{\partial \tilde{\theta}_t}{\partial \tilde{\omega}_t} &= \frac{\partial}{\partial \tilde{\omega}_t}\left(\theta_{t-1} + \omega_t \Delta t\right) = \Delta t
\end{aligned} \tag{12}$$

As hinted in the question, $\tilde{x}_t$, $\tilde{y}_t$, as well as their partial derivatives have ω_t in the denominator and are thus indeterminate in their current form as $\omega_t \to 0$. However, these terms are composed from continuous functions; by inspection, V_t and ω_t are our continuous control variables, sin and cos are continuous. Therefore we may apply l'Hopital's rule to evaluate them at the limit where $\omega_t = 0$.

$$
\begin{aligned}
\lim_{\omega_t \to 0} x_t &= x_{t-1} + \lim_{\omega_t \to 0} \frac{V_t\big(\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\big)}{\omega_t} \\
&= x_{t-1} + \lim_{\omega_t \to 0} \frac{\frac{\partial}{\partial \omega_t} V_t\big(\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\big)}{\frac{\partial}{\partial \omega_t}\omega_t} \quad , \quad \text{l'Hopital's rule} \\
&= x_{t-1} + \lim_{\omega_t \to 0} \frac{V_t \Delta t \cos(\theta_{t-1} + \omega_t \Delta t)}{1} \\
&= x_{t-1} + V_t \Delta t \cos(\theta_{t-1} + \omega_t \Delta t)\Big|_{\omega_t = 0} \\
&= x_{t-1} + V_t \Delta t \cos(\theta_{t-1}) \\
\lim_{\omega_t \to 0} y_t &= y_{t-1} + \lim_{\omega_t \to 0} -\frac{V_t}{\omega_t}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1})\Big] \\
&= y_{t-1} + V_t \Delta t \sin(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{x}_t}{\partial \tilde{\theta}_t} &= \lim_{\omega_t \to 0} \frac{V_t}{\omega_t}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1})\Big] \\
&= -V_t \Delta t \sin(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{y}_t}{\partial \tilde{\theta}_t} &= \lim_{\omega_t \to 0} \frac{V_t}{\omega_t}\Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\Big] \\
&= V_t \Delta t \cos(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{x}_t}{\partial \tilde{V}_t} &= \lim_{\omega_t \to 0} \frac{1}{\omega_t}\Big[\sin(\theta_{t-1} + \omega_t \Delta t) - \sin(\theta_{t-1})\Big] \\
&= \Delta t \cos(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{y}_t}{\partial \tilde{V}_t} &= \lim_{\omega_t \to 0} -\frac{1}{\omega_t}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1})\Big] \\
&= \Delta t \sin(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{x}_t}{\partial \tilde{\omega}_t} &= \lim_{\omega_t \to 0} \frac{V_t}{\omega_t^2}\Big[\sin(\theta_{t-1}) - \sin(\theta_{t-1} + \omega_t \Delta t) + \omega_t \Delta t \cos(\theta_{t-1} + \omega_t \Delta t)\Big] \\
&= \lim_{\omega_t \to 0} \frac{V_t}{2\omega_t}\Big[\underbrace{-\Delta t \cos(\theta_{t-1} + \omega_t \Delta t) + \Delta t \cos(\theta_{t-1} + \omega_t \Delta t)}_{=0} - \omega_t \Delta_t^2 \sin(\theta_{t-1} + \omega_t \Delta t)\Big] \\
&= \frac{V_t}{2}\Big[-\Delta t^2 \sin(\theta_{t-1} + \omega_t \Delta t) - \omega_t \Delta_t^3 \cos(\theta_{t-1} + \omega_t \Delta t)\Big]\Big|_{\omega_t = 0} \\
&= -\frac{V_t \Delta t^2}{2}\sin(\theta_{t-1}) \\
\lim_{\omega_t \to 0} \frac{\partial \tilde{y}_t}{\partial \tilde{\omega}_t} &= \lim_{\omega_t \to 0} \frac{V_t}{\omega_t^2}\Big[\cos(\theta_{t-1} + \omega_t \Delta t) - \cos(\theta_{t-1}) + \omega_t \Delta t \sin(\theta_{t-1} + \omega_t \Delta t)\Big] \\
&= \lim_{\omega_t \to 0} \frac{V_t}{2\omega_t}\Big[\underbrace{-\Delta t \sin(\theta_{t-1} + \omega_t \Delta t) + \Delta t \sin(\theta_{t-1} + \omega_t \Delta t)}_{=0} + \omega_t \Delta t^2 \cos(\theta_{t-1} + \omega_t \Delta t)\Big] \\
&= \frac{V_t}{2}\Big[\Delta t^2 \cos(\theta_{t-1} + \omega_t \Delta t) - \omega_t \Delta t^3 \sin(\theta_{t-1} + \omega_t \Delta t)\Big]\Big|_{\omega_t = 0} \\
&= \frac{V_t \Delta t^2}{2}\cos(\theta_{t-1})
\end{aligned}
\tag{13}
$$

(ii) (code) Following the pset, since the EKF assumes Gaussian belief, our unobservable state $\mathbf{x}_t$ can be assumed to be distributed as $\mathbf{x}_t \sim \mathcal{N}(\mathbf{x}_{t-1}, \Sigma_{t-1})$. Modeling the noise resulting from approximating the dynamics as $\upsilon \sim \mathcal{N}(0, R)$, our best prediction $\bar{\mathbf{x}}_t$ i.e. the mean of $\mathbf{x}_t$ is thus represented in the following EKF assignment step:

$$\begin{aligned} \bar{\mathbf{x}}_t = g(\mathbf{x}_{t-1}, \mathbf{u}_t) \leftarrow \tilde{\mathbf{x}} = g(\tilde{\mathbf{x}}_{t-1}, \tilde{\mathbf{u}}_t) \\ \bar{\Sigma}_t \leftarrow G_{x,t}\Sigma_{t-1}G_{x,t}^{\mathsf{T}} + \Delta t \cdot G_{u,t}RG_{u,t}^{\mathsf{T}} \end{aligned} \tag{14}$$

(iii) (code)

(iv) (code)

(v) (code)

(vi) (code)

(vii) (code)

(viii) TODO

Problem 2: EKF SLAM

(i) (code)

(ii) (code)

(iii) TODO

Extra Credit: Monte Carlo Localization

(i) (code)

(ii) (code)

(iii) (code)

(iv) TODO

(v) TODO