GPI Final Exam (2021/6/18)

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$1_{(14\text{ points})}$

Central field problem:

In the classical model of tritium (Isotope of H, with 1 proton, 2 neutron), the nucleus charge is $+1e$, and electron (charge $-e$) moves around nuclei in a circular orbit with radius $= r_0$. Suddenly a nuclei reaction happened and the one neutron becomes proton inside the nuclei to bring the charge of it to $+2e$, the tritium becomes a Helium ion. The electron in the old orbit is facing a new situation:

(The force between nuclei charge and e is Coulomb attraction: $F = -\frac{c}{r^2}; C = \frac{Ze^2}{4\pi\varepsilon_0}; Z$: number of proton; ε_0 is a constant; use m_e for reduced mass)

1. (4 points) Express the total energy E_0, Coulomb potential energy U, kinetic energy K and angular momentum L_0 of e in tritium in the original circular orbit, in terms of C, r_0; i.e. $E_0, U, K, L_0 = ?$
2. (4 points) In the new situation (helium ion), what is the energy E and L for electron, in terms of E_0, L_0
3. (4 points) Find the new orbit parameter r_0', ε; closest and furthest $r_{\max}$; $r_{\min}$ in terms of r_0 and other constant.
4. (2 points) Find the long (2a) and short axis (2b) of the new elliptical orbit in terms of r_0

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(1) $U = \int_{r_0}^{+\infty} F(r)dr = -\frac{C}{r_0}, m\frac{v^2}{r_0} = \frac{C}{r_0^2}, K = \frac{1}{2}mv^2 = \frac{C}{2r_0}$

total energy is $E_0 = K - \frac{C}{r_0} = -\frac{C}{2r_0}$ and the angular momentum is

$L = mvr_0 = mr_0\sqrt{\frac{C}{mr_0}} = \sqrt{Cmr_0}$

(2) At the moment the reaction happened, the kinetic energy and radius don't change

$$E = K - \frac{C'}{r_0} = K - \frac{2C}{r_0} = -\frac{3}{2}\frac{C}{r_0} = 3E_0, L = mvr_0 = L_0$$

(3) According to elliptical orbit energy $E = -\frac{C'}{2a} = -\frac{2C}{2a} = -\frac{3C}{2r_0} \Longrightarrow a = \frac{2}{3}r_0$

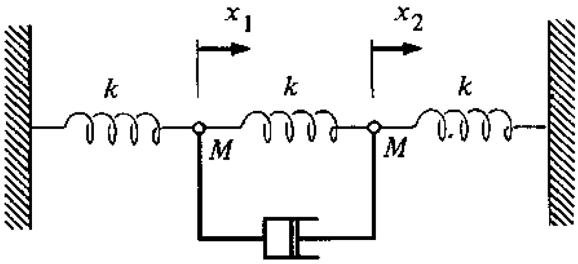
$r_{\max} = r_0, r_{\min} + r_{\max} = 2a = \frac{4}{3}r_0 \Longrightarrow r_{\min} = \frac{1}{3}r_0$

(4) $a + c = r_0, a - c = \frac{1}{3}r_0 \Longrightarrow a = \frac{2}{3}r_0, c = \frac{1}{3}r_0, b = \sqrt{a^2 - c^2} = \frac{r_0}{\sqrt{3}}$

$2a = \frac{4}{3}r_0 . 2b = \frac{2\sqrt{3}}{3}r_0$

$2_{(16\text{ points})}$

Two identical mass points, mass $= m$, are connected through identical springs (spring constant=k) to the fixed walls as shown in figure:



There is also a damper (whose mass can be neglected) between the two m. It provides a resisting force trying to slow the faster moving particle, its magnitude is bv, where b is a constant and $\mathrm{v} = |\dot{x_1} - \dot{x_2}|; x_1, x_2$ are displacement from equilibrium.

1. (4 points) Please write out the equations of motion for the points, i.e. $\ddot{x}_1, \ddot{x}_2$ has to obey what equations, using parameters of $\omega_0 = \sqrt{k/m}$ and $\gamma = \dfrac{b}{m}$
2. (4 points) Above equations may be coupled and hard to solve. We can simplify (decouple) them by replacing variables, such as $y_2 = x_1 - x_2$ etc. Let's call new variables y_1, y_2, please list out relation between y_1, y_2 and x_1, x_2; also list out equations on $\ddot{y}_1, \ddot{y}_2$, in terms of $\omega_0; \gamma$
3. (4 points) What are the general solutions for y_1, y_2 (under weak damping), which parameters will depend on initial conditions?
4. (4 points) If the initial conditions are, at $t = 0$, the two points are at equilibrium positions, and point1 (m represented by x_1) receive a quick blow and gain velocity of v_0. After a long enough time, the motion of the two points will become $x_1 = x_2 = \dfrac{v_0}{2\omega} f(t)$ What is the function form of $f(t)$? and what is the ω (in terms of given parameters)

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(1) Newton's second law $\begin{cases} k(x_2 - x_1) - kx_1 - b(\dot{x_1} - \dot{x_2}) = m\ddot{x}_1 \\ -k(x_2 - x_1) - kx_2 + b(\dot{x_1} - \dot{x_2}) = m\ddot{x}_2 \end{cases}$ replace $\sqrt{\dfrac{k}{m}} = \omega_0, \gamma = \dfrac{b}{m}$

$$\begin{cases} \ddot{x_1} = \omega_0^2(x_2 - 2x_1) + \gamma(\dot{x_2} - \dot{x_1}) \\ \ddot{x_2} = \omega_0^2(x_1 - 2x_2) + \gamma(\dot{x_1} - \dot{x_2}) \end{cases}$$

(2) Add and subtract two equations and set $\begin{cases} y_1 = x_1 + x_2 \\ y_2 = x_1 - x_2 \end{cases}$ therefore

$$\begin{cases} \ddot{y_1} = -\omega_0^2 y_1 \\ \ddot{y_2} = -3\omega_0^2 y_2 - 2\gamma\dot{y_2} \end{cases}$$

(3) Since the damping is weak, $\lambda_{1,2} = -\gamma \pm i\sqrt{3\omega_0^2 - \gamma^2}$ so

$$\begin{cases} y_1(t) = A_1 \cos(\omega_0 t + \varphi_1) \\ y_2(t) = A_2 e^{-\gamma t} \cos\left(\sqrt{3\omega_0^2 - \gamma^2} t + \varphi_2\right) \end{cases}$$

(4) After a long enough time, $e^{-\gamma t} \to 0, y_2 \to 0$ And for $y_1(t)$

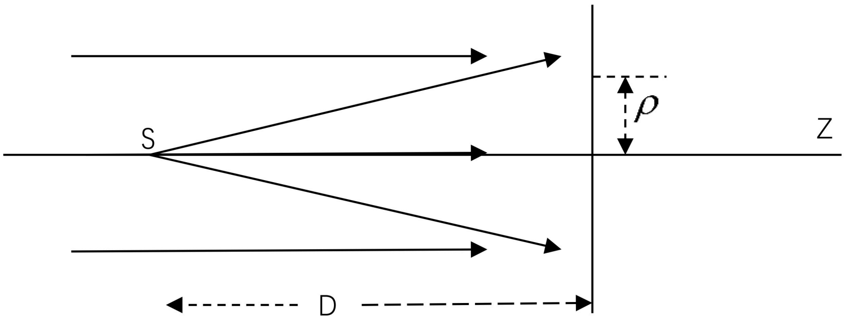
$y_1(0) = x_1(0) + x_2(0) = 0, y_1'(0) = x_1'(0) + x_2'(0) = v_0$ So $y_1(t) = \dfrac{v_0}{\omega_0} \sin(\omega_0 t) = x_1(t) + x_2(t)$

when $t \to +\infty$ $x_1(t) = x_2(t) = \frac{1}{2}y_1(t) = \frac{v_0}{2\omega_0}\sin(\omega_0 t)$

So $f(t) = \sin(\omega_0 t), \omega = \omega_0$

3 (12 points)

A plane wave of wavelength λ travels along the z direction, excite the atoms at point S. The emitting light from the atoms will be **in phase** with the plane wave and has same frequency, and its emission can be treated as spherical wave. At distance D from the atoms we put the observing screen. The amplitudes for the plane wave and spherical wave on the screen are both A (This is true under paraxial approximation, $D >> \rho$), ρ is the distance to the center of screen. Compute the intensity distribution on the screen $\mathrm{I(x,y)}$, in terms of ρ, D, λ and A



For the plane wave, $\tilde{A_1} = A\cos(\omega t)$, and under paraxial approximation, the phase difference is

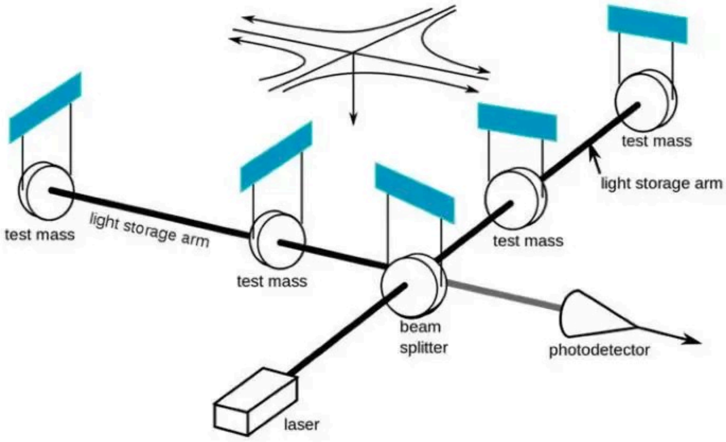
$$\Delta\varphi = \frac{2\pi}{\lambda}(\sqrt{D^2+x^2+y^2} - D) \approx \frac{2\pi D}{\lambda}((1+\frac{1}{2}\frac{x^2+y^2}{D^2}) - 1) = \frac{\pi(x^2+y^2)}{\lambda D} = \frac{\pi\rho^2}{\lambda D}$$

so $\tilde{A_2} = A\cos(\omega t + \Delta\varphi)$, $\tilde{A}_{tot} = \tilde{A_1} + \tilde{A_2} = 2A\cos\left(\frac{\Delta\varphi}{2}\right)\cos\left(\omega t + \frac{\Delta\varphi}{2}\right)$

$$I(\rho) = |\tilde{A_{tot}}^2| = 4A^2\cos^2(\frac{\Delta\varphi}{2}) \cdot \frac{1}{2} = A^2(1+\cos(\Delta\varphi)) = A^2(1+\cos\left(\frac{\pi\rho^2}{\lambda D}\right))$$

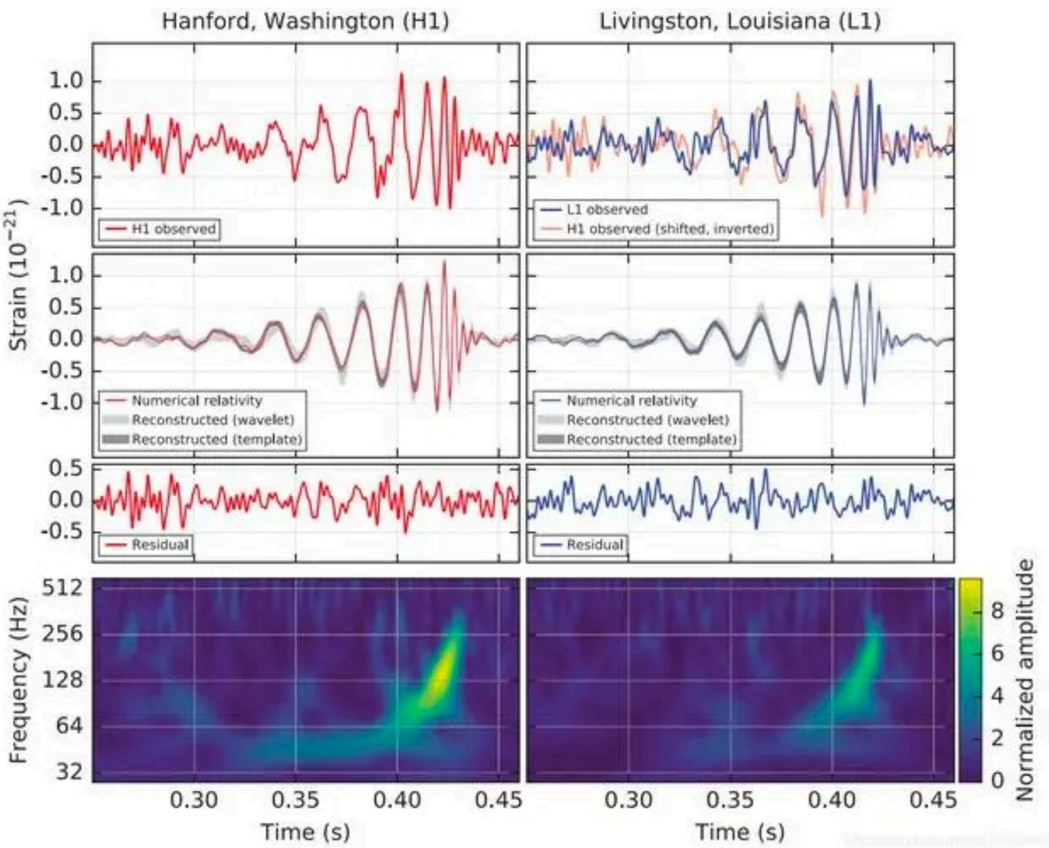
4 (14 points)

Interference:
LIGO (Laser Interferometer Gravitation-Wave Observatory) is using interference to observe the expansion-contraction if space due to gravitation wave.



As shown in the figure: It is basically a big Michelson Interferometer, two arm length L_0 is 4000 meter $(4\ \mathrm{km})$ each. Two extra mirrors inside will make light being reflected backwardforward $N = 300$ times, effectively prolong the length of each arm. The light from each arm finally meets at the beam splitter and creates interference pattern. When gravitation wave (indicated by the curve on top) passes, one arm length will be lengthened and the other be compressed. This effect is expressed by a strain coefficient δ, it expresses the length difference in the two arms by: $\Delta L = \delta L_0$

When there is no gravitation wave, the two arms are with equal length, and due to halfwavelength difference effect, light from two paths cancels each other and the interference is dark; when the gravitation wave creates a difference in arm length, the interference intensity will change, and from such change δ can be measured.



The measured results are shown above. Vertical is the calculated δ from intensity change, horizontal is time. Two groups of data on the left and right are from two observatories located at different places (Hanford at Washington State and Livingston at Louisiana). The top row is the measurement; the middle is theoretical simulation and the bottom is noise level.

1. From data, what is the order of magnitude of δ? How big is the length difference ΔL? (2 points)
2. At such order of magnitude δ, what is the phase difference between the light that passes the two arms of the interferometer then reaches the detector? The center wavelength of laser in use is $\lambda = 1064\ \mathrm{nm}$ (3 points)
3. At such phase difference, what is the change of light intensity from zero? Needed parameter: The total intensity of the laser is $I_0 = 1000\ \mathrm{W}$ (5 points)
4. For such small intensity change, and the signal is really not too much above the noise, why the result is not due to some seismic-environmental changes (such as seismic activity, local temperature fluctuation, or even just student drop a wrench on the floor $\cdots$) ? (4 points)

$(1)\ 10^{-21} \sim 10^{-22}$, $\Delta L_{\text{max}} = 4 \times 10^{-18}\ \mathrm{m}$

$(2)\ \Delta\varphi = \frac{2\pi}{\lambda}\Delta L \cdot 2N = 1.42 \times 10^{-8}\ \mathrm{rad}$

$(3)\ \tilde{A}_1 = A_0 \cos{(\omega t)}, \tilde{A}_2 = A_0 \cos{(\omega t + \pi + \Delta\varphi)}$ and the total intensity is $I_0 = \overline{\tilde{A}_1^2} + \overline{\tilde{A}_2^2} = \frac{A_0^2}{2} + \frac{A_0^2}{2} = A_0^2$

the final amplitude is
$\tilde{A} = \tilde{A}_1 + \tilde{A}_2 = A_0(\cos{(\omega t)} - \cos{(\omega t + \Delta\varphi)}) = 2A_0 \sin\left(\frac{\Delta\varphi}{2}\right)\sin\left(\omega t + \frac{\Delta\varphi}{2}\right)$

So the intensity is $I = \overline{\tilde{A}^2} = 4A_0^2 \sin^2(\frac{\Delta\varphi}{2}) \cdot \frac{1}{2} \approx 2A_0^2(\frac{\Delta\varphi}{2})^2 = \frac{1}{2}I_0(\Delta\varphi)^2 = 1.00 \times 10^{-13}\ \mathrm{W}$

(4) Two different experienments in different places show the same trend of the wave.

5 (12 points)

A spaceship flying at constant speed v towards $+x$ away from earth. On board it has a light transmitter and detector. **The following data before questions are all referring to ship's frame** (S ' frame): The ship emit out a light signal with frequency ω_0 towards the earth, part of it will be reflected by earth and comes back to the ship (as how radar works). 20 s after the emission, the detector on the ship picks up the reflected signal from earth, and the frequency of the reflected light ω is found to be half of ω_0, i.e., $\omega = \frac{1}{2}\omega_0$.

1. (2 points) At the time when the emitted light just reaches the earth and been reflected, what is the position of the earth in S' (ship's) frame?
2. (4 points) What is the velocity v of ship relative to the earth?
3. (6 points) At the time when the reflected light reaches the ship and been detected, how far away of the ship from the earth in earth's frame? (Write the answer in γ, β first in case your v value is wrong in 2) and you still be able getting most credit for this part;

$(1)\ \Delta x' = \frac{c\Delta t}{2} = 10\mathrm{ls}$

(2) According to the Doppler effect of light $(\sqrt{\frac{1-\beta}{1+\beta}})^2 = \frac{1}{2} \Longrightarrow \beta = \frac{1}{3}, v = \frac{1}{3}c$

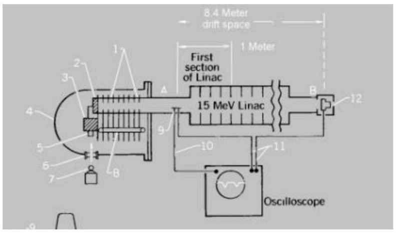
(3) According to time dilation effect, in the S frame, $\Delta t=\gamma\Delta t'$. Set their initial distance is l

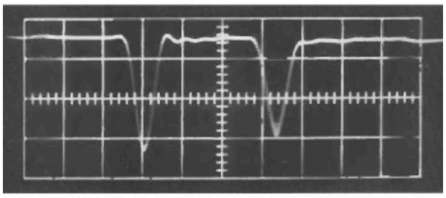
$$\Delta t=\frac{l}{c}+\frac{(l+\frac{l}{c}\cdot v)}{c-v}=\frac{l}{c}\frac{2}{1-\beta},l'=l+\frac{l}{c}\cdot v+\frac{(l+\frac{l}{c}\cdot v)}{c-v}\cdot v=l\cdot\frac{1+\beta}{1-\beta}$$

So $l'=\frac{1+\beta}{1-\beta}l=\frac{1+\beta}{1-\beta}\frac{c\Delta t(1-\beta)}{2}=\frac{1+\beta}{2}\gamma c\Delta t=\frac{c\Delta t}{2}\sqrt{\frac{1+\beta}{1-\beta}}=10\sqrt{2}\mathrm{ls}$

6 (16 points)

In the famous Bertozzi's Ultimate speed experiment, stationary electrons were passing an accelerating electric field with potential of negative millions of volts (MeV in energy). The accelerating electrons then travel in vacuum tube and their terminal speed can be measured with time-of-flight method. The general setup and results are shown in the figure: (top 2 are setup and signal on oscilloscope; bottom 2 are results)





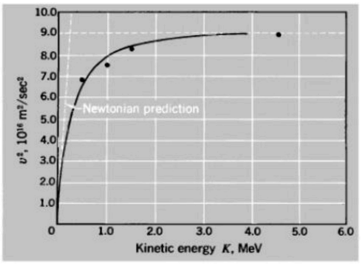


TABLE 1-1

Kinetic energy K*, MeV*	*Flight time* t*,* $\times 10^{-8}$ *sec*	*Electron speed* v*,* $\times 10^{8}$ *m/sec*	v^2*,* $\times 10^{16}$ *m²/sec²*
0.5	3.23	2.60	6.8
1.0	3.08	2.73	7.5
1.5	2.92	2.88	8.3
4.5	2.84	2.96	8.8
15	2.80	3.00	9.0

1. (4 points) Please write out the formular for relation between v^2 and K (kinetics energy), of which the curve in the figure above should follow.
2. (4 points) Compute the v^2 at $\mathrm{K}=1.0\mathrm{MeV}$ and $\mathrm{K}=4.5\mathrm{MeV}$ from the formula, taking the rest mass of electron to be $m_e=0.5\mathrm{MeV}c^2=9\times10^{16}(\mathrm{~m/s})^2$.
3. (2 points) There may be a difference between the calculation in 2) with measured results in table 1.1. Suppose the measured results are accurate, what do you think is the major reason that cause the difference.
4. (5 points) Now we use electrons with $\mathrm{K}=4.5\mathrm{MeV}; m_e=0.5\mathrm{MeV}$ to bombard a stationary proton (rest mass $m_p=940\mathrm{MeV}$). If they form a " H " atom, what will be the rest mass (in MeV) of "H" atom?
5. (1 point) How many energy needs to be released in order to form a stable H atom?

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(1) $K=\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}-mc^2\Longrightarrow v^2=c^2(1-(\frac{mc^2}{K+mc^2})^2)$

(2) $v^2(K=1\ \mathrm{MeV})=8\times10^{16}\ (\mathrm{m/s})^2, v^2(K=4.5\ \mathrm{MeV})=8.91\times10^{16}\ (\mathrm{m/s})^2$

(3) The velocity of radiation decreases as electrons move.

(4) $E = K + m_e c^2 + m_p c^2 = 945 \text{ MeV}, p = \frac{1}{c}\sqrt{(K+m_e c^2)^2 - (m_e c^2)^2} = 4.975 \text{ MeV/c}$

For the new atom $E^2 = (pc)^2 + E_0^2 \implies E_0 = \sqrt{E^2 - (pc)^2} = 944.987 \text{ MeV}$

(5) $\Delta E = 4.4987 \text{ MeV}$

7 (16 points)

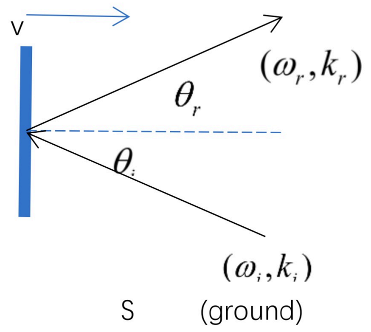
In this problem we use transformation relation to work out reflection of light by a moving mirror.

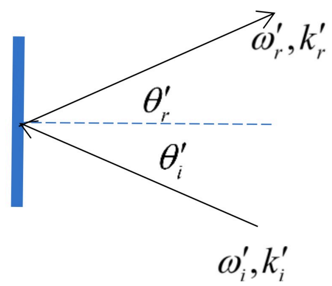
We know that energy-momentum is a 4-vector, i.e. $(E/c, \vec{p})$ transform obeying Lorentz between inertial frames. Using the relation $E = \hbar\omega, \vec{p} = \hbar\vec{k}, \omega$ is the angular frequency and $\vec{k}$ wave vector of light wave, $\hbar$ is a universal constant; then the $\underset{\rightarrow}{K} \equiv (\omega/c, \vec{k})$ is also a 4-vector.

1. (3 points, 1 for each)
   a) For light, what is the length of this $\underset{\rightarrow}{K}$, i.e. $|\underset{\rightarrow}{K}|^2$ equals what?

b) This means $\frac{\omega}{|k|}$ equals what? $|k| = \sqrt{k_x^2 + k_y^2 + k_z^2}$ is the length of wave vector. c) Is this $\frac{\omega}{|k|}$ value invariant between frames?

Now take a look for light reflection by a moving mirror, the mirror (S') is moving with v relative to the ground S (the source of light is stationary on the ground), the ground view and mirror view are shown in the figure below:





In the mirror frame (S' view), it just sees a light with frequency ω' coming in with angle θ'_i, and it will reflect the incoming light. The reflection law in the mirror frame is just the familiar one: $\omega'_r = \omega'_i, \theta'_r = \theta'_i$. But for the ground observer, the reflected light may have different frequency and out-going angle compared with incoming one:

2. (7 points) For a given incoming light in S frame, i.e. ω_i, k_i (k_i is the magnitud of incoming wave vector) and angle θ_i is known, please find out the ω'_i and $\cos\theta'_i$, the frequency and incoming angle in the mirror (S') frame, express them in terms of $c, \beta, \gamma, \omega_i, \theta_i$
3. (6 points) Using the reflection law in mirror frame, find out the frequency and angle of reflected light in ground frame: ω_r and $\cos\theta_r$, express them in terms of $c, \beta, \gamma, \omega_i, \theta_i$

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(1) $|\underset{\rightarrow}{K}| = 0, \frac{\omega}{|k|} = c$, $\frac{\omega}{|k|}$ is invariant

$(2)\ \underset{\rightarrow}{K} = (\frac{\omega}{c}, k_x, k_y, k_z)$ satisfies Lorentz transformation.

$$\begin{pmatrix} \frac{\omega'}{c} \\ k'_x \\ k'_y \\ k'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix}$$

$$\omega' = \gamma(\omega - \beta c k_x) = \gamma(\omega - \beta c(-\frac{\omega}{c} \cdot \cos\theta_i)) = \gamma\omega(1 + \beta\cos\theta_i)$$

$$k'_x = \gamma(k_x - \beta \cdot \frac{\omega}{c}) = \gamma(-\frac{\omega}{c} \cdot \cos\theta_i - \beta \cdot \frac{\omega}{c}) = -\frac{\omega}{c}\gamma(\cos\theta_i + \beta)$$

$$\cos\theta'_i = \frac{k'_x}{|k'|} = \frac{k'_x}{|\frac{\omega'}{c}|} = \frac{\cos\theta_i + \beta}{1 + \beta\cos\theta_i} \Longrightarrow \theta'_i = \arccos\left(\frac{\cos\theta_i + \beta}{1 + \beta\cos\theta_i}\right)$$

$(3)\ \omega'_r = \omega'_i$, then transform to S frame

$$\begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega'}{c} \\ k'_x \\ k'_y \\ k'_z \end{pmatrix}$$

$$\omega_r = \gamma(\omega' + \beta c k'_x) = \gamma(\gamma\omega(1 + \beta\cos\theta_i) + \beta c(\frac{\omega}{c}\gamma(\beta + \cos\theta_i))) = \frac{1 + \beta^2 + 2\beta\cos\theta_i}{1 - \beta^2}\omega$$

$$k_{rx} = \gamma(k'_x + \beta\frac{\omega'}{c}) = \gamma(\gamma\frac{\omega}{c}(\cos\theta_i + \beta) + \beta\frac{\gamma\omega(1 + \beta\cos\theta_i)}{c}) = \frac{\beta^2\cos\theta_i + 2\beta + \cos\theta_i}{1 - \beta^2}\,\frac{\omega}{c}$$

$$\theta_r = \arccos\left(\frac{k_{rx}}{\frac{\omega_r}{c}}\right) = \arccos\left(\frac{\beta^2\cos\theta_i + 2\beta + \cos\theta_i}{1 + 2\beta\cos\theta_i + \beta^2}\right)$$