AA 274A: Principles of Robot Autonomy I
Problem Set X

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Problem 1: Trajectory Generation via Differential Flatness

(i) We are given initial and final conditions in terms of variables $\{x, y, V, \theta\}$. The equations are:

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & t_f & t_f^2 & t_f^3 \\
0 & 1 & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4
\end{bmatrix}
=
\begin{bmatrix}
x(0) \\ \dot{x}(0) \\ x(t_f) \\ \dot{x}(t_f)
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & t_f & t_f^2 & t_f^3 \\
0 & 1 & 2t_f & 3t_f^2
\end{bmatrix}
\begin{bmatrix}
y_1 \\ y_2 \\ y_3 \\ y_4
\end{bmatrix}
=
\begin{bmatrix}
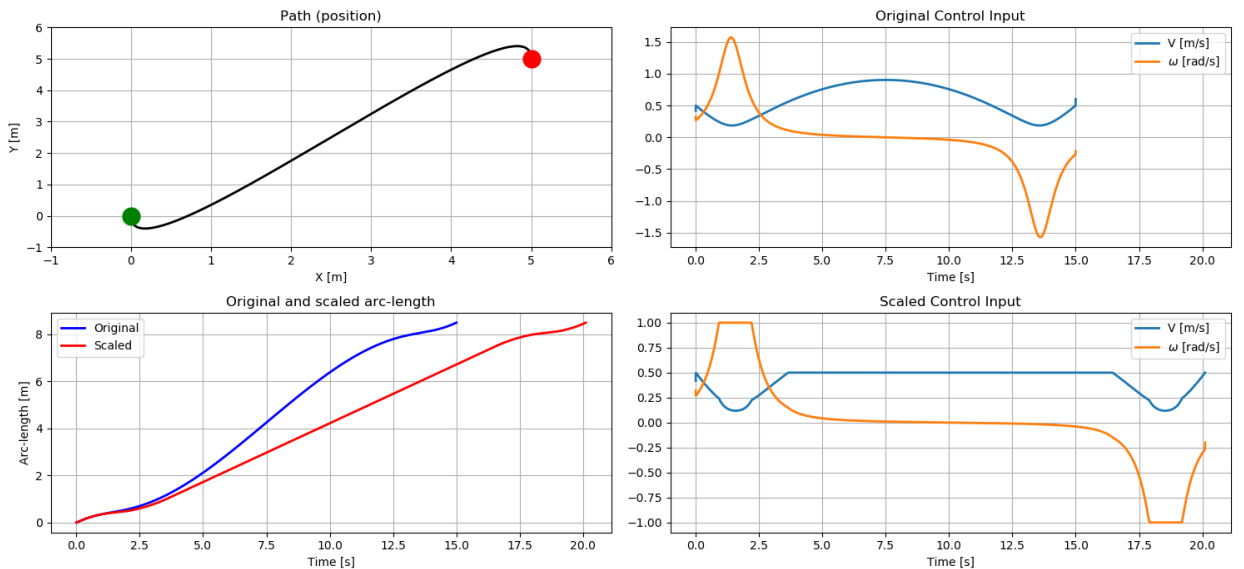y(0) \\ \dot{y}(0) \\ y(t_f) \\ \dot{y}(t_f)
\end{bmatrix}
$$

where $\dot{x}(t) = V \cos\theta$ and $\dot{y}(t) = V \sin\theta$ as given by the robot's kinematic model.

(ii) Since $\det(\boldsymbol{J}) = V$, $V > 0 \; \forall t$ is a sufficient and necessary condition for the matrix $\boldsymbol{J}$ to be invertible.
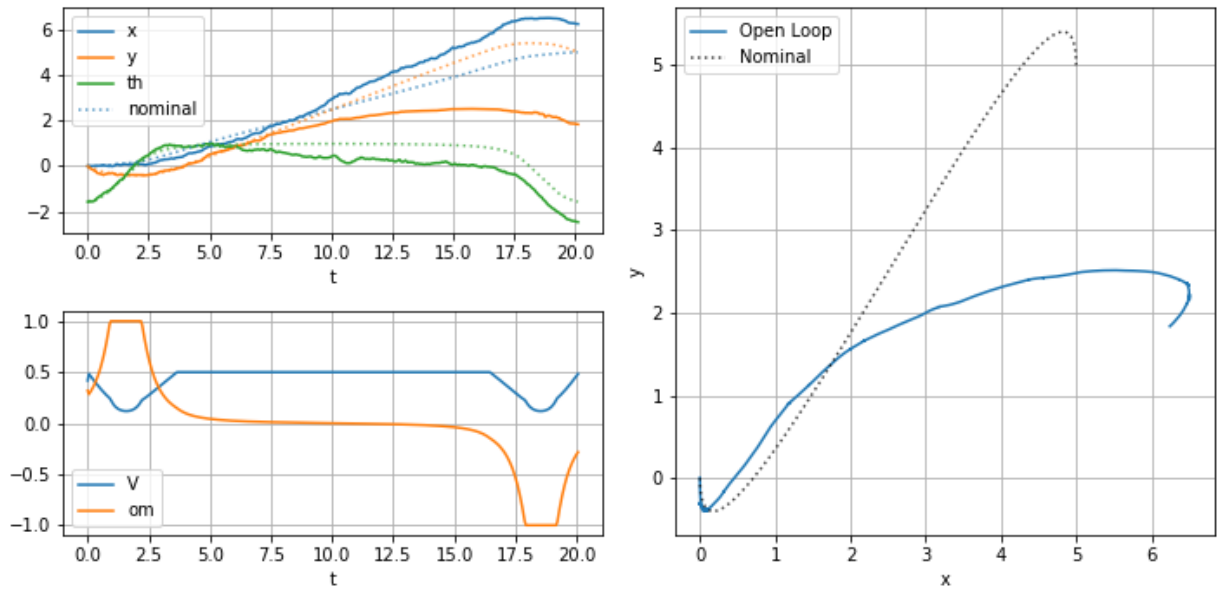
(iii) (code)

(iv) (code)

(v) _____

Trajectory of unicycle model in absence of noise. Initial and final conditions as given.
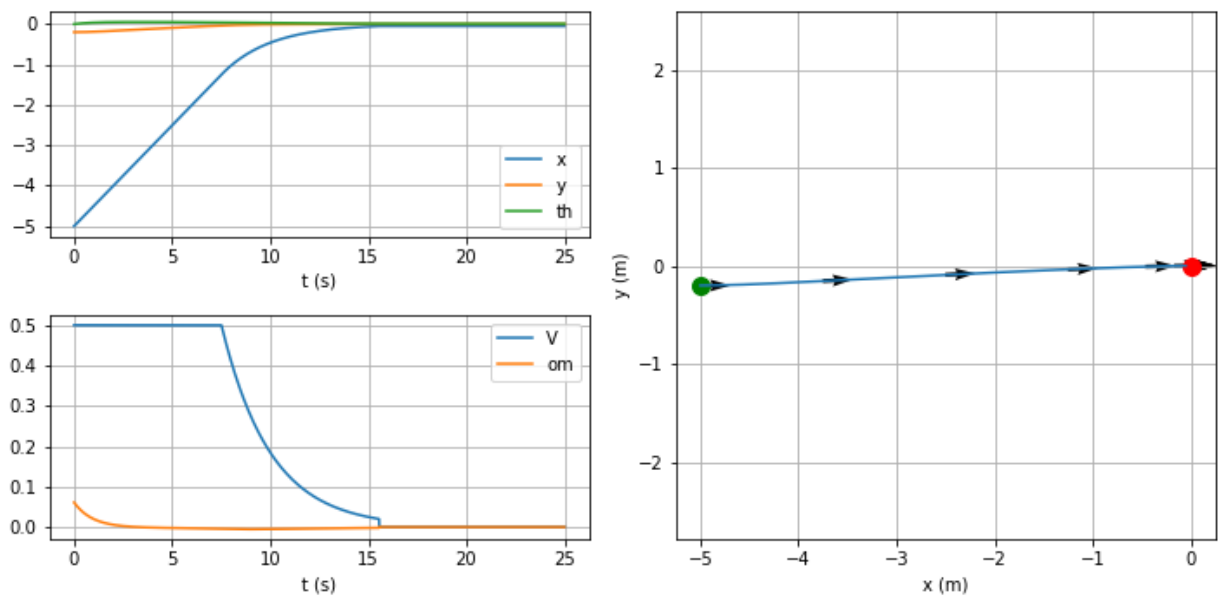
(vi)

Trajectory of unicycle model where control vector $u_{\text{noisy}} = u + \epsilon$ where ϵ is simulated isotropic Gaussian noise.
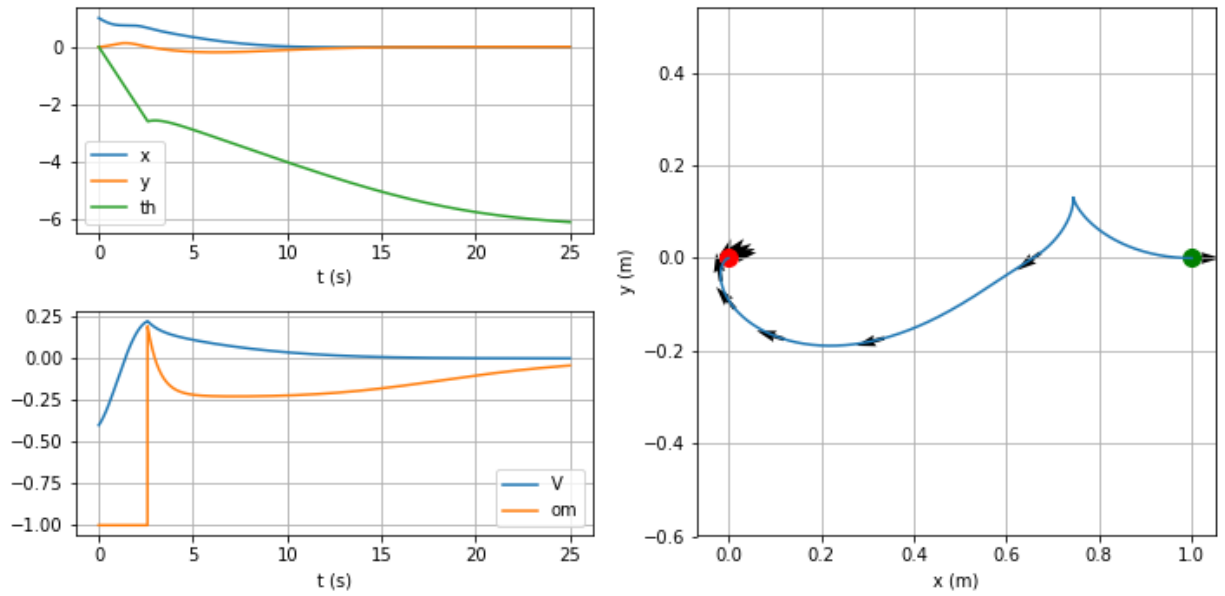
Problem 2: Pose Stabilization
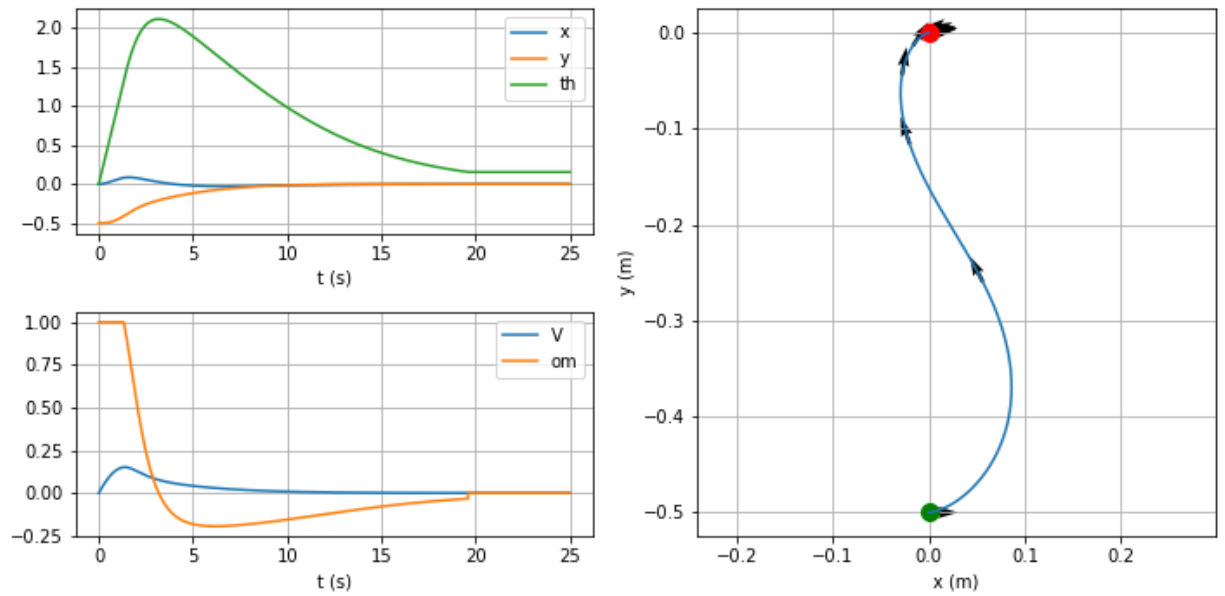
(i) (code)

(ii) (code)

(iii)

Forward parking

Reverse parking

Parallel parking

Problem 3: Trajectory Tracking

(i) Starting from our extended unicycle model, the given kinematic equations are:

$$
\begin{aligned}
\dot{x}(t) &= V\cos(\theta) \\
\dot{y}(t) &= V\sin(\theta) \\
\dot{V}(t) &= a(t) \\
\dot{\theta}(t) &= \omega(t)
\end{aligned}
\tag{1}
$$

$$
\underbrace{\begin{bmatrix} \ddot{x}(t) \\ \ddot{y}(t) \end{bmatrix}}_{\ddot{\boldsymbol{z}} = \boldsymbol{z}^{(q+1)}} = \underbrace{\begin{bmatrix} \cos(\theta) & -V\sin(\theta) \\ \sin(\theta) & V\cos(\theta) \end{bmatrix}}_{:=\boldsymbol{J}} \begin{bmatrix} a \\ \omega \end{bmatrix} := \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\boldsymbol{w}}
\tag{2}
$$

Equation (2) is in the form of a linear ODE $\boldsymbol{z}^{(q+1)} = \boldsymbol{w}$. For the unicycle model, q is known to be 1. We want to design a control law for the virtual input term $\boldsymbol{w}$.

Below is the exact linearization scheme in lecture 4. Where superscript (j) denotes the jth derivative w.r.t. t, subscript d denotes the 'desired' open-loop trajectory that we wish to track, we define ith component of the linear tracking error to be:

$$
\begin{aligned}
e_i(t) &:= z_i(t) - z_{i,d}(t) \\
e_i^{(q+1)}(t) &= z_i^{(q+1)}(t) - z_{i,d}^{(q+1)}(t) \\
&= w_i - w_{i,d} \\
\boldsymbol{e}(t) &= \boldsymbol{w}(t) - \boldsymbol{w}_d(t)
\end{aligned}
\tag{3}
$$

, which is a second-order ODE for our system.

Following lecture 4 notes, we consider a closed-loop control law of the form:

$$
w_i(t) = w_{i,d}(t) - \sum_{j=0}^{q} k_{i,j} e_i^{(j)}(t)
\tag{4}
$$

which results in closed-loop dynamics of the form:

$$
\boldsymbol{z}^{(q+1)} = \boldsymbol{w}_d - \sum_{j=0}^{q} K_j \boldsymbol{e}^{(j)}
\tag{5}
$$

, where K_j is a diagonal matrix containing elements $k_{i,j}$. Since $\boldsymbol{z}_d^{(q+1)} = \boldsymbol{w}_d$, we finally get:

$$
\boldsymbol{e}^{(q+1)} + \sum_{j=0}^{q} K_j \boldsymbol{e}^{(j)} = 0
\tag{6}
$$

For our system, (6) becomes:

$$
\begin{aligned}
(\ddot{x} - \ddot{x}_d) + k_{dx}(\dot{x} - \dot{x}) &\quad + k_{px}(x_d - x) = 0 \\
\underbrace{(\ddot{y} - \ddot{y}_d)}_{\ddot{e}} + k_{dy}\underbrace{(\dot{y} - \dot{y})}_{\dot{e}} &\quad + k_{py}\underbrace{(y_d - y)}_{e} = 0
\end{aligned}
\tag{7}
$$

This is a 2nd order ODE in $\boldsymbol{e}$ in standard form. We know that for the system to be stable, the real parts of both roots of its characteristic equation must be > 0. We also know that the system is critically-damped when the roots are equal. This tells us what the values of K should be.

Rearranging (7), since $\ddot{x} = u_1$ and $\ddot{y} = u_2$ from (2), we get:

$$
\begin{aligned}
u_1 &= \ddot{x}_d &+ k_{px}(x_d - x) + k_{dx}(\dot{x} - \dot{x}) = 0 \\
u_2 &= \ddot{y}_d &+ k_{py}(y_d - y) + k_{dy}(\dot{y} - \dot{y}) = 0
\end{aligned}
\tag{8}
$$

Finally, to recover our real controls $\{a, \omega\}$, we invert J in (2) and solve for:
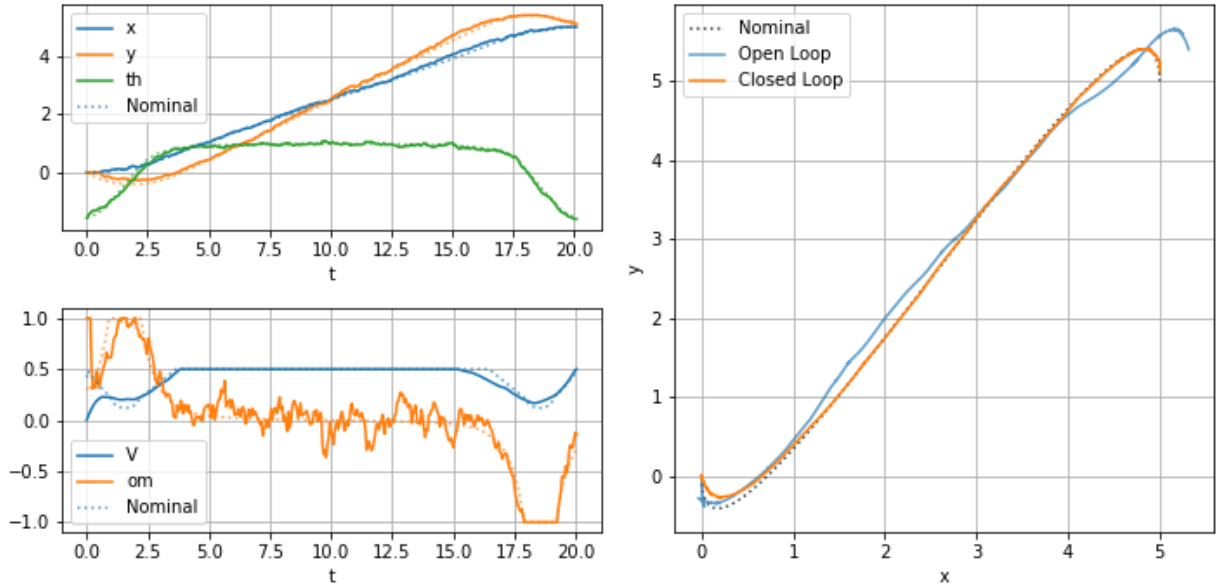
$$
\begin{aligned}
\begin{bmatrix} a \\ w \end{bmatrix} &= J^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \ , \ V > 0 \\
\begin{bmatrix} a \\ w \end{bmatrix} &= \frac{1}{V} \begin{bmatrix} V\cos(\theta) & V\sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
\end{aligned}
\tag{9}
$$

To recover V as the problem requested,

$$
\begin{aligned}
V(T) &= \int_{t_0}^{T} a(t)\, dt \\
V_t &= V_{t-1} + a_t \Delta t
\end{aligned}
\tag{10}
$$

(ii) (code)

(iii) ——

Trajectory of unicycle under two-part tracking controller.

Extra Problem: Optimal Control and Trajectory Optimization

(i) To minimize:

$$J = \int_0^{t_f} \underbrace{[\lambda + V^2 + \omega^2]}_{g} \, dt \, , \quad \lambda \text{ constant} \tag{11}$$

Subject to:

$$\begin{aligned}
\dot{x} &= V \cos\theta \\
\dot{y} &= V \sin\theta \\
\dot{\theta} &= \omega
\end{aligned} \tag{12}$$

Boundary conditions:

$$\begin{aligned}
x(0) &= 0 & y(0) &= 0 & \theta(0) &= -\tfrac{\pi}{2} \\
x(t_f) &= 5 & y(t_f) &= 5 & \theta(t_f) &= -\tfrac{\pi}{2}
\end{aligned} \tag{13}$$

We follow the procedure outlined in lecture 5.

Let:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} \, , \quad \boldsymbol{u} = \begin{bmatrix} V \\ \omega \end{bmatrix} \, , \quad \boldsymbol{f} = \begin{bmatrix} V\cos\theta \\ V\sin\theta \\ \omega \end{bmatrix} \tag{14}$$

1. First, we need the Hamiltonian for the system:

$$\begin{aligned}
\mathcal{H} &= g + \boldsymbol{p}^\mathsf{T} \boldsymbol{f} \\
\mathcal{H} &= \lambda + V^2 + \omega^2 + p_1 V \cos\theta + p_2 V \sin\theta + p_3 \omega
\end{aligned} \tag{15}$$

2. Next, we find the NOCs, which are defined by:

$$\begin{aligned}
\dot{\boldsymbol{x}}^* &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \\
\dot{\boldsymbol{p}}^* &= -\frac{\partial \mathcal{H}}{\partial \boldsymbol{x}} \\
0 &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{u}}
\end{aligned} \tag{16}$$

For our system, these correspond to $2n + m$ equations. To solve for t_f, we additionally define a dummy state variable r such that $r = t_f$ and $\dot{r} = 0$.

The NOCs for our system are:

$$\begin{array}{lll}
\dot{x}^* = V \cos\theta & \dot{p}_1^* = 0 & 0 = \frac{\partial \mathcal{H}}{\partial V} \\
\dot{y}^* = V \sin\theta & \dot{p}_2^* = 0 & 0 = \frac{\partial \mathcal{H}}{\partial \omega} \\
\dot{\theta}^* = \omega & \dot{p}_3^* = p_1 V \sin(\theta) - p_2 V \cos(\theta) & \\
\dot{r}^* = 0 & &
\end{array} \tag{17}$$

Since

$$\frac{\partial \mathcal{H}}{\partial \omega} = 2\omega + p_3 = 0 \quad , \quad \frac{\partial \mathcal{H}}{\partial V} = 2V + p_1 \cos(\theta) + p_2 \sin(\theta)$$

$$\omega = -\frac{p_3}{2} \quad , \quad V = -\frac{p_1 \cos(\theta) + p_2 \sin(\theta)}{2} \tag{18}$$

3. We need to have $2n + 1$ boundary conditions to solve for the $2n$ differential equations plus 1 free variable t_f. With x_f fixed and t_f free, these are governed by:

$$
\begin{aligned}
x^*(t_0) &= x_0 \\
x^*(t_f) &= x_f \\
\mathcal{H} + \frac{\partial h}{\partial t} &= 0
\end{aligned}
\tag{19}
$$

In our case, $h = 0$ as we don't have initial cost. These correspond to:

$$
\begin{aligned}
x^*(0) &= 0 & x^*(t_f) &= 5 \\
y^*(0) &= 0 & y^*(t_f) &= 5 \\
\theta^*(0) &= -\tfrac{\pi}{2} & \theta^*(t_f) &= -\tfrac{\pi}{2} \\
\lambda + V^2 + \omega^2 & + p_1 V \cos\theta + p_2 V \sin\theta + p_3\omega = 0
\end{aligned}
\tag{20}
$$

4. Now we need to rescale for time. Let τ be our new time scale such that:

$$
\begin{aligned}
\tau &= \frac{t}{t_f} \ , \ \tau \in [0,1] \\
\frac{d}{d\tau} &= \frac{dt}{d\tau} \cdot \frac{d}{dt} = t_f \frac{d}{dt}
\end{aligned}
\tag{21}
$$

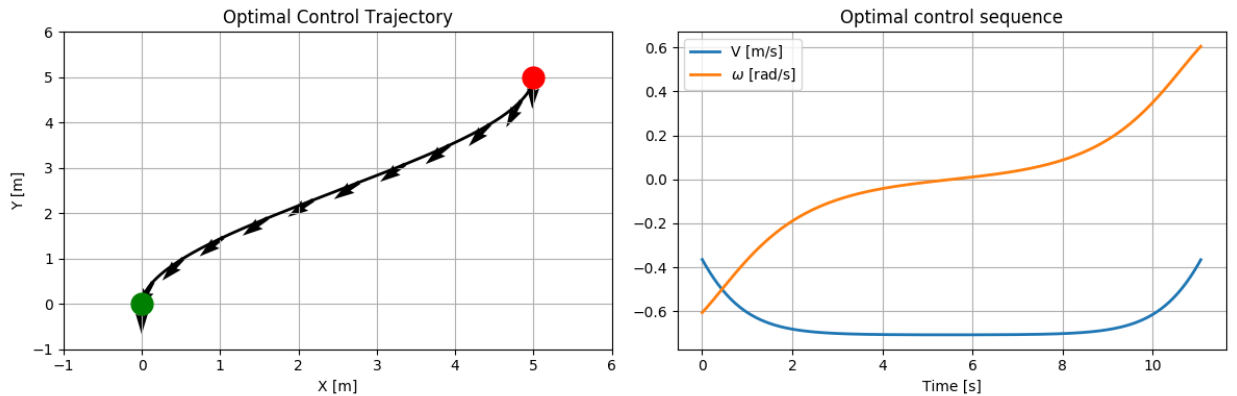Denoting $\frac{d}{d\tau}$ as $[.]'$, the NOCs become:

$$
\begin{aligned}
x'^* &= rV\cos\theta & p_1'^* &= 0 & 0 &= \frac{\partial\mathcal{H}}{\partial V} \\
y'^* &= rV\sin\theta & p_2'^* &= 0 & 0 &= \frac{\partial\mathcal{H}}{\partial\omega} \\
\theta'^* &= r\omega & p_4'^* &= rp_1 V\sin(\theta) - rp_2 V\cos(\theta) \\
r'^* &= 0
\end{aligned}
\tag{22}
$$

The boundary conditions do not have t terms, so the expressions remain the same.
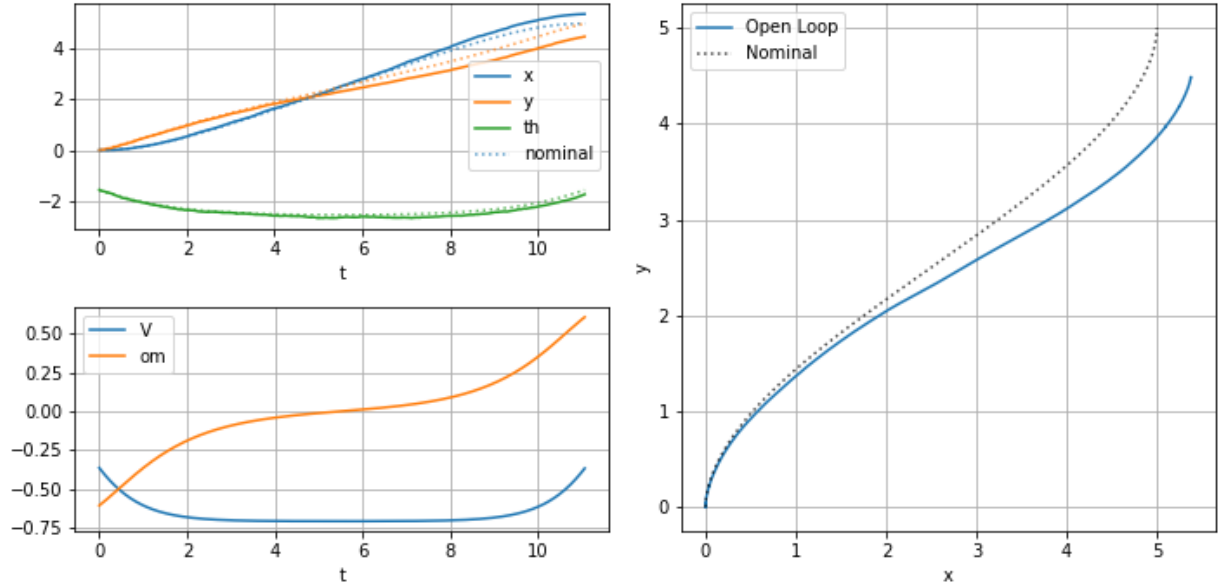
(ii) (code)

(iii)

Trajectory of unicycle model as solved by nonlinear optimization.

(iv) The larger λ is, the larger the emphasis the solver puts on minimizing time.

(v)

Comparison of open-loop trajectory and optimized trajectory in presence of isotropic Gaussian noise.