GPI Final Exam (2021/1/4)

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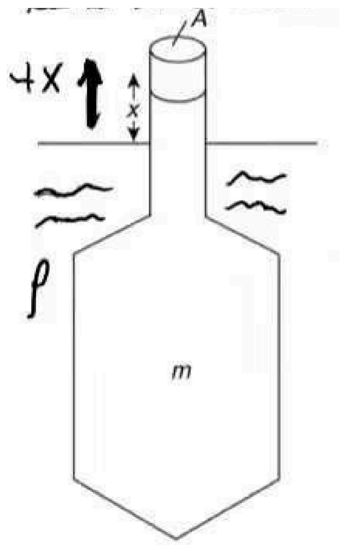
(In questions asking for the expression of certain relation, you need to express them in terms of given and physical constants. Of course, you can combine them to write simple formula, but do explicitly list out your regrouping. For example: $a = A/m, A = vc/b$,V,c,b,m are the constants or terms provided. 题目中所要你求的关系表达式，需要用给定量 和物理常数表达。当然你可以组合定义以简便表达，但一定明确给出你的重组)

$1_{(16 \text{ points})}$

(16 points) The device below can be used to measure the density and viscosity of a liquid using a so-called hydrometer. The hydrometer is shown in the figure below:The hydrometer has total mass $= m$; Its bottom is heavier so that it can stably float in liquid (it can move up-down, but will not rock side-ways). The long neck part has uniform cross area A. The equilibrium position of the hydrometer in a liquid is taken as $x = 0$. Now if the hydrometer is displaced from its equilibrium by x as shown in the figure (the x is always within the neck region), it will start to oscillate. From measuring this oscillation frequency and amplitude decay, we can deduce the density and viscosity of the liquid.



(1) (5 points) Neglect any friction caused by the viscosity of the liquid, A) write out the equation of motion for x of the hydrometer. B) Express its natural frequency ω_0 in terms of ρ the density of liquid; hydrometer mass m; g gravitation constant etc.

(2) (3 points) If the liquid is viscous, it will introduce a resistance force $\frac{F_{\text{friction}}}{m} = -\gamma\dot{x}$; γ is called damping or viscous coefficient; If we would like to have the hydrometer be back to the stable equilibrium as quick as possible, what is the γ in term of the ω_0 ?

(3) (8 points) In application of measurement of the density ρ and viscosity γ of the liquid, the liquid should not be too viscous so that the hydrometer can oscillate. We can directly measure the displacement x (for example let the top of the hydrometer be the cathode of a capacitor, while another fixed plate (the dashed line of in the figure) be the anode, and we can measure the voltage difference between them which is proportional to x). The measured result is shown in the figure below:

The time between the 1^{st} peak and 9^{th} peak is 4 seconds; and the 9^{th} peak is about e^{-1} of the first peak. From measurement, please calculate the following values:

a) Period T of the damped oscillation; b) angular frequency ω; c) The value of natural frequency ω_0 and the viscosity γ ?; d) Quality factor Q for this oscillator.[(a), b) are 1 point each; (c) 4 points; (d) 2 points]

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(1) Initial state equilibrium $mg = \rho V g$, when it moves x, according to Newton's second law

$$\rho(V - xA)g - mg = m\ddot{x} = -xAg$$

So $\ddot{x} = -\dfrac{\rho A g}{m}x = -\omega_0^2 x$, then $\omega_0 = \sqrt{\dfrac{\rho A g}{m}}$

(2) Compare to (1), add $F_{\text{friction}} = -m\gamma\dot{x}$, according to Newton's seconde law

$$\rho(V - xA)g - mg - m\gamma\dot{x} = m\ddot{x} = -xAg - m\gamma\dot{x} \Longrightarrow \ddot{x} + \gamma\dot{x} + \frac{\rho A g}{m}x = 0$$

its eigenvalues are $\lambda_{1,2} = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$. If $\gamma^2 - 4\omega_0^2 \geq 0$, the hydrometer will be back as quick as possible.

$$\gamma \geq 2\omega_0$$

(3) a) $4 = (9-1)\cdot T \Longrightarrow T = 0.5s$ b) $T = \dfrac{2\pi}{\omega} \Longrightarrow \omega = 4\pi \text{ rad/s}$

c) Obviously $\gamma < 2\omega_0$, so $x(t) = x_0 e^{-\frac{\gamma}{2}t}\cos\left(\sqrt{\omega_0^2 - \dfrac{1}{4}\gamma^2}t + \varphi\right)$, $e^{-\frac{\gamma}{2}\cdot 4} = e^{-1} \Longrightarrow \gamma = 0.5 \text{ s}^{-1}$

solve $\sqrt{\omega_0^2 - \dfrac{1}{4}\gamma^2} = \omega$ then $\omega_0 = \sqrt{\omega_0^2 + \dfrac{1}{4}\gamma^2} = 12.56886 \text{ rad/s}$

d) $Q = 2\pi\dfrac{E}{\Delta E} = 2\pi\dfrac{A_0^2}{A_0^2(1 - (e^{-\frac{\gamma}{2}T})^2)} \approx \dfrac{2\pi}{\gamma T} = \dfrac{\omega}{\gamma} = 25.13771$

2 (12 points)

Central Field. Below you need to Calculate lowest orbit energy, angular momentum, radius and speed of an electron at the ground orbit in hydrogen (H) atom, using Bohr's orbit model and de Broglie matter wave hypothesis. Bohr proposed electron like a particle moves around nuclei in a circular orbit subject to Coulomb force. It has orbit radius R (called Bohr radius), energy E, angular momentum L and speed v to be determined. Using de Broglie matter wave, the longest wavelength can stably exist on the orbit must satisfy: $\lambda = 2\pi R$ (If you understand the standing wave picture behind this formula, that's great; If not, just treat it as a given condition), and you may know his famous relation between momentum and wavelength: $p = \dfrac{h}{\lambda}, h = 6.63 \times 10^{-34}$ (SI unit). The Coulomb potential energy in hydrogen atom between electron and proton is: $U = -\dfrac{\text{A}}{r}$ A is a natural constant $A = 2.3 \times 10^{-28}$ (SI unit). Mass of electron is $m_e = 9.11 \times 10^{-31}$ kg (the reduced mass can be treated equal to this since proton is 1800 time massive). Using the provided information, calculate the

A) Angular momentum L in terms of h.
B) Bohr Radius R in units of $\AA$ $\left(1\AA = 10^{-10}\ \mathrm{m}\right)$
C) Energy E in $\mathrm{eV} 1\mathrm{eV} = 1.6 \times 10^{-19}\ \mathrm{J}$
D) Orbital speed $V = \alpha C, C$ is speed of light, express α in terms of physical constants (h, m_e, C and Bohr radius R) and find its value (3 points each for A) to D))

A) $\lambda = 2\pi R = \frac{h}{p} \Longrightarrow L = pR = \frac{h}{2\pi} = \hbar$

B) $F = -\frac{dU}{dr} = -\frac{A}{r^2}, \frac{A}{R^2} = m\frac{v^2}{R} \Longrightarrow \begin{cases} mv^2 R = A \\ mvR = \frac{h}{2\pi} \end{cases} \Longrightarrow R = \frac{h^2}{4\pi^2 mA} = 0.531\AA$

C) $E = \frac{1}{2}mv^2 - \frac{A}{R} = -\frac{A}{2R} = -\frac{2\pi^2 mA^2}{h^2} = -13.52\ \mathrm{eV}$

D) $\alpha = \frac{V}{C} = \frac{2\pi A}{hC} = 7.26 \times 10^{-3} \approx \frac{1}{137}$

3 (14 points)

Interference:
LIGO (Laser Interferometer Gravitation-Wave Observatory) is using interference to observe the expansion-contraction if space due to gravitation wave.

As shown in the figure: It is basically a big Michelson Interferometer, two arm length L_0 is 4000 meter $(4\ \mathrm{km}$) each. Two extra mirrors inside will make light being reflected backwardforward $N = 300$ times, effectively prolong the length of each arm. The light from each arm finally meets at the beam splitter and creates interference pattern. When gravitation wave (indicated by the curve on top) passes, one arm length will be lengthened and the other be compressed. This effect is expressed by a strain coefficient δ, it expresses the length difference in the two arms by: $\Delta L = \delta L_0$
When there is no gravitation wave, the two arms are with equal length, and due to halfwavelength difference effect, light from two paths cancels each other and the interference is dark; when the gravitation wave creates a difference in arm length, the interference intensity will change, and from such change δ can be measured.

The measured results are shown above. Vertical is the calculated δ from intensity change, horizontal is time. Two groups of data on the left and right are from two observatories located at different places (Hanford at Washington State and Livingston at Louisiana). The top row is the measurement; the middle is theoretical simulation and the bottom is noise level.

1. From data, what is the order of magnitude of δ ? How big is the length difference ΔL ? (2 points)
2. At such order of magnitude δ, what is the phase difference between the light that passes the two arms of the interferometer then reaches the detector? The center wavelength of laser in use is $\lambda = 1064\ \mathrm{nm}$ (3 points)
3. At such phase difference, what is the change of light intensity from zero? Needed parameter: The total intensity of the laser is $I_0 = 1000\ \mathrm{W}$ (5 points)
4. For such small intensity change, and the signal is really not too much above the noise, why the result is not due to some seismic-environmental changes (such as seismic activity, local temperature

fluctuation, or even just student drop a wrench on the floor $\cdots$) ? (4 points)

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(1) $10^{-21} \sim 10^{-22}$, $\Delta L_{\text{max}} = 4 \times 10^{-18}$ m

(2) $\Delta\varphi = \frac{2\pi}{\lambda}\Delta L \cdot 2N = 1.42 \times 10^{-8}$ rad

(3) $\tilde{A}_1 = A_0 \cos(\omega t), \tilde{A}_2 = A_0 \cos(\omega t + \pi + \Delta\varphi)$ and the total intensity is
$I_0 = \overline{\tilde{A}_1^2} + \overline{\tilde{A}_2^2} = \frac{A_0^2}{2} + \frac{A_0^2}{2} = A_0^2$

the final amplitude is
$\tilde{A} = \tilde{A}_1 + \tilde{A}_2 = A_0(\cos(\omega t) - \cos(\omega t + \Delta\varphi)) = 2A_0 \sin\left(\frac{\Delta\varphi}{2}\right) \sin\left(\omega t + \frac{\Delta\varphi}{2}\right)$

So the intensity is $I = \overline{\tilde{A}^2} = 4A_0^2 \sin^2(\frac{\Delta\varphi}{2}) \cdot \frac{1}{2} \approx 2A_0^2(\frac{\Delta\varphi}{2})^2 = \frac{1}{2}I_0(\Delta\varphi)^2 = 1.00 \times 10^{-13}$ W

(4) Two different experienments in different places show the same trend of the wave.

4 (10 points)

(10 points) For a give wedge-shaped thin film (the top and bottom surfaces are flat planes, intersect and has very small angle α) shown in the figure:

At the beginning it is air $(n = 1)$ between the top-bottom surface, the incoming light with vacuum wavelength 500 nm illuminates the top at normal angle.

1. Give a simple description or sketch of the interference fringe at the top surface of the thin film (no calculation needed). (2 points)
2. Now we start to fill some gas $(n > 1)$ into the space between the top-bottom surfaces, will the fringe moves towards or away from the intersection? (2 points)
3. If the spacing between the fringes (the distance between the adjacent maxima or minima) changes from original (air case) 1 mm to 0.8 mm (unknown gas), what is the index of refraction n of the unknown gas? (6 points)

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(1) uniform interference fringe

(2) towards

(3) $\Delta x \cdot 2\alpha = \lambda \Longrightarrow \Delta x = \frac{\lambda}{2\alpha}$ and after filling the gas $\lambda' = \frac{\lambda}{n}$ $\Delta x' = \frac{\lambda}{2\alpha n}$ So $n = \frac{\Delta x}{\Delta x'} = 1.25$

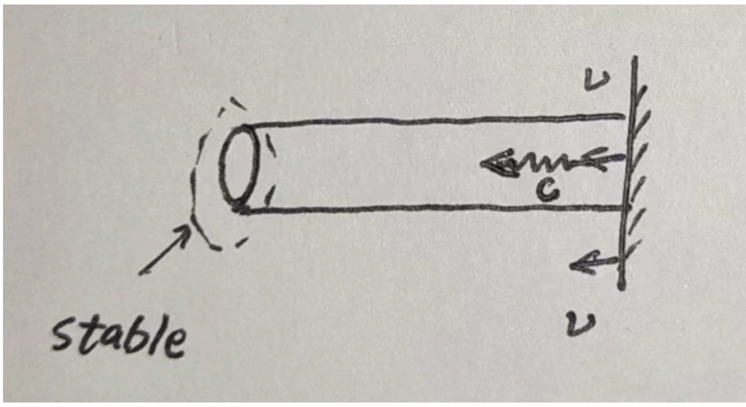
5 (12 points)

As shown in the figure:

A rod with proper length L_0 is moving with constant high velocity v along x direction in lab frame (frame S). The head of it H will hit a stiff wall and be stopped, and at meantime the stop signal (the curly curve in figure) will transmit from H through the rod with speed of light c (the position of wall is at $x=0$). When the stop signal reaches the tail of the rod T, only then it stops moving. There are also marks on the ground in lab with spacing $\Delta d=1$ unit length. When the tail of the rod stops, it will reach certain marks n. (fig.b)

1. (4 points) $\ln \mathrm{S}$ view, calculate the value of n, in terms of L_0, β
2. (4 points) In the rod view (call it frame S') in which the tail of rod T is stationary. please draw the figure in S', corresponding to the figure a) above. In the figure, please specify the length of the rod; the spacing of the mark $\Delta d'$, which parts are moving and their speed.
3. (4 points) In frame S', calculate the stop marks n', in terms of L_0, β.

$(1)\ \Delta t=\frac{\Delta L'}{v+c}=\frac{L_0\sqrt{1-\beta^2}}{v+c}$ and the value of n equals to $c\Delta t=L_0\sqrt{\frac{1-\beta}{1+\beta}}$

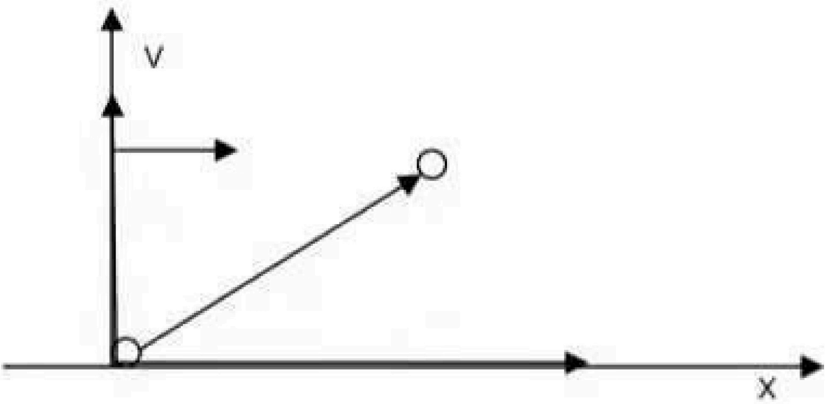
(2) 

its length is L, $\Delta d'=\sqrt{1-\beta^2}$

$(3)\ \Delta x=L_0-\frac{L_0}{c}\cdot v=(1-\beta)L_0$ and the value is $n'=\frac{\Delta x}{\Delta d'}=L_0\sqrt{\frac{1-\beta}{1+\beta}}$

6 (18 points)

In the ground frame, $A(x=y=0)$ fires a bullet at $t=0$ towards a stationary target ball at $(x,y)=(4ls, 3ls)$ (no friction or resistance). The bullet and target ball both have same rest mass m; the bullet has speed $u=0.5c$ (with respect to A), the bullet will sink into ball and travel with it. For another observer B whose speed $v=0.8c$ relative to A, and $x'=0, y'=0, t'=0$ overlap those of A's:



1. (5 points) For observer B, what are the space-time coordinates for the bullet hits the target ball?

2. (3 points) Suppose B does not know u (the bullet speed relative to A), but B can measure the velocity of the bullet in his frame using the results of 1), what is the velocity of bullet $\left(u_x', u_y'\right)$ in B's frame from results in (a)?
3. (2 points) For A, since he knows $u = 0.5c$, and he can deduce bullet's velocity measured by B to be $\left(u_x', u_y'\right)$, please do a calculation on $\left(u_x', u_y'\right)$ from A's view.
4. (4 points) After bullet hits ball and moves together with it. What will be the final momentum (express the momentum in its x, y components) and energy in A's frame
5. (4 points) Answer same questions in 4) but in B's frame.

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(1) the hit event happens at $\begin{cases} x = 4ls \\ y = 3ls \\ t = 10s \end{cases} \xRightarrow{v=0.8c} t' = \dfrac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \dfrac{34}{3}\,\mathrm{s}$

(2) Use the time that B measures $v'_{By} = \dfrac{\Delta y}{t'} = \dfrac{9}{34}c$ and the v_x difference between A and B is

$$v_{Ax} - v_{Bx} = \frac{\Delta x'}{t'} = \frac{x \cdot \sqrt{1-\beta^2}}{t'} = \frac{18}{85}c = 0.8c - v_{Bx}$$

So $v_{Bx} = \dfrac{10}{17}c$, $\vec{v}_B = (-\dfrac{10}{17}c, \dfrac{9}{34}c)$

(3) Use relativistic velocity transformation $v_x' = \dfrac{v_x - v}{1 - \dfrac{v_x v}{c^2}} = -\dfrac{10}{17}c, v_y' = \dfrac{v_y \cdot \sqrt{1-\beta^2}}{1 - \dfrac{v_x v}{c^2}} = \dfrac{9}{34}c$

(4) $|\vec{p}| = \dfrac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \dfrac{\sqrt{3}}{3}mc$. According to momentum conservation $\vec{p'} = \dfrac{mc}{\sqrt{3}}(\dfrac{4}{5}, \dfrac{3}{5})$

add the energy together $E = mc^2 + \dfrac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} = (1 + \dfrac{2\sqrt{3}}{3})mc^2$

(5) Apply relativistic energy momentum transformation

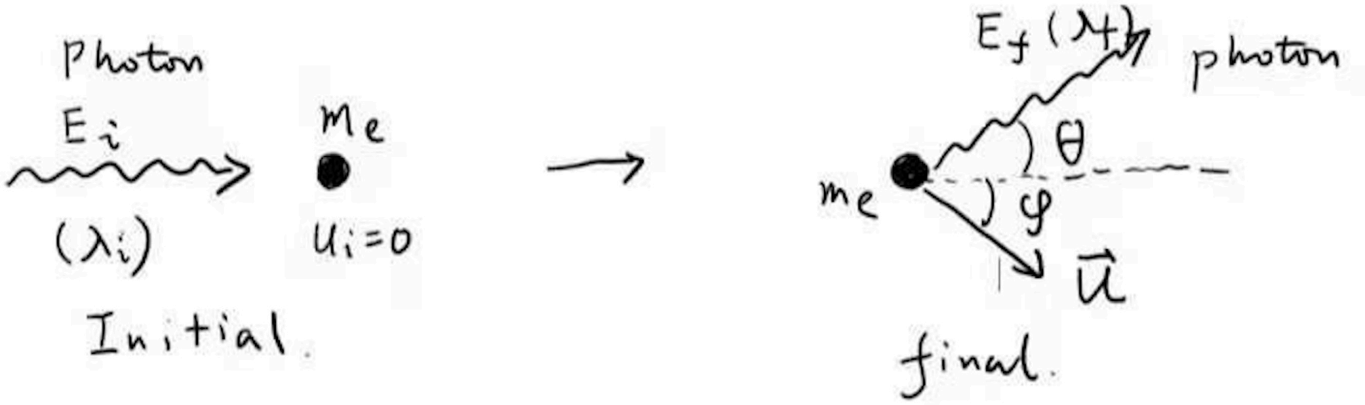
$$\begin{pmatrix} \dfrac{E'}{c} \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

So $E' = \gamma(E - \beta c p_x) = \dfrac{75 + 34\sqrt{3}}{45}mc^2, p_x' = \gamma(p_x - \beta\dfrac{E}{c}) = -(\dfrac{4}{3} + \dfrac{4\sqrt{3}}{9})mc$

$p_y' = p_y = \dfrac{\sqrt{3}}{5}mc, \vec{p'} = (-\dfrac{12 + 4\sqrt{3}}{9}mc, \dfrac{\sqrt{3}}{5}mc), E' = \dfrac{75 + 34\sqrt{3}}{45}mc^2$

7 (18 points)

Compton Scattering of photon by stationary electron:



The initial photon with energy E_i (and wavelength $\lambda_i, E=\dfrac{hc}{\lambda}$) along $\mathbf{x}$-direction interacts with a stational electron (rest mass $m_e, m_e=9.1\times10^{-31}\ \mathrm{kg}\sim 0.51\mathrm{MeV}$). The scattered photon will be along direction θ, with energy E_f and corresponding wavelength λ_f. Compton effect is how the wavelength change of the photon $\Delta\lambda=\lambda_f-\lambda_i$ relates to scattering angle θ

1. (6 points) Write out the conservation relation of the process:
   a) Using 4-vector symbol (state each 4 -vector corresponding to which)

b) Write out energy conservation; momentum (components) conservation respectively.

2. (6 points) Find the relation between the $\Delta\lambda$ and θ
3. (4 points) The relation would be like: $\Delta\lambda=\lambda_C(1-\cos\theta), \lambda_C$ is a natural constant called Compton wavelength (for electron), please express it in terms of other constant, and calculate its value.
4. (2 points) If the photon interacts with a proton whose rest mass $m_p=1800m_e$, what is the value of Compton wavelength in this case?

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(1) a) $\mathcal{P}=\begin{pmatrix}\dfrac{E}{c}\\ p_x\\ p_y\\ p_z\end{pmatrix}$ so $\mathcal{P}_0=\begin{pmatrix}\dfrac{h}{\lambda_i}+m_ec\\ \dfrac{h}{\lambda_i}\\ 0\\ 0\end{pmatrix}, \mathcal{P}_1=\begin{pmatrix}\dfrac{h}{\lambda_f}+\dfrac{E}{c}\\ \dfrac{h}{\lambda_f}\cos\theta+p\cos\varphi\\ \dfrac{h}{\lambda_f}\sin\theta-p\sin\varphi\\ 0\end{pmatrix}$

b) Energy conservation

$$\frac{h}{\lambda_i}+m_ec=\frac{h}{\lambda_f}+\frac{E}{c}$$

momentum (components) conservation

$$\begin{cases}\dfrac{h}{\lambda_i}=\dfrac{h}{\lambda_f}\cos\theta+p\cos\varphi\\ 0=\dfrac{h}{\lambda_f}\sin\theta-p\sin\varphi\end{cases}$$

(2) Eliminate φ,

$$(p\cos\varphi)^2+(p\sin\varphi)^2=(\frac{h}{\lambda_i}-\frac{h}{\lambda_f}\cos\theta)^2+(\frac{h}{\lambda_f}\sin\theta)^2=h^2(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_f^2}-\frac{2\cos\theta}{\lambda_i\lambda_f})=p^2$$

For the photon $p^2c^2+E_0^2=E^2=h^2c^2(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_f^2}-\frac{2\cos\theta}{\lambda_i\lambda_f})+E_0^2$ And we have

$$E=hc(\frac{1}{\lambda_i}-\frac{1}{\lambda_f})+E_0$$

$$\therefore h^2c^2(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_f^2}-\frac{2}{\lambda_i\lambda_f})+E_0^2+2hcE_0(\frac{1}{\lambda_i}-\frac{1}{\lambda_f})=h^2c^2(\frac{1}{\lambda_i^2}+\frac{1}{\lambda_f^2}-\frac{2\cos\theta}{\lambda_i\lambda_f})+E_0^2$$

$$\therefore 2hcE_0\frac{\Delta\lambda}{\lambda_i\lambda_f}=2h^2c^2\frac{1-\cos\theta}{\lambda_i\lambda_f}\implies\Delta\lambda=\frac{h}{m_ec}(1-\cos\theta)$$

$(3)\ \lambda_C=\frac{h}{m_ec}=2.43\times10^{-12}\ \mathrm{m}$

$(4)\ \lambda'=\frac{h}{m_pc}=\frac{h}{1800m_ec}=\frac{\lambda_C}{1800}=1.35\times10^{-15}\ \mathrm{m}$