

AA 274A: Principles of Robot Autonomy I

Problem Set X

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Problem 1: Trajectory Generation via Differential Flatness

(i) We are given initial and final conditions in terms of variables $\{x, y, V, \theta\}$. The equations are:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \\ x(t_f) \\ \dot{x}(t_f) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ y(t_f) \\ \dot{y}(t_f) \end{bmatrix}$$

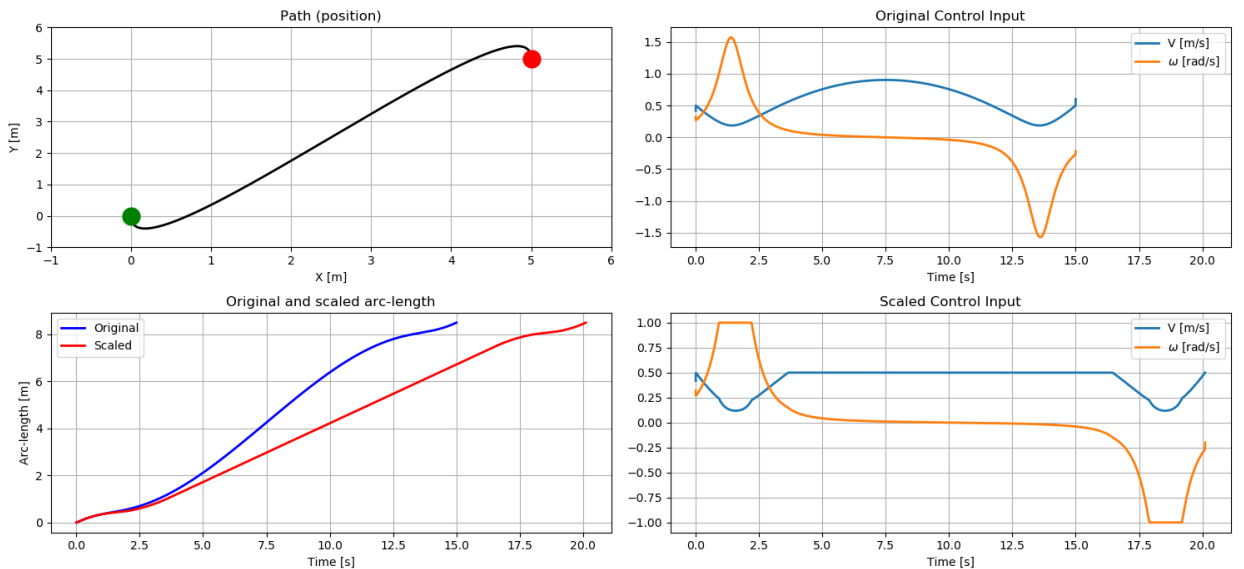
where $\dot{x}(t) = V \cos \theta$ and $\dot{y}(t) = V \sin \theta$ as given by the robot's kinematic model.

(ii) Since $\det(\mathbf{J}) = V$, $V > 0 \forall t$ is a sufficient and necessary condition for the matrix \mathbf{J} to be invertible.

(iii) (code)

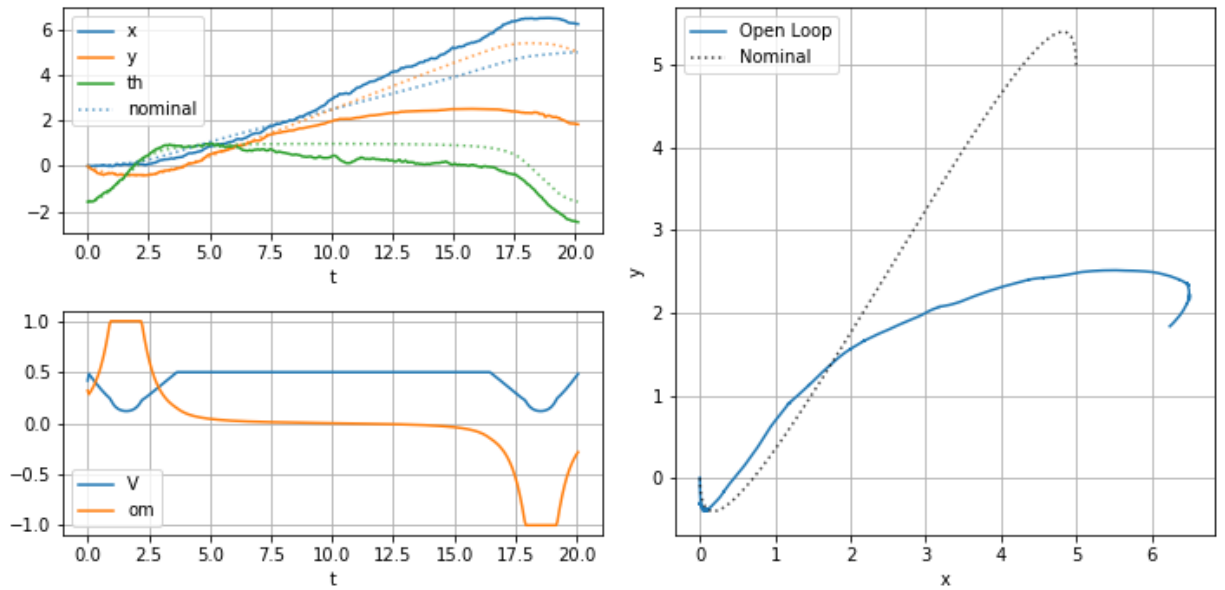
(iv) (code)

(v)



Trajectory of unicycle model in absence of noise. Initial and final conditions as given.

(vi)



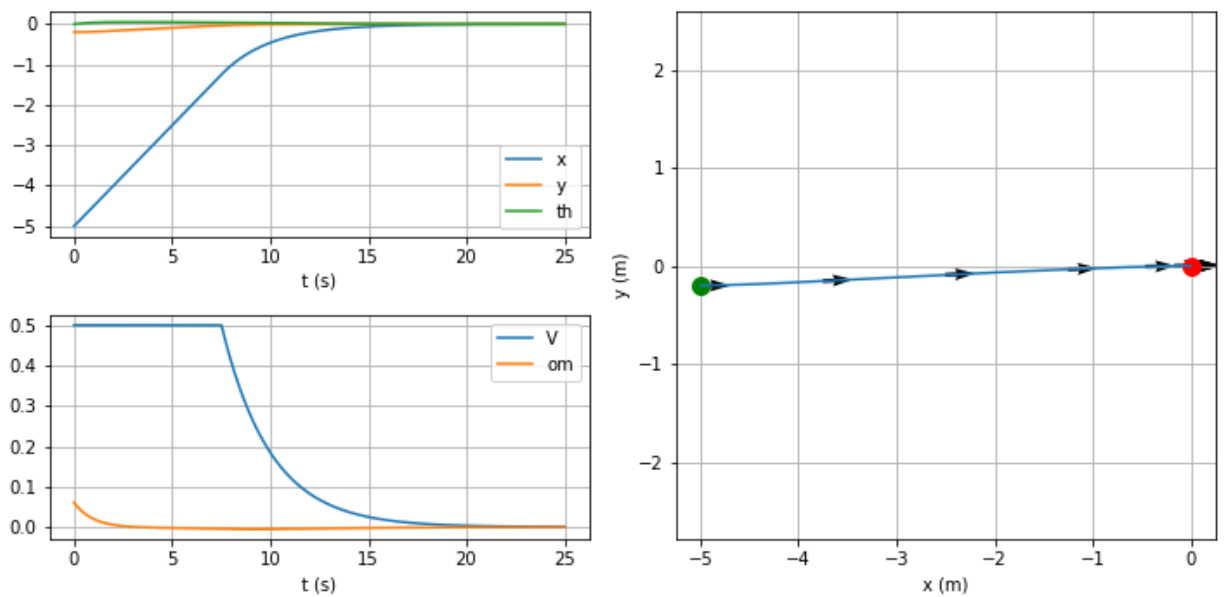
Trajectory of unicycle model where control vector $u_{\text{noisy}} = u + \epsilon$ where ϵ is simulated isotropic Gaussian noise.

Problem 2: Pose Stabilization

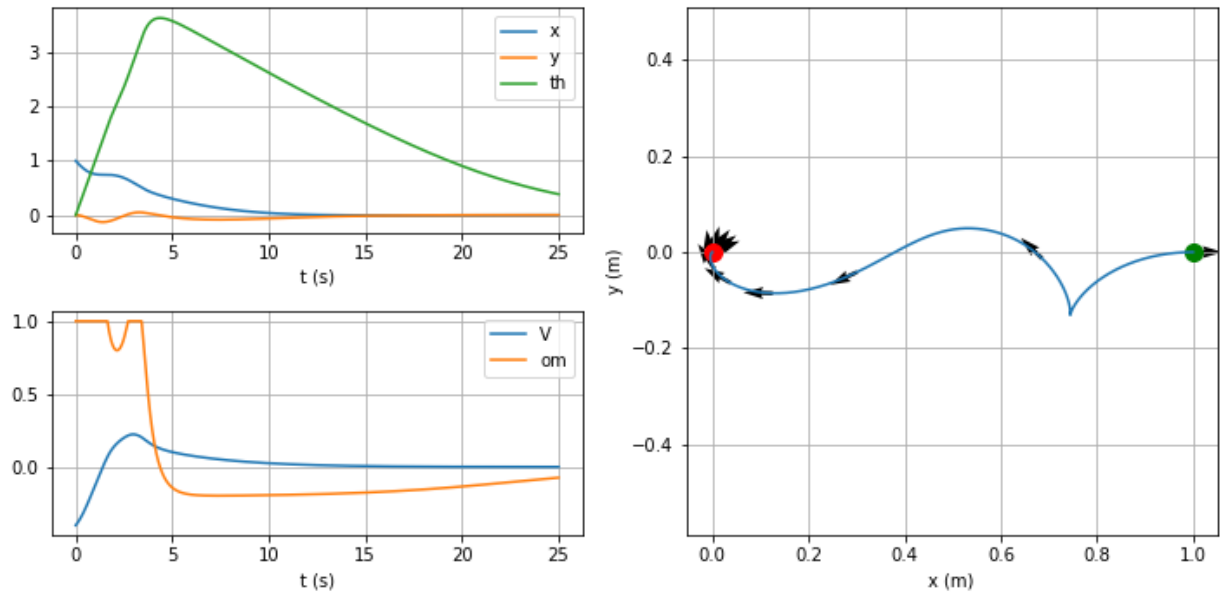
(i) (code)

(ii) (code)

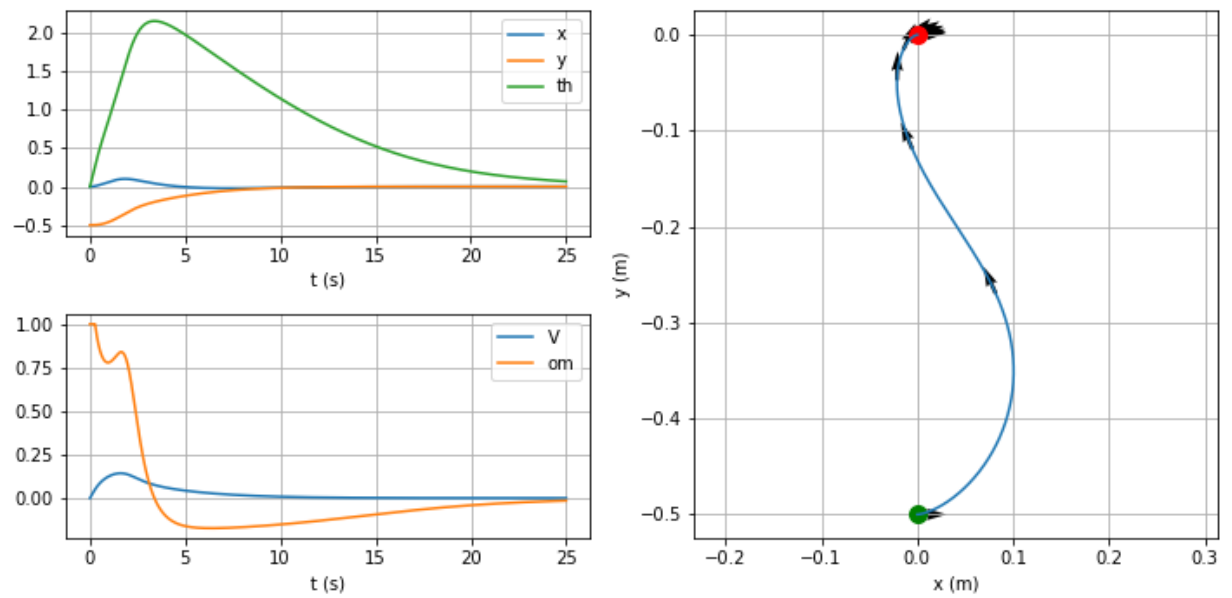
(iii)



Forward parking



Reverse parking



Parallel parking

Problem 3: Trajectory Tracking

(i) Starting from our extended unicycle model, the given kinematic equations are:

$$\begin{aligned}\dot{x}(t) &= V \cos(\theta) \\ \dot{y}(t) &= V \sin(\theta) \\ \dot{V}(t) &= a(t) \\ \dot{\theta}(t) &= \omega(t)\end{aligned}\tag{1}$$

$$\underbrace{\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix}}_{\mathbf{\ddot{z}}=\mathbf{z}^{(q+1)}} = \underbrace{\begin{bmatrix} \cos(\theta) & -V \sin(\theta) \\ \sin(\theta) & V \cos(\theta) \end{bmatrix}}_{:=\mathbf{J}} \underbrace{\begin{bmatrix} a \\ \omega \end{bmatrix}}_{\mathbf{w}} := \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{w}}\tag{2}$$

Equation (2) is in the form of a linear ODE $\mathbf{z}^{(q+1)} = \mathbf{w}$. For the unicycle model, q is known to be 1. We want to design a control law for the virtual input term \mathbf{w} .

Below is the exact linearization scheme in lecture 4. Where superscript (j) denotes the j th derivative w.r.t. t , subscript d denotes the 'desired' open-loop trajectory that we wish to track, we define i th component of the linear tracking error to be:

$$\begin{aligned}e_i(t) &:= z_i(t) - z_{i,d}(t) \\ e_i^{(q+1)}(t) &= z_i^{(q+1)}(t) - z_{i,d}^{(q+1)}(t) \\ &= w_i - w_{i,d} \\ \mathbf{e}(t) &= \mathbf{w}(t) - \mathbf{w}_d(t)\end{aligned}\tag{3}$$

, which is a second-order ODE for our system.

Following lecture 4 notes, we consider a closed-loop control law of the form:

$$w_i(t) = w_{i,d}(t) - \sum_{j=0}^q k_{i,j} e_i^{(j)}(t)\tag{4}$$

which results in closed-loop dynamics of the form:

$$\mathbf{z}^{(q+1)} = \mathbf{w}_d - \sum_{j=0}^q \mathbf{K}_j \mathbf{e}^{(j)}\tag{5}$$

, where \mathbf{K}_j is a diagonal matrix containing elements $k_{i,j}$. Since $\mathbf{z}_d^{(q+1)} = \mathbf{w}_d$, we finally get:

$$\mathbf{e}^{(q+1)} + \sum_{j=0}^q \mathbf{K}_j \mathbf{e}^{(j)} = \mathbf{0}\tag{6}$$

For our system, (6) becomes:

$$\begin{aligned}(\ddot{x} - \ddot{x}_d) + k_{dx}(\dot{x} - \dot{x}_d) + k_{px}(x_d - x) &= 0 \\ \underbrace{(\ddot{y} - \ddot{y}_d)}_{\ddot{e}} + k_{dy}(\dot{y} - \dot{y}_d) + k_{py}(\underbrace{y_d - y}_e) &= 0\end{aligned}\tag{7}$$

This is a 2nd order ODE in \mathbf{e} in standard form. We know that for the system to be stable, the real parts of both roots of its characteristic equation must be > 0 . We also know that the system is critically-damped when the roots are equal. This tells us what the values of K should be.

Rearranging (7), since $\ddot{x} = u_1$ and $\ddot{y} = u_2$ from (2), we get:

$$\begin{aligned} u_1 &= \ddot{x}_d + k_{px}(x_d - x) + k_{dx}(\dot{x} - \dot{x}) = 0 \\ u_2 &= \ddot{y}_d + k_{py}(y_d - y) + k_{dy}(\dot{y} - \dot{y}) = 0 \end{aligned} \quad (8)$$

Finally, to recover our real controls $\{a, \omega\}$, we invert J in (2) and solve for:

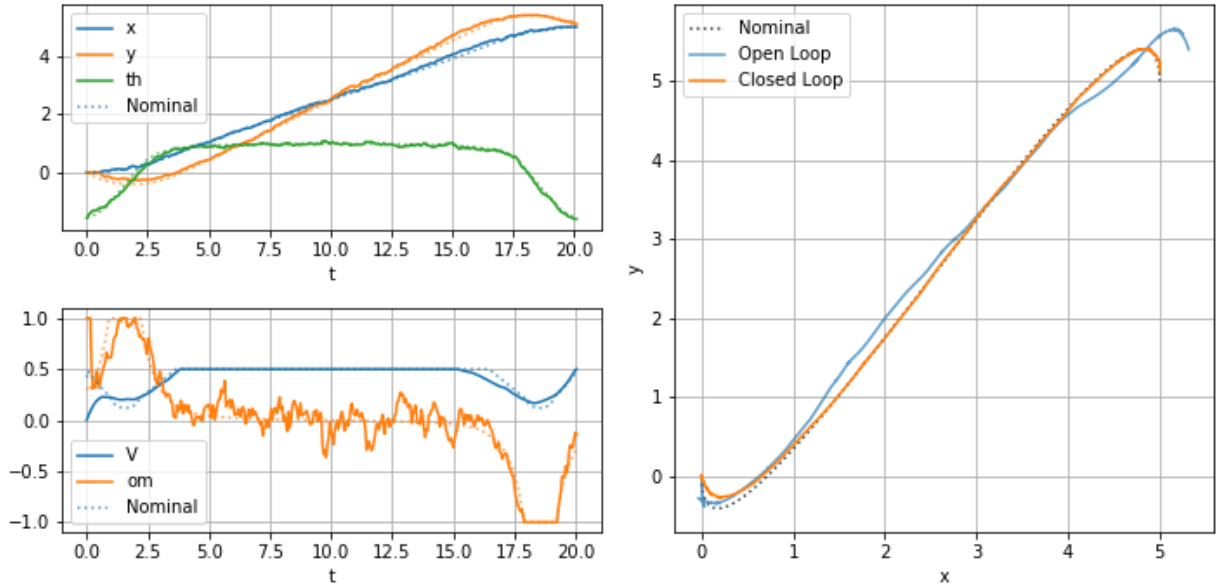
$$\begin{aligned} \begin{bmatrix} a \\ w \end{bmatrix} &= J^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad V > 0 \\ \begin{bmatrix} a \\ w \end{bmatrix} &= \frac{1}{V} \begin{bmatrix} V \cos(\theta) & V \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \quad (9)$$

To recover V as the problem requested,

$$\begin{aligned} V(T) &= \int_{t_0}^T a(t) dt \\ V_t &= V_{t-1} + a_t \Delta t \end{aligned} \quad (10)$$

(ii) (code)

(iii)



Trajectory of unicycle under two-part tracking controller.

Extra Problem: Optimal Control and Trajectory Optimization

(i) To minimize:

$$J = \int_0^{t_f} \underbrace{[\lambda + V^2 + \omega^2]}_g dt, \quad \lambda \text{ constant} \quad (11)$$

Subject to:

$$\begin{aligned} \dot{x} &= V \cos \theta \\ \dot{y} &= V \sin \theta \\ \dot{\theta} &= \omega \end{aligned} \quad (12)$$

Boundary conditions:

$$\begin{aligned} x(0) &= 0 & y(0) &= 0 & \theta(0) &= -\frac{\pi}{2} \\ x(t_f) &= 5 & y(t_f) &= 5 & \theta(t_f) &= -\frac{\pi}{2} \end{aligned} \quad (13)$$

We follow the procedure outlined in lecture 5.

Let:

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} V \\ \omega \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} V \cos \theta \\ V \sin \theta \\ \omega \end{bmatrix} \quad (14)$$

1. First, we need the Hamiltonian for the system:

$$\begin{aligned} \mathcal{H} &= g + \mathbf{p}^\top \mathbf{f} \\ \mathcal{H} &= \lambda + V^2 + \omega^2 + p_1 V \cos \theta + p_2 V \sin \theta + p_3 \omega \end{aligned} \quad (15)$$

2. Next, we find the NOCs, which are defined by:

$$\begin{aligned} \dot{\mathbf{x}}^* &= \frac{\partial \mathcal{H}}{\partial \mathbf{p}} \\ \dot{\mathbf{p}}^* &= -\frac{\partial \mathcal{H}}{\partial \mathbf{x}} \\ 0 &= \frac{\partial \mathcal{H}}{\partial \mathbf{u}} \end{aligned} \quad (16)$$

For our system, these correspond to $2n + m$ equations. To solve for t_f , we additionally define a dummy state variable r such that $r = t_f$ and $\dot{r} = 0$.

The NOCs for our system are:

$$\begin{aligned} \dot{x}^* &= V \cos \theta & \dot{p}_1^* &= 0 & 0 &= \frac{\partial \mathcal{H}}{\partial V} \\ \dot{y}^* &= V \sin \theta & \dot{p}_2^* &= 0 & 0 &= \frac{\partial \mathcal{H}}{\partial \omega} \\ \dot{\theta}^* &= \omega & \dot{p}_3^* &= p_1 V \sin(\theta) - p_2 V \cos(\theta) \\ \dot{r}^* &= 0 \end{aligned} \quad (17)$$

Since

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial \omega} &= 2\omega + p_3 = 0 \\ \omega &= -\frac{p_3}{2} \end{aligned} \quad (18)$$

3. We need to have $2n + 1$ boundary conditions to solve for the $2n$ differential equations plus 1 free variable t_f . With x_f fixed and t_f free, these are governed by:

$$\begin{aligned} x^*(t_0) &= x_0 \\ x^*(t_f) &= x_f \\ \mathcal{H} + \frac{\partial h}{\partial t} &= 0 \end{aligned} \quad (19)$$

In our case, $h = 0$ as we don't have initial cost. These correspond to:

$$\begin{aligned} x^*(0) &= 0 & x^*(t_f) &= 5 \\ y^*(0) &= 0 & y^*(t_f) &= 5 \\ \theta^*(0) &= -\frac{\pi}{2} & \theta^*(t_f) &= -\frac{\pi}{2} \\ \lambda + V^2 + \omega^2 + p_1 V \cos \theta + p_2 V \sin \theta + p_3 \omega &= 0 \end{aligned} \quad (20)$$

4. Now we need to rescale for time. Let τ be our new time scale such that:

$$\begin{aligned} \tau &= \frac{t}{t_f}, \tau \in [0, 1] \\ \frac{d}{d\tau} &= \frac{dt}{d\tau} \cdot \frac{d}{dt} = t_f \frac{d}{dt} \end{aligned} \quad (21)$$

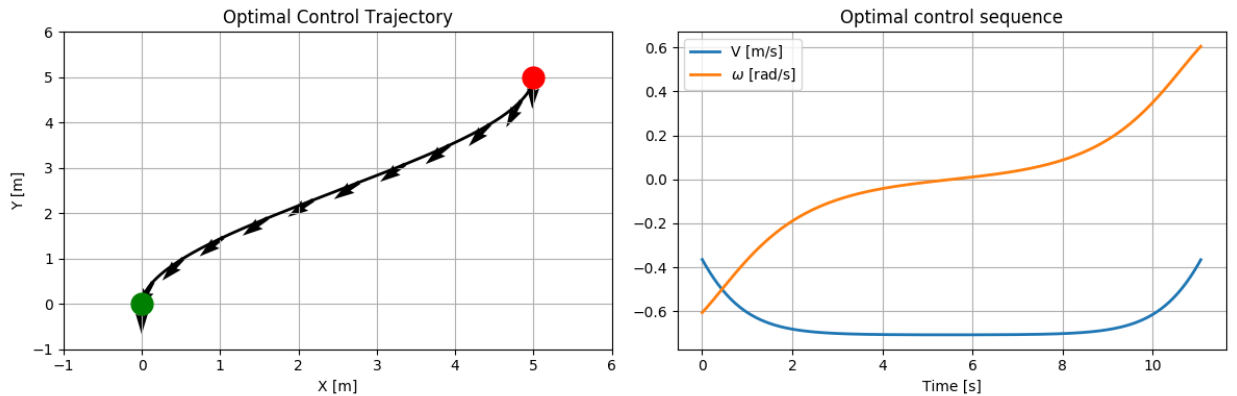
Denoting $\frac{d}{d\tau}$ as $[\cdot]'$, the NOCs become:

$$\begin{aligned} x'^* &= rV \cos \theta & p_1'^* &= 0 & 0 &= \frac{\partial \mathcal{H}}{\partial V} \\ y'^* &= rV \sin \theta & p_2'^* &= 0 & 0 &= \frac{\partial \mathcal{H}}{\partial \omega} \\ \theta'^* &= r\omega & p_4'^* &= rp_1 V \sin(\theta) - rp_2 V \cos(\theta) \\ r'^* &= 0 \end{aligned} \quad (22)$$

The boundary conditions do not have t terms, so the expressions remain the same.

(ii) (code)

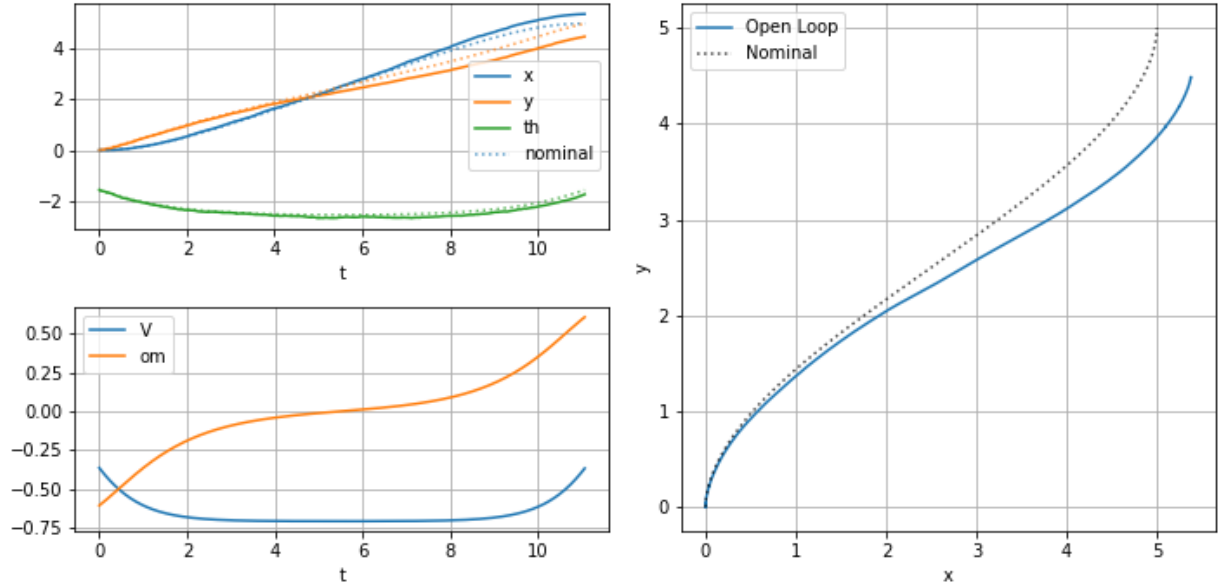
(iii)



Trajectory of unicycle model as solved by nonlinear optimization.

(iv) The larger λ is, the larger the emphasis the solver puts on minimizing time.

(v)



Comparison of open-loop trajectory and optimized trajectory in presence of isotropic Gaussian noise.